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Robust Observer Design for Hybrid Dynamical Systems with Linear Maps and Approximately Known Jump Times

Pauline Bernard and Ricardo G. Sanfelice

Abstract—This paper proposes a general framework for the state estimation of plants given by hybrid systems with linear flow and jump maps, in the favorable case where their jump events can be detected (almost) instantaneously. A candidate observer consists of a copy of the plant’s hybrid dynamics with continuous-time and/or discrete-time correction terms multiplied by two constant gains, and with jumps triggered by those of the plant. Assuming that the time between successive jumps is known to belong to a given closed set allows us to formulate an augmented system with a timer which keeps track of the time elapsed between successive jumps and facilitates the analysis. Then, since the jumps of the plant and of the observer are synchronized, the error system has time-invariant linear flow and jump maps, and a Lyapunov analysis leads to sufficient conditions for the design of the observer gains for uniform asymptotic stability in three different settings: continuous and discrete updates, only discrete updates, and only continuous updates. These conditions take the form of matrix inequalities, which we solve in examples including cases where the time between successive jumps is unbounded or tends to zero (Zeno behavior), and cases where either both the continuous and discrete dynamics, only one of them, or neither of them are detectable. Finally, we study the robustness of this approach when the jumps of the observer are delayed with respect to those of the plant. We show that if the plant’s trajectories are bounded and the time between successive jumps is lower-bounded away from zero, the estimation error is bounded, and arbitrarily small outside the delay intervals between the plant’s and the observer’s jumps.

Index Terms—observer, hybrid systems, impulsive systems

I. INTRODUCTION

A. Background

In many applications, estimating the state of a system is crucial, whether it be for control, supervision, or fault diagnosis purposes. Unfortunately, different from the linear time-invariant continuous-time setting, the problem of designing observers for hybrid systems is unsolved, even when the flow/jump maps are linear. The lack of general tools for such systems is mainly due to the fact that hybrid systems combine both continuous-time and discrete-time dynamics, which in general leads to solutions from nearby initial conditions that have different jump times. Such a mismatch of time domains makes the formulation of observability/detectability and, in

turn, observer design very challenging. In particular, the notions of *observability* (reconstruction of the initial condition) and *determinability* (reconstruction of the final condition) are no longer equivalent when the jump map is not invertible.

When the plant’s jump times are unknown, the error system approach does not apply since the jumps of the observer and of the plant are not necessarily synchronized. Unfortunately, very few observer results exist for such a setting. This problem is overcome in a particular case in [1], thanks to the fact that the jump map g is such that $g \circ g$ is the identity map, and in a slightly more general setting in [2], thanks to a change of coordinates transforming the jump map into the identity map. In the latter case, the observer problem reduces to the design of an observer for a continuous-time system, but no result concerning the existence and construction of such a transformation is available yet. Another path explored in the particular setting of switched systems is to estimate the plant’s switching signal: its observability has been studied in [3], [4] and some designs exist based on *mode location observers* to detect and identify mode switches (see [5], [6], [7], [8], [9]).

Impulsive systems consist of a class of continuous-time dynamical systems with state jumps that occur at pre-specified times, which, in most articles in the literature, are separated by nonzero periods of flow (in particular, to avoid Zeno behavior). The impulsive systems literature is rich and includes a variety of models of impulsive systems. In particular, models of impulsive systems in which the state includes a logic variable that selects the right-hand side of the differential equation governing the dynamics in between impulses are referred to as switched impulsive systems, or also as switched systems with known jump times. In that setting, the difficulties due to a possible mismatch of the trajectories’ domains disappear since the jump times are assumed known. Observability and determinability thus reduce to comparing inputs with same time domain and have been extensively studied with geometric/algebraic conditions given in [10], [11], [12], [13], [14]. As for observer design, results first appeared assuming each mode is observable [15], and then more generally in [16] (resp. in [14]), for impulsive systems (resp. switched impulsive systems) that are observable (resp. determinable) for any impulse time sequence containing more than a known finite number N of jumps. In other words, the information available during a single flow interval is no sufficient to reconstruct the full state, but it becomes sufficient after N jumps. In [16], the observer consists of an impulsive system synchronized with the plant, with innovation terms at jumps only. Those

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innovations are linear in the error, with a time-varying gain that is related to a weighted observability Grammian over the past N jumps. In [14], the authors develop an observation procedure based on the continuous-time estimation of the observable states of each of the past N modes: after some time, putting together the information given by each mode enables to reconstruct the whole state. Note however that [16], [14], the time elapsed between successive jumps must be lower bounded away from zero and upper bounded.

Another important class of hybrid systems for which observer results exist is when the system itself has continuous-time dynamics, but the measurements are available intermittently at specific time instances. For such a class of systems and other systems with sporadic events, observers have been designed under specific assumptions on the time elapsed between successive events or, in the case of periodic events, the sampling period. From [17], convergence of an impulsive observer with linear innovation terms triggered by the measurement events is guaranteed when the sampling period is sufficiently small. This design is extended in [18] to any constant sampling period provided that appropriate matrix inequalities are satisfied, and further extended in [19] to the case of sporadic measurements, i.e., when the time elapsed between sampling events varies in a known interval. In [17], [18], [19] though, the “inter-jump” duration must be lower bounded away from zero and upper bounded by known constants.

B. Contributions

In this paper, we consider general hybrid systems as in [20] with linear flow and jump maps, and possibly an input whose value is considered known at all times. In this paper we make the following contributions

- *Definition of hybrid observer:* Under the assumption that the plant’s jumps are detected instantaneously, a candidate observer is a hybrid system that jumps at the same time as the plant does, and is fed with the known input and linear correction terms in either the flow map, the jump map, or both.
- *Broad class of hybrid plants:* Our results only assume that the time between successive jumps belongs to a known (possibly unbounded) closed set. This allows us to formulate (Section II) an augmented hybrid system with a timer that keeps track of the time elapsed between successive jumps.
- *General design conditions:* The proposed augmented system is employed to derive sufficient conditions for the design of the gains defining the observer’s correction terms, so as to ensure uniform global asymptotic stability in three different settings: both continuous-time and discrete-time updates (Section III), only discrete-time updates (Section IV), and finally, only continuous-time updates (Section V).
- *Constructive design conditions:* The conditions take the form of matrix inequalities, which depend on the information available on the flow time between successive jumps. A key difference with the literature of impulsive

systems cited above is that no assumption of lower/upper boundedness of the “inter-jump” duration is required in our approach. In particular, the Zeno phenomenon is allowed.

- *Semiglobal practical estimation with mismatch of jump times:* Finally, in Section VI, we treat the more challenging case where the jumps of the plant and of the observer are not perfectly synchronized. We show the robustness of our observer in so far as semi-global practical stability is achieved if the plant’s trajectories and inputs are bounded and if the time elapsed between successive jumps is lower-bounded away from zero. More precisely, the observer with sufficiently small delay in the jumps provides an arbitrarily precise estimate, except during the delay intervals between the plant’s jumps and the observer’s where peaking occurs.

Preliminary results were given in [21], but restricted to the case were at least either the continuous dynamics or the discrete dynamics are detectable. We give their proofs here and complete them by sufficient conditions in the case were neither the continuous nor the discrete dynamics of the plant are detectable (but the plant as a whole is), and by a robustness analysis with respect to delays in the triggering of the observer’s jumps.

Notation. \mathbb{R} (resp. \mathbb{N}) denotes the set of real numbers (resp. integers), $\mathbb{R}_{\geq 0} = [0, +\infty)$, $\mathbb{R}_{> 0} = (0, +\infty)$, and $\mathbb{N}_{> 0} = \mathbb{N} \setminus \{0\}$. The components of a square matrix P are denoted p_{ij} , and $\lambda_m(P)$ (resp. $\lambda_M(P)$) stands for its smallest (resp. largest) eigenvalue. The symbol \star in a matrix denotes the symmetric blocks. \mathbb{B} stands for a closed Euclidian ball of appropriate dimension, of radius 1 and centered at 0.

II. HYBRID OBSERVER

A. Problem statement

In this paper, we consider hybrid plants of the form

$$\mathcal{H}_u \begin{cases} \dot{x} = A_c x + B_c u_c & x \in C \\ x^+ = A_d x + B_d u_d & x \in D \\ y_c = H_c x & x \in C \\ y_d = H_d x & x \in D \end{cases} \quad (1)$$

with state $x \in \mathbb{R}^n$, input u being the collection of a continuous-time input $u_c : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^{m_c}$ and a discrete-time input $u_d : \mathbb{N} \rightarrow \mathbb{R}^{m_d}$, and output $y = (y_c, y_d) \in \mathbb{R}^{p_c} \times \mathbb{R}^{p_d}$. For this class of hybrid systems, we are interested in estimating the state of the plant (1) when its solutions are initialized in a given subset \mathcal{X}_0 of \mathbb{R}^n . To formally state this problem, we first introduce the notion of solution and other needed definitions related to hybrid systems.

A solution x to a hybrid system is given by a *hybrid arc* which is a function defined on a *hybrid time domain*, denoted $\text{dom } x$. A hybrid time domain $\text{dom } x$ is a subset of $\mathbb{R}_{\geq 0} \times \mathbb{N}$ such that for any (T', J') in $\text{dom } x$, there exist $J \in \mathbb{N}$ and a sequence of times $0 = t_0 \leq t_1 \leq \dots \leq t_J$ such that

$$\text{dom } x \cap ([0, T'] \times \{0, 1, \dots, J'\}) = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j) .$$

Then, $x : \text{dom } x \rightarrow \mathbb{R}^n$ is a hybrid arc if $\text{dom } x$ is a hybrid time domain and $t \mapsto x(t, j)$ for each $j \in \mathbb{N}$ is absolutely continuous. For a hybrid arc x , we denote $\text{dom}_t x$ (resp. $\text{dom}_j x$) the projection of $\text{dom } x$ on the first (resp. second) dimension, $T(x) = \sup \text{dom}_t x$, $J(x) = \sup \text{dom}_j x$, $t_j(x)$ the time stamp associated to jump j which is uniquely characterized by

$$(t_j(x), j-1) \in \text{dom } x \quad , \quad (t_j(x), j) \in \text{dom } x$$

and $\mathcal{T}(x) = \{t_j(x) : j \in \text{dom}_j x \cap \mathbb{N}_{>0}\}$. We say that x is

- *complete* if $\text{dom } x$ is unbounded
- *eventually continuous* (resp. *eventually discrete*) if $J(x) < +\infty$ and $T(x) > t_J(x)$ (resp. $T(x) < +\infty$ and $\text{dom } x \cap (T(x) \times \mathbb{N})$ contains at least two points)
- *Zeno* if x is complete and $T(x) < +\infty$.

