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SIEGEL MODULAR FORMS OF DEGREE THREE AND INVARIANTS OF TERNARY QUARTICS

REYNALD LERCIER AND CHRISTOPHE RITZENTHALER

Abstract. We determine the structure of the graded ring of Siegel modular forms of degree 3. It is generated by 19 modular forms, among which we identify a homogeneous system of parameters with 7 forms of weights 4, 12, 12, 14, 18, 20 and 30. We also give a complete dictionary between the Dixmier-Ohno invariants of ternary quartics and the above generators.

1. Introduction and main results

Let $g \geq 1$ be an integer and let $\mathbf{R}_g(\Gamma_g)$ denote the $\mathbb{C}$-algebra of *modular forms of degree* g for the symplectic group $\mathrm{Sp}_{2g}(\mathbb{Z})$ (see Section 2 for a precise definition). It is a normal and integral domain of finite type over $\mathbb{C}$, closely related to the moduli space of principally polarized abelian varieties over $\mathbb{C}$. But even generators of these algebras are only known for small values of g: $g = 1$ is usually credited to Klein [Kle90, FK65] and Poincaré [Poi05, Poi11], $g = 2$ to Igusa [Igu62] and $g = 3$ to Tsuyumine [Tsu86]. In the latter, Tsuyumine gives 34 generators and asks if they form a minimal set of generators. We answer by the negative and prove in the present paper that there exists a subset of 19 of them which is still generating the algebra and which is minimal (Theorem 3.1). As a by-product we also exhibit a (possibly incomplete) set of 55 relations and use them to obtain a homogeneous system of parameters for this algebra (Theorem 3.3).

Unlike Tsuyumine, we extensively use algebra softwares since we base our strategy on evaluation/interpolation which leads to computing ranks and invert large dimensional matrices. Still, a naive application would have forced us to work with complex numbers, which would have been bad for efficiency but also to certificate our computations. Hence, in order to perform exact arithmetic computations, we make a detour through the beautiful geometry of smooth plane quartics and Weber's formula [Web76] which allows us to express values (of quotients) of the theta constants and ultimately modular forms as rational numbers (up to a fourth root of unity). The strategy could be interesting for future investigations for $g = 4$ as those theta constants can be computed in a similar way [Çel19].

We then move on to a second task in the continuation of the famous Klein's formula, see [Kle90, Eq. 118, p. 462] and [LRZ10, MV13, Ich18a]. This formula relates a certain modular form of weight 18, namely $\boldsymbol{\chi}_{18}$, to the square of the discriminant of plane quartics. A complete dictionary between modular forms and invariants was only known for $g = 1$ and $g = 2$. For $g = 3$, these formulas can come with two flavors: restricting to the the image of the hyperelliptic locus in the Jacobian locus, one gets expressions of the modular forms in terms of Shioda invariants for binary octics, see [Tsu86] and [LG19]; considering the generic case, one gets expressions in

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terms of Dixmier-Ohno invariants for ternary quartics, see Proposition 4.3. Extra care was taken in making these formulas as normalized as possible using the background of [LRZ10] and also to eliminate as much as possible parasite coefficients coming from relations between the invariants. As a striking example, the modular form $\boldsymbol{\chi}_{28}$ is equal to $-2^{171} \cdot 3^3\, I_{27}^3\, I_3$ (the exponent of 2 is large because the normalization chosen by Dixmier for I_{27} is not optimal at 2). We finally give formulas in the opposite direction and express all Dixmier-Ohno invariants as quotients of modular forms by powers of I_{27}, see Proposition 4.5. We hope that such formulas may eventually lead to a set of generators for the ring of invariants of ternary quartics with good arithmetic properties. Indeed, theta constants have intrinsically good "reduction properties modulo primes" (in the sense that they often have a primitive Fourier expansion) and may help guessing such a set of generators.

The full list of expressions for the 19 Siegel modular forms either in terms of the theta constants or in terms of curve invariants, the expressions of Dixmier-Ohno invariants in terms of Siegel modular forms and the 55 relations in the algebra, are available at [LR19].

