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ANALYSIS OF A SUGIMOTO'S MODEL OF NONLINEAR ACOUSTICS IN AN ARRAY OF HELMHOLTZ RESONATORS

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A coupled system involving a nonlinear scalar PDE and a linear ODE is theoretically investigated. This hypebolic system with relaxation models the propagation of nonlinear waves in a waveguide connected to Helmholtz resonators, this device being an example of a nonlinear acoustic metamaterial. In a previous paper [Sugimoto, J. Fluid. Mech. 1992], it has been shown that this device allows also the propagation of acoustic solitons. In the present paper, the mathematical properties of the coupled system are analysed: formation of singularity in finite time, existence of global smooth solutions for small data, existence of entropy solutions in fractional BV spaces and uniqueness with a single family of entropies. New results are also deduced about weakly coupled systems. Numerical simulations illustrate these findings.

Keywords: nonlinear waves, hyperbolic systems, acoustic solitons, fractional BV spaces, balance laws, shocks.

AMS Subject Classification: 26A45, 35L65, 35L67, 76N10

1. Introduction

In a series of papers,[36,37,38,39] Sugimoto and coauthors proposed a model to describe the propagation of nonlinear acoustic waves in a tube connected to a set of Helmholtz resonators (Fig. 1). The original motivation of this work was to describe the high-amplitude waves generated by high-speed trains in tunnels, and to propose resonators as a mean to counterbalance the effect of shock waves. The Sugimoto's model writes as a coupled one-dimensional system of PDE and ODE ($x \in \mathbb{R}$, $t > 0$):

$$\begin{cases} \partial_t u + \partial_x \left(a u + b \, \dfrac{u^2}{2} \right) = -\Omega^2 \, \partial_t p, & \text{(1.1a)} \\ \partial_t^2 p + \varepsilon \, \partial_t p + \omega_0^2 \, p = u, & \text{(1.1b)} \end{cases}$$

with the coefficients $a > 0$, $b > 0$, $\varepsilon \geq 0$, the characteristic frequencies ω_0 and Ω, and the initial data

$$u(x,0) = u_0(x), \qquad p(x,0) = p_0(x), \qquad \partial_t p(x,0) = p_1(x). \tag{1.2}$$

Subsequently, the Sugimoto's model has led to numerical, experimental and theoretical works.[24,33,29] Enrichment of this model, in particular by taking into account nonlinear attenuation mechanisms in (1.1b), have led to the first experimental observation of acoustic solitons.[33]

Besides its physical relevance, the Sugimoto's model is of mathematical interest. It can be put in the form of a first-order 3×3 diagonal hyperbolic system. For systems of this form with N equations, the existence of solutions in BV spaces is known by using $N = 3$ families of entropies.[30] In the homogeneous scalar case ($N = 1$), existence was proven in the fractional BV^s spaces.[5] A similar result was subsequently obtained in the case of a 2×2 system issued from chemical engineering.[6] One objective here is to extend existence of BV^s solutions to the case of system (1.2). Another objective is to prove the formation of shocks and then, in the framework of weak entropy solutions, to get uniqueness with only one family of entropies.

The article is organized as follows. Sec. 2 recalls the physical modeling underlying the system (1.1), which is rewritten as a first-order 3×3 diagonal hyperbolic system. In Sec. 3, the breakdown of regular solutions for large initial data is investigated (Theorems 3.1 and 3.2). Existence of global smooth solutions for small data is expected physically. Therefore it is studied near equilibrium, based on the Shizuta-Kawashima coupling condition.[35] Numerical experiments illustrate the discussion. In Sec. 4, Theorem 4.1 states the existence of a global weak entropy solution for initial data in BV^s. Contrary to existing results,[30] one family of entropies is sufficient to prove the uniqueness of the entropy solution. Sec. 5 makes a link with the theory of weakly coupled systems, for which existence and uniqueness results are already known.[30] Theorem 5.2 extends these results to the case of BV^s solutions. Lastly, future directions of research are outlined in Sec. 6.

2. Problem statement

2.1. *Sugimoto's model*

Let us consider an air-filled tube connected with a network of Helmholtz resonators (Fig. 1). The cylindrical resonators are uniformly distributed along the tube. The geometrical parameters are the radius of the guide R, the axial spacing between resonators D, the radius of the neck r, the length of the neck L, the radius of the cavity r_h and the height of the cavity H, which may vary depending on the resonator. Hence, the cross-sectional area of the guide is $A = \pi R^2$ and the volume of each resonator is $V = \pi r_h^2 H$.

The physical parameters are the ratio of specific heats at constant pressure and volume γ, the pressure at equilibrium p_0, the density at equilibrium ρ_0, and a phe-

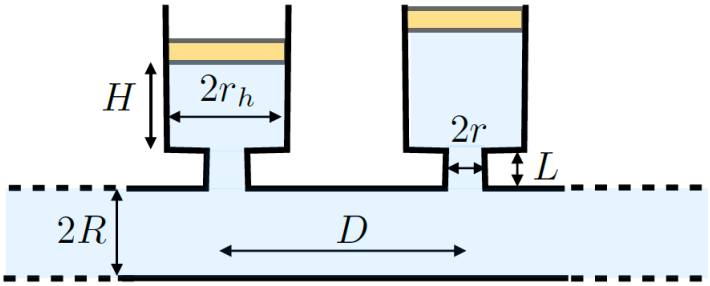

Fig. 1. sketch of the guide connected with Helmholtz resonators.

nomenological term modeling the losses in the necks of the resonators ϵ. The other dissipation mechanisms are neglected: the sound diffusivity is negligible, and modeling the boundary layer in the tube involves fractional derivatives, which complicates the analysis. It yields the linear sound speed a_0 and the natural angular frequencies ω_0 and Ω:

$$a_0 = \sqrt{\frac{\gamma\, p_0}{\rho_0}}, \qquad \omega_0 = a_0 \sqrt{\frac{B}{L\,V}} = a_0\, \frac{r}{r_h} \frac{1}{\sqrt{L\,H}}, \qquad \Omega = \sqrt{\frac{V}{2\,\rho_0\, a_0\, A\, D}}. \tag{2.1}$$

The following assumptions are made about the waves of angular frequency ω and wavelength λ propagating in the tube:

- low-frequency regime ($\omega < \omega^* \approx \frac{1.84\, a_0}{R}$), so that only the plane mode propagates and the 1D approximation is valid;[8]
- weak acoustic nonlinearity in the tube (small Mach number);[13]
- continuous distribution of resonators (wavelength $\lambda \gg D$);
- linear response of the resonators, no turbulence.

Under these assumptions, the system can be described by a one-dimensional configuration with a surfacic distribution of resonators, where the unknowns are the axial velocity of the gas u and the excess pressure in the cavity (compared to the guide) p. The right-going simple wave is then modeled by a coupled PDE-ODE system initially proposed by Sugimoto in Ref. 36:

$$\begin{cases} \partial_t u + \partial_x \left(a_0 u + \dfrac{\gamma+1}{2}\, \dfrac{u^2}{2} \right) = -\Omega^2\, \partial_t p, & \text{(2.2a)} \\ \partial_t^2 p + \epsilon\, \partial_t p + \omega_0^2\, p = \omega_0^2\, \dfrac{\gamma\, p_0}{a_0}\, u. & \text{(2.2b)} \end{cases}$$

Defining the normalization parameter $\theta = \omega_0^2 \frac{\gamma p_0}{a_0}$, the scaled coordinate $t \to t\sqrt{\theta}$ and the scaled parameters

$$a = \frac{a_0}{\sqrt{\theta}}, \qquad b = \frac{\gamma+1}{2\sqrt{\theta}}, \qquad \varepsilon := \frac{\epsilon}{\sqrt{\theta}}, \qquad \omega_0 := \frac{\omega_0}{\sqrt{\theta}}, \tag{2.3}$$

then the system (2.2) recovers (1.1). The parameters satisfy $a > 0$, $b > 0$, $c > 0$, $\omega_0 > 0$ and $\varepsilon \geq 0$, and they may vary with the position x. It is noted that the system originally proposed incorporates additional terms of attenuation.[36] The full system is given in Sec. 6.