For $u_c : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^{m_c}$ and $u_d : \mathbb{N} \rightarrow \mathbb{R}^{m_d}$, we say that a hybrid arc x is solution to \mathcal{H}_u with output $y = (y_c, y_d)$ if for all $j \in \mathbb{N}$,

- for all $t \in (t_j(x), t_{j+1}(x))$, $x(t, j) \in C$
- for almost all t in $(t_j(x), t_{j+1}(x))$, we have $\dot{x}(t, j) = A_c x(t, j) + B_c u_c(t)$
- for all t in $[t_j(x), t_{j+1}(x)]$, $y_c(t, j) = H_c x(t, j)$

and for all $(t, j) \in \text{dom } x$ such that $(t, j+1) \in \text{dom } x$, $x(t, j) \in D$, $y_d(t, j) = H_d x(t, j)$, and

$$x(t, j+1) = A_d x(t, j) + B_d u_d(j).$$

A solution x is maximal if it cannot be continued into a solution with larger domain. We denote $\mathcal{S}_{\mathcal{H}_u}(\mathcal{X}_0)$ the set of maximal solutions of \mathcal{H}_u with initial condition in \mathcal{X}_0 and input u . We will also need the following definition.

Definition II.1. For a closed subset \mathcal{I} of $\mathbb{R}_{\geq 0}$, an input u , and a subset \mathcal{X}_0 of \mathbb{R}^n , we will say that $\mathcal{C}_{\mathcal{H}_u}(\mathcal{X}_0, \mathcal{I})$ holds if for any solution $x \in \mathcal{S}_{\mathcal{H}_u}(\mathcal{X}_0)$,

- $0 \leq t - t_j(x) \leq \sup \mathcal{I} \quad \forall (t, j) \in \text{dom } x$
- $t_{j+1}(x) - t_j(x) \in \mathcal{I}$ holds
 - $\forall j \in \mathbb{N}_{>0}$ if $J(x) = +\infty$
 - $\forall j \in \{1, \dots, J(x) - 1\}$ if $J(x) < +\infty$

In other words, the set \mathcal{I} describes the possible lengths of the flow intervals between successive jumps. The role of the first item in Definition II.1 is to bound the length of the intervals of flow which are not covered by the second item, namely possibly the first one, which is $[0, t_1(x)]$, and the last one, which is $\text{dom}_t x \cap [t_{J(x)}(x), +\infty)$ (when defined). We are now ready to state the observer problem of interest.

Problem 1. Design an observer assuming we know

- the value of the input u at all times,
- when the plant's jumps occur,
- the outputs y_c during flows and/or y_d at jumps,
- some information about the flow time between successive jumps, namely a closed subset \mathcal{I} of $\mathbb{R}_{\geq 0}$ such that $\mathcal{C}_{\mathcal{H}_u}(\mathcal{X}_0, \mathcal{I})$ holds.

The existence of a set \mathcal{I} such that $\mathcal{C}_{\mathcal{H}_u}(\mathcal{X}_0, \mathcal{I})$ holds is not a problem because it always holds for $\mathcal{I} = \mathbb{R}_{\geq 0}$. But as we will see later, it is advantageous to select \mathcal{I} as tight as possible,

namely it is convenient to have as much information about the duration of flow between successive jumps as possible. The following example shows how \mathcal{I} can be chosen depending on \mathcal{X}_0 .

Example II.2. Consider a bouncing ball with gravity coefficient $\mathbf{g} > 0$ and restitution coefficient $\lambda > 0$, modelled as system (1) with^{1,2}

$$\begin{aligned} A_c &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad , \quad A_d = \begin{pmatrix} -1 & 0 \\ 0 & -\lambda \end{pmatrix} & (2) \\ C &= \mathbb{R}_{\geq 0} \times \mathbb{R} \quad , \quad D = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0, x_2 \leq 0\} \\ B_c &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad , \quad B_d = 0 \quad , \quad u_c \equiv -\mathbf{g} \end{aligned}$$

If $\lambda < 1$, any maximal solution x is such that³ $T < +\infty$ and $J = +\infty$. The time between two successive jumps $t_{j+1} - t_j$ tends to zero when j tends to $+\infty$, and its upper bound increases with $|x(0, 0)|$. So we can take $\mathcal{I} = [0, \tau_M]$ with $\tau_M \geq 0$, if \mathcal{X}_0 is bounded. Otherwise, $\mathcal{I} = \mathbb{R}_{\geq 0}$.

If now $\lambda > 1$, any maximal solution x initialized in $\mathbb{R}^2 \setminus \{(0, 0)\}$ is such that $T = +\infty$, $J = +\infty$. The time between two successive jumps $t_{j+1} - t_j$ tends to $+\infty$ when j tends to $+\infty$, and its lower bound decreases with $|x(0, 0)|$. Therefore, if there exists $\delta > 0$ such that \mathcal{X}_0 is a subset of $\mathbb{R}^n \setminus \delta\mathbb{B}$, one can take $\mathcal{I} = [\tau_m, +\infty)$ with $\tau_m > 0$. Otherwise, $\mathcal{I} = \mathbb{R}_{\geq 0}$.

Finally if $\lambda = 1$, any maximal solution x initialized in $\mathbb{R}^2 \setminus \{(0, 0)\}$, is such that $T = +\infty$, $J = +\infty$, and the time between two successive jumps $t_{j+1} - t_j$ is constant for all $j \geq 1$, and increases with $|x(0, 0)|$. The maximal solution initialized at $(0, 0)$ is discrete, i.e., $T = 0$ and $J = +\infty$. We can take \mathcal{I} of the form:

- $\mathcal{I} = [0, \tau_M]$ with $\tau_M \geq 0$, if \mathcal{X}_0 is bounded.
- $\mathcal{I} = [\tau_m, +\infty)$ with $\tau_m > 0$, if there exists $\delta > 0$ such that \mathcal{X}_0 is a subset of $\mathbb{R}^2 \setminus \delta\mathbb{B}$.
- $\mathcal{I} = [\tau_m, \tau_M]$ with $\tau_m > 0$ and $\tau_M > 0$, if there exists $\delta > 0$ such that \mathcal{X}_0 is a bounded subset of $\mathbb{R}^2 \setminus \delta\mathbb{B}$.
- otherwise, $\mathcal{I} = \mathbb{R}_{\geq 0}$. △

B. Proposed hybrid observer

Since the plant's jump times and the value of the input are assumed to be known, we propose to use an observer

$$\hat{\mathcal{H}}_{u,y} \begin{cases} \dot{\hat{x}} = A_c \hat{x} + B_c u_c + L_c (y_c - H_c \hat{x}) & \text{when } \mathcal{H}_u \text{ flows} \\ \hat{x}^+ = A_d \hat{x} + B_d u_d + L_d (y_d - H_d \hat{x}) & \text{when } \mathcal{H}_u \text{ jumps} \end{cases} \quad (3)$$

that is synchronized with the plant. Problem 1 thus reformulate as:

Problem 2. For a set of initial conditions \mathcal{X}_0 and a closed subset \mathcal{I} , design gains $L_c \in \mathbb{R}^{n \times p_c}$ and $L_d \in \mathbb{R}^{n \times p_d}$ such that for any input u such that $\mathcal{C}_{\mathcal{H}_u}(\mathcal{X}_0, \mathcal{I})$ holds, every maximal

¹The coefficient -1 in A_d is arbitrary because $x_1 = 0$ in the jump set. We use -1 because, numerically, if x_1 is negative when the jump condition is detected, it is useful to change its sign after the jump in order for the flow condition to be verified at the next iteration.

²Several definitions of H_c and H_d will be considered later.

³To simplify the notation, we write T , J and t_j for $T(x)$, $J(x)$, $t_j(x)$.

solution x of \mathcal{H}_u initialized in \mathcal{X}_0 and every maximal solution \hat{x} of $\hat{\mathcal{H}}_{u,y}$ are complete and verify

$$\lim_{t+j \rightarrow \infty} x(t, j) - \hat{x}(t, j) = 0 .$$

To use the hybrid framework introduced in [20] and express the fact that $\mathcal{C}_{\mathcal{H}_u}(\mathcal{X}_0, \mathcal{I})$ is satisfied, we will consider the augmentation of \mathcal{H}_u in (1) given by the hybrid system

$$\mathcal{H}_u^\tau \left\{ \begin{array}{l} \dot{x} = A_c x + B_c u_c \\ \dot{\tau} = 1 \end{array} \right\} (x, \tau) \in C^\tau \quad (4)$$

$$\left\{ \begin{array}{l} x^+ = A_d x + B_d u_d \\ \tau^+ = 0 \end{array} \right\} (x, \tau) \in D^\tau$$

$$\left\{ \begin{array}{l} y_c = H_c x \\ y_d = H_d x \end{array} \right\} \begin{array}{l} (x, \tau) \in C^\tau \\ (x, \tau) \in D^\tau \end{array}$$

with, denoting $\tau_M = \sup \mathcal{I}$,

$$C^\tau = \mathbb{R}^n \times ([0, \tau_M] \cap \mathbb{R}_{\geq 0}) \quad , \quad D^\tau = \mathbb{R}^n \times \mathcal{I} \quad (5)$$

and the cascade interconnection of \mathcal{H}_u^τ with $\hat{\mathcal{H}}_{u,y}$ resulting in the hybrid system

$$\hat{\mathcal{H}}_u^\tau \left\{ \begin{array}{l} \dot{x} = A_c x + B_c u_c \\ \dot{\hat{x}} = A_c \hat{x} + B_c u_c + L_c (H_c x - H_c \hat{x}) \\ \dot{\tau} = 1 \end{array} \right\} (x, \hat{x}, \tau) \in \hat{C}^\tau$$

$$\left\{ \begin{array}{l} x^+ = A_d x + B_d u_d \\ \hat{x}^+ = A_d \hat{x} + B_d u_d + L_d (H_d x - H_d \hat{x}) \\ \tau^+ = 0 \end{array} \right\} (x, \hat{x}, \tau) \in \hat{D}^\tau \quad (6)$$

with

$$\hat{C}^\tau = \mathbb{R}^n \times \mathbb{R}^n \times ([0, \tau_M] \cap \mathbb{R}_{\geq 0}) \quad , \quad \hat{D}^\tau = \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{I} \quad (7)$$

The models \mathcal{H}_u^τ and $\hat{\mathcal{H}}_u^\tau$ are such that the timer τ has to reach \mathcal{I} before a jump can occur and is forced to jump when reaching τ_M (if finite). This enables to relate the behavior of \mathcal{H}_u , $\hat{\mathcal{H}}_{u,y}$, \mathcal{H}_u^τ , and $\hat{\mathcal{H}}_u^\tau$ as follows.

