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Robust Observer Design for Hybrid Dynamical Systems with Linear Maps and Approximately Known Jump Times

Pauline Bernard and Ricardo G. Sanfelice

Abstract—This paper proposes a general framework for the state estimation of plants given by hybrid systems with linear flow and jump maps when their jump events can be detected (almost) instantaneously. A candidate observer consists of a copy of the plant’s hybrid dynamics with continuous-time and/or discrete-time correction terms multiplied by two constant gains, and with jumps triggered by those of the plant. Assuming that the time between successive jumps is known to belong to a given closed set allows us to formulate an augmented system with a timer which keeps track of the time elapsed between successive jumps and facilitates the analysis. Then, since the jumps of the plant and of the observer are synchronized, the error system has time-invariant linear flow and jump maps, and a Lyapunov analysis leads to sufficient conditions for the design of the observer gains for uniform asymptotic stability in three different settings: continuous and discrete updates, only discrete updates, and only continuous updates. These conditions take the form of matrix inequalities, which we solve in examples including cases where the time between successive jumps is unbounded or tends to zero (Zeno behavior), and cases where either both the continuous and discrete dynamics, only one of them, or neither of them are detectable. Finally, we study the robustness of this approach when the jumps of the observer are delayed with respect to those of the plant. We show that if the plant’s trajectories are bounded and the time between successive jumps is lower-bounded away from zero, the estimation error is bounded, and arbitrarily small outside the delay intervals between the plant’s and the observer’s jumps.

Index Terms—observer, hybrid systems, impulsive systems

I. INTRODUCTION

Unlike for linear time-invariant systems, the problem of designing observers for hybrid systems is unsolved, even when the flow/jump maps are linear. The lack of general tools for such systems is mainly due to the fact that hybrid systems combine both continuous-time and discrete-time dynamics, which in general leads to solutions from nearby initial conditions that have different jump times. Such a mismatch of time domains makes the formulation of observability/detectability and, in turn, observer design very challenging. In particular, the notions of *observability* (reconstruction of the initial condition) and *determinability* (reconstruction of the final condition) are no longer equivalent when the jump map is not invertible.

When the plant’s jump times are unknown, the error system approach does not apply since the jumps of the observer and of the plant are not necessarily synchronized. Very few observer results exist apart from particular settings as in [1], thanks to the fact that the jump map g is such that $g \circ g$ is the identity map, and in [2], thanks to a change of coordinates transforming the jump map into the identity map. Another path explored in the particular setting of switched systems is to estimate the plant’s switching signal: its observability has been studied in [3], [4] and some designs exist based on *mode location observers* to detect and identify mode switches (see [5], [6], [7], [8], [9]).

On the other hand, impulsive systems consist of a class of continuous-time dynamical systems with state jumps that occur at pre-specified times, which are usually assumed to be separated by nonzero periods of flow (in particular, to avoid Zeno behavior). The impulsive systems literature is rich and includes a variety of models of impulsive systems. In particular, models of impulsive systems in which the state includes a logic variable that selects the right-hand side of the differential equation governing the dynamics in between impulses are referred to as switched impulsive systems, or also as switched systems with known jump times. In that setting, the difficulties due to a possible mismatch of the trajectories’ domains disappear since the jump times are assumed known. Observability and determinability thus reduce to comparing inputs with same time domain and have been extensively studied with geometric/algebraic conditions given in [10], [11], [12], [13], [14]. As for observer design, results first appeared assuming each mode is observable [15], and then more generally in [16] (resp. in [14]), for impulsive systems (resp. switched impulsive systems) that are observable (resp. determinable) for any impulse time sequence containing more than a known finite number N of jumps. In other words, the information available during a single flow interval is not sufficient to reconstruct the full state, but it becomes sufficient after N jumps. In [16], the observer consists of an impulsive system synchronized with the plant, with innovation terms at jumps only. Those innovations are linear in the error, with a time-varying gain that is related to a weighted observability Gramian over the past N jumps. In [14], the authors develop an observation procedure based on the continuous-time estimation of the observable states of each of the past N modes: after some time, putting together the information given by each mode enables to reconstruct the whole state.

Another important class of hybrid systems for which ob-

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server results exist is when the system itself has continuous-time dynamics, but the measurements are available intermittently at specific time instances. For such a class of systems with sporadic events, observers have been designed under specific assumptions on the time elapsed between successive events or, in the case of periodic events, the sampling period. From [17], convergence of an impulsive observer with linear innovation terms triggered by the measurement events is guaranteed when the sampling period is sufficiently small. This design is extended in [18] to any constant sampling period provided that appropriate matrix inequalities are satisfied, and further extended in [19], [20] to the case of sporadic measurements, i.e., when the time elapsed between sampling events varies in a known interval.

Following [21], we consider general hybrid systems as in [22] with linear flow and jump maps, and possibly an input whose value is considered known at all times. Under the assumption that the plant's jumps are detected instantaneously, a candidate observer is a hybrid system that jumps at the same time as the plant does, and is fed with the known input and linear correction terms in either the flow map, the jump map, or both. Our results only assume that the time between successive jumps belongs to a known (possibly unbounded) closed set, thus allowing for *Zeno*, eventually discrete, and eventually continuous trajectories in a unified framework, unlike the previously cited [14]-[20]. This enables us to formulate (Section II) an augmented hybrid system with a timer that keeps track of the time elapsed between successive jumps. We derive sufficient conditions for the design of the gains defining the observer's correction terms, so as to ensure uniform global asymptotic stability in three different settings: both continuous-time and discrete-time updates (Section III), only discrete-time updates (Section IV), and only continuous-time updates (Section V). Finally, in Section VI, we show the robustness of the observer when the jumps of the plant and of the observer are not perfectly synchronized. More precisely, the observer with sufficiently small delay in the jumps provides an arbitrarily precise estimate, except during the delay intervals between the plant's jumps and the observer's where peaking occurs.

Preliminary results were given in [21], but restricted to the case where at least either the continuous dynamics or the discrete dynamics are detectable. We complete them by more general sufficient conditions in the case where neither the continuous nor the discrete dynamics of the plant are detectable (but the plant as a whole is), and by a robustness analysis with respect to delays in the triggering of the observer's jumps.

Notation. \mathbb{R} (resp. \mathbb{N}) denotes the set of real numbers (resp. integers), $\mathbb{R}_{\geq 0} = [0, +\infty)$, $\mathbb{R}_{> 0} = (0, +\infty)$, and $\mathbb{N}_{> 0} = \mathbb{N} \setminus \{0\}$. For a square matrix P , $\text{eig}(P)$ denotes the set of its eigenvalues, and $\lambda_m(P)$ (resp. $\lambda_M(P)$) stands for its smallest (resp. largest) eigenvalue. The symbol \star in a matrix denotes the symmetric blocks. \mathbb{B} stands for a closed Euclidian ball of appropriate dimension, of radius 1 and centered at 0. We consider hybrid dynamical systems of the form (see [22])

$$\dot{x} = F(x), \quad x \in C \quad x^+ = G(x), \quad x \in D$$

where F (resp. G) is the flow (resp. jump) map, and C (resp. D) is the flow (resp. jump) set. Solutions to such systems are defined

on so-called *hybrid time-domains*. A subset E of $\mathbb{R}_{\geq 0} \times \mathbb{N}$ is a *compact hybrid time-domain* if $E = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \leq t_1 \leq \dots \leq t_J$, and it is a hybrid time domain if for any $(T, J) \in E$, $E \cap [0, T] \times \{0, \dots, J\}$ is a compact hybrid time domain. For a solution $(t, j) \mapsto x(t, j)$ (see [22, Definition 2.6]), we denote $\text{dom } x$ its domain, $\text{dom}_t x$ (resp. $\text{dom}_j x$) its projection on the time (resp. jump) component, $T(x) := \sup \text{dom}_t x$, $J(x) := \sup \text{dom}_j x$, and for a positive integer j , $t_j(x)$ the only time defined by $(t_j, j) \in \text{dom } x$ and $(t_j, j-1) \in \text{dom } x$. When no ambiguity is possible, we will omit x and write T, J, t_j . We say that x is *complete*, resp. *t-complete*, resp. *j-complete*, if $\text{dom } x$, resp. $\text{dom}_t x$, resp. $\text{dom}_j x$ is unbounded; x is *eventually continuous* (resp. *eventually discrete*) if $J < +\infty$ and $T > t_J$ (resp. $T < +\infty$ and $\text{dom } x \cap (T \times \mathbb{N})$ contains at least two points); x is *Zeno* if it is complete and $T < +\infty$.

II. HYBRID OBSERVER

A. Problem statement

We consider a hybrid plant of the form

$$\mathcal{H}_u \begin{cases} \dot{x} = A_c x + B_c u_c, & y_c = H_c x, & x \in C \\ x^+ = A_d x + B_d u_d, & y_d = H_d x, & x \in D \end{cases} \quad (1)$$

with state $x \in \mathbb{R}^n$, input u being the collection of a continuous-time input $u_c : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^{m_c}$ and a discrete-time input $u_d : \mathbb{N} \rightarrow \mathbb{R}^{m_d}$, and output $y = (y_c, y_d) \in \mathbb{R}^{p_c} \times \mathbb{R}^{p_d}$. For this class of hybrid systems, we are interested in estimating the state of the plant (1) when its solutions are initialized in a given subset \mathcal{X}_0 of \mathbb{R}^n . We denote $\mathcal{S}_{\mathcal{H}_u}(\mathcal{X}_0)$ the set of maximal solutions of \mathcal{H}_u with initial condition in \mathcal{X}_0 and input u .

Definition II.1. For a closed subset \mathcal{I} of $\mathbb{R}_{\geq 0}$, an input u , and a subset \mathcal{X}_0 of \mathbb{R}^n , we will say that $\mathcal{C}_{\mathcal{H}_u}(\mathcal{X}_0, \mathcal{I})$ holds if for any solution $x \in \mathcal{S}_{\mathcal{H}_u}(\mathcal{X}_0)$,

- $0 \leq t - t_j(x) \leq \sup \mathcal{I} \quad \forall (t, j) \in \text{dom } x$
- $t_{j+1}(x) - t_j(x) \in \mathcal{I}$ holds
 - $\forall j \in \mathbb{N}_{> 0}$ if $J(x) = +\infty$
 - $\forall j \in \{1, \dots, J(x) - 1\}$ if $J(x) < +\infty$

In other words, the set \mathcal{I} describes the possible lengths of the flow intervals between successive jumps. The role of the first item in Definition II.1 is to bound the length of the intervals of flow which are not covered by the second item, namely possibly the first one, which is $[0, t_1(x)]$, and the last one, which is $\text{dom}_t x \cap [t_{J(x)}(x), +\infty)$ (when defined). We are now ready to state the observer problem of interest. Our goal is now to design an observer assuming we know: 1) the value of the input u at all times, 2) when the plant's jumps occur, 3) the outputs y_c during flows and/or y_d at jumps, 4) some information about the flow time between successive jumps, namely a closed subset \mathcal{I} of $\mathbb{R}_{\geq 0}$ such that $\mathcal{C}_{\mathcal{H}_u}(\mathcal{X}_0, \mathcal{I})$ holds.