2.2. *First-order hyperbolic system*

Introducing the new variable $\varphi := \partial_t p$, the Sugimoto's system (1.1) is put in the form of a first-order system

$$\begin{cases} \partial_t u + \partial_x \left(a u + b \, \dfrac{u^2}{2} \right) = -\Omega^2 \, \varphi, & \text{(2.4a)} \\ \partial_t \varphi = u - \varepsilon \varphi - \omega_0^2 \, p, & \text{(2.4b)} \\ \partial_t p = \varphi, & \text{(2.4c)} \end{cases}$$

with initial data

$$u(x,0) = u_0(x), \qquad \varphi(x,0) = \varphi_0(x), \qquad p(x,0) = p_0(x). \tag{2.5}$$

Compatibility between these initial data and those of the 2×2 Sugimoto's system will be examined further in (3.3). Let us consider smooth solutions of (2.4). Multiplying (2.4a) by u, (2.4b) by $\Omega^2 \, \varphi$ and (2.4c) by $\Omega^2 \, \omega_0^2 \, p$, adding the terms and integrating with respect to x yields

$$\frac{d}{dt} \frac{1}{2} \int_{\mathbb{R}} \left(u^2 + \Omega^2 \varphi^2 + \Omega^2 \, \omega_0^2 \, p^2 \right) \, dx = - \int_{\mathbb{R}} \varepsilon \, \Omega^2 \, \varphi^2 \, dx. \tag{2.6}$$

The energy in l.h.s. in (2.6) is convex. If $\varepsilon > 0$, the system is dissipative, whereas $\varepsilon = 0$ describes a conservative system. For weak solutions of (2.8), the equality in (2.6) becomes an inequality. Setting

$$\mathbf{U} = (u, \varphi, p)^\top, \qquad \mathbf{F}(\mathbf{U}) = \left(a u + b \, \frac{u^2}{2}, 0, 0 \right)^\top, \qquad \mathbf{S} = \begin{pmatrix} 0 & -\Omega^2 & 0 \\ 1 & -\varepsilon & -\omega_0^2 \\ 0 & 1 & 0 \end{pmatrix}, \tag{2.7}$$

the system (2.4) writes as a balance laws

$$\partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = \mathbf{S} \, \mathbf{U}. \tag{2.8}$$

The flux $\mathbf{F}$ in (2.7) is diagonal. The Jacobian matrix $\mathbf{A}(\mathbf{U}) = \nabla_{\mathbf{U}} \mathbf{F}(\mathbf{U})$ has 2 real eigenvalues:

- the eigenvalue $a + b \, u$ is genuinely nonlinear with the eigenvector $e_1 = (1,0,0)^\top$;
- the eigenvalue 0 is linearly degenerate of multiplicity 2, with an eigenspace generated by $e_2 = (0,1,0)^\top$ and $e_3 = (0,0,1)^\top$.

Without the source term, the system is diagonal, hence u, φ and p are the Riemann invariants. The eigenvalues of $\mathbf{S}$ are

$$\mathrm{Sp}(\mathbf{S}) = \left\{0,\ \frac{1}{2}\left(-\varepsilon + \left(\varepsilon^2 - 4\left(\Omega^2 + \omega_0^2\right)\right)^{1/2}\right),\ \frac{1}{2}\left(-\varepsilon - \left(\varepsilon^2 - 4\left(\Omega^2 + \omega_0^2\right)\right)^{1/2}\right)\right\}. \tag{2.9}$$

If $\varepsilon \neq 2\sqrt{\Omega^2 + \omega_0^2}$, then $\mathbf{S}$ is diagonalizable.

2.3. *Numerical methods*

The numerical resolution of (2.8) is detailed in Ref. 24. It relies on a splitting strategy. The non-homogeneous system (2.8) is split into a homogeneous hyperbolic PDE

$$\partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = \mathbf{0}$$

which is solved by a finite-volume scheme with flux limiters, and a linear ODE

$$\partial_t \mathbf{U} = \mathbf{S}\,\mathbf{U}$$

which is solved exactly. The coupling between the successive resolutions is done by the second-order Strang splitting.[21] The choice of the numerical set-up (mesh size Δx, variable time step Δt) ensure reference solutions, not polluted by numerical artefacts. The numerical value of a variable ϕ at $x_i = i\Delta x$ and at time $t_n = t_{n-1} + \Delta t$ is denoted ϕ_i^n.

To detect the emergence of a shock in the conservative case $\varepsilon = 0$, we use the fact that energy in (2.6) is conserved in the case of smooth solutions. For this purpose, we define a discrete counterpart of the energy in (2.6):

$$\mathcal{E}^n = \sqrt{\frac{\Delta x}{2} \sum_i \left((u_i^n)^2 + \Omega^2 (\varphi_i^n)^2 + \Omega^2\, \omega_0^2\, (p_i^n)^2\right)}. \tag{2.10}$$

As long as the solution is smooth, $\mathcal{E}^n$ remains constant. When a shock emerges, then $\mathcal{E}^n$ decreases. Obviously, this numerical criterion is valid when the numerical dissipation of the numerical scheme is sufficiently small, which will be the case here.

3. Formation of singularities

In the case of large data, we prove sufficient conditions for the breakdown of regular solutions in finite time. For this purpose, the 2×2 system (1.1) is first transformed into a scalar equation with a source term (Sec. 3.1). Then one considers successively the dissipative case $\varepsilon > 0$ (Sec. 3.2) and the conservative case $\varepsilon = 0$ (Sec. 3.3). In the case of small data, global smooth solutions are expected: the Sugimoto's model has been defined to prevent from the occurence of shocks and to propagate acoustic solitons.[36] The Shizuta-Kawashima is invocated to study the possible existence of global smooth solutions near equilibrium (Sec. 3.4). Numerical experiments are proposed to illustrate these properties (Sec. 3.5).

3.1. *Burgers equation with memory*

Equation (1.1b) is differentiated in terms of t. Using $\varphi = \partial_t p$ yields:

$$\begin{cases} \partial_t u + \partial_x \left(a u + b \, \dfrac{u^2}{2} \right) = -\Omega^2 \, \varphi, & \text{(3.1a)} \\ \dfrac{d^2\varphi}{dt^2} + \varepsilon \dfrac{d\varphi}{dt} + \omega_0^2 \varphi = \partial_t u \equiv \sigma, & \text{(3.1b)} \end{cases}$$

with the initial data

$$u(x,0) = u_0(x), \qquad \varphi(x,0) = \varphi_0(x), \qquad \frac{d\varphi}{dt}(x,0) = \varphi_1(x). \tag{3.2}$$

Smooth solutions of (1.1) are solutions of (3.1), provided the initial data (3.2) satisfy the compatibility condition

$$\varphi_1(x) = u_0(x) - \varepsilon \, \varphi_0(x) - \omega_0^2 \, p_0(x). \tag{3.3}$$

Conversely, integrating (3.1b) with respect to t recovers (1.1) under the condition (3.3). The explicit solution of (3.1b) is

$$\varphi(x,t) = \varphi_0(x) \, J_0(t) + \varphi_1(x) \, J_1(t) + \int_0^t J_1(t-s) \, \sigma(x,s) \, ds, \tag{3.4}$$

where J_0 and J_1 depend on the coefficients of (3.1b). Setting

$$\Delta^2 = \varepsilon^2 - (2\,\omega_0)^2, \qquad \delta = \sqrt{|\Delta|}, \tag{3.5}$$

calculations yield the following 4 cases:

- $\varepsilon = 0$, $\omega_0 = 0$:

$$J_0(t) = 1, \qquad J_1(t) = t. \tag{3.6}$$

- $0 \le \varepsilon < 2\,\omega_0$:

$$\begin{aligned} J_0(t) &= \exp(-\varepsilon\, t/2) \left(\cos(\delta t/2) + \frac{\varepsilon}{\delta} \, \sin(\delta t/2) \right) \underset{\varepsilon \to 0}{\equiv} \cos \omega_0 t, \\ J_1(t) &= \frac{2}{\delta} \exp(-\varepsilon\, t/2) \sin(\delta t/2) \underset{\varepsilon \to 0}{\equiv} \frac{1}{\omega_0} \, \sin \omega_0 t. \end{aligned} \tag{3.7}$$

- $\varepsilon = 2\,\omega_0$:

$$J_0(t) = \left(1 + \frac{\varepsilon}{2}\, t \right) \exp(-\varepsilon\, t/2), \qquad J_1(t) = t \, \exp(-\varepsilon\, t/2). \tag{3.8}$$

- $\varepsilon > 2\,\omega_0$:

$$\begin{aligned} J_0(t) &= \frac{1}{2\,\delta} \left((\varepsilon + \delta) \, \exp\left(-(\varepsilon - \delta)\, t/2\right) - (\varepsilon - \delta) \, \exp\left(-(\varepsilon + \delta)\, t/2\right) \right), \\ J_1(t) &= \frac{1}{\delta} \left(\exp\left(-(\varepsilon - \delta)\, t/2\right) - \exp\left(-(\varepsilon + \delta)\, t/2\right) \right). \end{aligned} \tag{3.9}$$