2. Review of Tsuyumine's construction of Siegel modular forms

We recall here the definition of the 34 generators for the $\mathbb{C}$-algebra of modular forms of degree 3 built by Tsuyumine. Surprisingly, they all are polynomials in theta constants with rational coefficients: one knows that when $g \geq 5$, there exists modular forms which are not in the algebra generated by theta constants [Man86], while answer for $g = 4$ is still pending [OPY08]. We take special care of the multiplicative constant involved in each expression.

2.1. Theta functions and theta constants.

Let $g \geq 1$ be an integer and $\mathbb{H}_g = \{\tau \in \mathbf{M}_g(\mathbb{C}),\ {}^t\tau = \tau,\ \operatorname{Im}\tau > 0\}$.

Definition 2.1. The *theta function* with characteristics $\left[\begin{smallmatrix}\varepsilon_1\\ \varepsilon_2\end{smallmatrix}\right] \in \mathbf{M}_{2,g}(\mathbb{Z})$ is given, for $z \in \mathbb{C}^g$ and $\tau \in \mathbb{H}_g$, by

$$\theta\left[\begin{smallmatrix}\varepsilon_1\\ \varepsilon_2\end{smallmatrix}\right](z,\tau) = \sum_{n\in\mathbb{Z}^g} \exp(\, i\pi\ (n+\varepsilon_1/2)\,\tau\,{}^t(n+\varepsilon_1/2)\,)\ \exp(\,2i\pi\ (n+\varepsilon_1/2)\,{}^t(z+\varepsilon_2/2)\,).$$

The *theta constant* (with characteristic $\left[\begin{smallmatrix}\varepsilon_1\\ \varepsilon_2\end{smallmatrix}\right]$) is the function of τ defined as $\theta\left[\begin{smallmatrix}\varepsilon_1\\ \varepsilon_2\end{smallmatrix}\right](\tau) = \theta\left[\begin{smallmatrix}\varepsilon_1\\ \varepsilon_2\end{smallmatrix}\right](0,\tau)$.

Proposition 2.2. *Let $z \in \mathbb{C}^g$, $\tau \in \mathbb{H}_g$, $\left[\begin{smallmatrix}\varepsilon_1\\ \varepsilon_2\end{smallmatrix}\right] \in \mathbf{M}_{2,g}(\mathbb{Z})$, then*

$$\theta\left[\begin{smallmatrix}\varepsilon_1\\ \varepsilon_2\end{smallmatrix}\right](-z,\tau) = \theta\left[\begin{smallmatrix}-\varepsilon_1\\ -\varepsilon_2\end{smallmatrix}\right](z,\tau)\,, \tag{2.1}$$

and

$$\forall \left[\begin{smallmatrix}\delta_1\\ \delta_2\end{smallmatrix}\right] \in \mathbf{M}_{2,g}(2\,\mathbb{Z}), \quad \theta\left[\begin{smallmatrix}\varepsilon_1+\delta_1\\ \varepsilon_2+\delta_2\end{smallmatrix}\right](z,\tau) = \exp(i\pi\ \varepsilon_1{}^t\delta_2/2)\ \theta\left[\begin{smallmatrix}\varepsilon_1\\ \varepsilon_2\end{smallmatrix}\right](z,\tau)\,. \tag{2.2}$$

Combining these two equations shows that $z \mapsto \theta\left[\begin{smallmatrix}\varepsilon_1\\ \varepsilon_2\end{smallmatrix}\right](z,\tau)$ is even if $\varepsilon_1\ {}^t\varepsilon_2 \equiv 0 \pmod 2$, and odd otherwise. The characteristics $\left[\begin{smallmatrix}\varepsilon_1\\ \varepsilon_2\end{smallmatrix}\right]$ are then said to be even and odd, respectively. In the following, we only make use of theta constants with characteristics with coefficients in $\{0,1\}$ because of Eq. (2.2).

To lighten notations, we number the theta constants as, for instance, done in [KLL+18]. We write $\theta_n := \theta\left[\begin{smallmatrix}\delta_0\ \delta_1\ \dots\ \delta_{g-1}\\ \varepsilon_0\ \varepsilon_1\ \dots\ \varepsilon_{g-1}\end{smallmatrix}\right]$ where $0 \leq n < 2^{2g}$ is the integer whose binary expansion is "$\delta_0\delta_1\cdots\delta_{g-1}\varepsilon_0\varepsilon_1\dots\varepsilon_{g-1}$" (with the convention $\theta_{2^{2g}} := \theta_0$). In genus 3, there are 36 even theta constants (the odd ones are all 0). We give in Table 1 the correspondence between their numbering as done in [Tsu86, pp.789–790] and our binary numbering.