The existence of a set \mathcal{I} such that $\mathcal{C}_{\mathcal{H}_u}(\mathcal{X}_0, \mathcal{I})$ holds is not a problem because it always holds for $\mathcal{I} = \mathbb{R}_{\geq 0}$. But as we will see later, it is advantageous to select \mathcal{I} as tight as possible, namely it is convenient to have as much information about the duration of flow between successive jumps as possible.

Example II.2. Consider a bouncing ball with gravity coefficient $\mathbf{g} > 0$ and restitution coefficient $\lambda > 0$, modelled as system (1) with

$$\begin{aligned} A_c &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, & A_d &= \begin{pmatrix} -1 & 0 \\ 0 & -\lambda \end{pmatrix} \\ C &= \mathbb{R}_{\geq 0} \times \mathbb{R}, & D &= \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0, x_2 \leq 0\} \\ B_c &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}, & B_d &= 0, & u_c &\equiv -\mathbf{g} \end{aligned} \quad (2)$$

If $\lambda < 1$, any maximal solution x is such that $T < +\infty$ and $J = +\infty$. The time between two successive jumps $t_{j+1} - t_j$ tends to zero when j tends to $+\infty$, and its upper bound increases with $|x(0, 0)|$. So we can take $\mathcal{I} = [0, \tau_M]$ with $\tau_M \geq 0$, if \mathcal{X}_0 is bounded. Otherwise, $\mathcal{I} = \mathbb{R}_{\geq 0}$.

If now $\lambda > 1$, any maximal solution x initialized in $\mathbb{R}^2 \setminus \{(0, 0)\}$ is such that $T = +\infty$, $J = +\infty$. The time between two successive jumps $t_{j+1} - t_j$ tends to $+\infty$ when j tends to $+\infty$, and its lower bound decreases with $|x(0, 0)|$. Therefore, if there exists $\delta > 0$ such that \mathcal{X}_0 is a subset of $\mathbb{R}^n \setminus \delta\mathbb{B}$, one can take $\mathcal{I} = [\tau_m, +\infty)$ with $\tau_m > 0$. Otherwise, $\mathcal{I} = \mathbb{R}_{\geq 0}$.

Finally if $\lambda = 1$, any maximal solution x initialized in $\mathbb{R}^2 \setminus \{(0, 0)\}$, is such that $T = +\infty$, $J = +\infty$, and the time between two successive jumps $t_{j+1} - t_j$ is constant for all $j \geq 1$, and increases with $|x(0, 0)|$. The maximal solution initialized at $(0, 0)$ is discrete, i.e., $T = 0$ and $J = +\infty$. We can take \mathcal{I} of the form:

- $\mathcal{I} = [0, \tau_M]$ with $\tau_M \geq 0$, if \mathcal{X}_0 is bounded.
- $\mathcal{I} = [\tau_m, +\infty)$ with $\tau_m > 0$, if there exists $\delta > 0$ such that \mathcal{X}_0 is a subset of $\mathbb{R}^2 \setminus \delta\mathbb{B}$.
- $\mathcal{I} = [\tau_m, \tau_M]$ with $\tau_m > 0$ and $\tau_M > 0$, if there exists $\delta > 0$ such that \mathcal{X}_0 is a bounded subset of $\mathbb{R}^2 \setminus \delta\mathbb{B}$.
- otherwise, $\mathcal{I} = \mathbb{R}_{\geq 0}$. \triangle

Remark II.3. The results obtained in this paper under assumptions on the set \mathcal{I} , namely on the time elapsed between successive jumps, could also be obtained in the framework of average dwell-time as done in [23]. Using Lyapunov tools from [24], [23], assumptions of the type “ $\min \mathcal{I} > 0$ ” (i.e. dwell-time) could typically be replaced by average dwell-time (ADT), and assumptions of the type “ \mathcal{I} bounded” by a reverse average dwell-time (rADT). However, unlike for switched systems, it is generally difficult to be aware of an ADT or rADT for a particular hybrid system based on the sole knowledge of A_c, A_d, C, D, u_c, u_d . Rather, we can infer some information on the time between successive jumps, for instance by computation of the distance between the values of x after the jumps and the jump set D . That is why we base our designs on some knowledge on the set \mathcal{I} only. It is also important to stress that unlike [24], [23], we do not restrict our study to dwell-time trajectories, since we allow $0 \in \mathcal{I}$, i.e. Zeno and discrete trajectories.

B. Proposed hybrid observer

Since the plant’s jump times and the value of the input are assumed to be known, we propose to use an observer

$$\hat{\mathcal{H}}_{u,y} \begin{cases} \dot{\hat{x}} = A_c \hat{x} + B_c u_c + L_c (y_c - H_c \hat{x}) & \text{when } \mathcal{H}_u \text{ flows} \\ \hat{x}^+ = A_d \hat{x} + B_d u_d + L_d (y_d - H_d \hat{x}) & \text{when } \mathcal{H}_u \text{ jumps} \end{cases} \quad (3)$$

that is synchronized with the plant. We then reformulate our problem as the following:

Problem 1. Given a closed subset \mathcal{I} of $\mathbb{R}_{\geq 0}$, design gains $L_c \in \mathbb{R}^{n \times p_c}$ and $L_d \in \mathbb{R}^{n \times p_d}$ such that there exist $\gamma > 0$ and $\theta > 0$ such that for any set of initial conditions \mathcal{X}_0 and any input u such that $\mathcal{C}_{\mathcal{H}_u}(\mathcal{X}_0, \mathcal{I})$ holds, every maximal solution x of \mathcal{H}_u initialized in \mathcal{X}_0 and every maximal solution \hat{x} of $\hat{\mathcal{H}}_{u,y}$ are complete and verify

$$|x(t, j) - \hat{x}(t, j)| \leq \gamma |x(0, 0) - \hat{x}(0, 0)| e^{-\theta(t+j)} \quad (4)$$

for all $(t, j) \in \text{dom } x (= \text{dom } \hat{x})$.

To use the hybrid framework introduced in [22] and express the fact that $\mathcal{C}_{\mathcal{H}_u}(\mathcal{X}_0, \mathcal{I})$ is satisfied, we consider the augmented error system

$$\mathcal{H}_\varepsilon^\tau \begin{cases} \dot{\varepsilon} = (A_c - L_c H_c) \varepsilon \\ \dot{\tau} = 1 \\ \varepsilon^+ = (A_d - L_d H_d) \varepsilon \\ \tau^+ = 0 \end{cases} \begin{cases} (\varepsilon, \tau) \in C^\tau \\ (\varepsilon, \tau) \in D^\tau \end{cases} \quad (5)$$

with, denoting $\tau_M = \sup \mathcal{I}$,

$$C^\tau = \mathbb{R}^n \times ([0, \tau_M] \cap \mathbb{R}_{\geq 0}), \quad D^\tau = \mathbb{R}^n \times \mathcal{I}. \quad (6)$$

The model $\mathcal{H}_\varepsilon^\tau$ is such that the timer τ has to reach \mathcal{I} before a jump can occur and is forced to jump when reaching τ_M (if finite). This enables to relate its behavior with that of \mathcal{H}_u and $\hat{\mathcal{H}}_{u,y}$ as follows.

Lemma II.4. Consider a subset \mathcal{X}_0 of \mathbb{R}^n , a closed subset \mathcal{I} of $\mathbb{R}_{\geq 0}$ and denote $\tau_M = \sup \mathcal{I} \leq +\infty$. For any input u such that $\mathcal{C}_{\mathcal{H}_u}(\mathcal{X}_0, \mathcal{I})$ holds, for any maximal solution x of \mathcal{H}_u initialized in \mathcal{X}_0 , and for any maximal solution \hat{x} of $\hat{\mathcal{H}}_{u,y}$, we have $\text{dom } x = \text{dom } \hat{x} =: \mathcal{D}$, and there exists a function τ defined on \mathcal{D} such that $(\hat{x} - x, \tau)$ is a solution to $\mathcal{H}_\varepsilon^\tau$. Furthermore, L_c and L_d solve Problem 1 for \mathcal{I} if there exist $\gamma > 0$ and $\theta > 0$ such that any trajectory of $\mathcal{H}_\varepsilon^\tau$ verifies

$$|\varepsilon(t, j)| \leq \gamma |\varepsilon(0, 0)| e^{-\theta(t+j)} \quad \forall (t, j) \in \text{dom } \varepsilon. \quad (7)$$

Proof. According to (3), x and \hat{x} have the same time domain. Besides, since $\mathcal{C}_{\mathcal{H}_u}(\mathcal{X}_0, \mathcal{I})$ holds, the function τ defined on \mathcal{D} by $\tau(t, j) := t - t_j(x)$ is such that $(\hat{x} - x, \tau)$ is solution to $\mathcal{H}_\varepsilon^\tau$. The rest follows directly by definition of UGES. \square

We conclude that in order to solve Problem 1, we just have to choose L_c and L_d such that (7) holds.

Example II.5. The proposed framework also applies to the case where the plant itself has continuous-time dynamics

$$\dot{x} = A x + B u, \quad y = H x$$

but the output y is only available at discrete times t_j , which do not necessarily occur periodically. In that case, one can use an observer given in (3) with $L_c = 0$, $A_c = A$, $B_c = B$, $A_d = I$, $B_d = 0$, $u_c = u$, $u_d = 0$, $H_d = H$, and L_d to be designed. Then, the subset \mathcal{I} models the knowledge we have about the time elapsed between two successive sampling events. For instance, \mathcal{I} is a singleton in the case of periodic sampling, and \mathcal{I} is a compact interval of $\mathbb{R}_{> 0}$ in the case of aperiodic

sampling considered in [19], [20]. In fact, depending on the class of events of interest, the set \mathcal{I} could be discrete, contain a finite or infinite number of elements, be a collection of intervals of $\mathbb{R}_{\geq 0}$, etc. \triangle

III. HYBRID OBSERVER WITH INNOVATION TERMS ON FLOWS AND JUMPS

The following theorem gives our first sufficient condition to ensure global exponential stability of the observer.