The solution (3.4) is injected into (3.1a). Integration by parts and the property $J_1(0) = 0$ allow to transform the 2×2 system (1.1) into the nonlocal scalar PDE

$$\begin{aligned} \partial_t u + \partial_x \left(a u + b \frac{u^2}{2} \right) &= \varphi_0(x)\, K_0(t) + (\varphi_1(x) - u_0(x))\; K_1(t) + \mathcal{L}u(x,t), \\ &= \varphi_0(x)\, (K_0(t) - \varepsilon\, K_1(t)) - \omega_0^2\, p_0(x)\, K_1(t) + \mathcal{L}u(x,t) \end{aligned} \tag{3.10}$$

where the compatibility condition (3.3) has been used, and with

$$\begin{aligned} &K_0(t) = -\Omega^2\, J_0(t), \qquad K_1(t) = -\Omega^2\, J_1(t), \qquad \mathcal{K}(t) = -\Omega^2\, J_1^{'}(t), \\ &\mathcal{L}u(x,t) = \int_0^t \mathcal{K}(t-s)\, u(x,s)\, ds. \end{aligned} \tag{3.11}$$

In the case where the resonators are initially at rest $p_0(x) = 0$ and $p_1(x) = 0$, then (3.10) reduces to the scalar PDE with memory

$$\partial_t u + \partial_x \left(a u + b \frac{u^2}{2} \right) = \int_0^t \mathcal{K}(t-s)\, u(x,s)\, ds. \tag{3.12}$$

This is a Burgers equation with a non-local source term as in Ref. 7.

3.2. *Dissipative case*

Based on (3.10)-(3.11), some norms are introduced:

$$\begin{aligned} &C_0 = \|\varphi_0^{'}\|_\infty \|K_0 - \varepsilon K_1\|_\infty + \omega_0^2\, \|p_0^{'}\|_\infty \|K_1\|_\infty, \\ &C = \int_0^{+\infty} |\mathcal{K}(t)|\, dt, \\ &C_\sharp = \frac{C + \sqrt{C^2 + 4\, b\, C_0}}{2\, b} \geq C. \end{aligned} \tag{3.13}$$

In the dissipative case $\varepsilon > 0$, C in (3.13) can be computed analytically from (3.7)-(3.9) and (3.11). Setting $t^* = \frac{1}{\delta} \arctan \frac{\delta}{\varepsilon}$, one obtains:

$$C = \begin{cases} \dfrac{2\,\Omega^2\,\varepsilon}{\varepsilon^2 + 4\,\delta^2} \left(-1 + \dfrac{4\,\sqrt{1 + (\delta/\varepsilon)^2}}{1 - \exp\left(-(\varepsilon\,\pi)/(2\,\delta)\right)} \exp\left(-\varepsilon\, t^*/2\right) \right) & \text{if } 0 < \varepsilon < 2\,\omega_0, \\ \dfrac{4\,\Omega^2}{\varepsilon\, \exp(1)} & \text{if } \varepsilon = 2\,\omega_0, \\ \dfrac{4\,\Omega^2}{\varepsilon + \delta} \left(\dfrac{\varepsilon - \delta}{\varepsilon + \delta} \right)^{\dfrac{\varepsilon - \delta}{2\,\delta}} & \text{if } \varepsilon > 2\,\omega_0. \end{cases} \tag{3.14}$$

One introduces also the straightforward lemma.

Lemma 3.1. *If $m < -C_\sharp$, the inequality $-b\,m^2 + C_0 + C|m| < 0$ holds. Similarly, $M > C_\sharp$ implies $-b\,M^2 + C_0 + C|M| < 0$.*

Sufficient conditions are now stated for the formation of singularity with smooth initial data.

Theorem 3.1 (Shock in finite time: case $\varepsilon > 0$). *Let us assume that the initial data (u_0, φ_0, p_0) are smooth with compact support. If $\inf u_0^{'}(x) < -C_\sharp$ and $\sup u_0^{'}(x) \leq C_\sharp$, then a shock appears in finite time.*

Proof. Following the idea of Lax,[19] we deduce from (3.10) the Riccati equation satisfied by $U = \partial_x u$ with initial condition $U_0 = u_0^{'}$:

$$\begin{aligned} \partial_t U + (a + bu)\,\partial_x U + b\,U^2 &= \varphi_0^{'}(x)(K_0(t) - \varepsilon K_1(t)) - \omega_0^2\, p_0^{'}(x)\, K_1(t) + \mathcal{L}\,U, \\ &\equiv k(x,t) + \mathcal{L}\,U. \end{aligned} \tag{3.15}$$

As long as the solution is smooth, equation (3.15) is rewritten along the characteristics:

$$\frac{d}{dt}U = -b\,U^2 + k(x,t) + \mathcal{L}\,U. \tag{3.16}$$

The integration is now along characteristic curves, so that the operator $\mathcal{L}$ is modified; nevertheless, the notation $\mathcal{L}$ is kept for simplicity. In the case of the homogeneous Burgers equation, the Riccati ODE is simply

$$\frac{d}{dt}U = -b\,U^2,$$

which blows-up at time $T^* = -1/(b\,U(0))$ if $U(0) < 0$. When a linear dissipative source term $-\lambda\,u$ is added to the Burgers equation, a similar analysis yields

$$\frac{d}{dt}U = -b\,U^2 - \lambda\,U,$$

whose threshold is $\inf U_0 < -\lambda/b$. The analysis of (3.16) is more complex due to the nonlocal source term $\mathcal{L}\,U$, which can accelerate or moderate the blow-up of U. Roughly speaking, the competition between the nonlinear part and the linear part favors a blow-up if $\frac{d}{dt}U < 0$ is large enough. Let us prove it under the assumptions of the theorem. At any x, equation (3.16) and the notations (3.13) give

$$\frac{d}{dt}U \leq -b\,U^2 + C_0 + C \sup_{(y,s)\in\mathbb{R}\times[0,t]} |U(y,s)|. \tag{3.17}$$

One introduces the suprema $m(t) = \inf U(.,t) = U(x_0,t_0)$ and $M(t) = \sup U(.,t)$. The existence of x_0 is ensured by the fact that the solution of (3.16) remains with compact support at all time (Sec. 4.2). Under the two conditions

$$m(t_0) < -C\sharp, \qquad |m(t_0)| > M(t_0), \tag{3.18}$$

the Lemma 3.1 ensures that the r.h.s of (3.17) is negative at (x_0,t_0). Similarly, $|M| > C_\sharp$ implies that $M(t) < C_\sharp$ for all time: $C_\sharp$ is a *barrier* for $M(t)$.[16]

It remains to prove (3.18) for all t. These two conditions are valid at $t_0 = 0$ from the assumptions of the theorem. Assuming that (3.18) is true at a given t_0, $U(x_0,t)$

is decreasing at least locally for $t < t_0$ and $t \simeq t_0$. Thus, for $h > 0$ small enough,

$$\begin{aligned} m(t_0+h) &\le U(x_0, t_0+h) = U(x_0,t_0) + h\dot{U}(x_0,t_0) + \mathcal{O}(h^2), \\ &\le m(t_0) + h\dot{U}(x_0,t_0) + \mathcal{O}(h^2), \\ &\le m(t_0) + h\left(-b\, m(t_0)^2 + C_0 + C|m(t_0)|\right) + \mathcal{O}(h^2). \end{aligned} \tag{3.19}$$

Denoting $\dot{m}$ the generalized right derivative of m:[9]

$$\dot{m}(t) = \limsup_{h>0, h\to 0} \frac{m(t+h) - m(t)}{h}, \tag{3.20}$$

then m satisfies

$$\dot{m}(t_0) \le -b\, m(t_0)^2 + C_0 + C|m(t_0)|. \tag{3.21}$$

It follows $m(t) \le y(t)$, where y satisfies

$$\begin{cases} \dot{y} = -b\, y^2 + C_0 - Cy, \\ y(0) = y_0 := m(0) < -C_\sharp. \end{cases} \tag{3.22}$$

The solution of (3.22) is explicitly known and blows-up in finite time T^*:

$$T^* = -\frac{-\ln \alpha}{\sqrt{C^2 + 4\, b\, C_0}} > 0, \qquad \alpha = \frac{C + \sqrt{C^2 + 4\, b\, C_0} + 2\, b\, y_0}{C - \sqrt{C^2 + 4\, b\, C_0} + 2\, b\, y_0}, \tag{3.23}$$

with $0 < \alpha < 1$. Since y is an upper solution of m, it follows that m blows up in finite time. The lifetime of $U = \partial_x u$ is smaller or equal than T^*. □

Remark 3.1. If the second condition $\sup u_0'(x) \le C_\sharp$ is not fullfilled in Theorem 3.1, a global smooth solution may exist.