Lemma II.3. Consider a subset \mathcal{X}_0 of \mathbb{R}^n , a closed subset \mathcal{I} of $\mathbb{R}_{\geq 0}$ and denote $\tau_M = \sup \mathcal{I} \leq +\infty$. For any input u such that $\mathcal{C}_{\mathcal{H}_u}(\mathcal{X}_0, \mathcal{I})$ holds, for any maximal solution x of \mathcal{H}_u initialized in \mathcal{X}_0 , and for any maximal solution \hat{x} of $\hat{\mathcal{H}}_{u,y}$, we have $\text{dom } x = \text{dom } \hat{x} =: \mathcal{D}$, and there exists a function τ defined on \mathcal{D} such that (x, τ) is solution to \mathcal{H}_u^τ and (x, \hat{x}, τ) is solution to $\hat{\mathcal{H}}_u^\tau$.

Proof. By definition of $\mathcal{T}(x)$, x and \hat{x} have the same time domain $\mathcal{D} = \text{dom } x = \text{dom } \hat{x}$. Besides, since $\mathcal{C}_{\mathcal{H}_u}(\mathcal{X}_0, \mathcal{I})$ holds, the function τ defined by $\tau(t, j) := t - t_j(x)$ for all (t, j) in \mathcal{D} gives the result. \square

We conclude that any property obtained for \mathcal{H}_u^τ or $\hat{\mathcal{H}}_u^\tau$ will be extendable to \mathcal{H}_u and the cascade $\mathcal{H}_u - \hat{\mathcal{H}}_{u,y}$, respectively, as long as \mathcal{H}_u is initialized in \mathcal{X}_0 and $\mathcal{C}_{\mathcal{H}_u}(\mathcal{X}_0, \mathcal{I})$ holds.

Example II.4. As mentioned in the introduction, the proposed framework also applies to the case where the plant itself has continuous-time dynamics

$$\dot{x} = A x + B u \quad , \quad y = H x$$

but the output y is only available at discrete times t_j , which do not necessarily occur periodically. In that case, one can use an observer given in (3) with $L_c = 0$,

$$A_c = A \quad , \quad B_c = B \quad , \quad A_d = I \quad , \quad B_d = 0$$

$$u_c = u \quad , \quad u_d = 0 \quad , \quad H_d = H \quad ,$$

and L_d to be designed. If we know that the time elapsed between two successive sampling events is in a closed subset \mathcal{I} of $\mathbb{R}_{\geq 0}$, then the interconnection between the system and the observer can be modelled exactly by $\hat{\mathcal{H}}_u^\tau$. For instance, \mathcal{I} is a singleton in the case of periodic sampling, and \mathcal{I} is a compact interval of $\mathbb{R}_{> 0}$ in the case of aperiodic sampling considered in [19]. In fact, depending on the class of events of interest, the set \mathcal{I} could be discrete, contain a finite or infinite number of elements, be a collection of intervals of $\mathbb{R}_{\geq 0}$, etc. \triangle

III. HYBRID OBSERVER WITH INNOVATION TERMS ON FLOWS AND JUMPS

The following theorem gives our first sufficient condition to ensure global exponential stability of the observer.

Theorem III.1. Consider a subset \mathcal{X}_0 of \mathbb{R}^n and a closed subset \mathcal{I} of $\mathbb{R}_{\geq 0}$. Assume there exist scalars a_c and a_d , matrices $L_c \in \mathbb{R}^{n \times p_c}$ and $L_d \in \mathbb{R}^{n \times p_d}$, and a positive definite symmetric matrix $P \in \mathbb{R}^{n \times n}$ such that:

$$(A_c - L_c H_c)^\top P + P(A_c - L_c H_c) \leq a_c P \quad (8a)$$

$$(A_d - L_d H_d)^\top P(A_d - L_d H_d) \leq e^{a_d} P \quad (8b)$$

$$a_c \tau + a_d < 0 \quad \forall \tau \in \mathcal{I} \quad (8c)$$

Then, there exist $\gamma > 0$ and $\theta > 0$ such that for any input u such that $\mathcal{C}_{\mathcal{H}_u}(\mathcal{X}_0, \mathcal{I})$ holds, every maximal solution x of \mathcal{H}_u initialized in \mathcal{X}_0 and every maximal solution \hat{x} of $\hat{\mathcal{H}}_{u,y}$ are complete and verify

$$\left| x(t, j) - \hat{x}(t, j) \right| \leq \gamma \left| x(0, 0) - \hat{x}(0, 0) \right| e^{-\theta(t+j)} \quad (9)$$

for all $(t, j) \in \text{dom } x (= \text{dom } \hat{x})$.

Proof. First observe that there always exists⁴ a positive scalar a such that

$$a_c \tau + a_d \leq -a(\tau + 1) \quad \forall \tau \in \mathcal{I} . \quad (10)$$

Let us introduce the continuously differentiable function V defined on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ by

$$V(x, \hat{x}, \tau) = (\hat{x} - x)^\top P(\hat{x} - x)$$

To show (9), we apply [20, Proposition 3.29]. For that, we will first prove that there exists M such that for any solution $\phi = (x, \hat{x}, \tau)$ to $\hat{\mathcal{H}}_u^\tau$, we have

$$a_c t + a_d j \leq M - a(t + j) \quad (t, j) \in \text{dom } \phi . \quad (11)$$

⁴Either \mathcal{I} is a compact set and this is satisfied for a sufficiently small positive number a thanks to (8c); or \mathcal{I} is unbounded and then necessarily a_c is negative, and by taking $a < |a_c|$ sufficiently small, the inequality holds.

We have for all $(t, j) \in \text{dom } \phi$,

$$a_c t + a_d j = a_c t_1 + \sum_{i=1}^{j-1} (a_c(t_{i+1} - t_i) + a_d) \\ a_c(t - t_j) + a_d .$$

Fix $j \in \text{dom}_j \phi$. By definition of \hat{C}^τ and \hat{D}^τ in $\hat{\mathcal{H}}_u^\tau$,

$$t_{i+1} - t_i \in \mathcal{I} \quad \forall i \in \{1, \dots, j-1\}, \quad t - t_j \in [0, \tau_M] \cap \mathbb{R}_{\geq 0}$$

so that according to (10),

$$a_c t_1 + \sum_{i=1}^{j-1} (a_c(t_{i+1} - t_i) + a_d) \\ \leq a_c t_1 - a \sum_{i=1}^{j-1} (t_{i+1} - t_i + 1) \\ \leq -a(t_j + j - 1) + (a_c + a)t_1$$

and for $\tau_m = \min \mathcal{I}$,

$$a_c(t - t_j) + a_d = a_c(t - t_j - \tau_m) + a_c \tau_m + a_d \\ \leq a_c(t - t_j - \tau_m) - a(\tau_m + 1) \\ = (a_c + a)(t - t_j - \tau_m) - a(t - t_j + 1) .$$

This yields

$$a_c t + a_d j \leq -a(t + j) + (a_c + a)t_1 + (a_c + a)(t - t_j - \tau_m) .$$

A bound for $(a_c + a)(t - t_j - \tau_m)$ is obtained with the following arguments:

- if $0 \leq t - t_j \leq \tau_m$, we get $(a_c + a)(t - t_j - \tau_m) \leq |a_c + a|\tau_m$
- else if $t - t_j \geq \tau_m$, either $\tau_M < +\infty$ and

$$(a_c + a)(t - t_j - \tau_m) \leq |a_c + a|(\tau_M + \tau_m)$$

or $\tau_M = +\infty$, then necessarily from (10), $a_d \leq -a$, and

$$(a_c + a)(t - t_j - \tau_m) \leq 0$$

because $t - t_j \geq \tau_m$.

As for the term $(a_c + a)t_1$, similarly, either $\tau_M < +\infty$ and

$$(a_c + a)t_1 \leq |a_c + a|\tau_M$$

or $\tau_M = +\infty$, then necessarily from (10), $a_d \leq -a$, and

$$(a_c + a)t_1 \leq 0 .$$

We conclude that there exists M such that (11) holds. According to [20, Proposition 3.29], we have

$$V(\phi(t, j)) \leq e^M e^{-a(t+j)} V(\phi(0, 0)) \quad \forall (t, j) \in \text{dom } \phi$$

and because for all (x, \hat{x}) in $\mathbb{R}^n \times \mathbb{R}^n$,

$$\lambda_m(P)|x - \hat{x}|^2 \leq V(x, \hat{x}, \tau) \leq \lambda_M(P)|x - \hat{x}|^2$$

we finally get (9) for all (t, j) in $\text{dom } \phi$ with

$$\theta = \frac{a}{2}, \quad \gamma = e^{\frac{M}{2}} \sqrt{\frac{\lambda_M(P)}{\lambda_m(P)}} .$$

Applying Lemma II.3 concludes the proof since the above holds for any solution of \mathcal{H}_u^τ . \square

Remark III.2. From conditions (8a)-(8c), we recover the fact that if $0 \in \mathcal{I}$, namely there are Zeno or eventually discrete solutions, then a_d must be negative, i.e., the innovation term in the discrete dynamics of the observer must make the error contractive at jumps; similarly if $\sup \mathcal{I} = +\infty$, then a_c must be negative, i.e., the innovation term in the continuous dynamics must make the error contractive during flow.

It is important to note that the set of initial conditions \mathcal{X}_0 is used to choose the set \mathcal{I} such that $\mathcal{C}_{\mathcal{H}_u}(\mathcal{X}_0, \mathcal{I})$ holds. Therefore, it possibly impacts conditions (8a)-(8c), but only through (8c).

The interesting property of conditions (8a)-(8c) is that they are affine (and thus convex) in τ , which means that it is sufficient to check them at the boundaries of the set \mathcal{I} only. This fact is formalized in the next result.

Corollary III.3. Consider a closed subset \mathcal{I} of $\mathbb{R}_{\geq 0}$. Let $\tau_m = \min \mathcal{I}$ and $\tau_M = \sup \mathcal{I}$. Assume there exist scalars a_c and a_d , matrices $L_c \in \mathbb{R}^{n \times p_c}$ and $L_d \in \mathbb{R}^{n \times p_d}$, and a positive definite symmetric matrix $P \in \mathbb{R}^{n \times n}$ such that (8a)-(8b) are satisfied. (8a)-(8c) hold if any of the following conditions is verified

- 1) $a_c \leq 0$ and $a_d < 0$,
- 2) $a_c < 0$ and $a_c \tau_m + a_d < 0$,
- 3) $a_c > 0$, $\tau_M < +\infty$, and $a_c \tau_M + a_d < 0$.

It is shown in [21, Example 3.3] how Conditions (8a)-(8c) can be solved analytically with $a_c < 0$ and $a_d < 0$ (item 1. of the previous corollary) for a bouncing ball modelled by (2) with a restitution coefficient $\lambda < 1$, and x_1 measured at all (hybrid) times, thus giving a global observer.