Tsuyumine	①	②	③	④	⑤	⑥	⑦	⑧	⑨	⑩	⑪	⑫	⑬	⑭	⑮	⑯	⑰	⑱
Binary	θ_{31}	θ_{27}	θ_{56}	θ_{48}	θ_{49}	θ_{59}	θ_{24}	θ_{16}	θ_{17}	θ_{28}	θ_{20}	θ_{21}	θ_{55}	θ_{54}	θ_{62}	θ_{47}	θ_{12}	θ_4

Tsuyumine	⑲	⑳	㉑	㉒	㉓	㉔	㉕	㉖	㉗	㉘	㉙	㉚	㉛	㉜	㉝	㉞	㉟	㊱
Binary	θ_5	θ_8	θ_{64}	θ_1	θ_{35}	θ_{34}	θ_{42}	θ_{40}	θ_{32}	θ_{33}	θ_3	θ_2	θ_{10}	θ_7	θ_6	θ_{14}	θ_{45}	θ_{61}

Table 1: Tsuyumine's numbering of even theta constants

The modular group $\Gamma_g := \mathrm{Sp}_{2g}(\mathbb{Z})$ acts on $\mathbb{H}_g$ by

$$\tau \to M.\tau := (\mathbf{A}\tau + \mathbf{B})\,(\mathbf{C}\tau + \mathbf{D})^{-1} \quad \text{for } M = \left(\begin{smallmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{smallmatrix}\right),$$

and this results in the following action of Γ_g on theta constants.

Proposition 2.3 (Transformation formula [Igu72, Chap. 5, Th. 2] [Cos11, Prop. 3.1.24])**.** *Let $\tau \in \mathbb{H}_g$, $\left[\begin{smallmatrix}\varepsilon_1\\ \varepsilon_2\end{smallmatrix}\right] \in \mathbf{M}_{2,g}(\mathbb{R})$ and $M \in \Gamma_g$, then*

$$\theta\left[\begin{smallmatrix}\varepsilon_1\\ \varepsilon_2\end{smallmatrix}\right](M.\tau) = \zeta_M \sqrt{\det(\mathbf{C}\tau + \mathbf{D})}\, \exp(-i\pi\,\sigma/4)\ \theta\left[\begin{smallmatrix}\delta_1\\ \delta_2\end{smallmatrix}\right](\tau) \tag{2.3}$$

with ζ_M an eighth root of unity depending only on M, $\left[\begin{smallmatrix}\delta_1\\ \delta_2\end{smallmatrix}\right] = M.\left[\begin{smallmatrix}\varepsilon_1\\ \varepsilon_2\end{smallmatrix}\right]$ where the action of M on a characteristic is defined by

$$\left[\begin{smallmatrix}\varepsilon_1\\ \varepsilon_2\end{smallmatrix}\right] \to M.\left[\begin{smallmatrix}\varepsilon_1\\ \varepsilon_2\end{smallmatrix}\right] = (\varepsilon_1 \frown \varepsilon_2)\, M + ({}^{\mathrm{t}}\mathbf{A}\,\mathbf{C})_\Delta \frown ({}^{\mathrm{t}}\mathbf{B}\,\mathbf{D})_\Delta$$

and

$$\sigma = \varepsilon_1\ \mathbf{A}\,{}^{\mathrm{t}}\mathbf{B}\ {}^{\mathrm{t}}\varepsilon_1 + 2\,\varepsilon_1\ \mathbf{B}\,{}^{\mathrm{t}}\mathbf{C}\ {}^{\mathrm{t}}\varepsilon_2 + \varepsilon_2\ \mathbf{C}\,{}^{\mathrm{t}}\mathbf{D}\ {}^{\mathrm{t}}\varepsilon_2 + (\,2\,\varepsilon_1\ \mathbf{A} + 2\,\varepsilon_2\ \mathbf{C} + ({}^{\mathrm{t}}\mathbf{A}\,\mathbf{C})_\Delta)\ {}^{\mathrm{t}}({}^{\mathrm{t}}\mathbf{B}\,\mathbf{D})_\Delta\,.$$

Here, "$\frown$" denotes the concatenation of two row vectors, and "$(.)_\Delta$" denotes the row vector equal to the diagonal of the square matrix given in argument.