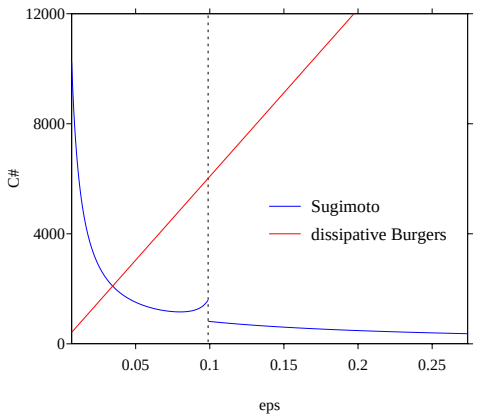

Fig. 2. Evolution of $C_\sharp$ (3.13) in terms of the dissipation ε. The vertical dashed line denotes $\varepsilon = 2\,\omega_0$. The parameters are given in Table 1.

In the case of the scalar Burgers equation with linear dissipation

$$\begin{cases} \partial_t u + \partial_x \left(a\, u + b\, \dfrac{u^2}{2} \right) = -\varepsilon\, u, \\ u(x,0) = u_0(x), \end{cases} \tag{3.24}$$

shock occurs iff $\inf u_0^{'}(x) < -\varepsilon/b \equiv C_\sharp$. This minimal slope of the initial data increases linearly with the dissipation parameter. In the case of Sugimoto's system, on the contrary, the shock limit $C_\sharp$ in (3.14) does not evolve monotonically with ε. It is recalled, nevertheless, that it is only a sufficient condition (and not necessary). The evolution of $C_\sharp$ with ε for both models is illustrated in Fig. 2.

3.3. *Conservative case*

Theorem 3.1 is now extended to the conservative case $\varepsilon = 0$ with $\omega_0 \neq 0$. Any $t > 0$ can be written $t = n\,\pi/(2\,\omega_0) + r$, with $0 \leq r < \pi/(2\,\omega_0)$. From (3.7) and (3.11), one obtains

$$\int_0^t |\mathcal{K}(\tau)|\, d\tau = \frac{2\,\Omega^2}{\pi} \left(t - r + \frac{\pi}{2\,\omega_0}\, \sin \omega_0\, r \right),$$

and hence

$$\|\mathcal{L}(.,t)\|_\infty \leq C(1+t), \qquad \text{with } C = \frac{2\,\Omega^2}{\pi}\, \max\left(1,\, \frac{\pi}{2\,\omega_0}\right). \tag{3.25}$$

Theorem 3.2 (Shock in finite time: case $\varepsilon = 0$). *Let consider smooth initial data (u_0, φ_0, p_0) with compact support. There exist two constants $C^\pm > 0$ depending on (φ_0, p_0) such that if $\inf u_0^{'} < -C^-$ and $\sup u_0^{'} \leq C^+$, then a shock appears in finite time.*

Proof. We follow the proof of Theorem 3.1 with the same notations. The main modification is that C has to be replaced by $C\,t$ in (3.22) since the norm of the linear operator is now of order t, as seen in (3.25):

$$\dot{y} = -b\, y^2 + C_0 - C(1+t)\, y. \tag{3.26}$$

Contrary to the dissipative case, the ODE (3.26) has not separable variables yielding an explicit solution. To prove the theorem, it suffices to get an upper solution blowing up in finite time $T^* < 1$. Setting $D = \max(C_0,\, 2C)$, these solutions satisfy $y < 0$ and

$$\dot{y} \leq f(y) := -b\, y^2 + D\,(1 + |y|). \tag{3.27}$$

Let z be the solution of $\dot{z} = f(z)$ and z^- be the negative root of $f(z) = 0$. If $z_0 < z^-$, then z is an upper solution of y; it is also decreasing and its lifespan is $T(z_0) = \int_{z_0}^{+\infty} \frac{dz}{-b\, z^2 + D(1+|z|)} \to 0$ when $z_0 \to -\infty$. Thus, there exists a constant C^- such that for $z_0 \leq -C^-$, then $T(z_0) < 1$. Consequently, the solution y of (3.26) with $y_0 \leq -C^-$ blows up before time 1. The constant C^+ is chosen as in the proof of Theorem 3.1 to keep the r.h.s. of (3.26) negative, which concludes the proof. □

3.4. *Case of small solutions*

If $\varepsilon \neq 0$, the hyperbolic system (2.4) is dissipative. In this case, Theorem 3.1 gives sufficient conditions of singularity formation for large smooth initial data. But due to the dissipation, global smooth solutions are also expected if the initial data are smooth and sufficiently small.

The so-called Shizuta-Kawashima coupling condition, or [SK]-condition, provides sufficient conditions for the existence of small global smooth solutions near equilibrium for partially dissipative hyperbolic system.[35,1] This condition is easy to characterize: no eigenvector of $\mathbf{A}$ must belong to $\ker(\mathbf{S})$. Application of this characterization to (2.4) yields the following proposition.

Proposition 3.1. *The [SK]-condition is fulfilled at all non-zero equilibrium parametrized by $p_e \neq 0$:*

$$(u_e, \varphi_e, p_e)^\top = (\omega_0^2\, p_e, 0, p_e)^\top. \tag{3.28}$$

On the contrary, the [SK] condition is not satisfied at rest:

$$(u_e, \varphi_e, p_e)^\top = (0, 0, 0)^\top. \tag{3.29}$$

Proof. When $u_e \neq 0$, $\ker(\mathbf{S}) = \{s\,(\omega_0^2, 0, 1)^\top,\ s \in \mathbb{R}\}$ is a one-dimensional subspace defined by $\varphi = 0$, $u = \omega_0^2\, p$. The first eigenvector of $\mathbf{A}$ associated with the non-null eigenvalue, $e_1 = (1, 0, 0) \notin \ker(\mathbf{S})$. The two-dimensional eigenspace associated to the eigenvalue 0 of $\mathbf{A}$ clearly does not belong to the kernel of $\mathbf{S}$.

On the contrary, if $u_e = 0$, the Jacobian matrix writes $\mathbf{A} = a\,\mathbf{I}$ where $\mathbf{I}$ is the 3×3 identity matrix. The eigenspace of $\mathbf{A}$ encompasses the full space, so the [SK]-condition is not satisfied by (3.29). □

The [SK]-condition is only sufficient for partially dissipative systems.[35] [SK] is not a necessary condition to have global smooth solutions.[28,41]

3.5. *Numerical experiments*

R (m)	D (m)	r (m)	L (m)	r_h (m)	H (m)
0.025	0.1	0.01	0.02	0.0215	0.1
γ	p_0 (Pa)	ρ_0 (kg/m^3)	ϵ (s^{-1})		
1.403	$1.01\,10^5$	1.177	5000		
a	b	ε	Ω	ω_0	
$4.75\,10^{-3}$	$1.64\,10^{-5}$	$6.84\,10^{-2}$	$3.00\,10^{-2}$	$4.94\,10^{-2}$	

Table 1. Geometrical, physical and scaled parameters in Sugimoto's model (2.4).

The numerical values of the geometrical and physical parameters (Sec. 2.1) are issued from Ref. 33. Based on (2.3), il follows the scaled parameters (2.4). The full set of parameters is given in Table 1.

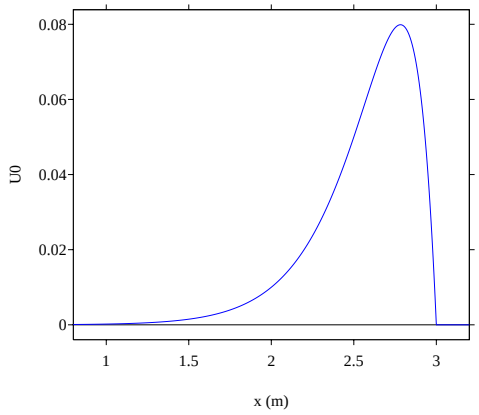

Fig. 3. Initial data $u_0(x)$ given in (3.30).