Theorem III.1. *Consider a closed subset \mathcal{I} of $\mathbb{R}_{\geq 0}$. Assume there exist scalars a_c and a_d , matrices $L_c \in \mathbb{R}^{n \times p_c}$ and $L_d \in \mathbb{R}^{n \times p_d}$, and a positive definite symmetric matrix $P \in \mathbb{R}^{n \times n}$ such that:*

$$(A_c - L_c H_c)^\top P + P(A_c - L_c H_c) \leq a_c P \quad (8a)$$

$$(A_d - L_d H_d)^\top P(A_d - L_d H_d) \leq e^{a_d} P \quad (8b)$$

$$a_c \tau + a_d < 0 \quad \forall \tau \in \mathcal{I} \quad (8c)$$

Then, L_c and L_d solve Problem 1 for \mathcal{I} .

Proof. First, there always exists¹ a positive scalar a such that

$$a_c \tau + a_d \leq -a(\tau + 1) \quad \forall \tau \in \mathcal{I}. \quad (9)$$

Let us introduce the continuously differentiable function V defined on $\mathbb{R}^n \times \mathbb{R}$ by $V(\varepsilon, \tau) = \varepsilon^\top P \varepsilon$. To show (7), we apply [22, Proposition 3.29]. For that, we prove that there exists M such that for any solution $\phi = (\varepsilon, \tau)$ to $\mathcal{H}_\varepsilon^\tau$,

$$a_c t + a_d j \leq M - a(t + j) \quad (t, j) \in \text{dom } \phi. \quad (10)$$

We have for all $(t, j) \in \text{dom } \phi$, $a_c t + a_d j = a_c t_1 + \sum_{i=1}^{j-1} (a_c(t_{i+1} - t_i) + a_d) a_c(t - t_j) + a_d$. Fix $j \in \text{dom}_j \phi$. By definition of \hat{C}^τ and \hat{D}^τ in (6)

$$t_{i+1} - t_i \in \mathcal{I} \quad \forall i \in \{1, \dots, j-1\}, \quad t - t_j \in [0, \tau_M] \cap \mathbb{R}_{\geq 0}$$

so that according to (9),

$$\begin{aligned} a_c t_1 + \sum_{i=1}^{j-1} (a_c(t_{i+1} - t_i) + a_d) &\leq a_c t_1 - a \sum_{i=1}^{j-1} (t_{i+1} - t_i + 1) \\ &\leq -a(t_j + j - 1) + (a_c + a)t_1 \end{aligned}$$

and for $\tau_m = \min \mathcal{I}$,

$$\begin{aligned} a_c(t - t_j) + a_d &= a_c(t - t_j - \tau_m) + a_c \tau_m + a_d \\ &\leq a_c(t - t_j - \tau_m) - a(\tau_m + 1) \\ &\leq (a_c + a)(t - t_j - \tau_m) - a(t - t_j + 1). \end{aligned}$$

This yields

$$a_c t + a_d j \leq -a(t + j) + (a_c + a)t_1 + (a_c + a)(t - t_j - \tau_m).$$

A bound for $(a_c + a)(t - t_j - \tau_m)$ is obtained by noticing that: 1) if $0 \leq t - t_j \leq \tau_m$, we get $(a_c + a)(t - t_j - \tau_m) \leq |a_c + a| \tau_m$ 2) else if $t - t_j \geq \tau_m$, either $\tau_M < +\infty$ and $(a_c + a)(t - t_j - \tau_m) \leq |a_c + a|(\tau_M + \tau_m)$ or $\tau_M = +\infty$, then necessarily from (9),

¹Either \mathcal{I} is a compact set and this is satisfied for a sufficiently small positive number a thanks to (8c); or \mathcal{I} is unbounded and then necessarily a_c is negative, and by taking $a < |a_c|$ sufficiently small, the inequality holds.

$a_d \leq -a$, and $(a_c + a)(t - t_j - \tau_m) \leq 0$. Similarly, either $\tau_M < +\infty$ and $(a_c + a)t_1 \leq |a_c + a| \tau_M$, or $\tau_M = +\infty$, then necessarily from (9), $a_d \leq -a$, and $(a_c + a)t_1 \leq 0$. We conclude that there exists M such that (10) holds. According to [22, Proposition 3.29], we have

$$V(\phi(t, j)) \leq e^M e^{-a(t+j)} V(\phi(0, 0)) \quad \forall (t, j) \in \text{dom } \phi$$

and because for all ε in \mathbb{R}^n ,

$$\lambda_m(P)|\varepsilon|^2 \leq V(\varepsilon, \tau) \leq \lambda_M(P)|\varepsilon|^2$$

we finally get UGES with $\theta = \frac{a}{2}$ and $\gamma = e^{\frac{M}{2}} \sqrt{\frac{\lambda_M(P)}{\lambda_m(P)}}$. Applying Lemma II.4 gives the result. \square

Remark III.2. *From conditions (8a)-(8c), we recover the fact that if $0 \in \mathcal{I}$, namely there are Zeno or eventually discrete solutions, then a_d must be negative, i.e., the innovation term in the discrete dynamics of the observer must make the error contractive at jumps; similarly if $\sup \mathcal{I} = +\infty$, then a_c must be negative, i.e., the innovation term in the continuous dynamics must make the error contractive during flow.*

It is important to note that the set of initial conditions \mathcal{X}_0 is used to choose the set \mathcal{I} such that $\mathcal{C}_{\mathcal{H}_u}(\mathcal{X}_0, \mathcal{I})$ holds. Therefore, it possibly impacts conditions (8a)-(8c), but only through (8c). The interesting property of conditions (8a)-(8c) is that they are affine (and thus convex) in τ , which means that it is sufficient to check them at the boundaries of the set \mathcal{I} only. This fact is formalized in the next result.

Corollary III.3. *Consider a closed subset \mathcal{I} of $\mathbb{R}_{\geq 0}$. Let $\tau_m = \min \mathcal{I}$ and $\tau_M = \sup \mathcal{I}$. Assume there exist scalars a_c and a_d , matrices $L_c \in \mathbb{R}^{n \times p_c}$ and $L_d \in \mathbb{R}^{n \times p_d}$, and a positive definite symmetric matrix $P \in \mathbb{R}^{n \times n}$ such that (8a)-(8b) are satisfied. (8a)-(8c) hold if any of the following conditions is verified*

- 1) $a_c \leq 0$ and $a_d < 0$,
- 2) $a_c < 0$ and $a_c \tau_m + a_d < 0$,
- 3) $a_c > 0$, $\tau_M < +\infty$, and $a_c \tau_M + a_d < 0$.

[21, Example 3.3] shows how Conditions (8a)-(8c) can be solved analytically with $a_c < 0$ and $a_d < 0$ for a bouncing ball modelled by (2) with a restitution coefficient $\lambda < 1$, and x_1 measured at all (hybrid) times, thus giving a global observer. This system is not of dwell time-type: it exhibits Zeno. But this is not problematic to the design of the proposed observers as such a case can be seamlessly handled by a proper choice of the set \mathcal{I} such that $0 \in \mathcal{I} = [0, \tau_M]$.

Remark III.4. *In the favorable case where both the continuous and the discrete dynamics are detectable (such as [21, Example 3.3]), it is not sufficient to choose independently $A_c - L_c H_c$ Hurwitz and $A_d - L_d H_d$ Schur. Indeed, their descent directions could be incompatible: jumps could destroy what has been achieved during flow, or vice versa. Take for instance $\mathcal{I} = \{\tau^*\}$ with $\tau^* \geq 0$. A necessary condition for convergence of the observer is that the error sampled at each jump converges to zero: this implies that the origin of the discrete system*

$$\varepsilon_{k+1} = (A_d - L_d H_d) \exp\left((A_c - L_c H_c)\tau_k\right) \varepsilon_k$$

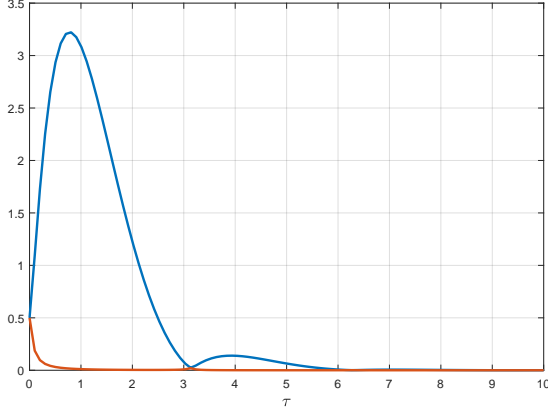


Fig. 1. Absolute value of the eigenvalues of $(A_d - L_d H_d) \exp((A_c - L_c H_c)\tau)$ for $A_c = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $A_d = 5A_c$, $H_c = H_d = (1, 0)$, with L_c (resp. L_d) chosen such that $A_c - L_c H_c$ (resp. $A_d - L_d H_d$) is Hurwitz (resp. Schur) with eigenvalues $(-1 + i, -1 - i)$ (resp. $(0.5, -0.5)$).

has to be asymptotically stable for $k \mapsto \tau_k \in \mathcal{I}$. For a given $\tau^* \geq 0$, this is not verified for every choice of $A_d - L_d H_d$ Schur and $A_c - L_c H_c$ Hurwitz, as illustrated on Figure 1: $(A_d - L_d H_d) \exp((A_c - L_c H_c)\tau)$ is Schur only if $\tau^* \notin [0.1, 2]$. To avoid this phenomenon, (8a) and (8b) should be solved with the same P , and $a_c \leq 0$ and $a_d < 0$. By the Schur complement, this is equivalent to solving the LMIs

$$\begin{aligned} A_c^\top P + P A_c - (\tilde{L}_c H_c + H_c^\top \tilde{L}_c^\top) &< 0 \\ \begin{pmatrix} P & (P A_d - \tilde{L}_d H_d)^\top \\ \star & P \end{pmatrix} &> 0 \end{aligned} \quad (11)$$

in $(P, \tilde{L}_c, \tilde{L}_d)$ and take $L_c = P^{-1} \tilde{L}_c$ and $L_d = P^{-1} \tilde{L}_d$. Note that the problem of finding common quadratic Lyapunov functions for several continuous-time or several discrete-time systems has been studied in the context of switched systems and quadratic stabilization (see e.g. [25]). But we are not aware of any result concerning the existence of a common quadratic function for a continuous-time system and a discrete-time system.

A drawback of Theorem III.1 is that (8c) requires at least a_c or a_d to be negative: either the continuous or the discrete dynamics have to be detectable. But take for instance the hybrid system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = 0 \\ \dot{x}_3 = 0 \end{cases}, \quad \begin{cases} x_1^+ = x_1 \\ x_2^+ = x_2 \\ x_3^+ = x_1 \end{cases} \quad (12)$$

with some arbitrary, but nonempty flow and jump sets. Suppose $H_c = H_d = (1 \ 0 \ 0)$. Neither the continuous nor the discrete dynamics is detectable, so Theorem III.1 cannot apply. Nevertheless, this hybrid system as a whole is detectable if there is at least one jump and one interval of flow. Indeed, if $y(t, j) = x_1(t, j) = 0$ for all (t, j) in the domain, the continuous part gives $x_2(t, j) = 0$ as soon as $[t, t + \tau) \times \{j\} \in \text{dom } x$ for some $\tau > 0$, and the discrete part gives $x_3(t, j) = 0$ for all j such that $(t, j - 1)$ and (t, j) are in the domain. Given

this detectability property, we would like to be able to write an observer for this system. We will see in the next section that it is possible.