2.2. **Siegel modular forms.** Let $\Gamma_g(\ell)$ denote the principal congruence subgroup of level ℓ, *i.e.* $\{M \in \Gamma_g \mid M \equiv \mathbf{1}_{2g} \bmod \ell\}$, and let $\Gamma_g(\ell, 2\ell)$ denote the congruence subgroup $\{M \in \Gamma_g(\ell) \mid ({}^{\mathrm{t}}\mathbf{A}\,\mathbf{C})_\Delta \equiv ({}^{\mathrm{t}}\mathbf{B}\,\mathbf{D})_\Delta \equiv 0 \bmod 2\,\ell\}$.

For a congruence subgroup $\Gamma \subset \Gamma_g$, let $\mathbf{R}_{g,h}(\Gamma)$ be the $\mathbb{C}$-vector space of *analytic Siegel modular forms of weight h and degree g for* Γ, consisting of complex holomorphic functions f on $\mathbb{H}_g$ satisfying for all $M \in \Gamma$,

$$f(M.\tau) = \det(\mathbf{C}\tau + \mathbf{D})^h \cdot f(\tau)$$

(for $g = 1$, one also requires that f is holomorphic at "infinity" but we will not look at this case here). We also denote the $\mathbb{C}$-*algebra of Siegel modular forms of degree g for* Γ by $\mathbf{R}_g(\Gamma) := \bigoplus \mathbf{R}_{g,h}(\Gamma)$. The modular group acts on $\mathbf{R}_{g,h}(\Gamma_g)$ by

$$f \to M.f := \det(\mathbf{C}\tau + \mathbf{D})^{-h} \cdot f(M.\tau)\,.$$

In particular, $f \in \mathbf{R}_{g,h}(\Gamma)$ if and only if $M.f = f$ for all $M \in \Gamma$.

A strategy to build modular forms for Γ_3 is first to construct a form $F \in \mathbf{R}_3(\Gamma_3(2))$, and then average over the finite quotient $\Gamma_g/\Gamma_g(2)$ to get a modular form $f \in \mathbf{R}_3(\Gamma_3)$, namely

$$f = \sum_{M \in \Gamma_3/\Gamma_3(2)} M.F\,. \tag{2.4}$$

All forms F are polynomials in the theta constants, and are of even weight. Hence, given an F, a careful application of the transformation formula (Prop. 2.3) gives all summands, where we do not care about the choice of the square root as it is raised to an even power.

Tsuyumine gives some of the building blocks Fs in terms of maximal syzygetic sets of even characteristics [Tsu86, Sec. 21]. Multiplying the theta constants in a given set is an element of $\mathbf{R}_3(\Gamma(2))$. The quotient $\Gamma_3/\Gamma_3(2)$ acts transitively on these 135 sets numbered from ((1)) to ((135)) by Tsuyumine. Among them, 33 are actually used to defines a set of generators for $\mathbf{R}_3(\Gamma_3)$. We give their expressions in Table 2.