Remark III.4. It is important to note that in the favorable case where both the continuous and the discrete dynamics are detectable (such as [21, Example 3.3]), it is not sufficient to choose independently $A_c - L_c H_c$ Hurwitz and $A_d - L_d H_d$ Schur. Indeed, their descent directions could be incompatible: jumps could destroy what has been achieved during flow, or vice versa. Take for instance $\mathcal{I} = \{\tau^*\}$ with $\tau^* \geq 0$. A necessary condition for convergence of the observer is that the error sampled at each jump converges to zero: this implies that the origin of the discrete system

$$e^+ = (A_d - L_d H_d) \exp \left((A_c - L_c H_c) \tau \right) e$$

has to be asymptotically stable. For a given $\tau^* \geq 0$, this is not verified for every choice of $A_d - L_d H_d$ Schur and $A_c - L_c H_c$ Hurwitz, as illustrated on Figure 1 : $(A_d - L_d H_d) \exp \left((A_c - L_c H_c) \tau \right)$ is Schur only if $\tau^* \notin [0.1, 2]$. To avoid this phenomenon, (8a) and (8b) should be solved with the same P , and $a_c \leq 0$ and $a_d < 0$. By the Schur complement, this is equivalent to solving the LMIs

$$A_c^\top P + P A_c - (\tilde{L}_c H_c + H_c^\top \tilde{L}_c^\top) < 0 \\ \left(\begin{array}{c} P \quad (P A_d - \tilde{L}_d H_d)^\top \\ \star \quad P \end{array} \right) > 0 \quad (12)$$

in $(P, \tilde{L}_c, \tilde{L}_d)$ and take $L_c = P^{-1} \tilde{L}_c$ and $L_d = P^{-1} \tilde{L}_d$. Note that the problem of finding common quadratic Lyapunov

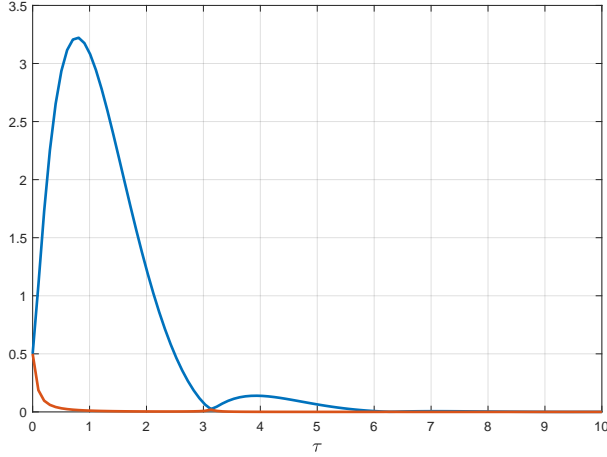


Fig. 1. Absolute value of the eigenvalues of $(A_d - L_d H_d) \exp((A_c - L_c H_c)\tau)$ for $A_c = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $A_d = 5A_c$, $H_c = H_d = (1, 0)$, with L_c (resp. L_d) chosen such that $A_c - L_c H_c$ (resp. $A_d - L_d H_d$) is Hurwitz (resp. Schur) with eigenvalues $(-1 + i, -1 - i)$ (resp. $(0.5, -0.5)$).

functions for several continuous-time or several discrete-time systems has been studied in the context of switched systems and quadratic stabilization (see e.g. [22]). But we are not aware of any result concerning the existence of a common quadratic function for a continuous-time system and a discrete-time system.

A drawback of Theorem III.1 is that (8c) requires at least a_c or a_d to be negative: either the continuous or the discrete dynamics have to be detectable. But take for instance the hybrid system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = 0 \\ \dot{x}_3 = 0 \end{cases}, \quad \begin{cases} x_1^+ = x_1 \\ x_2^+ = x_2 \\ x_3^+ = x_1 \end{cases} \quad (13)$$

with some arbitrary, but nonempty flow and jump sets. Suppose $H_c = H_d = (1 \ 0 \ 0)$. Neither the continuous nor the discrete dynamics is detectable, so Theorem III.1 cannot apply. Nevertheless, this hybrid system as a whole is detectable if there is at least one jump and one interval of flow. Indeed, if $y(t, j) = x_1(t, j) = 0$ for all (t, j) in the domain, the continuous part gives $x_2(t, j) = 0$ as soon as $[t, t + \tau) \times \{j\} \in \text{dom } x$ for some $\tau > 0$, and the discrete part gives $x_3(t, j) = 0$ for all j such that $(t, j - 1)$ and (t, j) are in the domain. Given this detectability property, we would like to be able to write an observer for this system. We will see in the next section that it is possible.

IV. HYBRID OBSERVER WITH INNOVATION TERMS ON JUMPS ONLY

We now consider the case where only y_d is known, namely the measurements from the plant are only available at jump times. Therefore, we build an observer with $L_c = 0$. Due to the lack of measurements during flow, and without the assumption that A_c is already Hurwitz, eventually continuous solutions are

not allowed. Hence, \mathcal{I} has to be bounded. The following result follows from combining Theorem III.1 and Corollary III.3.

Corollary IV.1. [Update at jumps] Consider a subset \mathcal{X}_0 of \mathbb{R}^n and a compact subset \mathcal{I} of $\mathbb{R}_{\geq 0}$. Assume there exist scalars $a_c \in \mathbb{R}$ and $a_d < 0$, a matrix $L_d \in \mathbb{R}^{n \times p_d}$, and a positive definite matrix $P \in \mathbb{R}^{n \times n}$ such that

$$A_c^\top P + P A_c \leq a_c P \quad (14a)$$

$$(A_d - L_d H_d)^\top P (A_d - L_d H_d) \leq e^{a_d} P \quad (14b)$$

$$a_c \tau_M + a_d < 0 \quad (14c)$$

with $\tau_M = \max \mathcal{I}$. Then, there exist $\gamma > 0$ and $\theta > 0$ such that for any input u such that $\mathcal{C}_{\mathcal{H}_u}(\mathcal{X}_0, \mathcal{I})$ holds, every maximal solution x of \mathcal{H}_u initialized in \mathcal{X}_0 , and every maximal solution \hat{x} of $\hat{\mathcal{H}}_{u,y}$, with $L_c = 0$ and L_d as above, are complete and verify

$$\begin{aligned} |x(t, j) - \hat{x}(t, j)| &\leq \gamma |x(0, 0) - \hat{x}(0, 0)| e^{-\theta(t+j)} \\ \forall (t, j) \in \text{dom } x (= \text{dom } \hat{x}) \end{aligned} \quad (15)$$

In [21, Example 4.2], we showed how conditions (14a)-(14c) can be solved analytically for \mathcal{I} of the form $[0, \tau_M]$ for a bouncing ball modelled by (2) with $\lambda < 1$, and with the measurement x_1 only available at jumps, thus giving a global observer for any compact set of the plant's initial conditions.

However, as we were observing in Section V, a limitation of conditions (14a)-(14c) is that they require the discrete part of the system to be detectable. But as we saw for system (13), it may happen that neither the continuous nor the discrete parts are detectable, and yet, the whole system is detectable. Motivated by these facts, the following result gives a sufficient condition that is weaker than (14a)-(14c) to write an observer with $L_c = 0$.

Theorem IV.2. Consider a subset \mathcal{X}_0 of \mathbb{R}^n and a compact subset \mathcal{I} of $\mathbb{R}_{\geq 0}$. Assume there exist a positive definite matrix $P \in \mathbb{R}^{n \times n}$ and a gain vector $L_d \in \mathbb{R}^{n \times p_d}$ such that

$$\begin{aligned} (\exp(A_c \tau))^\top (A_d - L_d H_d)^\top P (A_d - L_d H_d) \exp(A_c \tau) &< P \\ \forall \tau \in \mathcal{I} \end{aligned} \quad (16)$$

Then, there exist $\gamma > 0$ and $\theta > 0$ such that for any input u such that $\mathcal{C}_{\mathcal{H}_u}(\mathcal{X}_0, \mathcal{I})$ holds, every maximal solution x of \mathcal{H}_u initialized in \mathcal{X}_0 , and every maximal solution \hat{x} of $\hat{\mathcal{H}}_{u,y}$, with $L_c = 0$ and L_d verifying (16), are complete and verify

$$|x(t, j) - \hat{x}(t, j)| \leq \gamma |x(0, 0) - \hat{x}(0, 0)| e^{-\theta(t+j)} \quad (17)$$

for all (t, j) in $\text{dom } x (= \text{dom } \hat{x})$.

Proof. Take an input u , and solutions x and \hat{x} as in Theorem IV.2. According to Lemma II.3, there exists a function τ defined on $\text{dom } x = \text{dom } \hat{x}$ such that (x, \hat{x}, τ) is solution to $\hat{\mathcal{H}}_u^\tau$. It follows that $\phi = (\hat{x} - x, \tau)$ is a solution to the error hybrid system

$$\mathcal{H}_\varepsilon^\tau \begin{cases} \dot{\varepsilon} = A_c \varepsilon \\ \dot{\tau} = 1 \end{cases} \left\{ (\varepsilon, \tau) \in C^\tau \right. \\ \left. \begin{cases} \varepsilon^+ = (A_d - L_d H_d) \varepsilon \\ \tau^+ = 0 \end{cases} \right\} (\varepsilon, \tau) \in D^\tau \quad (18)$$

with C^τ and D^τ defined in (5). Let us introduce the differentiable function V defined on $\mathbb{R}^n \times [0, \tau_M]$ by

$$V(\varepsilon, \tau) = \varepsilon^\top (\exp(-A_c \tau))^\top P \exp(-A_c \tau) \varepsilon$$

Since \mathcal{I} is compact, τ_M is finite, and there exist positive scalars α_1 and α_2 such that

$$\alpha_1 |\varepsilon|^2 \leq V(\varepsilon, \tau) \leq \alpha_2 |\varepsilon|^2 \quad \forall (\varepsilon, \tau) \in \mathbb{R}^n \times [0, \tau_M].$$

It is straightforward to check that along the solutions, we have for all (ε, τ) in C^τ , $\overline{V(\varepsilon, \tau)} = 0$. Besides, since \mathcal{I} is compact, (16) implies that there exists $\beta \in (0, 1)$ such that for all τ in \mathcal{I} ,

$$(\exp(A_c \tau))^\top (A_d - L_d H_d)^\top P (A_d - L_d H_d) \exp(A_c \tau) - P \leq -\beta P.$$

So for all (ε, τ) in D^τ , with $g = ((A_d - L_d H_d)\varepsilon, 0)$,

$$V(g) \leq (1 - \beta)V(\varepsilon, \tau).$$

Integrating along the solution ϕ to $\mathcal{H}_\varepsilon^\tau$,

$$V(\phi(t, j)) \leq (1 - \beta)^j V(\phi(0, 0)) \quad \forall (t, j) \in \text{dom } \phi.$$

But we know that for all (t, j) in $\text{dom } \phi$, $t - t_j \leq \tau_M$ and $t_j - t_{j-1} \leq \tau_M$ for $j \geq 1$, so that $t_j \leq \tau_M j$ and $t \leq \tau_M(j+1)$. Thus, for any positive real number σ ,

$$-\sigma(t + j) \geq -\sigma(\tau_M + 1)j - \sigma \tau_M$$

and taking $\sigma = -\frac{\ln(1-\beta)}{\tau_M+1}$, we get

$$V(\phi(t, j)) \leq e^{-\sigma(t+j)} e^{\sigma \tau_M} V(\phi(0, 0)) \quad \forall (t, j) \in \text{dom } \phi.$$

Finally, for all (t, j) in $\text{dom } \phi$, (17) holds with

$$\theta = \frac{\sigma}{2}, \quad \gamma = e^{\frac{\sigma}{2} \tau_M} \sqrt{\frac{\alpha_2}{\alpha_1}}.$$

□

Remark IV.3. Condition (16) is exactly the one obtained in [19, Equation (12)] with $A_d = I$ and \mathcal{I} a compact interval of $\mathbb{R}_{>0}$, in the context of a continuous-time system with sporadic measurements, as in Example II.4.