IV. HYBRID OBSERVER WITH INNOVATION TERMS ON JUMPS ONLY

We now consider the case where only y_d is known, namely the measurements from the plant are only available at jump times. Therefore, we build an observer with $L_c = 0$. Due to the lack of measurements during flow, and without the assumption that A_c is already Hurwitz, eventually continuous solutions are not allowed. Hence, \mathcal{I} has to be bounded. From Theorem III.1 and Corollary III.3, we deduce that it is sufficient to find scalars $a_c \in \mathbb{R}$ and $a_d < 0$, a matrix $L_d \in \mathbb{R}^{n \times p_d}$, and a positive definite matrix $P \in \mathbb{R}^{n \times n}$ such that

$$A_c^\top P + P A_c \leq a_c P \quad (13a)$$

$$(A_d - L_d H_d)^\top P (A_d - L_d H_d) \leq e^{a_d} P \quad (13b)$$

$$a_c \tau_M + a_d < 0 \quad (13c)$$

with $\tau_M = \max \mathcal{I}$.

In [21, Example 4.2], we showed how conditions (13a)-(13c) can be solved analytically for \mathcal{I} of the form $[0, \tau_M]$ for a bouncing ball exhibiting Zeno trajectories modelled by (2) with $\lambda < 1$, and with the measurement x_1 only available at jumps, thus giving a global observer for any compact set of the plant's initial conditions.

However, a limitation of conditions (13a)-(13c) is that they are nonlinear in the unknowns and require the discrete part of the system to be detectable. But as we saw for system (12), it may happen that neither the continuous nor the discrete parts are detectable, and yet, the whole system is detectable. In fact, when $L_c = 0$, the study can be reduced to an equivalent discrete-time system

$$\varepsilon_{k+1} = (A_d - L_d H_d) \exp(A_c \tau_k) \varepsilon_k \quad (14)$$

with input $k \mapsto \tau_k \in \mathcal{I}$, which captures the evolution of $\mathcal{H}_\varepsilon^\tau$ after each jump.

Lemma IV.1. Consider a compact subset \mathcal{I} of $\mathbb{R}_{\geq 0}$. Assume there exist $L_d \in \mathbb{R}^{n \times p_d}$, $\gamma > 0$ and $\rho > 0$ with $|\rho| < 1$, such that for any sequence $(\tau_k)_{k \in \mathbb{N}}$ of elements of \mathcal{I} , any solution ε to (14) with input $k \mapsto \tau_k$ verifies $|\varepsilon_k| \leq \gamma \rho^k |\varepsilon_0|$ for all $k \in \mathbb{N}$. Then, $L_c = 0$ and L_d solve Problem 1 for \mathcal{I} .

Proof. See Appendix A. \square

When L_c is chosen to be zero, it is thus sufficient to choose the gain L_d such that the discrete-time system (14) with input in \mathcal{I} is exponentially stable. A first sufficient condition, that is weaker than (13a)-(13c), is given in the following theorem.

Theorem IV.2. Consider a compact subset \mathcal{I} of $\mathbb{R}_{\geq 0}$. Assume there exist a positive definite matrix $P \in \mathbb{R}^{n \times n}$ and a gain vector $L_d \in \mathbb{R}^{n \times p_d}$ such that

$$(\exp(A_c \tau))^\top (A_d - L_d H_d)^\top P (A_d - L_d H_d) \exp(A_c \tau) < P \quad \forall \tau \in \mathcal{I}. \quad (15)$$

Then, $L_c = 0$ and L_d verifying (15) solve Problem 1 for \mathcal{I} .

Proof. Follows directly from Lemma IV.1 by considering the Lyapunov function $V(\varepsilon) = \varepsilon^\top P \varepsilon$. \square

Remark IV.3. Condition (15) is exactly the one obtained in [19, Equation (12)] with $A_d = I$ and \mathcal{I} a compact interval of $\mathbb{R}_{>0}$, in the context of a continuous-time system with sporadic measurements, as in Example II.5.

The existence of the matrix P verifying (15) for a given τ is equivalent to $(A_d - L_d H_d) \exp(A_c \tau)$ being Schur for some gain L_d , which in turn is equivalent to the detectability of the discrete-time system

$$z_{k+1} = A_d \exp(A_c \tau) z_k, \quad y_k = H_d z_k \quad (16)$$

(since $\exp(A_c \tau)$ is invertible). This implies that system (1) with $u \equiv 0$ and sampled at a constant sampling period $\tau \in \mathcal{I}$ must be detectable. Thus, having (15) for any $\tau \in \mathcal{I}$ requires detectability of (16) for any $\tau \in \mathcal{I}$. It is not sufficient, however, because (15) must be verified with the same L_d and P for all $\tau \in \mathcal{I}$. So (15) requires in fact the detectability of the LPV discrete-time system

$$z_{k+1} = A_d \exp(A_c \tau_k) z_k, \quad y_k = H_d z_k, \quad (17)$$

with input τ_k in the compact set \mathcal{I} . Actually, (15) is stronger because it requires a quadratic Lyapunov function with a matrix P , that is independent from the sequence $k \mapsto \tau_k$. This property is sometimes called ‘‘quadratic detectability’’ (see [26], [27]).

Remark IV.4. By the Schur complement, finding P and L_d satisfying (15) is equivalent to finding P and \tilde{L}_d satisfying the LMIs

$$\begin{pmatrix} P & \exp(A_c \tau)^\top (P A_d - \tilde{L}_d H_d)^\top \\ \star & P \end{pmatrix} > 0 \quad \forall \tau \in \mathcal{I} \quad (18)$$

with $\tilde{L}_d = P L_d$. In the case where \mathcal{I} has infinitely many elements, an infinite number of LMIs must be solved which is not desirable. However, it is shown in [19] that it is always possible to compute numerically a polytopic decomposition of $\exp(A_c \tau)$, namely a finite number of matrices $\{M_1, \dots, M_\nu\}$ such that $\exp(A_c \tau)$ is in the convex hull of those matrices whenever $\tau \in \mathcal{I}$. Since (18) is convex in $\exp(A_c \tau)$, it is then sufficient to solve the finite number of LMIs

$$\begin{pmatrix} P & M_i^\top (P A_d - \tilde{L}_d H_d)^\top \\ \star & P \end{pmatrix} > 0 \quad \forall i \in \{1, \dots, \nu\} \quad (19)$$

with common P and \tilde{L}_d . In particular, if A_c is nilpotent of order N , we have $\exp(A_c \tau) = \sum_{k=0}^{N-1} \frac{\tau^k}{k!} A_c^k$ so that for all τ in a compact subset \mathcal{I} of $\mathbb{R}_{>0}$, $\exp(A_c \tau)$ is in the convex hull of the $\nu = 2^{N-1}$ matrices $\left\{ I + \sum_{k=1}^{N-1} \frac{\tau_k^k}{k!} A_c^k \right\}$ with $\tau_k \in \{\tau_m, \tau_M\}$ for all k , with $\tau_m = \min \mathcal{I}$ and $\tau_M = \max \mathcal{I}$.

Example IV.5. Consider the system (12) where $A_c = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $A_d = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, and $H_d = H_c = (1 \ 0 \ 0)$. Neither the continuous pair (A_c, H_c) nor the discrete pair (A_d, H_d) is detectable, so conditions (8a)-(8c) cannot be solved. However, $A(\tau) := A_d \exp(A_c \tau) = \begin{pmatrix} 1 & \tau & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is such that the discrete

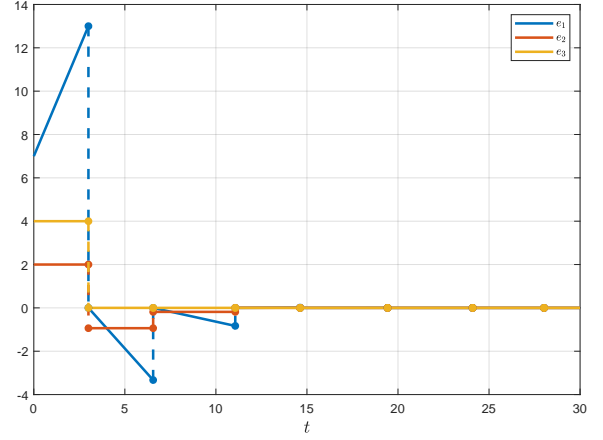


Fig. 2. Error between a trajectory of system (12) with flow interval length ranging in $\mathcal{I} = [2, 5]$ and a trajectory observer (3) with $L_c = 0$ and $L_d = (1, 0.2259, 1)^\top$.

pair $(A(\tau), H_d)$ is detectable for any nonzero τ . Therefore, if $\mathcal{I} = \{\tau\}$, there exists P and L_d such that (15) is satisfied. Otherwise, since A_c is nilpotent of order 2, according to Remark IV.4, for any \mathcal{I} compact subset of $\mathbb{R}_{>0}$, it is enough to solve the two LMIs given by

$$\begin{pmatrix} P & (I + \tau A_c)^\top (P A_d - \tilde{L}_d C)^\top \\ \star & P \end{pmatrix} > 0 \quad (20)$$

for $\tau = \tau_m = \min \mathcal{I} > 0$ and $\tau = \tau_M = \max \mathcal{I}$. If there exist solutions to (20), then by Theorem IV.2, we obtain an observer. For instance, when choosing $\tau_m = 2$ and $\tau_M = 5$ and solving the LMIs via Yalmip for P and \tilde{L}_d , we get $L_d = P \tilde{L}_d = (1, 0.2259, 1)^\top$. The error between a trajectory of system (12) (with jumps triggered randomly such that the flow intervals last between 2 and 5 units of time) and a trajectory of the observer is plotted on Figure 2. \triangle

Example IV.6. Consider again the bouncing ball (2) with restitution coefficient $\lambda < 1$, with measurements at jumps only, namely $H_c = 0$ and $H_d = (1, 0)$. As observed in Example II.2, for any compact set of initial conditions \mathcal{X}_0 , there exists $\tau_M > 0$ such that $\mathcal{C}_{\mathcal{H}_u}(\mathcal{X}_0, \mathcal{I})$ holds with $\mathcal{I} = [0, \tau_M]$. According to Theorem IV.2, it is enough to satisfy (15) instead of (13a)-(13c). Since A_c is nilpotent of order 2, we get again from Remark IV.4 that it is enough to solve (20) for $\tau = 0$ and $\tau = \tau_M$. With $\lambda = 0.8$ and $\tau_M = 5$, we obtain $L_d = (-1, -0.1487)$. The result of a simulation with initial condition $x_0 = (5, 0)$, $\hat{x}_0 = (10, 1)$ is shown on Figure 3. \triangle