In Theorem 3.1, the sufficient condition for the formation of shock involves different bounds for the maximum and minimum slopes. To place ourselves in the hypotheses of the theorem, we use as initial data

$$u_0(x) = \begin{cases} g(x - x_0) & \text{if } x < x_0, \\ 0 & \text{else,} \end{cases} \qquad \text{with } g(x) = A\, x\, \exp(-x/\sigma). \tag{3.30}$$

The other initial conditions are null: $\varphi_0 = p_0 \equiv 0$. In (3.30), one chooses $x_0 = 4$ m and $\sigma = 0.427$ m, which corresponds roughly to a wavelength $\lambda = 4$ m (Fig. 3). The maximal and minimal slopes of the initial data (3.30) are deduced: $\sup u_0'(x) = u_0'(x_0 - 2\sigma) = A/(\exp(1))^2$ and $\inf u_0'(x) = u_0'(x_0) = -A$.

Fig. 4 illustrates the conservative case $\varepsilon = 0$, for which no bounds are explicitly known about the slopes $\inf u_0'(x)$ and $\sup u_0'(x)$ (Theorem 3.2). We represent u at different instants as well as the temporal evolution of the discrete energy (2.10), for two values of the amplitude: $A = 100$ and $A = 5000$. If $A = 100$, the discrete energy $\mathcal{E}$ is constant, which indicates that no shock appears. If $A = 5000$ on the contrary, the discrete energy drops from the first moments, then reaches a plateau. It indicates the formation of a shock visible in at $t = 200$ s. Once the shock is formed, a smooth soliton emerges and propagates without deformation.

The dissipative case is illustrated in Fig. 5. The discrete energy is no longer shown, since it decreases independantly from the occurence of shocks, and thus it does not give an indicator of singularity formation. The initial data yield $C_0 = 0$ and $C_\sharp = 1215.55$ in (3.13). If $A = 100$, then $\sup u_0'(x) = 13.53 < C_\sharp$ but $\inf u_0'(x) = -100 > C_\sharp$: the sufficient condition of singularity formation is no more satisfied. No

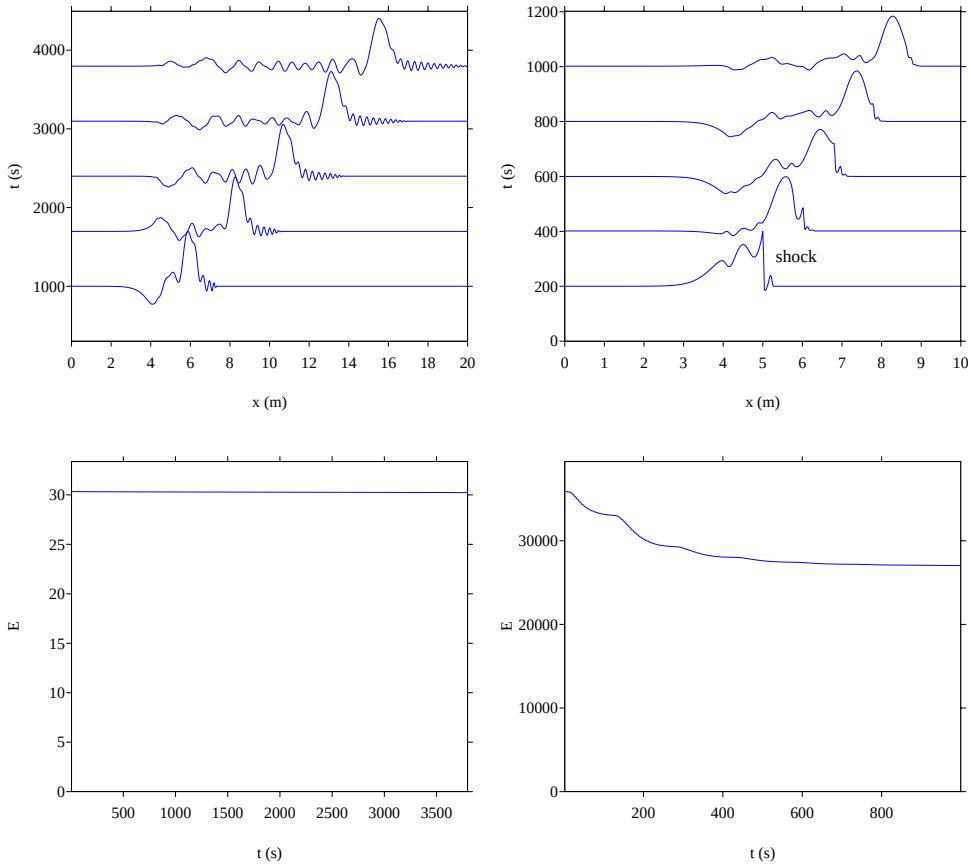

Fig. 4. Snapshots of u at various instants (top) and time evolution of the discrete energy (bottom) in the conservative case $\varepsilon = 0$. Left row: $A = 100$, right row: $A = 5000$.

shock is observed (left row). If $A = 5000$, on the contrary, then $\sup u_0^{'}(x) = 676 < C_\sharp$ and $\inf u_0^{'}(x) = -5000 < C_\sharp$: the assumptions of Theorem 3.1 are satisfied. Indeed, a shock is observed during the first instants (right row).

4. Large global BV^s solutions

4.1. *Definitions and main result*

Since shocks can occur, weak solutions are now considered.

Definition 4.1 (Weak entropy solutions). Let consider the system (2.8) with initial data $\mathbf{U}_0$. A function $\mathbf{U} \in L^\infty_{loc}([0,+\infty[\times\mathbb{R},\mathbb{R}^3)$ is a weak solution if, for all smooth vectorial test functions $\mathbf{\Phi} \in C^\infty_c(\mathbb{R}\times[0,+\infty[,\mathbb{R}^3)$ with compact support in

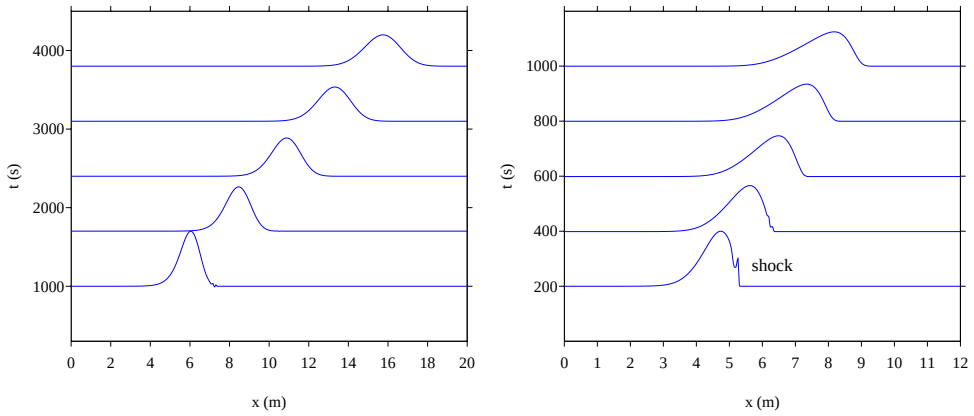

Fig. 5. Snapshots of u at various instants in the dissipative case $\varepsilon \neq 0$. Left row: $A = 100$, right row: $A = 5000$.

$\mathbb{R} \times [0, +\infty[$, one has:

$$\int_0^{+\infty} \int_{\mathbb{R}} \left(\mathbf{U} \cdot \partial_t \mathbf{\Phi} + \mathbf{F}(\mathbf{U}) \cdot \partial_x \mathbf{\Phi} + \mathbf{\Phi}^\top \cdot \mathbf{S}\,\mathbf{U} \right) dx\, dt + \int_{\mathbb{R}} \mathbf{U}_0(x) \cdot \mathbf{\Phi}(x,0) dx = 0. \tag{4.1}$$

Moreover, if $\mathbf{U} \in C^0([0, +\infty[, L^1_{loc}(\mathbb{R}, \mathbb{R}^3)$ and satisfies for all convex entropy $\eta \in C^2(\mathbb{R}, \mathbb{R})$ with the associated entropy flux $\psi'(u) = (a + b\,u)\,\eta'(u)$ and for all nonnegative scalar test functions $\phi \in C_c^\infty(\mathbb{R} \times [0, +\infty[, \mathbb{R}^+)$:

$$\int_0^{+\infty} \int_{\mathbb{R}} \left(\eta(u)\, \partial_t \phi + \psi(u)\, \partial_x \phi + \eta'(u)\, \Omega^2\, \varphi\, \phi \right) dx\, dt + \int_{\mathbb{R}} \eta(u_0(x))\, \phi(x,0)\, dx \geq 0, \tag{4.2}$$

then $\mathbf{U}$ is called an entropy solution.