Remark IV.4. The existence of the matrix P verifying (16) for a given τ is equivalent to $(A_d - L_d H_d) \exp(A_c \tau)$ being Schur for some gain L_d , which in turn is equivalent to the detectability of the discrete-time system

$$z^+ = A_d \exp(A_c \tau) z, \quad y = H_d z \quad (19)$$

(since $\exp(A_c \tau)$ is invertible). This implies that system (1) with $u \equiv 0$ and sampled at a constant sampling period $\tau \in \mathcal{I}$ must be detectable. Thus, having (16) for any $\tau \in \mathcal{I}$ requires detectability of (19) for any $\tau \in \mathcal{I}$. It may not be sufficient, however, because (16) must be verified with the same L_d and P for all $\tau \in \mathcal{I}$. So (16) is in fact related to the stronger property of detectability of the difference inclusion

$$z^+ \in A_d \exp(A_c \mathcal{I}) z, \quad y = H_d z.$$

Remark IV.5. By the Schur complement, finding P and L_d satisfying (16) is equivalent to finding P and \tilde{L}_d satisfying the LMIs

$$\begin{pmatrix} P & \exp(A_c \tau)^\top (P A_d - \tilde{L}_d H_d)^\top \\ \star & P \end{pmatrix} > 0 \quad \forall \tau \in \mathcal{I} \quad (20)$$

with $\tilde{L}_d = P L_d$. In the case where \mathcal{I} has infinitely many elements, an infinite number of LMIs must be solved which is not desirable. However, it is shown in [19] via a polytopic decomposition that $\exp(A_c \tau)$ is in the convex hull of a finite number of matrices. Therefore, since (20) is convex in $\exp(A_c \tau)$, it is sufficient to solve this LMI for this finite set of matrices. Note also that when A_c is nilpotent of order N , we have

$$\exp(A_c \tau) = \sum_{k=0}^{N-1} \frac{\tau^k}{k!} A_c^k$$

so that for all τ in a compact subset \mathcal{I} of $\mathbb{R}_{\geq 0}$, $\exp(A_c \tau)$ is in the convex hull of the $\nu = 2^{N-1}$ matrices

$$\left\{ M_1, \dots, M_\nu \right\} = \left\{ I + \sum_{k=1}^{N-1} \frac{\tau_k^k}{k!} A_c^k : \forall k \in \{1, \dots, N-1\}, \tau_k \in \{\tau_m, \tau_M\} \right\}$$

with $\tau_m = \min \mathcal{I}$ and $\tau_M = \max \mathcal{I}$. Therefore, it is enough to solve the finite number of LMIs

$$\begin{pmatrix} P & M_i^\top (P A_d - \tilde{L}_d H_d)^\top \\ \star & P \end{pmatrix} > 0 \quad \forall i \in \{1, \dots, \nu\}$$

with common P and \tilde{L}_d .

Example IV.6. Consider the system (13) where

$$A_c = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_d = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$H_d = H_c = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}.$$

Neither the continuous pair (A_c, H_c) nor the discrete pair (A_d, H_d) is detectable, so conditions (8a)-(8c) cannot be solved. However, following Remark IV.4,

$$A(\tau) := A_d \exp(A_c \tau) = \begin{pmatrix} 1 & \tau & 0 \\ 0 & 1 & 0 \\ 1 & \tau & 0 \end{pmatrix}$$

is such that the discrete pair $(A(\tau), H_d)$ is detectable for any nonzero τ . Therefore, if $\mathcal{I} = \{\tau\}$, there exists P and L_d such that (16) is satisfied. Otherwise, since A_c is nilpotent of order 2, according to Remark IV.5, for any \mathcal{I} compact subset of $\mathbb{R}_{>0}$, it is enough to solve the two LMIs given by

$$\begin{pmatrix} P & (I + \tau A_c)^\top (P A_d - \tilde{L}_d C)^\top \\ \star & P \end{pmatrix} > 0 \quad (21)$$

for $\tau = \tau_m = \min \mathcal{I} > 0$ and $\tau = \tau_M = \max \mathcal{I}$. If there exist solutions to (21), then by Theorem IV.2, we obtain an observer. For instance, when choosing $\tau_m = 2$ and $\tau_M = 5$ and solving the LMIs via Yalmip for P and \tilde{L}_d , we get $L_d = P \tilde{L}_d =$

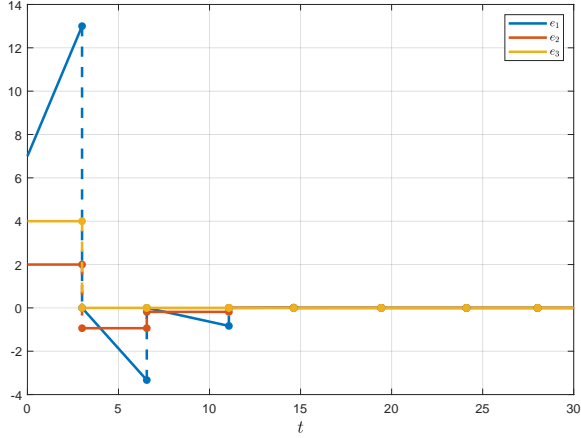


Fig. 2. Error between a trajectory of system (13) with flow interval length ranging in $\mathcal{I} = [2, 5]$ and a trajectory observer (3) with $L_c = 0$ and $L_d = (1, 0.2259, 1)^\top$.

$(1, 0.2259, 1)^\top$. The error between a trajectory of system (13) (with jumps triggered randomly such that the flow intervals last between 2 and 5 units of time) and a trajectory of the observer is plotted on Figure 2. \triangle

Since the observer with discrete updates ($L_c = 0$) given by Theorem IV.2 is a particular case of the observer with continuous and discrete updates given by Theorem III.1, it is natural to wonder whether there is a link between the conditions obtained in each of those results, namely between (8a)-(8c) and (16).

Lemma IV.7. Assume there exist (a_c, a_d) in \mathbb{R}^2 , $(L_c$ in $\mathbb{R}^{n \times p_c}$, L_d in $\mathbb{R}^{n \times p_d}$, and a positive definite matrix P in $\mathbb{R}^{n \times n}$ such that (8a)-(8b) hold. Then, for any τ in $\mathbb{R}_{\geq 0}$,

$$(\exp(M_c \tau))^\top M_d^\top P M_d \exp(M_c \tau) \leq e^{a_c \tau + a_d P} \quad (22)$$

with $M_c = A_c - L_c H_c$ and $M_d = A_d - L_d H_d$. If, in addition, $L_c = 0$ and (8c) holds, then (16) holds.

Proof. See Lemma A.1. \square

It follows that conditions (14a)-(14c) imply (16) in the case where $L_c = 0$.

Example IV.8. Consider again the bouncing ball (2) with restitution coefficient $\lambda < 1$, with measurements at jumps only, namely $H_c = 0$ and $H_d = (1, 0)$. As observed in Example II.2, for any compact set of initial conditions \mathcal{X}_0 , there exists $\tau_M > 0$ such that $\mathcal{C}_{\mathcal{H}_u}(\mathcal{X}_0, \mathcal{I})$ holds with $\mathcal{I} = [0, \tau_M]$. According to Theorem IV.2, it is enough to satisfy (16) instead of (14a)-(14c). Since A_c is nilpotent of order 2, we get from Remark IV.5 that it is enough to solve the two LMIs

$$\begin{pmatrix} P & (I + \tau A_c)^\top (P A_d - \tilde{L}_d H_d)^\top \\ \star & P \end{pmatrix} > 0$$

for $\tau = 0$ and $\tau = \tau_M$. With $\lambda = 0.8$ and $\tau_M = 5$, we obtain $L_d = (-1, -0.1487)$. The result of a simulation with initial condition $x_0 = (5, 0)$, $\hat{x}_0 = (10, 1)$ is shown on Figure (3). \triangle

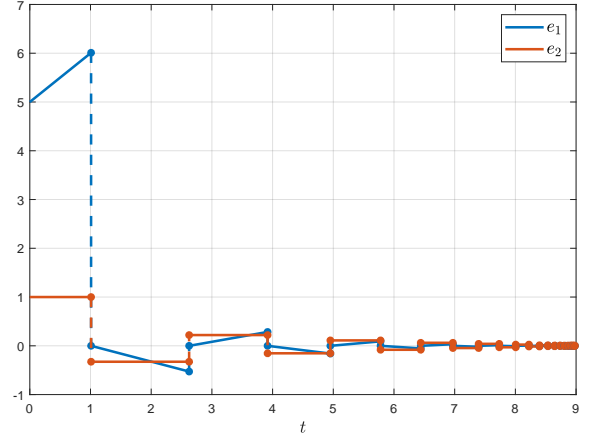


Fig. 3. Error between a Zeno trajectory of system (2) with $\lambda = 0.8$ and a trajectory of the synchronized observer (3) with $L_c = 0$ and $L_d = (-1, -0.1487)^\top$.

V. HYBRID OBSERVER WITH INNOVATION TERMS ON FLOWS ONLY

When \mathcal{I} is unbounded, it is not possible to implement an observer with discrete updates only: continuous updates are necessary. And when the continuous dynamics are detectable, it may be sufficient to use only continuous updates (with $L_d = 0$). The following corollary follows from Theorem III.1 and Corollary III.3.