The advantage of using a constant gain L_d is that it is sufficient to compute once the vertices M_i of the polytopic decomposition of $\exp(A_c \tau)$ for $\tau \in \mathcal{I}$ and solve offline the finite number of LMIs (19). However, as mentioned above, those LMIs might not be solvable since they require a stronger property than detectability of (17). In that case, we may allow L_d to be time-varying, by adapting L_d to τ , as done in the particular case of sampled-data observers in [20]. Indeed, the component τ of $\mathcal{H}_\varepsilon^\tau$ represents the time elapsed since the previous jump and can therefore be considered known to the

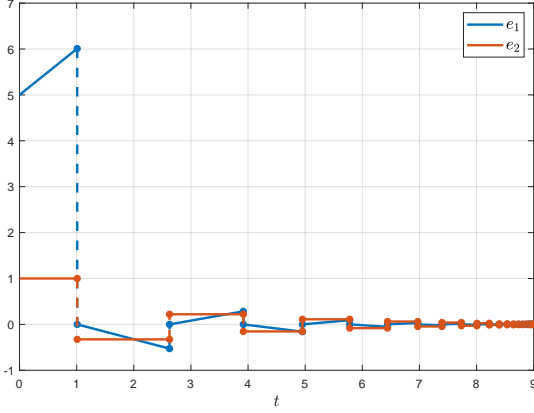


Fig. 3. Error between a Zeno trajectory of system (2) with $\lambda = 0.8$ and a trajectory of the synchronized observer (3) with $L_c = 0$ and $L_d = (-1, -0.1487)^\top$.

observer: at each jump, the correction gain L_d of (3) is adapted to the length of the previous interval of flow, namely L_d is replaced by $L_d(\tau)$. Then, reproducing the same reasoning, we obtain (14) with $L_d(\tau_k)$ instead of L_d . Since the matrix $A_d \exp(A_c \tau_k)$ admits a polytopic decomposition, we recover exactly the framework of [28], [29], [30]. In particular, it is shown that if $A_d \exp(A_c \tau) = \sum_{i=1}^{\nu} \xi_i(\tau) M_i$ with $\xi_i(\tau) \geq 0$, $\sum_{i=1}^{\nu} \xi_i(\tau) = 1$, then L_d and P can be chosen of the form $L_d(\tau) = \sum_{i=1}^{\nu} \xi_i(\tau) L_i$ and $P(\tau) = \sum_{i=1}^{\nu} \xi_i(\tau) P_i$, where $L_i := G_i^{-1} F_i$ and P_i are obtained by solving the ν^2 LMIs

$$\begin{pmatrix} P_i & M_i^\top (G_i A_d - F_i H_d)^\top \\ \star & G_i^\top + G_i - P_j \end{pmatrix} > 0 \quad \forall (i, j) \in \{1, \dots, \nu\}^2 \quad (21)$$

The advantage of this approach is that the LMIs (21) are relaxed compared to (19) (which is recovered by taking $P_i = P_j = G_i$) and are therefore feasible for a larger class of systems. However, although (21) can be solved offline as before, this method now requires to compute the full polytopic decomposition of $A_d \exp(A_c \tau)$ at each jump time. Alternatively, a simpler approach is to consider (17) as a LTV system and design $L_d(\tau)$ in (14) as the gain of a standard discrete-time recursive Kalman filter.

Finally, we finish this part by showing that the condition (15) given by Theorem III.1 is indeed weaker than (8a)-(8c) given by Theorem IV.2 when $L_c = 0$.

Lemma IV.7. *Assume there exist (a_c, a_d) in \mathbb{R}^2 , L_c in $\mathbb{R}^{n \times p_c}$, L_d in $\mathbb{R}^{n \times p_d}$, and a positive definite matrix P in $\mathbb{R}^{n \times n}$ such that (8a)-(8b) hold. Then, for any τ in $\mathbb{R}_{\geq 0}$,*

$$(\exp(M_c \tau))^\top M_d^\top P M_d \exp(M_c \tau) \leq e^{a_c \tau + a_d P} \quad (22)$$

with $M_c = A_c - L_c H_c$ and $M_d = A_d - L_d H_d$. If, in addition, $L_c = 0$ and (8c) holds, then (15) holds.

Proof. See Appendix A.1. \square

V. HYBRID OBSERVER WITH INNOVATION TERMS ON FLOWS ONLY

When \mathcal{I} is unbounded, it is not possible to implement an observer with discrete updates only: continuous updates are

necessary. And when the continuous dynamics are detectable, it may be sufficient to use only continuous updates (with $L_d = 0$). From Theorem III.1 and Corollary III.3, we know it is sufficient to find scalars $a_d \in \mathbb{R}$ and $a_c < 0$, a matrix L_c in $\mathbb{R}^{n \times p_c}$, and a positive definite matrix $P \in \mathbb{R}^{n \times n}$ such that

$$(A_c - L_c H_c)^\top P + P(A_c - L_c H_c) \leq a_c P \quad (23a)$$

$$A_d^\top P A_d \leq e^{a_d P} \quad (23b)$$

$$a_c \tau_m + a_d < 0 \quad (23c)$$

where $\tau_m = \min \mathcal{I}$. In [21, Example 5.2], we have seen how the conditions (23a)-(23c) can be analytically solved for \mathcal{I} of the form $[\tau_m, +\infty)$ for the bouncing ball (2) with a restitution coefficient $\lambda \geq 1$, thus giving a global observer for the plant initialized in $\mathbb{R}^2 \setminus \delta \mathbb{B}$ for $\delta > 0$.

When the pair (A_c, H_c) is observable, the Lyapunov equation (23a) can be solved for any negative number a_c . To solve (23c), it is tempting to take $|a_c|$ very large, but this has to be done with care since P depends on a_c and thus a_d in (23b) does too. The following theorem shows that this is always possible by choosing the eigenvalues of $A_c - L_c H_c$ sufficiently large, namely through a high-gain design. For that, we define $\mathcal{V} \in \mathbb{R}^{n \times n}$ a change of coordinates transforming (A_c, H_c) into a block-diagonal observable form, namely such that

$$\mathcal{V} A_c \mathcal{V}^{-1} = \mathbf{A} + \mathbf{D} \mathbf{H} \quad , \quad H_c \mathcal{V}^{-1} = \mathbf{H}$$

with $\mathbf{A} := \text{blkdiag}(A_1, \dots, A_{p_c})$, $\mathbf{D} := \text{blkdiag}(D_1, \dots, D_{p_c})$, $\mathbf{C} := \text{blkdiag}(C_1, \dots, C_{p_c})$,

$$A_i = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & & \\ \vdots & \ddots & \ddots & \\ 0 & & 1 & 0 & 0 \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix} \in \mathbb{R}^{p_{c,i} \times p_{c,i}}$$

$$C_i = (0 \quad \dots \quad 0 \quad 1) \in \mathbb{R}^{1 \times p_{c,i}} \quad ,$$

$D_i \in \mathbb{R}^{p_{c,i} \times 1}$, and $p_{c,i}$ integers such that $\sum_{i=1}^{p_c} p_{c,i} = p_c$.

Theorem V.1. *Assume $\min \mathcal{I} > 0$ and (A_c, H_c) observable. Given \mathcal{V} , \mathbf{A} , \mathbf{D} and \mathbf{H} defined above, consider vectors L_i such that $A_i - L_i C_i$ is Hurwitz, and for a positive scalar ℓ define $\mathcal{L}_i(\ell) := \text{diag}(\ell^{n_{o,i}-1}, \dots, \ell, 1)$. Then, there exists ℓ^* such that for any $\ell \geq \ell^*$, taking*

$$L_c = \mathcal{V}^{-1} (\mathbf{D} + \ell \mathcal{L}(\ell) \mathbf{L}) \quad , \quad L_d = 0 \quad , \quad (24)$$

with $\mathbf{L} := \text{blkdiag}(L_1, \dots, L_{p_c})$, $\mathcal{L} := \text{blkdiag}(\mathcal{L}_1, \dots, \mathcal{L}_{p_c})$, solves Problem 1.

Proof. By construction of \mathcal{V} , the dynamics of $\mathcal{H}_\varepsilon^\tau$ in the coordinates $(z, \tau) = (\mathcal{V} \varepsilon, \tau)$ read

$$\begin{cases} \dot{z} = (\mathbf{A} - \ell \mathcal{L}(\ell) \mathbf{L}) z \\ \dot{\tau} = 1 \end{cases} \quad \begin{cases} z^+ = \mathcal{V} \mathcal{J} \mathcal{V}^{-1} z \\ \tau^+ = 0 \end{cases}$$

with the same flow/jump maps. By construction of \mathbf{L} , there exist a positive definite matrix \mathbf{P} and $\lambda > 0$ such that $(\mathbf{A} - \mathbf{L} \mathbf{C})^\top \mathbf{P} + \mathbf{P}(\mathbf{A} - \mathbf{L} \mathbf{C}) \leq -\lambda \mathbf{P}$. Define $\mathcal{D}(\ell) = \text{blkdiag}(\mathcal{D}_1(\ell), \dots, \mathcal{D}_{p_c}(\ell))$ with $\mathcal{D}_i(\ell) =$

$\text{diag}(1, \ell, \dots, \ell^{n_{o,i}-1})$, and the differentiable function V defined on $\mathbb{R}^n \times [0, \tau_M]$ by

$$V(z, \tau) = e^{\ell \frac{\lambda}{2} \tau} z^\top \mathcal{D}(\ell)^\top \mathbf{P} \mathcal{D}(\ell) z.$$