For weakly coupled systems, entropies are usually tested on each equation to get uniqueness results,[30] as recalled in the forthcoming Sec. 5. On the contrary, the entropy condition in Definition 4.1 is only tested on the first equation (2.4a) of the Sugimoto's system. This choice is motivated by the fact that the only nonlinearity (generating shock waves) occurs in the first equation. As seen further, using a single family of entropies suffices to prove the uniqueness of the solution to (2.4). The fractional spaces BV^s, $0 < s < 1$, are recalled:

Definition 4.2 (BV^s function). Let TV^s be the total variation, also called the $p-$variation with $p = 1/s$:

$$TV^s(u) = \sup_{n \in \mathbb{N},\ x_0 < x_1 < \ldots < x_n} \sum_{0 \leq i < n} |u(x_{i+1}) - u(x_i)|^{1/s}. \tag{4.3}$$

The BV^s scalar functions on $\mathbb{R}$ are defined by $\mathrm{BV}^s(\mathbb{R}) = \{u,\ TV^s(u) < +\infty\}$. A vectorial function is in BV^s if all its components belong to BV^s. Equivalently, the TV^s variation can be taken with a vectorial norm instead of the absolute value.

Now we state the main theorem of this paper, concerning the existence of BV^s solutions of the Sugimoto's system.

Theorem 4.1 (Large $\mathbf{BV}^s$ solutions). *Let the initial data* $\mathbf{U}_0 = (u_0, \varphi_0, p_0)^\top$ *belong to* $BV^s(\mathbb{R}, \mathbb{R}^3)$, $0 < s < 1$. *Then the system* (2.4) *admits an unique global weak entropy solution in*

$$L^\infty_{loc}([0, +\infty[, BV^s(\mathbb{R}, \mathbb{R}^3) \cap C^s_{loc}([0, +\infty[, L^{1/s}_{loc}(\mathbb{R}, \mathbb{R}^3).$$

Moreover, the solution satisfies the energy inequality:

$$\frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}} \left(u^2 + \Omega^2\,\varphi^2 + \Omega^2\,\omega_0^2\,p^2\right)\,dx \leq -\int \varepsilon\,\Omega^2\,\varphi^2\,dx. \tag{4.4}$$

Remark 4.1 (BV solutions). For $s = 1$, the usual $\mathrm{BV}=\mathrm{BV}^1$ solutions for scalar conservation laws are recovered by the Sugimotos' system where C^s has to be replaced by Lipschitz: if $\mathbf{U}_0$ belongs to $\mathrm{BV}(\mathbb{R}, \mathbb{R}^3)$ then

$$\mathbf{U} \in L^\infty_{loc}([0, +\infty[, \mathrm{BV}(\mathbb{R}, \mathbb{R}^3) \cap \mathrm{Lip}_{loc}([0, +\infty[, L^1_{loc}(\mathbb{R}, \mathbb{R}^3).$$

Remark 4.2 (L^∞ solutions). For $s = 0$, the functional space is $\mathrm{BV}^0 = L^\infty$. For weakly coupled systems with initial data in L^∞, results similar to 4.1 are already known,[30] as seen in Sec. 5. In this particular case, Theorem 4.1 yields a new result: one whole family of scalar entropy $\eta \in C^2(\mathbb{R}, \mathbb{R})$ depending only on u is sufficient to prove uniqueness.

The rest of Sec. 4 is devoted to the proof of Theorem 4.1, which amounts to prove the propagation of the BV^s regularity in space for all positive time. The existence of BV^s solutions relies on a splitting scheme. The uniqueness follows from the classical Kruzkov proof of doubling of variables. Such a result is known for weakly coupled systems[30] using as many families of entropy as the size of the system; we check below that one single family of entropies is sufficient for the Sugimoto's system, which constitutes a slight improvement. Finally, it ends by the proof of the energy inequality.

4.2. *Splitting scheme*

The BV^s, BV and L^∞ estimates are obtained by a splitting scheme which splits the system (2.8) in two parts: first, the Burgers equation, second the differential systems. These two parts are studied independently.

Let $\Delta t > 0$, $\mathbf{U}^0(x) = \mathbf{U}_0(x)$, $\mathbf{U}^n = (u^n, \varphi^n, p^n)^\top$ stands for the approximate solution built by the splitting scheme at time $t_n = n\Delta t$. The half-step value $\mathbf{U}^{n+1/2}$ is the solution of the homogeneous hyperbolic PDE

$$\begin{cases} \partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = \mathbf{0}, & (x,t) \in \mathbb{R}\times]t_n, t_{n+1}], \\ \mathbf{U}(x, t_n) = \mathbf{U}^n. \end{cases} \tag{4.5}$$

Then $\mathbf{U}^{n+1}$ is given by the solution of the ODE parametrized by x,

$$\begin{cases} \dfrac{d}{dt}\mathbf{U} = \mathbf{S}\,\mathbf{U}, \qquad t \in]t_n, t_{n+1}] \\ \mathbf{U}(.,t_n) = \mathbf{U}^{n+1/2}. \end{cases} \tag{4.6}$$

The splitting scheme can be performed in various ways, for instance a wave front tracking[10] for Burgers and an approximate solution for the ODE. We choose here the most precise splitting scheme. The hyperbolic part is only a Burgers equation which is solved exactly. The linear ODE is solved explicitly:

(1) $u^{n+1/2}$ is the *exact* solution at time $t_n + \Delta t$ of the Burgers equation with initial data u^n. The other components are constant: $\varphi^{n+1/2} = \varphi^n$ and $p^{n+1/2} = p^n$;
(2) $\mathbf{U}^{n+1} = \exp(t\,\mathbf{S})\,\mathbf{U}^{n+1/2}$ is the *exact* solution of the ODE.

Burgers equation. It is well known that both the L^∞ norm and the total variation of the entropy solution u of the Burgers equation are non increasing.[10] A similar result has been obtained recently concerning the fractional total variation: $TV^s(u)$ is non increasing with respect to time.[5]

Differential system. The ODE (4.6) writes

$$\begin{cases} \dfrac{du}{dt} = -\Omega^2\,\varphi, & \text{(4.7a)} \\ \dfrac{d\varphi}{dt} = u - \varepsilon\,\varphi - \omega_0^2\,p, & \text{(4.7b)} \\ \dfrac{dp}{dt} = \varphi. & \text{(4.7c)} \end{cases}$$

The Euclidean norm associated to this differential system is

$$|\mathbf{U}|_2^2 = u^2 + \Omega^2\,\varphi^2 + \Omega^2\,\omega_0^2\,p^2. \tag{4.8}$$

The L^∞ norm and the total variation w.r.t. x are chosen in accordance with (4.8):

$$\begin{aligned} &\|\mathbf{U}(x)\|_{2,\infty} = \sup_{x\in\mathbb{R}} |\mathbf{U}(x)|_2, \\ &TV_2^s(\mathbf{U}) = \sup_{n\in\mathbb{N},\ x_0<\ldots<x_n} \sum_{0\le i<n} |\mathbf{U}(x_{i+1}) - \mathbf{U}(x_i)|_2^{1/s}. \end{aligned} \tag{4.9}$$

Proposition 4.1 (Decaying energy and total variation). *The solution of* (4.6) *satisfies*

$$\begin{cases} \dfrac{1}{2}\dfrac{d}{dt}|\mathbf{U}|_2^2 = -\varepsilon\,\Omega^2\,\varphi^2 \le 0, & \text{(4.10a)} \\ TV_2^s(\mathbf{U}(.,t)) \le TV_2^s(\mathbf{U}(.,0)). & \text{(4.10b)} \end{cases}$$

Proof. The derivation of (4.10a) is usual. It follows the same computation as for the global space energy decay (2.6), except there is no space variable. Mutiplying (4.7a) by u, (4.7b) by $\Omega^2\,\varphi$, (4.7c) by $\Omega^2\,\omega_0^2\,p$ and adding all the terms yields

$$\frac{1}{2}\frac{d}{dt}\left(u^2+\Omega^2\,\varphi^2+\Omega^2\,\omega_0^2\,p^2\right)=-\varepsilon\,\Omega^2\,\varphi^2.$$

To study the total variation in space, let $x<y$. The differential system is linear with respect to the initial data $\mathbf{U}(x,0)-\mathbf{U}(y,0)$, thus $\mathbf{U}(x,t)-\mathbf{U}(y,t)$ satisfies also (4.6). Thus, the inequality $|\mathbf{U}(x,t)-\mathbf{U}(y,t)|_2\le|\mathbf{U}(x,0)-\mathbf{U}(y,0)|_2$ is a direct consequence of the pointwise decay of the energy (4.10a). Summing up the variation and taking the supremum yields (4.10b). □

Concerning the asymptotic behavior of the solution $\mathbf{U}(t)$ of the differential system (4.7) with initial data $(u_0,\varphi_0,p_0)^\top$, the equilibrium are the constant states $(u,\varphi,p)=(\omega_0^2\,p_0,0,p_0)^\top$ which belong to $\ker(\mathbf{S})$. The matrix $\mathbf{S}$ is diagonalizable, except for one value of ε (2.9). Using a basis diagonalizing $\mathbf{S}$ or the Jordan normal form, one writes $\mathbf{U}_0=\mathbf{U}_1+\mathbf{U}_2$ where $\mathbf{U}_1$ belongs to $\ker(\mathbf{S})$ and $\mathbf{U}_2$ belongs to the plane P in the range of $\mathbf{S}$. Two cases occur:

(1) the dissipative case $\varepsilon>0$. Then $\mathbf{U}(t)\to\mathbf{U}_1$ exponentially when $t\to+\infty$.
(2) the conservative case $\varepsilon=0$. Then $\mathbf{U}(t)-\mathbf{U}_1$ is an harmonic oscillator in the plane P.