Corollary V.1. [Continuous update] Consider a subset \mathcal{X}_0 of \mathbb{R}^n and a closed subset \mathcal{I} of $\mathbb{R}_{\geq 0}$. Assume there exist scalars $a_d \in \mathbb{R}$ and $a_c < 0$, a matrix L_c in $\mathbb{R}^{n \times p_c}$, and a positive definite matrix $P \in \mathbb{R}^{n \times n}$ such that

$$(A_c - L_c H_c)^\top P + P(A_c - L_c H_c) \leq a_c P \quad (23a)$$

$$A_d^\top P A_d \leq e^{a_d P} \quad (23b)$$

$$a_c \tau_m + a_d < 0 \quad (23c)$$

where $\tau_m = \min \mathcal{I}$. Then, there exist $\gamma > 0$ and $\theta > 0$ such that for any input u such that $\mathcal{C}_{\mathcal{H}_u}(\mathcal{X}_0, \mathcal{I})$ holds, every maximal solution x of \mathcal{H}_u initialized in \mathcal{X}_0 , and every maximal solution \hat{x} of $\hat{\mathcal{H}}_{u,y}$, with L_c as above and $L_d = 0$, are complete and verify

$$|x(t, j) - \hat{x}(t, j)| \leq \gamma |x(0, 0) - \hat{x}(0, 0)| e^{-\theta(t+j)} \quad \forall (t, j) \in \text{dom } x (= \text{dom } \hat{x}). \quad (24)$$

In [21, Example 5.2], we have seen how the conditions (23a)-(23c) can be analytically solved for \mathcal{I} of the form $[\tau_m, +\infty)$ for the bouncing ball (2) with a restitution coefficient $\lambda \geq 1$, thus giving a global observer for the plant initialized in $\mathbb{R}^2 \setminus \delta \mathbb{B}$ for $\delta > 0$.

When the pair (A_c, H_c) is observable, the Lyapunov equation (23a) can be solved for any negative number a_c . To solve (23c), it is tempting to take $|a_c|$ very large, but this has to be done with care since P depends on a_c and thus a_d in (23b) does too. The following lemma clarifies this dependence, in the single-output case ($p_c = 1$) to simplify the presentation.

Lemma V.2. Assume the pair (A_c, H_c) is observable, $p_c = 1$, and the eigenvalues $(\lambda_1, \dots, \lambda_n)$ of $A_c - L_c H_c$ are real and with negative real part. Then, there exist P , a_c and a_d verifying (23a)-(23b)-(23c) if

$$M_\lambda^\top M_\lambda < e^{2 \min |\lambda_i| \tau_m} I \quad (25)$$

with $\tau_m = \min \mathcal{I}$ and

$$M_\lambda = V_\lambda M_o A_d M_o^{-1} V_\lambda^{-1}, \quad (26)$$

where V_λ is the Vandermonde matrix

$$V_\lambda = \begin{pmatrix} 1 & \lambda_1 & \dots & \lambda_1^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & \lambda_n & \dots & \lambda_n^{n-1} \end{pmatrix},$$

and M_o is an invertible matrix in $\mathbb{R}^{n \times n}$ such that the change of coordinates $z = M_o x$ transforms the continuous dynamics

$$\dot{x} = A_c x, \quad y = H_c x$$

into the observable Brunovski form

$$\dot{z} = A_o z + N_o y, \quad y = H_o z$$

with

$$A_o = \begin{pmatrix} 0 & \dots & \dots & 0 \\ 1 & & & \\ 0 & \ddots & & \vdots \\ \vdots & & 1 & \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix}, \quad H_o = (0, \dots, 0, 1),$$

and a matrix N_o in $\mathbb{R}^{n \times 1}$.

Proof. The existence of M_o is guaranteed by observability of the pair (A_c, H_c) . By definition of the change of coordinates, $H_o = H_c M_o^{-1}$. It follows that $M_o(A_c - L_c H_c)M_o^{-1}$ is in companion form and can be diagonalized thanks to the Vandermonde matrix V_λ , namely

$$V_\lambda M_o (A_c - L_c H_c) M_o^{-1} V_\lambda^{-1} = \text{diag}(\lambda_1, \dots, \lambda_n).$$

Thanks to this diagonality, we deduce that (23a) is verified with⁵ $P_\lambda = M_o^\top V_\lambda^\top V_\lambda M_o$ and $a_c = -2 \min |\lambda_i|$. It follows that (23b)-(23c) are equivalent to (25). \square

Lemma V.2 says that a possible way of selecting a gain L_c such that (23a)-(23b)-(23c) are solvable is to choose the n negative eigenvalues of $A_c - L_c H_c$ so that (25) holds. The advantage of this approach is that the size of the problem is reduced from $\frac{n(n+1)}{2} + np_c + 2$ to n . In the multi-output case where $p_c > 1$, the target Brunovski form is made of blocks of the type (A_o, H_o) , so a similar result can be obtained by reasoning blockwise.

Note that in [21, Example 5.2], we used the fact that M_λ is homogeneous of degree 0 in λ to say that (25) holds for λ_i sufficiently large.

⁵We write P_λ instead of P to highlight the matrices that depend on the eigenvalues λ_i .

VI. ROBUSTNESS WITH RESPECT TO DELAYS IN JUMPS

We now study how the observer convergence is impacted if the observer jumps are delayed with respect to the plant's, thus leading to a mismatch between the observer jump times and those of the plant. For this, we suppose $\min \mathcal{I} > 0$, and we choose to study the particular case where the value of the innovation term implemented in the observer at the delayed jump is the one that would be computed at the actual plant's jump time. This covers the following situations:

- The measurement and computation of the innovation $A_d \hat{x} + B_d u_d + L_d (y_d - H_d \hat{x})$ are instantaneous, but the implementation of the jump in the observer is delayed.
- The measurement takes a known amount of time δ to arrive to the observer (the measurement has a time stamp), and the update of \hat{x} is chosen as $A_d \hat{x}(t - \delta) + B_d u_d + L_d (y_d - H_d \hat{x}(t - \delta))$, thanks to a buffer in \hat{x} or by backward integration of \hat{x} .

Inspired from [23], for any $\Delta \in [0, \min \mathcal{I}]$, this situation can be modelled as

$$\hat{\mathcal{H}}_u(\Delta) \left\{ \begin{array}{l} \left. \begin{array}{l} \dot{x} = A_c x + B_c u_c \\ \dot{\hat{x}} = A_c \hat{x} + B_c u_c + L_c H_c (x - \hat{x}) \\ \dot{\tau} = 1 \\ \dot{\mu} = 0 \\ \dot{\tau}_\delta = -\min\{\tau_\delta + 1, 1\} \end{array} \right\} \\ (x, \hat{x}, \tau, \mu, \tau_\delta) \in \hat{C}(\Delta), \\ \\ \left. \begin{array}{l} x^+ = A_d x + B_d u_d \\ \hat{x}^+ = \hat{x} \\ \tau^+ = 0 \\ \mu^+ = A_d \hat{x} + B_d u_d + L_d H_d (x - \hat{x}) \\ \tau_\delta^+ \in [0, \Delta] \end{array} \right\} \\ (x, \hat{x}, \tau, \mu, \tau_\delta) \in \hat{D}_{-1}(\Delta), \\ \\ \left. \begin{array}{l} x^+ = x \\ \hat{x}^+ = \mu \\ \tau^+ = \tau \\ \mu^+ = \mu \\ \tau_\delta^+ = -1 \end{array} \right\} \\ (x, \hat{x}, \tau, \mu, \tau_\delta) \in \hat{D}_0(\Delta), \end{array} \right. \quad (27)$$

with

$$\hat{C}(\Delta) = \hat{C} \times \mathbb{R}^n \times ([0, \Delta] \cup \{-1\}) \quad (28)$$

$$\hat{D}_{-1}(\Delta) = \hat{D} \times \mathbb{R}^n \times \{-1\} \quad (29)$$

$$\hat{D}_0(\Delta) = \mathbb{R}^n \times \mathbb{R}^n \times [0, \tau_M] \times \mathbb{R}^n \times \{0\} \quad (30)$$

where $\tau_M = \sup \mathcal{I}$ and the sets \hat{C} and \hat{D} are the flow and jump sets of $\hat{\mathcal{H}}_u$ defined in (7). Compared to $\hat{\mathcal{H}}_u$, we have added two variables μ and τ_δ evolving in \mathbb{R}^n and $[0, \Delta] \cup \{-1\}$ respectively. The state τ_δ is a timer modelling the delay between the plant's jump and the observer's jump. The role of μ is to store the update to be implemented in the observer at the end of the delay interval, when it actually jumps. More precisely, when $\tau_\delta = -1$ and τ is not in \mathcal{I} , the plant and the observer flow and τ_δ remains equal to -1 . If τ reaches \mathcal{I} and the plant jumps, then the update that should have been instantaneously implemented in the observer is stored in the memory state μ , and τ_δ is set to a number in $[0, \Delta]$ thus starting

a delay period: the plant and observer states then flow and the time τ_δ decreases, until it reaches 0. At this point, a delay interval of length smaller than or equal to Δ has elapsed, the observer jumps, and its state is updated with the content of μ .

Note that the plant's state is not allowed to jump again before the delay expressed by τ_δ has expired. That is why this model only works in the case where $\Delta < \min \mathcal{I}$, i.e., the maximal delay is smaller than the smallest possible time between successive jumps of the plant.

Assumption VI.1. We denote \mathcal{U} a set of inputs $u : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^{m_c} \times \mathbb{R}^{m_a}$ of interest. There exist compact subsets \mathcal{X} , \mathcal{X}_0 of \mathbb{R}^n , U_c of \mathbb{R}^{m_c} , and U_d of \mathbb{R}^{m_a} , such that any input u in \mathcal{U} , and any solution x to the plant (1) initialized in \mathcal{X}_0 with input u , verify $x(t, j) \in \mathcal{X}$ and $(u_c(t, j), u_d(t, j)) \in U_c \times U_d$ for all $(t, j) \in \text{dom } x$.