There exist an integer p and positive scalars $\underline{\lambda}, \bar{\lambda}$ such that for all $\ell \geq 1$, and all $z \in \mathbb{R}^n$, $\underline{\lambda} z^\top z \leq V(z) \leq \ell^p \bar{\lambda} z^\top z$. Then, along the dynamics of $\mathcal{H}_\varepsilon^\tau$, since

$$\mathcal{D}(\ell)(\mathbf{A} - \ell \mathbf{L} \mathcal{C}(\ell) \mathbf{L} \mathbf{C}) = \ell(\mathbf{A} - \mathbf{L} \mathbf{C}) \mathcal{D}(\ell) \quad (25)$$

we get

$$\begin{aligned} \dot{V}(z, \tau) &= \ell \frac{\lambda}{2} V(z, \tau) \\ &+ \ell e^{\ell \frac{\lambda}{2} \tau} z^\top \mathcal{D}(\ell)^\top ((\mathbf{A} - \mathbf{L} \mathbf{C})^\top \mathbf{P} + \mathbf{P}(\mathbf{A} - \mathbf{L} \mathbf{C})) \mathcal{D}(\ell) z \\ &\leq -\ell \frac{\lambda}{2} V(z) \end{aligned}$$

and for all (ε, τ) in D^τ , with $g = ((A_d - L_d H_d) \varepsilon, 0)$,

$$\begin{aligned} V(g) - V(\varepsilon, \tau) &= z^\top (\mathcal{V} \mathcal{J} \mathcal{V}^{-1})^\top \mathcal{D}(\ell)^\top \mathbf{P} \mathcal{D}(\ell) \mathcal{V} \mathcal{J} \mathcal{V}^{-1} z \\ &- e^{\ell \frac{\lambda}{2} \tau} z^\top \mathcal{D}(\ell)^\top \mathbf{P} \mathcal{D}(\ell) z \\ &\leq (\ell^p \bar{\lambda} |\mathcal{V} \mathcal{J} \mathcal{V}^{-1}|^2 - e^{\ell \frac{\lambda}{2} \tau} \underline{\lambda}) z^\top z. \end{aligned}$$

Taking ℓ sufficiently large to have $\ell^p \bar{\lambda} |\mathcal{V} \mathcal{J} \mathcal{V}^{-1}|^2 < e^{\ell \frac{\lambda}{2} \tau} \underline{\lambda}$, we get UGES. The result follows from Lemma II.4. \square

From (25), we deduce that choosing L_c as in (24) gives

$$\mathcal{D}(\ell) \mathcal{V} (A_c - L_c H_c) \mathcal{V}^{-1} \mathcal{D}(\ell)^{-1} = \ell(\mathbf{A} - \mathbf{L} \mathbf{C})$$

so that $\text{eig}(A_c - L_c H_c) = \ell \text{eig}(\mathbf{A} - \mathbf{L} \mathbf{C})$, so that the eigenvalues of $A_c - L_c H_c$ can be chosen as $(\ell \lambda_1, \dots, \ell \lambda_n)$, with λ_i with negative real part and ℓ sufficiently large. This is exactly the condition obtained in [21, Example 5.2] to solve conditions (23a)-(23c). In other words, if the observer converges sufficiently fast during flow, then the perturbation happening through A_d at the jumps is compensated, exploiting the property that exponential growth wins over polynomial growth. A similar idea is also used in the context of continuous-time nonlinear high-gain observers to compensate for Lipschitz nonlinearities [31].

VI. ROBUSTNESS WITH RESPECT TO DELAYS IN JUMPS

We now study how the observer convergence is impacted if the observer jumps are delayed with respect to the plant's, thus leading to a mismatch between the observer jump times and those of the plant. For this, we suppose $\min \mathcal{I} > 0$, and we choose to study the particular case where the value of the innovation term, implemented in the observer at the delayed jump, is the one that would have been computed at the actual plant's jump time if there had been no delay. This covers the situations where the measurement and computation of the innovation $A_d \hat{x} + B_d u_d + L_d (y_d - H_d \hat{x})$ are instantaneous, but the implementation of the jump in the observer is delayed; or the measurement takes a known amount of time δ to arrive to the observer, and the update of \hat{x} is chosen as $A_d \hat{x}(t - \delta) + B_d u_d + L_d (y_d - H_d \hat{x}(t - \delta))$, thanks to a buffer

in \hat{x} or by backward integration of \hat{x} . Inspired from [32], for any $\Delta \in [0, \min \mathcal{I}]$, this situation can be modelled as

$$\hat{\mathcal{H}}_u(\Delta) \left\{ \begin{array}{l} \dot{x} = A_c x + B_c u_c \\ \dot{\hat{x}} = A_c \hat{x} + B_c u_c + L_c H_c (x - \hat{x}) \\ \dot{\tau} = 1 \\ \dot{\mu} = 0 \\ \dot{\tau}_\delta = -\min\{\tau_\delta + 1, 1\} \end{array} \right\} x \in \hat{C}(\Delta)$$

$$\left\{ \begin{array}{l} x^+ = A_d x + B_d u_d \\ \hat{x}^+ = \hat{x} \\ \tau^+ = 0 \\ \mu^+ = A_d \hat{x} + B_d u_d + L_d H_d (x - \hat{x}) \\ \tau_\delta^+ \in [0, \Delta] \end{array} \right\} x \in \hat{D}_{-1}(\Delta)$$

$$\left\{ \begin{array}{l} x^+ = x, \quad \mu^+ = \mu, \quad \tau^+ = \tau \\ \hat{x}^+ = \mu, \quad \tau_\delta^+ = -1, \end{array} \right\} x \in \hat{D}_0(\Delta) \quad (26)$$

with $x = (x, \hat{x}, \tau, \mu, \tau_\delta)$, $\hat{C}(\Delta) = \hat{C}^\tau \times \mathbb{R}^n \times ([0, \Delta] \cup \{-1\})$, $\hat{D}_{-1}(\Delta) = \hat{D}^\tau \times \mathbb{R}^n \times \{-1\}$, $\hat{D}_0(\Delta) = \mathbb{R}^n \times \mathbb{R}^n \times [0, \tau_M] \times \mathbb{R}^n \times \{0\}$, and $\hat{C}^\tau = \mathbb{R}^n \times \mathbb{R}^n \times ([0, \tau_M] \cap \mathbb{R}_{\geq 0})$, $\hat{D}^\tau = \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{I}$. where $\tau_M = \sup \mathcal{I}$. $\hat{\mathcal{H}}_u(\Delta)$ contains two new variables μ and τ_δ evolving in \mathbb{R}^n and $[0, \Delta] \cup \{-1\}$ respectively. The state τ_δ is a timer modelling the delay between the plant's jump and the observer's jump. The role of μ is to store the update to be implemented in the observer at the end of the delay interval, when it actually jumps. More precisely, when $\tau_\delta = -1$ and τ is not in \mathcal{I} , the plant and the observer flow and τ_δ remains equal to -1 . If τ reaches \mathcal{I} and the plant jumps, then the update that should have been instantaneously implemented in the observer is stored in the memory state μ , and τ_δ is set to a number in $[0, \Delta]$ thus starting a delay period: the plant and observer states then flow and the time τ_δ decreases, until it reaches 0. At this point, a delay interval of length smaller than or equal to Δ has elapsed, the observer jumps, and its state is updated with the content of μ .

Note that the plant's state is not allowed to jump again before the delay expressed by τ_δ has expired. That is why this model only works in the case where $\Delta < \min \mathcal{I}$, i.e., the maximal delay is smaller than the smallest possible time between successive jumps of the plant.

Assumption VI.1. We denote \mathcal{U} a set of inputs $u : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^{m_c} \times \mathbb{R}^{m_a}$ of interest. There exist compact subsets $\mathcal{X}, \mathcal{X}_0$ of \mathbb{R}^n , U_c of \mathbb{R}^{m_c} , and U_d of \mathbb{R}^{m_a} , such that any input u in \mathcal{U} , and any solution x to the plant (1) initialized in \mathcal{X}_0 with input u , verify $x(t, j) \in \mathcal{X}$ and $(u_c(t, j), u_d(t, j)) \in U_c \times U_d$ for all $(t, j) \in \text{dom } x$.

Theorem VI.2. Suppose Assumption VI.1 holds. Consider a compact subset \mathcal{I} of $\mathbb{R}_{\geq 0}$ with $\min \mathcal{I} > 0$, vectors $L_c \in \mathbb{R}^{n \times p_c}$ and $L_d \in \mathbb{R}^{n \times p_a}$ such that the set $\{0\} \times ([0, \tau_M] \cap \mathbb{R})$ is uniformly globally asymptotically stable (UGAS) for $\mathcal{H}_\varepsilon^\tau$ defined in (5). Then, there exist a \mathcal{KL} function β and a scalar σ such that for any $\eta > 0$ and any $\epsilon > 0$, there exists $\Delta^* > 0$ such that for any $\Delta \in [0, \Delta^*]$ and any solution $\phi_\delta = (x, \hat{x}, \tau, \mu, \tau_\delta)$ to $\hat{\mathcal{H}}_u(\Delta)$ verifying

$$\varrho(0, 0) := |\hat{x}(0, 0) - x(0, 0)| + |\mu(0, 0) - x(0, 0)| \leq \eta,$$

we have, denoting $t_j = t_j(\phi_\delta)$ for simplicity, $\text{dom } \phi_\delta = \mathcal{D}_{-1} \cup \mathcal{D}_0$ with $\mathcal{D}_k = \left(\bigcup_{j \in J_k} [t_j, t_{j+1}] \times \{j\} \right)$, $k \in \{0, -1\}$,

$$J_{-1} = \{j \in \mathbb{N} : \tau_\delta(t, j) = -1 \quad \forall t \in [t_j, t_{j+1}]\}$$

$$J_0 = \{j \in \mathbb{N} : \tau_\delta(t, j) \in [0, \Delta] \quad \forall t \in [t_j, t_{j+1}]\},$$

such that for all j in J_0 , $t_{j+1} - t_j \leq \Delta$, and we have for all (t, j) in \mathcal{D}_{-1}

$$|\hat{x}(t, j) - x(t, j)| \leq \beta(\varrho(0, 0), t + j) + \epsilon, \quad (27)$$

and for all (t, j) in \mathcal{D}_0 ,

$$\begin{aligned} |\hat{x}(t, j) - x(t, j)| \leq e^{\sigma(t-t_j)} & \left(\beta(|\varrho(0, 0)|, t_j + j) + \epsilon \right. \\ & \left. + \max_{x \in \mathcal{X}, u_d \in U_d} |(I - A_d)x + B_d u_d| \right) \end{aligned} \quad (28)$$

Proof. The proof relies on [32]. See Appendix C. \square

In other words, if the trajectories of the plant and the input are bounded, we achieve

- semiglobal practical stability if $A_d = I$ and $B_d = 0$, namely the jump map is the identity;
- semiglobal practical stability except on the delay intervals (of maximal length Δ) otherwise.

Note that the parameter σ describing the behavior of the error during the delay intervals is related to the eigenvalues of $A_c - L_c H_c$. Indeed, if the latter matrix is Hurwitz, the mismatch tends to be corrected by the flow during the delay interval, namely $\sigma < 0$.