4.3. *BV estimate for the full system*

Now, we are able to obtain uniform estimates of $\mathbf{U}$ on any strip $[0,T]\times\mathbb{R}$ for any positive time T. Using the decrease of the norm and of the total variation in each part of the splitting, it is tempting to expect the same decay for the whole splitting scheme, but it is wrong. The reason is that the norms involved in the two steps of the splitting are not the same. For the Burgers part of the splitting, the norm $\max(\|u\|_\infty,\|\varphi\|_\infty,\|p\|_\infty)$ is not increasing, and for the ODE part this is the norm $\|U\|_\infty$ related to the Euclidean norm (4.8) which is non increasing. The following norm and total variation, well adapted to the Burgers' part of the splitting scheme, are now introduced:

$$\begin{aligned}&|\mathbf{U}|=\max(|u|,|\varphi|,|p|),\\&\|\mathbf{U}\|_\infty=\max(\|u\|_\infty,\|\varphi\|_\infty,\|p\|_\infty),\\&TV^s(\mathbf{U})=\max\left(TV^s(u),TV^s(\varphi),TV^s(p)\right).\end{aligned}\tag{4.11}$$

Based on these norms, the following proposition provides bounds sufficient to prove the convergence, as done in Ref. 10, 6.

Proposition 4.2 (L^∞ and $\mathbf{BV}^s$ estimates on $(0,T)\times\mathbb{R}$). *There exists a positive constant c such that for all n such that $n\,\Delta t\le T$,*

$$\begin{aligned}&\|\mathbf{U}^n\|_\infty\le\|\mathbf{U}^0\|_\infty\,\exp(c\,T),\\&TV^s(\mathbf{U}^n)\le TV^s(\mathbf{U}^0)\,\exp(c\,T).\end{aligned}\tag{4.12}$$

Proof. The two parts of the splitting are studied successively, by induction. First, solving the Burgers equation and thanks to the maximum principle, one gets $\|u^{n+1/2}\|_\infty \leq \|u^n\|_\infty$. Morerover $\varphi^{n+1/2} = \varphi^n$, $p^{n+1/2} = p^n$ are unchanged, so that $\|\mathbf{U}^{n+1/2}\|_\infty \leq \|\mathbf{U}^n\|_\infty$. Second, solving exactly the differential linear system (4.6) can increase the L^∞ norm. Let $c > 0$ be the matrix norm of $\mathbf{S}$ related to the vectorial norm $\|.\|_\infty$:

$$c = \sup_{|\mathbf{U}|=1} |\mathbf{S}\,\mathbf{U}| = |||\mathbf{S}|||_\infty,$$

then

$$|\mathbf{U}^{n+1}(x)| \leq \exp(c\,\Delta t)\,|\mathbf{U}^{n+1/2}(x)| \leq \exp(c\,\Delta t)\,\|\mathbf{U}^{n+1/2}\|_\infty,$$

and hence:

$$\|\mathbf{U}^{n+1}\|_\infty \leq \exp(c\,\Delta t)\,\|\mathbf{U}^{n+1/2}\|_\infty \leq \exp(c\,\Delta t)\,\|\mathbf{U}^n\|_\infty.$$

This is enough to get the L^∞ bound on $[0, T]$.

The TV bound is obtained in the same way. The total variation decays for the entropy solution of the Burgers equation and the variables (φ^n, p^n) are unchanged, so

$$TV^s(\mathbf{U}^{n+1/2}) \leq TV^s(\mathbf{U}^n).$$

The differential system (4.6) is linear, thus $\mathbf{V}_i^{n+1} = \mathbf{U}^{n+1}(x_{i+1}) - \mathbf{U}^{n+1}(x_i)$ satisfies the same system and

$$|\mathbf{U}^{n+1}(x_{i+1}) - \mathbf{U}^{n+1}(x_i)| \leq \exp(c\,\Delta t)\,|\mathbf{U}^{n+1/2}(x_{i+1}) - \mathbf{U}^{n+1/2}(x_i)|.$$

Adding on i and taking the supremum yields

$$TV^s(\mathbf{U}^{n+1}) \leq \exp(c\,\Delta t)\,TV^s(\mathbf{U}^{n+1/2}) \leq \exp(c\,\Delta t)\,TV^s(\mathbf{U}^n),$$

which concludes the proof. □

These BV bounds provide the compactness in space, and hence the compactness in space and time through a bound in $Lip_t L^1_x$. Then, passing to the limit, the same bound is obtained for an entropy solution. The passage to the limit is classic, see for instance Ref. 17, 26, 27, 32. In the case of fractional BV^s, the Lipschitz estimate in time is replaced by a Hölder estimate.[5,6,11]

4.4. *Uniqueness with only Burgers' entropies*

The uniqueness is a consequence of two stability results w.r.t the L^1 norm: first, the Kruzkov stability for a scalar equation with a source term, second, the stability of the solutions of a differential system. The general case for weakly coupled system is discussed in Sec. 5. In the particular case of the system (2.8), let us consider two initial data $\mathbf{U}_0$ and $\widetilde{\mathbf{U}}_0$ and the corresponding solutions $\mathbf{U}$ and $\widetilde{\mathbf{U}}$.

Using the entropies only for the first equation (2.4a), the Kruzkov method of doubling of variables[18,10] yields for a scalar equation with a source term:

$$\int_{\mathbb{R}} |u-\widetilde{u}|(x,t)dx \leq \int_{\mathbb{R}} |u_0-\widetilde{u}_0|(x)\,dx + \int_0^T \int_{\mathbb{R}} \Omega^2\,|\varphi-\widetilde{\varphi}|(x,t)\,dx\,dt. \tag{4.13}$$

Next, substracting φ and $\widetilde{\varphi}$ and also p and $\widetilde{p}$ in the two last equations of the linear system (2.4b)-(2.4c), we have directly for an explicit constant $C>0$:

$$\begin{aligned}\int_{\mathbb{R}} \left(|\varphi-\widetilde{\varphi}|+|p-\widetilde{p}|\right)(x,t)\,dx \leq \int_{\mathbb{R}} \left(|\varphi_0-\widetilde{\varphi}_0|+|p_0-\widetilde{p}_0|\right)(x,t)\,dx \\ +C\int_0^T \int_{\mathbb{R}} \left(|u-\widetilde{u}|+|\varphi-\widetilde{\varphi}|+|p-\widetilde{p}|\right)(x,t)\,dx\,dt.\end{aligned} \tag{4.14}$$

The two inequalities are added. Gronwall lemma provides the L^1 stability.

4.5. *Energy inequality*

Now, we turn to prove inequality (4.4).