Theorem VI.2. Suppose Assumption VI.1 holds. Consider a compact subset \mathcal{I} of $\mathbb{R}_{\geq 0}$ with $\min \mathcal{I} > 0$, vectors $L_c \in \mathbb{R}^{n \times p_c}$ and $L_d \in \mathbb{R}^{n \times p_a}$ such that there exists a \mathcal{KL} function β_0 such that for any input u in \mathcal{U} , any maximal solution $\phi = (x, \hat{x}, \tau)$ of $\hat{\mathcal{H}}_u^\tau$ defined in (6) initialized in $\mathcal{X}_0 \times \mathbb{R}^n \times [0, \tau_M]$ is complete and verifies

$$\left| x(t, j) - \hat{x}(t, j) \right| \leq \beta_0 \left(\left| x(0, 0) - \hat{x}(0, 0) \right|, t + j \right) \quad \forall (t, j) \in \text{dom } \phi. \quad (31)$$

Then, there exist a \mathcal{KL} function β and a scalar σ such that for any $\eta > 0$ and any $\varepsilon > 0$, there exists $\Delta^* > 0$ such that for any $\Delta \in [0, \Delta^*]$ and any solution $\phi_\delta = (x, \hat{x}, \tau, \mu, \tau_\delta)$ to $\hat{\mathcal{H}}_u(\Delta)$ verifying

$$\epsilon(0, 0) := |\hat{x}(0, 0) - x(0, 0)| + |\mu(0, 0) - x(0, 0)| \leq \eta,$$

we have, denoting $t_j = t_j(\phi_\delta)$ for simplicity, $\text{dom } \phi_\delta = \mathcal{D}_{-1} \cup \mathcal{D}_0$ with

$$\begin{aligned} \mathcal{D}_k &= \left(\bigcup_{j \in J_k} [t_j, t_{j+1}] \times \{j\} \right) \quad k \in \{0, -1\} \\ J_{-1} &= \{j \in \mathbb{N} : \tau_\delta(t, j) = -1 \quad \forall t \in [t_j, t_{j+1}]\} \\ J_0 &= \{j \in \mathbb{N} : \tau_\delta(t, j) \in [0, \Delta] \quad \forall t \in [t_j, t_{j+1}]\}, \end{aligned}$$

such that for all j in J_0 , $t_{j+1} - t_j \leq \Delta$, and we have for all (t, j) in \mathcal{D}_{-1}

$$|\hat{x}(t, j) - x(t, j)| \leq \beta(\epsilon(0, 0), t + j) + \varepsilon, \quad (32)$$

and for all (t, j) in \mathcal{D}_0 ,

$$\begin{aligned} |\hat{x}(t, j) - x(t, j)| &\leq e^{\sigma(t-t_j)} \left(\beta(|\epsilon(0, 0)|, t_j + j) + \varepsilon \right. \\ &\quad \left. + \max_{x \in \mathcal{X}, u_d \in U_d} |(I - A_d)x + B_d u_d| \right) \quad (33) \end{aligned}$$

Proof. The proof relies on [23]. See Appendix A. \square

In other words, if the trajectories of the plant and the input are bounded, we achieve

- semiglobal practical stability if $A_d = I$ and $B_d = 0$, namely the jump map is the identity;
- semiglobal practical stability except on the delay intervals (of maximal length Δ) otherwise.

Note that the parameter σ describing the behavior of the error during the delay intervals is related to the eigenvalues of $A_c - L_c H_c$. Indeed, if the latter matrix is Hurwitz, the mismatch tends to be corrected by the flow during the delay interval, namely $\sigma < 0$.

In fact, this mismatch cannot be prevented if the jump map is not the identity. This well-known phenomenon, called *peaking*, was reported in the context of observation [1], but also more generally output-feedback and tracking [24]. This suggests that the Euclidian distance to evaluate the observer error is not appropriate and more general distances could be designed [25]. In particular here, if $B_d = 0$, semi-global practical stability could be obtained with the generalized distance

$$d(x, \hat{x}) = \min \left\{ |x - \hat{x}|, |A_d x - \hat{x}| \right\}.$$

Note that in the limit case where $0 \in \mathcal{I}$, namely the plant's jumps could happen arbitrarily fast, then a delay in the observer jumps (however small) could lead to several jumps of delay, namely, one could consider the distance

$$d(x, \hat{x}) = \inf_{k \in \mathbb{N}} |A_d^k x - \hat{x}|.$$

However, not much could be done if an infinite number of jumps happened during the delay interval.

Example VI.3. We come back to Example IV.6 and redo the simulation presented on Figure 2 with a delay in the triggering of the observer's jump. The results are presented in Figures 4-5-6 with delays of $\Delta = 0.05$, $\Delta = 0.1$ and $\Delta = 0.5$ respectively. Note that in this example, the assumption of boundedness of the plant's trajectory is not verified since x_1 and x_3 diverge. We can still see that the smaller the delay, the smaller the error outside the delay intervals. It could also happen in that case that the mismatch during the delay intervals grows larger and larger, although this is not the case here.

VII. CONCLUSION

Under the assumption that the plant's jumps can be detected, we have given sufficient conditions for asymptotic convergence of an observer for general hybrid systems with linear flow/jump maps. Those conditions take the form of matrix inequalities which can often be solved thanks to LMI solvers. The obtained observer must be synchronized with the plant but we have shown its robustness with respect to delays in its jumps.

Further research is necessary to develop observer designs that do not require the knowledge or detection of the plant's jumps. This case is more complex because the error system is no longer time-invariant and the Lyapunov analysis can no longer be carried out with Euclidian distances.

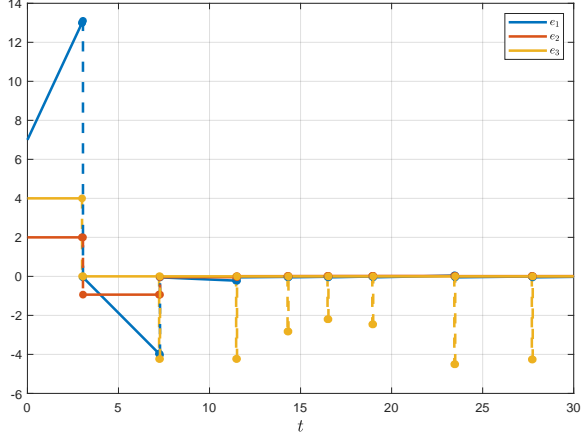


Fig. 4. Error between a trajectory of system (13) with random interjump intervals in $\mathcal{I} = [2, 5]$ and observer (3) with $L_c = 0$ and $L_d = (1, 0.2259, 1)^\top$, and jumps triggered with a delay $\Delta = 0.05$.

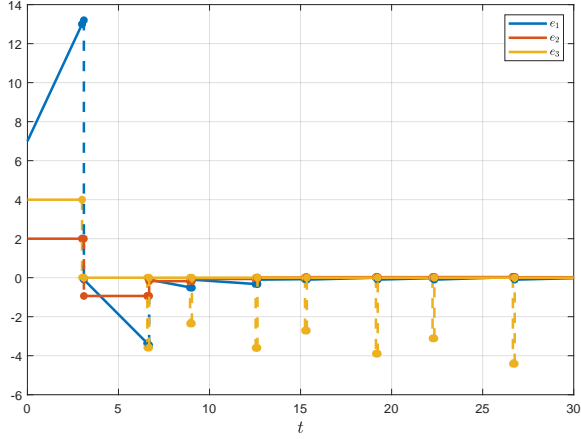


Fig. 5. Error between a trajectory of system (13) with random interjump intervals in $\mathcal{I} = [2, 5]$ and observer (3) with $L_c = 0$ and $L_d = (1, 0.2259, 1)^\top$, and jumps triggered with a delay $\Delta = 0.1$.

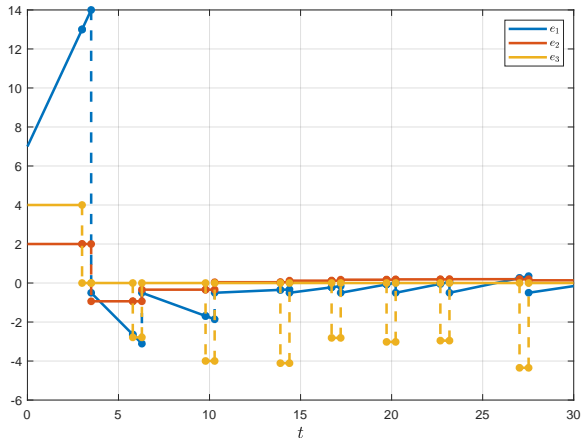


Fig. 6. Error between a trajectory of system (13) with random interjump intervals in $\mathcal{I} = [2, 5]$ and observer (3) with $L_c = 0$ and $L_d = (1, 0.2259, 1)^\top$, and jumps triggered with a delay $\Delta = 0.5$.

APPENDIX

Lemma A.1. Assume there exist a positive definite matrix P , matrices A_1 and A_2 , and scalars a_1 and a_2 such that

$$A_1^\top P + P A_1 \leq a_1 P \quad (34)$$

$$A_2^\top P A_2 \leq e^{a_2} P \quad (35)$$

Then, for any τ in $\mathbb{R}_{\geq 0}$,

$$(\exp(A_1 \tau))^\top A_2^\top P A_2 \exp(A_1 \tau) \leq e^{a_1 \tau + a_2} P. \quad (36)$$

Proof. Directly from (35), we get

$$\begin{aligned} (\exp(A_1 \tau))^\top A_2^\top P A_2 \exp(A_1 \tau) \\ \leq e^{a_2} (\exp(A_1 \tau))^\top P \exp(A_1 \tau). \end{aligned}$$

Take e in \mathbb{R}^n . Define the function $f_e : \mathbb{R} \rightarrow \mathbb{R}$ by

$$f_e(\tau) = e^\top (\exp(A_1 \tau))^\top P \exp(A_1 \tau) e$$

With (34), we have

$$\begin{aligned} \frac{df_e}{d\tau}(\tau) &= e^\top (\exp(A_1 \tau))^\top [A_1^\top P + P A_1] \exp(A_1 \tau) e \\ &\leq a_1 f_e(\tau). \end{aligned}$$

It follows that for all $\tau \geq 0$, $f_e(\tau) \leq e^{a_1 \tau} f_e(0)$ and since this is valid for all e in \mathbb{R}^n , we get

$$(\exp(A_1 \tau))^\top P \exp(A_1 \tau) \leq e^{a_1 \tau} P$$

and (36) follows. \square

Take a solution $\phi_\delta = (x, \hat{x}, \tau, \mu, \tau_\delta)$ to $\hat{\mathcal{H}}_u(\Delta)$ for some $\Delta > 0$. Given the definition of the jump map, it is straightforward to observe that for any (t, j) in $\text{dom } \phi_\delta$, j either belongs to J_{-1} or J_0 . Now, $(x, e, \tau, \mu_e, \tau_\delta)$ with

$$e = \hat{x} - x, \quad \mu_e = \mu - x$$

is solution to the hybrid system

$$\mathcal{H}_u^e(\Delta) \left\{ \begin{array}{l} \dot{x} = A_c x + B_c u_c \\ \dot{e} = (A_c - L_c H_c) e \\ \dot{\tau} = 1 \\ \dot{\mu}_e = -(A_c x + B_c u_c) \\ \dot{\tau}_\delta = -\min(\tau_\delta + 1, 1) \end{array} \right\} \begin{array}{l} (x, e, \tau, \mu_e, \tau_\delta) \in \hat{C}(\Delta) \\ \\ \\ \\ \\ \end{array} \quad (37)$$

$$\mathcal{H}_u^e(\Delta) \left\{ \begin{array}{l} x^+ = A_d x + B_d u_d \\ e^+ = e + (I - A_d) x - B_d u_d \\ \tau^+ = 0 \\ \mu_e^+ = (A_d - L_d H_d) e \\ \tau_\delta^+ \in [0, \Delta] \end{array} \right\} \begin{array}{l} (x, e, \tau, \mu_e, \tau_\delta) \in \hat{D}_{-1}(\Delta) \\ \\ \\ \\ \\ \end{array}$$

$$\mathcal{H}_u^e(\Delta) \left\{ \begin{array}{l} x^+ = x \\ e^+ = \mu_e \\ \tau^+ = \tau \\ \mu_e^+ = \mu_e \\ \tau_\delta^+ = -1 \end{array} \right\} \begin{array}{l} (x, e, \tau, \mu_e, \tau_\delta) \in \hat{D}_0(\Delta) \end{array}$$