In fact, this mismatch cannot be prevented if the jump map is not the identity. This well-known phenomenon, called *peaking*, was reported in the context of observation [1], but also more generally output-feedback and tracking [33]. This suggests that the Euclidian distance to evaluate the observer error is not appropriate and more general distances could be designed [34]. In particular here, if $B_d = 0$, semi-global practical stability could be obtained with the generalized distance

$$d(x, \hat{x}) = \min \left\{ |x - \hat{x}|, |A_d x - \hat{x}| \right\}.$$

Note that in the limit case where $0 \in \mathcal{I}$, namely the plant's jumps could happen arbitrarily fast, then a delay in the observer jumps (however small) could lead to several jumps of delay, namely, one could consider the distance

$$d(x, \hat{x}) = \inf_{k \in \mathbb{N}} |A_d^k x - \hat{x}|.$$

However, not much could be done if an infinite number of jumps happened during the delay interval.

Example VI.3. We come back to Example IV.5 and redo the simulation presented on Figure 2 with a delay in the triggering of the observer's jump. The results are presented in Figures 4-5 with delays of $\Delta = 0.05$ and $\Delta = 0.5$ respectively. Note that in this example, the assumption of boundedness of the plant's trajectory is not verified since x_1 and x_3 diverge. We can still see that the smaller the delay, the smaller the error

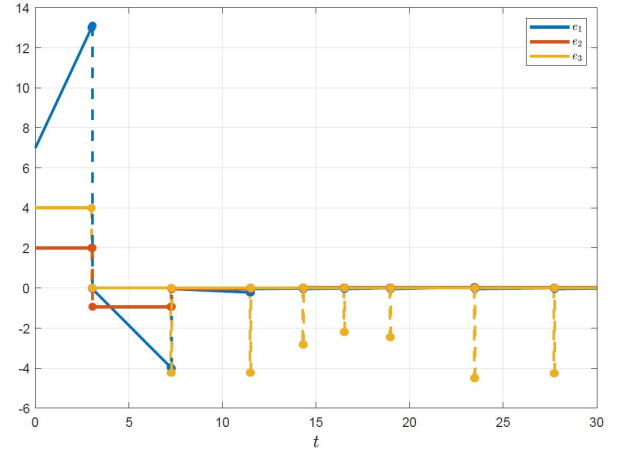


Fig. 4. Error between a trajectory of system (12) with random interjump intervals in $\mathcal{I} = [2, 5]$ and observer (3) with $L_c = 0$ and $L_d = (1, 0.2259, 1)^\top$, and jumps triggered with a delay $\Delta = 0.05$.

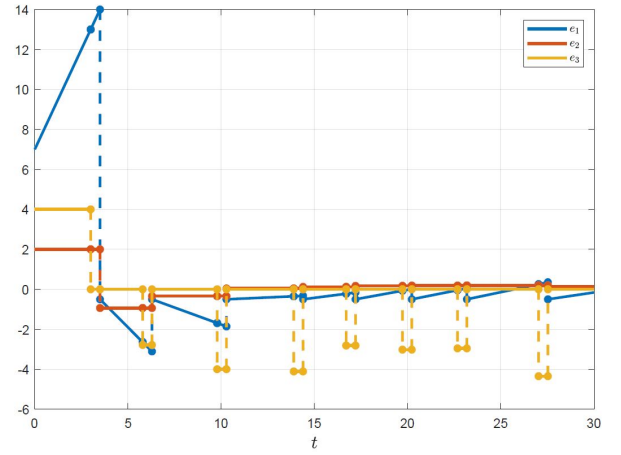


Fig. 5. Error between a trajectory of system (12) with random interjump intervals in $\mathcal{I} = [2, 5]$ and observer (3) with $L_c = 0$ and $L_d = (1, 0.2259, 1)^\top$, and jumps triggered with a delay $\Delta = 0.5$.

outside the delay intervals. It could also happen in that case that the mismatch during the delay intervals grows larger and larger, although this is not the case here.

VII. CONCLUSION

Under the assumption that the plant's jumps can be detected, we have given sufficient conditions for asymptotic convergence of an observer for general hybrid systems with linear flow/jump maps. Those conditions take the form of matrix inequalities which can often be solved thanks to LMI solvers. The obtained observer must be synchronized with the plant but we have shown its robustness with respect to delays in its jumps. Further research is necessary to develop observer designs that do not require the knowledge or detection of the plant's jumps. This case is more complex because the error system is no longer time-invariant and the Lyapunov analysis can no longer be carried out with Euclidian distances.

A. Proof of Lemma IV.1

From Lemma II.4, we know it is enough to prove that the set $\{0\} \times ([0, \tau_M] \cap \mathbb{R}_{\geq 0})$ is UGES for $\mathcal{H}_\varepsilon^\tau$. Take a solution ϕ of $\mathcal{H}_\varepsilon^\tau$. It is easy to see that $\tilde{\varepsilon} : \mathbb{N} \rightarrow \mathbb{R}^n$ defined by

$$\tilde{\varepsilon}_k = \varepsilon(t_k, k)$$

verifies (14) for all $k \geq 1$ with input $\tilde{\tau}$ defined by

$$\tilde{\tau}_k = \tau(t_{k+1}, k) \in \mathcal{I} \quad \forall k \geq 1.$$

It follows that for all $k \geq 1$,

$$|\varepsilon(t_k, k)| \leq \gamma \rho^k |\varepsilon(t_1, 1)| \leq c_1 \gamma \rho^k |\varepsilon(0, 0)|,$$

with $c_1 = \max_{\tau \in [0, \tau_M]} |(A_d - L_d H_d) \exp(A_c \tau)|$. We deduce that, for all $(t, j) \in \text{dom } \phi$ such that $j \geq 1$,

$$|\varepsilon(t, j)| = |\exp(A_c \tau(t, j)) \varepsilon(t_j, j)| \leq c_1 c_2 \gamma \rho^j |\varepsilon(0, 0)|$$

with $c_2 = \max_{\tau \in [0, \tau_M]} |\exp(A_c \tau)|$. But we know that for all (t, j) in $\text{dom } \phi$, $t - t_j \leq \tau_M$ and $t_j - t_{j-1} \leq \tau_M$ for $j \geq 1$, so that $t_j \leq \tau_M j$ and $t \leq \tau_M(j + 1)$. Thus, for any positive real number σ ,

$$-\sigma(t + j) \geq -\sigma(\tau_M + 1)j - \sigma\tau_M$$

and taking $\sigma = -\frac{\ln \rho}{\tau_M + 1}$, we get for all $j \geq 1$,

$$|\varepsilon(t, j)| \leq c_1 c_2 \gamma e^{-\sigma(t+j)} e^{\sigma\tau_M} |\varepsilon(0, 0)|.$$

Besides, we have

$$|\varepsilon(t, 0)| \leq c_2 |\varepsilon(0, 0)| \leq c_2 e^{\sigma\tau_M} e^{-\sigma t} |\varepsilon(0, 0)|$$

so that we conclude

$$|\varepsilon(t, j)| \leq \max\{c_1 \gamma, 1\} c_2 e^{-\sigma(t+j)} e^{\sigma\tau_M} |\varepsilon(0, 0)|.$$

for all $(t, j) \in \text{dom } \phi$, which proves UGES.

B. Technical lemma

Lemma A.1. Assume there exist a positive definite matrix P , matrices A_1 and A_2 , and scalars a_1 and a_2 such that

$$A_1^\top P + P A_1 \leq a_1 P \quad (29a)$$

$$A_2^\top P A_2 \leq e^{a_2} P \quad (29b)$$

Then, for any τ in $\mathbb{R}_{\geq 0}$,

$$(\exp(A_1 \tau))^\top A_2^\top P A_2 \exp(A_1 \tau) \leq e^{a_1 \tau + a_2} P. \quad (30)$$

Proof. Directly from (29b), we get

$$\begin{aligned} & (\exp(A_1 \tau))^\top A_2^\top P A_2 \exp(A_1 \tau) \\ & \leq e^{a_2} (\exp(A_1 \tau))^\top P \exp(A_1 \tau). \end{aligned}$$

Take e in \mathbb{R}^n . Define the function $f_e : \mathbb{R} \rightarrow \mathbb{R}$ by

$$f_e(\tau) = e^\top (\exp(A_1 \tau))^\top P \exp(A_1 \tau) e$$

With (29a), we have

$$\begin{aligned} \frac{df_e}{d\tau}(\tau) &= e^\top (\exp(A_1 \tau))^\top [A_1^\top P + P A_1] \exp(A_1 \tau) e \\ &\leq a_1 f_e(\tau). \end{aligned}$$

It follows that for all $\tau \geq 0$, $f_e(\tau) \leq e^{a_1 \tau} f_e(0)$ and since this is valid for all e in \mathbb{R}^n , we get

$$(\exp(A_1 \tau))^\top P \exp(A_1 \tau) \leq e^{a_1 \tau} P$$

and (30) follows. \square

C. Proof of Theorem VI.2

Take a solution $\phi_\delta = (x, \hat{x}, \tau, \mu, \tau_\delta)$ to $\hat{\mathcal{H}}_u(\Delta)$ for some $\Delta > 0$. Given the definition of the jump map, it is straightforward to observe that for any (t, j) in $\text{dom } \phi_\delta$, j either belongs to J_{-1} or J_0 . Now, $(x, \varepsilon, \tau, \mu_\varepsilon, \tau_\delta)$ with $\varepsilon = \hat{x} - x$ and $\mu_\varepsilon = \mu - x$ is solution to the hybrid system

$$\mathcal{H}_u^\varepsilon(\Delta) \left\{ \begin{array}{l} \dot{x} = A_c x + B_c u_c \\ \dot{\varepsilon} = (A_c - L_c H_c) \varepsilon \\ \dot{\tau} = 1 \\ \dot{\mu}_\varepsilon = -(A_c x + B_c u_c) \\ \dot{\tau}_\delta = -\min(\tau_\delta + 1, 1) \end{array} \right\} x \in \hat{C}(\Delta)$$

$$\mathcal{H}_u^\varepsilon(\Delta) \left\{ \begin{array}{l} x^+ = A_d x + B_d u_d \\ \varepsilon^+ = \varepsilon + (I - A_d)x - B_d u_d \\ \tau^+ = 0 \\ \mu_\varepsilon^+ = (A_d - L_d H_d) \varepsilon \\ \tau_\delta^+ \in [0, \Delta] \end{array} \right\} x \in \hat{D}_{-1}(\Delta)$$

$$\mathcal{H}_u^\varepsilon(\Delta) \left\{ \begin{array}{l} x^+ = x, \quad \mu_\varepsilon^+ = \mu_\varepsilon, \quad \tau^+ = \tau \\ \varepsilon^+ = \mu_\varepsilon, \quad \tau_\delta^+ = -1, \end{array} \right\} x \in \hat{D}_0(\Delta) \quad (31)$$