Proof. Let $e=\frac{1}{2}\left(u^2+\Omega^2\,\varphi^2+\Omega^2\,\omega_0^2\,p^2\right)$ be the density of energy and $E=\int_{\mathbb{R}} e\,dx$ be the total energy. The latter is written

$$E^{n+1}-E^n = \underbrace{E^{n+1/2}-E^n}_{I_1} + \underbrace{E^{n+1}-E^{n+1/2}}_{I_2}.$$

During the first part of the splitting, the entropy inequality for the exact solution of the Burgers equation satisfies in the sense of distributions

$$\partial_t u^2 + \partial_x u^3/6 \leq 0, \qquad u\in\mathcal{D}'.$$

By integration in space, one obtains $\int_{\mathbb{R}}\left(u^{n+1/2}\right)^2 dx \leq \int_{\mathbb{R}}\left(u^n\right)^2 dx$. Since φ and p are constant during this step, it follows $I_1\leq 0$. During the second part of the splitting, the solution of the differential system (4.7) satisfies

$$\partial_t e = -\varepsilon\,\Omega^2\,\varphi^2.$$

Integrating the latter equation over $[t_n,t_{n+1}]$ and approximating the r.h.s. yields

$$e^{n+1}-e^{n+1/2} = -\int_{t_n}^{t_n+\Delta t} \varepsilon\,\Omega^2\,\varphi^2\,dt \leq -\Delta t\,\varepsilon\,\Omega^2\left(\varphi^{n+1/2}\right)^2 + \mathcal{O}(\Delta t)^2.$$

Integration in space provides an inequality about I_2. Summing the contributions of I_1 and I_2 gives

$$\frac{E^{n+1}-E^n}{\Delta t} \leq -\int_{\mathbb{R}} \varepsilon\,\Omega^2\,\left(\varphi^{n+1/2}\right)^2 dx + \mathcal{O}(\Delta t). \tag{4.15}$$

Passing to the limit, the inequality (4.4) is obtained. □

5. Weakly coupled systems in BV^s

The system (2.4) studied along this paper is a particular case of weakly coupled hyperbolic systems:

$$\begin{cases} \partial_t u_i + \partial_x f_i(u_i) = g_i(\mathbf{U}), & \text{(5.1a)} \\ \mathbf{U}(x,0) = \mathbf{U}_0(x), & \text{(5.1b)} \end{cases}$$

where $i = 1, \dots, d$, $\mathbf{U} = (u_1, \dots, u_d)^\top$, $\mathbf{G}(\mathbf{U}) = (g_1(\mathbf{U}), \dots, g_d(\mathbf{U}))^\top$ is a Lipschitz function and $\mathbf{U}_0 \in L^\infty$. In the case of system (2.4), there is only one nonlinear flux $f_1(u_1) = u_1^2/2$, $f_i \equiv 0$ for $i > 1$, and a linear source term.

Classical results about weakly coupled systems in L^∞ are recalled. Then a new result about the propagation of BV^s regularity will be proven.

Definition 5.1 (Entropy solutions[30]). A function $\mathbf{U}$ in

$$L^\infty_{loc}(\mathbb{R} \times [0, +\infty[, \mathbb{R}^d) \cup C^0([0, +\infty[, L^1_{loc}(\mathbb{R}, \mathbb{R}^d)$$

is said to be an entropy solution of (5.1b) with initial data $\mathbf{U}_0 = (u_i^0, \cdots, u_d^0)^\top$ if for all diagonal convex entropy $\eta(\mathbf{U}) = (\eta_1(u_1), \dots, \eta_d(u_d))^\top$, η_i convex for all i, and for all diagonal nonnegative smooth function ϕ with compact support in $\mathbb{R} \times [0, +\infty[$, $\phi = (\phi_1, \dots, \phi_d)^\top$, $\phi_i(x,t) \geq 0$, we have for all $i = 1, \dots, d$:

$$\begin{aligned} &\int_0^{+\infty} \int_{\mathbb{R}} \Big(\eta_i(u_i)\, \partial_t \phi_i + q_i(u_i)\, \partial_x \phi_i + \eta^{'}(\mathbf{U})\, g_i(\mathbf{U})\, \phi \Big)\, dx\, dt \\ &+ \int_{\mathbb{R}} \eta_i(u_i^0(x))\, \phi_i(x,0)\, dx \geq 0, \end{aligned} \tag{5.2}$$

where $\mathbf{q} = (q_1, \cdots, q_d)^\top$ is the diagonal entropy-flux, with $q_i' = \eta_i' f_i'$.

Theorem 5.1 (Existence and uniqueness for weakly coupled system[30]). *The system* (5.1b) *with the initial data* $\mathbf{U}_0 \in L^\infty(\mathbb{R}, \mathbb{R}^d)$ *has one and only one entropy solution* $\mathbf{U} \in C^0([0, +\infty[, L^1_{loc}(\mathbb{R}, \mathbb{R}^d))$.

The reader is refered to Ref. 30 for the proof. The global existence is simply a consequence of the global Lipschitz assumption on $\mathbf{G}$. The extension to the multi-dimensional case with degenerate viscosity has been done in Ref. 34, 15 with many examples. In the case of system (2.4), weaker conditions than those of Theorem 5.1 were needed to prove existence and uniqueness of the entropy solution. In Sec. 4, Theorem 4.1 has been proven indeed with only one family of entropy.

Theorem 5.2 (BV^s entropy solution). *If the initial data* $\mathbf{U}_0$ *belongs to* $BV^s(\mathbb{R}, \mathbb{R}^d)$ *for* $0 < s \leq 1$*, then the unique entropy solution* $\mathbf{U}$ *of* (5.1b) *belongs to*

$$L^\infty_{loc}([0, +\infty[, BV^s(\mathbb{R}, \mathbb{R}^d)) \cap Lip^s_{loc}([0, +\infty[, L^{1/s}_{loc}(\mathbb{R}, \mathbb{R}^d)).$$

Proof. We follow the same lines than for the proof of Theorem 4.1. The Lipschitz regularity of $\mathbf{G}$ ensures linear estimates. Again, a splitting scheme is used. Let us explain on the first small interval of time $[0, \Delta t]$ how to proceed:

(1) $\mathbf{U}^{1/2}$ is the exact solution of the d decoupled scalar conservation laws:

$$\begin{cases} \partial_t u_i + \partial_x f_i(u_i) = 0, \\ u_i(x,0) = u_i^0(x), \end{cases} \qquad i = 1, \dots, d. \tag{5.3}$$

(2) $\mathbf{U}^1$ is the exact solution of the nonlinear ODE:

$$\begin{cases} \partial_t \mathbf{U} = \mathbf{G}(\mathbf{U}), \\ \mathbf{U}(x,0) = \mathbf{U}^{1/2}(x). \end{cases} \tag{5.4}$$

The only point is to check that, for some constant $c > 0$,

$$TV^s(\mathbf{U}^1(.,\Delta t)) \leq \exp(c\Delta t)\, TV^s(\mathbf{U}^{1/2}(.,\Delta t)) \leq \exp(c\Delta t)\, TV^s(\mathbf{U}^0(.,0)). \tag{5.5}$$

For the first inequality, the Lipschitz stability w.r.t. to the initial data is used. Let $x < y$ be fixed, then for some $c > 0$ depending only on the $\mathbf{G}$ Lipschitz norm, one has:

$$|\mathbf{U}^1(x,\Delta t) - \mathbf{U}^1(y,\Delta t)| \leq \exp(c\Delta t)\, |\mathbf{U}^1(x,0) - \mathbf{U}^1(y,0)|.$$

Thus $TV^s(\mathbf{U}^1(.,\Delta t)) \leq \exp(c\,p\,\Delta t)\, TV^s(\mathbf{U}^1(.,0))$ with $p = 1/s$. To conclude the first inequality of (5.5), one notices the equality $\mathbf{U}^1(.,0) = \mathbf{U}^{1/2}(.,\Delta t)$. The second inequality of (5.5) is simply the TV^s decay of all conservation laws.[5] Iterating the argument concludes the proof. □

6. Conclusion

Some questions remain open and deserve further mathematical investigations:

- Is only one strictly convex entropy η sufficient to caracterize the unique entropy solution of system (1.1)? This result has been proven for the first time by Panov[31] for homogeneous conservation laws with a convex flux.
- The inequality of energy (4.4) can be understood as an entropy inequality. This entropy does not depend only on u but on the three components. Is the energy inequality enough to caracterize the unique entropy solution of system (1.1)?
- The dissipative term in (1.1b) is purely phenomenological and it only accounts for losses in the resonators. A more detailed modeling is needed to lead to a quantitative agreement between simulations and experiments:[33]

$$\begin{cases} \partial_t u + \partial_x \left(au + b\dfrac{u^2}{2} \right) = c\, \partial_t^{-1/2} \partial_x u - \Omega^2\, \partial_t p, & \text{(6.1a)} \\ \partial_t^2 p + \varepsilon\, \partial_t^{3/2} p + \omega_0^2\, p - m \partial_t^2 p^2 + n\, |\partial_t p|\, \partial_t p = u. & \text{(6.1b)} \end{cases}$$

  The visco-thermal losses in the tube and in the resonators are modeled by fractional derivatives in time (coefficients c and ε). The nonlinear attenuation at the neck of the resonators is also taken into account (coefficient n). The analysis of (6.1) requires further efforts.

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