If we had $\dot{\mu}_e = 0$, $A_d = I$ and $B_d = 0$, we could write an independent error system in $(e, \tau, \mu_e, \tau_\delta)$ without the state

x . Besides, this system would be exactly the delayed version introduced in [23] of the error system

$$\mathcal{H}_e^\tau \left\{ \begin{array}{l} \dot{e} = (A_c - L_c H_c) e \\ \dot{\tau} = 1 \end{array} \right\} (e, \tau) \in C^\tau \quad (38)$$

$$\left\{ \begin{array}{l} e^+ = (A_d - L_d H_d) e \\ \tau^+ = 0 \end{array} \right\} (e, \tau) \in D^\tau$$

with C^τ and D^τ defined in (5). According to (31), we know that the set $\{0\} \times [0, \tau_M]$ is uniformly globally asymptotically stable (UGAS) for (38), and we could therefore deduce from [23] semi global practical stability of the delayed system. Our goal is thus to get (41) as close as possible to what (41) would be with $\dot{\mu}_e = 0$, $A_d = I$ and $B_d = 0$. The key idea here is to notice that the value taken by μ_e when $\tau_\delta = -1$, i.e. in the time intervals $[t_j, t_{j+1}]$ with j in J_{-1} , have no impact on the other states. Indeed, when $\tau_\delta = -1$, the flow and jump maps are independent from μ_e (and μ_e is reset at the jump to an arbitrary value). Therefore, $(x, e, \tau, \mu_e, \tau_\delta)$ would still be solution to $\mathcal{H}_u^e(\Delta)$ if μ_e was kept constant during the time intervals associated to J_{-1} . Let us now study the behavior of μ_e during the time where $\tau_\delta \in [0, \Delta]$, i.e. in the time intervals given by j in J_0 . The flow map is still independent from μ_e , but the jump map with $\tau_\delta = 0$ is not. Therefore, the only value of μ_e which has an impact on the other states is the value at the end of the interval, namely $\mu_e(t_{j+1}, j)$ for j in J_0 . Denote

$$m = \min_{x \in \mathcal{X}, u_c \in U_c} -(A_c x + B_c u_c)$$

$$M = \max_{x \in \mathcal{X}, u_c \in U_c} -(A_c x + B_c u_c)$$

so that

$$\dot{\mu}_e(t, j) \in [m, M] \quad \forall (t, j) \in \text{dom } \phi .$$

For any integer j in J_0 , $t_{j+1} - t_j \leq \Delta$, which yields

$$\mu_e(t_{j+1}, j) \in \mu_e(t_j, j) + [m, M]\Delta \quad \forall j \in J_0 . \quad (39)$$

So consider now the function $\bar{\mu}_e$ defined on $\text{dom } \phi$ by

$$\bar{\mu}_e(t, j) = \mu_e(t_j, j) \quad \forall (t, j) \in \text{dom } \phi .$$

It is constant during flow and from (39),

$$\mu_e(t_{j+1}, j) \in \bar{\mu}_e(t_{j+1}, j) + [m, M]\Delta \quad \forall j \in J_0 . \quad (40)$$

Besides, from the definition of the jump map of $\mathcal{H}_u^e(\Delta)$ on $\bar{D}_0(\Delta)$, for all j in J_0 ,

$$\bar{\mu}_e(t_{j+1}, j+1) = \mu_e(t_{j+1}, j+1) = \mu_e(t_{j+1}, j)$$

and

$$e(t_{j+1}, j+1) = \mu_e(t_{j+1}, j) \quad \forall j \in J_0 .$$

Therefore, $(e, \tau, \bar{\mu}_e, \tau_\delta)$ is solution to

$$\mathcal{H}^e(\Delta) \left\{ \begin{array}{l} \dot{e} = (A_c - L_c H_c) e \\ \dot{\tau} = 1 \\ \dot{\bar{\mu}}_e = 0 \\ \dot{\tau}_\delta = -\min(\tau_\delta + 1, 1) \end{array} \right\} (e, \tau, \bar{\mu}_e, \tau_\delta) \in C^e(\Delta)$$

$$\left\{ \begin{array}{l} e^+ \in e + (I - A_d)\mathcal{X} + B_d U_d \\ \tau^+ = 0 \\ \bar{\mu}_e^+ = (A_d - L_d H_d) e \\ \tau_\delta^+ \in [0, \Delta] \end{array} \right\} (e, \tau, \bar{\mu}_e, \tau_\delta) \in D_{-1}^e(\Delta)$$

$$\left\{ \begin{array}{l} e^+ \in \bar{\mu}_e + [m, M]\Delta \\ \tau^+ = \tau \\ \bar{\mu}_e^+ \in \bar{\mu}_e + [m, M]\Delta \\ \tau_\delta^+ = -1 \end{array} \right\} (e, \tau, \bar{\mu}_e, \tau_\delta) \in D_0^e(\Delta) \quad (41)$$

with

$$C^e(\Delta) = \mathbb{R}^n \times [0, \tau_M] \times \mathbb{R}^n \times ([0, \Delta] \cup \{-1\}) \quad (42)$$

$$D_{-1}^e(\Delta) = \mathbb{R}^n \times [0, \tau_M] \times \mathbb{R}^n \times \{-1\} \quad (43)$$

$$D_0^e(\Delta) = \mathbb{R}^n \times [0, \tau_M] \times \mathbb{R}^n \times \{0\} \quad (44)$$

Now, observe that the values taken by e during the intervals $[t_j, t_{j+1}]$ where $\tau_\delta \in [0, \Delta]$, i.e for j in J_0 , have no impact on the other states because their flow map and the jump map on $D_0^e(\Delta)$ are independent from e . In other words, there exists a function \bar{e} defined on $\text{dom } \phi$ such that

$$\bar{e}(t, j) = e(t, j) \quad \forall (t, j) \in \text{dom } \phi : j \in J_{-1} \quad (45)$$

and $(\bar{e}, \tau, \bar{\mu}_e, \tau_\delta)$ is solution to

$$\bar{\mathcal{H}}^e(\Delta) \left\{ \begin{array}{l} \dot{\bar{e}} = (A_c - L_c H_c) \bar{e} \\ \dot{\tau} = 1 \\ \dot{\bar{\mu}}_e = 0 \\ \dot{\tau}_\delta = -\min(\tau_\delta + 1, 1) \end{array} \right\} (\bar{e}, \tau, \bar{\mu}_e, \tau_\delta) \in C^e(\Delta)$$

$$\left\{ \begin{array}{l} \bar{e}^+ = \bar{e} \\ \tau^+ = 0 \\ \bar{\mu}_e^+ = (A_d - L_d H_d) \bar{e} \\ \tau_\delta^+ \in [0, \Delta] \end{array} \right\} (\bar{e}, \tau, \bar{\mu}_e, \tau_\delta) \in D_{-1}^e(\Delta) \quad (46)$$

$$\left\{ \begin{array}{l} \bar{e}^+ \in \bar{\mu}_e + [m, M]\Delta \\ \tau^+ = \tau \\ \bar{\mu}_e^+ \in \bar{\mu}_e + [m, M]\Delta \\ \tau_\delta^+ = -1 \end{array} \right\} (e, \tau, \bar{\mu}_e, \tau_\delta) \in D_0^e(\Delta)$$

From [19, Theorem 1] and [23, Proposition 3.8], we know that the set $\mathcal{A}^e = \mathcal{A}_{-1}^e \cup \mathcal{A}_0^e$ with

$$\mathcal{A}_0^e = \{0\} \times [0, \tau_M] \times \{0\} \times \{0\}$$

$$\mathcal{A}_{-1}^e = \{0\} \times [0, \tau_M] \times \mathcal{M} \times \{-1\}$$

and $\mathcal{M} = (A_d - L_d H_d)\mathbb{R}^n$, is UGAS for $\bar{\mathcal{H}}^e(0)$. Since \mathcal{A}_e is compact and $\bar{\mathcal{H}}^e(0)$ verifies the hybrid basic conditions, \mathcal{A}_e is semi globally practically robustly \mathcal{KL} asymptotically stable

for $\bar{\mathcal{H}}^e(0)$ according to [20, Lemma 7.20]. This means that there exists a \mathcal{KL} function β such that for any $\varepsilon > 0$ and any compact set K of $C^e(0) \cup D^e(0)$, there exists $\rho > 0$ such that any solution $\bar{\phi}_e = (\bar{e}, \tau, \bar{\mu}_e, \tau_\delta)$ to a ρ -perturbation of $\bar{\mathcal{H}}^e(0)$, initialized in K verifies

$$|\bar{\phi}_e(t, j)|_{\mathcal{A}_e} \leq \beta(|\bar{\phi}_e(0, 0)|_{\mathcal{A}_e}, t + j) + \varepsilon.$$

Since

$$|\bar{e}(t, j)| \leq |\bar{\phi}_e(t, j)|_{\mathcal{A}_e} \leq |\bar{e}(t, j)| + |\bar{\mu}_e(t, j)|$$

and $\mathcal{H}^e(\Delta)$ can be included in any outer-perturbation of $\mathcal{H}^e(0)$ by taking Δ sufficiently small, we obtain (32) thanks to (45).

Finally, let us bound the error on the intervals given by j in J_0 . Because of (32) and the jump map of $\mathcal{H}^e(\Delta)$ when $\tau_\delta = -1$, we have for all j in J_0 ,

$$|e(t_j, j)| \leq \beta(|\epsilon(0, 0)|, t_j + j) + \varepsilon + \max_{x \in \mathcal{X}, u_d \in U_d} |(I - A_d)x + B_d u_d|$$

so there exists a scalar σ depending on the eigenvalues of $A_c - L_c H_c$ such that (33) holds.

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