with $x = (x, \varepsilon, \tau, \mu_\varepsilon, \tau_\delta)$. If we had $\dot{\mu}_\varepsilon = 0$, $A_d = I$ and $B_d = 0$, we could write an independent error system in $(\varepsilon, \tau, \mu_\varepsilon, \tau_\delta)$ without the state x . Besides, this system would be exactly the delayed version introduced in [32] of the error system $\mathcal{H}_\varepsilon^\tau$ in (5). By assumption, we know that the set $\{0\} \times ([0, \tau_M] \cap \mathbb{R})$ is UGAS for (5), and we could therefore deduce from [32] semi global practical stability of the delayed system. Our goal is thus to get (31) as close as possible to what (31) would be with $\dot{\mu}_\varepsilon = 0$, $A_d = I$ and $B_d = 0$. The key idea here is to notice that the value taken by μ_ε when $\tau_\delta = -1$, i.e. in the time intervals $[t_j, t_{j+1}]$ with j in J_{-1} , have no impact on the other states. Indeed, when $\tau_\delta = -1$, the flow and jump maps are independent from μ_ε (and μ_ε is reset at the jump to an arbitrary value). Therefore, $(x, \varepsilon, \tau, \mu_\varepsilon, \tau_\delta)$ would still be solution to $\mathcal{H}_u^\varepsilon(\Delta)$ if μ_ε was kept constant during the time intervals associated to J_{-1} . Let us now study the behavior of μ_ε during the time where $\tau_\delta \in [0, \Delta]$, i.e. in the time intervals given by j in J_0 . The flow map is still independent from μ_ε , but the jump map with $\tau_\delta = 0$ is not. Therefore, the only value of μ_ε which has an impact on the other states is the value at the end of the interval, namely $\mu_\varepsilon(t_{j+1}, j)$ for j in J_0 . Denote

$$m = \min_{x \in \mathcal{X}, u_c \in U_c} -(A_c x + B_c u_c)$$

$$M = \max_{x \in \mathcal{X}, u_c \in U_c} -(A_c x + B_c u_c)$$

so that

$$\dot{\mu}_\varepsilon(t, j) \in [m, M] \quad \forall (t, j) \in \text{dom } \phi.$$

For any integer j in J_0 , $t_{j+1} - t_j \leq \Delta$, which yields

$$\mu_\varepsilon(t_{j+1}, j) \in \mu_\varepsilon(t_j, j) + [m, M]\Delta \quad \forall j \in J_0. \quad (32)$$

So consider now the function $\bar{\mu}_\varepsilon$ defined on $\text{dom } \phi$ by

$$\bar{\mu}_\varepsilon(t, j) = \mu_\varepsilon(t_j, j) \quad \forall (t, j) \in \text{dom } \phi.$$

It is constant during flow and from (32),

$$\mu_\varepsilon(t_{j+1}, j) \in \bar{\mu}_\varepsilon(t_{j+1}, j) + [m, M]\Delta \quad \forall j \in J_0. \quad (33)$$

Besides, from the definition of the jump map of $\mathcal{H}_u^\varepsilon(\Delta)$ on $\hat{D}_0(\Delta)$, for all j in J_0 ,

$$\bar{\mu}_\varepsilon(t_{j+1}, j+1) = \mu_\varepsilon(t_{j+1}, j+1) = \mu_\varepsilon(t_{j+1}, j)$$

and

$$\varepsilon(t_{j+1}, j+1) = \mu_\varepsilon(t_{j+1}, j) \quad \forall j \in J_0.$$

Therefore, $(\varepsilon, \tau, \bar{\mu}_\varepsilon, \tau_\delta)$ is solution to

$$\mathcal{H}^\varepsilon(\Delta) \left\{ \begin{array}{l} \dot{\varepsilon} = (A_c - L_c H_c) \varepsilon \\ \dot{\tau} = 1 \\ \dot{\bar{\mu}}_\varepsilon = 0 \\ \dot{\tau}_\delta = -\min(\tau_\delta + 1, 1) \end{array} \right\} (\varepsilon, \tau, \bar{\mu}_\varepsilon, \tau_\delta) \in C^\varepsilon(\Delta)$$

$$\mathcal{H}^\varepsilon(\Delta) \left\{ \begin{array}{l} \varepsilon^+ \in \varepsilon + (I - A_d) \mathcal{X} + B_d U_d \\ \tau^+ = 0 \\ \bar{\mu}_\varepsilon^+ = (A_d - L_d H_d) \varepsilon \\ \tau_\delta^+ \in [0, \Delta] \end{array} \right\} (\varepsilon, \tau, \bar{\mu}_\varepsilon, \tau_\delta) \in D_{-1}^\varepsilon(\Delta)$$

$$\mathcal{H}^\varepsilon(\Delta) \left\{ \begin{array}{l} \varepsilon^+ \in \bar{\mu}_\varepsilon + [m, M]\Delta \\ \tau^+ = \tau \\ \bar{\mu}_\varepsilon^+ \in \bar{\mu}_\varepsilon + [m, M]\Delta \\ \tau_\delta^+ = -1 \end{array} \right\} (\varepsilon, \tau, \bar{\mu}_\varepsilon, \tau_\delta) \in D_0^\varepsilon(\Delta)$$

with

$$C^\varepsilon(\Delta) = \mathbb{R}^n \times [0, \tau_M] \times \mathbb{R}^n \times ([0, \Delta] \cup \{-1\})$$

$$D_{-1}^\varepsilon(\Delta) = \mathbb{R}^n \times [0, \tau_M] \times \mathbb{R}^n \times \{-1\}$$

$$D_0^\varepsilon(\Delta) = \mathbb{R}^n \times [0, \tau_M] \times \mathbb{R}^n \times \{0\}$$

Now, observe that the values taken by ε during the intervals $[t_j, t_{j+1}]$ where $\tau_\delta \in [0, \Delta]$, i.e for j in J_0 , have no impact on the other states because their flow map and the jump map on $D_0^\varepsilon(\Delta)$ are independent from e . In other words, there exists a function $\bar{\varepsilon}$ defined on $\text{dom } \phi$ such that

$$\bar{\varepsilon}(t, j) = e(t, j) \quad \forall (t, j) \in \text{dom } \phi : j \in J_{-1} \quad (34)$$

and $(\bar{\varepsilon}, \tau, \bar{\mu}_\varepsilon, \tau_\delta)$ is solution to

$$\bar{\mathcal{H}}^\varepsilon(\Delta) \left\{ \begin{array}{l} \dot{\bar{\varepsilon}} = (A_c - L_c H_c) \bar{\varepsilon} \\ \dot{\tau} = 1 \\ \dot{\bar{\mu}}_\varepsilon = 0 \\ \dot{\tau}_\delta = -\min(\tau_\delta + 1, 1) \end{array} \right\} (\bar{\varepsilon}, \tau, \bar{\mu}_\varepsilon, \tau_\delta) \in C^\varepsilon(\Delta)$$

$$\bar{\mathcal{H}}^\varepsilon(\Delta) \left\{ \begin{array}{l} \bar{\varepsilon}^+ = \bar{\varepsilon} \\ \tau^+ = 0 \\ \bar{\mu}_\varepsilon^+ = (A_d - L_d H_d) \bar{\varepsilon} \\ \tau_\delta^+ \in [0, \Delta] \end{array} \right\} (\bar{\varepsilon}, \tau, \bar{\mu}_\varepsilon, \tau_\delta) \in D_{-1}^\varepsilon(\Delta)$$

$$\bar{\mathcal{H}}^\varepsilon(\Delta) \left\{ \begin{array}{l} \bar{\varepsilon}^+ \in \bar{\mu}_\varepsilon + [m, M]\Delta \\ \tau^+ = \tau \\ \bar{\mu}_\varepsilon^+ \in \bar{\mu}_\varepsilon + [m, M]\Delta \\ \tau_\delta^+ = -1 \end{array} \right\} (\bar{\varepsilon}, \tau, \bar{\mu}_\varepsilon, \tau_\delta) \in D_0^\varepsilon(\Delta) \quad (35)$$

From [19, Theorem 1] and [32, Proposition 3.8], we know that the set $\mathcal{A}^\varepsilon = \mathcal{A}_{-1}^\varepsilon \cup \mathcal{A}_0^\varepsilon$ with

$$\mathcal{A}_0^\varepsilon = \{0\} \times [0, \tau_M] \times \{0\} \times \{0\}$$

$$\mathcal{A}_{-1}^\varepsilon = \{0\} \times [0, \tau_M] \times \mathcal{M} \times \{-1\}$$

and $\mathcal{M} = (A_d - L_d H_d) \mathbb{R}^n$, is UGAS for $\bar{\mathcal{H}}^\varepsilon(0)$. Since \mathcal{A}_ε is compact and $\bar{\mathcal{H}}^\varepsilon(0)$ verifies the hybrid basic conditions, \mathcal{A}_ε is semi globally practically robustly \mathcal{KL} asymptotically stable for $\bar{\mathcal{H}}^\varepsilon(0)$ according to [22, Lemma 7.20]. This means that there exists a \mathcal{KL} function β such that for any $\varepsilon > 0$ and any compact set K of $C^\varepsilon(0) \cup D^\varepsilon(0)$, there exists $\rho > 0$ such that any solution $\bar{\phi}_\varepsilon = (\bar{\varepsilon}, \tau, \bar{\mu}_\varepsilon, \tau_\delta)$ to a ρ -perturbation of $\bar{\mathcal{H}}^\varepsilon(0)$, initialized in K verifies

$$|\bar{\phi}_\varepsilon(t, j)|_{\mathcal{A}_\varepsilon} \leq \beta(|\bar{\phi}_\varepsilon(0, 0)|_{\mathcal{A}_\varepsilon}, t + j) + \varepsilon.$$

Since $|\bar{\varepsilon}(t, j)| \leq |\bar{\phi}_\varepsilon(t, j)|_{\mathcal{A}_\varepsilon} \leq |\bar{\varepsilon}(t, j)| + |\bar{\mu}_\varepsilon(t, j)|$ and $\mathcal{H}^\varepsilon(\Delta)$ can be included in any outer-perturbation of $\mathcal{H}^\varepsilon(0)$ by taking Δ sufficiently small, we obtain (27) thanks to (34).

Finally, let us bound the error on the intervals given by j in J_0 . Because of (27) and the jump map of $\mathcal{H}^\varepsilon(\Delta)$ when $\tau_\delta = -1$, we have for all j in J_0 ,

$$|\varepsilon(t_j, j)| \leq \beta(|\varrho(0, 0)|, t_j + j) + \varepsilon + \max_{x \in \mathcal{X}, u_d \in U_d} |(I - A_d)x + B_d u_d|$$

so there exists a scalar σ depending on the eigenvalues of $A_c - L_c H_c$ such that (28) holds.

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