#	θ-monomial	#	θ-monomial	#	θ-monomial
((1))	$\theta_3\,\theta_{28}\,\theta_{31}\,\theta_{33}\,\theta_{34}\,\theta_{61}\,\theta_{62}\,\theta_{64}$	((38))	$\theta_5\,\theta_6\,\theta_{28}\,\theta_{31}\,\theta_{33}\,\theta_{34}\,\theta_{56}\,\theta_{59}$	((90))	$\theta_1\,\theta_6\,\theta_7\,\theta_{48}\,\theta_{49}\,\theta_{54}\,\theta_{55}\,\theta_{64}$
((2))	$-\theta_1\,\theta_2\,\theta_{28}\,\theta_{31}\,\theta_{32}\,\theta_{35}\,\theta_{61}\,\theta_{62}$	((39))	$-\theta_7\,\theta_8\,\theta_{16}\,\theta_{31}\,\theta_{32}\,\theta_{47}\,\theta_{55}\,\theta_{56}$	((99))	$-\theta_5\,\theta_8\,\theta_{17}\,\theta_{28}\,\theta_{34}\,\theta_{47}\,\theta_{54}\,\theta_{59}$
((3))	$-\theta_3\,\theta_8\,\theta_{20}\,\theta_{31}\,\theta_{33}\,\theta_{42}\,\theta_{54}\,\theta_{61}$	((43))	$\theta_7\,\theta_{12}\,\theta_{20}\,\theta_{31}\,\theta_{35}\,\theta_{40}\,\theta_{48}\,\theta_{59}$	((103))	$\theta_4\,\theta_8\,\theta_{12}\,\theta_{16}\,\theta_{20}\,\theta_{24}\,\theta_{28}\,\theta_{64}$
((4))	$\theta_1\,\theta_{10}\,\theta_{20}\,\theta_{31}\,\theta_{35}\,\theta_{40}\,\theta_{54}\,\theta_{61}$	((45))	$\theta_7\,\theta_{24}\,\theta_{31}\,\theta_{40}\,\theta_{47}\,\theta_{48}\,\theta_{55}\,\theta_{64}$	((111))	$\theta_1\,\theta_4\,\theta_5\,\theta_{16}\,\theta_{17}\,\theta_{20}\,\theta_{21}\,\theta_{64}$
((5))	$\theta_2\,\theta_8\,\theta_{21}\,\theta_{31}\,\theta_{32}\,\theta_{42}\,\theta_{55}\,\theta_{61}$	((47))	$-\theta_{14}\,\theta_{17}\,\theta_{31}\,\theta_{33}\,\theta_{47}\,\theta_{48}\,\theta_{62}\,\theta_{64}$	((115))	$-\theta_8\,\theta_{20}\,\theta_{28}\,\theta_{34}\,\theta_{42}\,\theta_{54}\,\theta_{62}\,\theta_{64}$
((18))	$-\theta_1\,\theta_4\,\theta_{17}\,\theta_{20}\,\theta_{40}\,\theta_{45}\,\theta_{56}\,\theta_{61}$	((51))	$-\theta_1\,\theta_6\,\theta_{24}\,\theta_{31}\,\theta_{40}\,\theta_{47}\,\theta_{49}\,\theta_{54}$	((118))	$-\theta_3\,\theta_{10}\,\theta_{21}\,\theta_{28}\,\theta_{33}\,\theta_{40}\,\theta_{55}\,\theta_{62}$
((31))	$\theta_3\,\theta_4\,\theta_7\,\theta_{24}\,\theta_{27}\,\theta_{28}\,\theta_{31}\,\theta_{64}$	((54))	$-\theta_3\,\theta_{12}\,\theta_{16}\,\theta_{31}\,\theta_{34}\,\theta_{45}\,\theta_{49}\,\theta_{62}$	((119))	$-\theta_1\,\theta_{20}\,\theta_{21}\,\theta_{34}\,\theta_{35}\,\theta_{54}\,\theta_{55}\,\theta_{64}$
((32))	$-\theta_1\,\theta_2\,\theta_5\,\theta_6\,\theta_{24}\,\theta_{27}\,\theta_{28}\,\theta_{31}$	((55))	$\theta_3\,\theta_{24}\,\theta_{27}\,\theta_{32}\,\theta_{35}\,\theta_{56}\,\theta_{59}\,\theta_{64}$	((131))	$\theta_1\,\theta_2\,\theta_3\,\theta_4\,\theta_5\,\theta_6\,\theta_7\,\theta_{64}$
((34))	$\theta_1\,\theta_5\,\theta_{10}\,\theta_{14}\,\theta_{16}\,\theta_{20}\,\theta_{27}\,\theta_{31}$	((73))	$\theta_8\,\theta_{16}\,\theta_{24}\,\theta_{32}\,\theta_{40}\,\theta_{48}\,\theta_{56}\,\theta_{64}$	((132))	$-\theta_4\,\theta_5\,\theta_6\,\theta_7\,\theta_{32}\,\theta_{33}\,\theta_{34}\,\theta_{35}$
((36))	$-\theta_4\,\theta_{10}\,\theta_{14}\,\theta_{17}\,\theta_{21}\,\theta_{27}\,\theta_{31}\,\theta_{64}$	((85))	$\theta_1\,\theta_{16}\,\theta_{17}\,\theta_{32}\,\theta_{33}\,\theta_{48}\,\theta_{49}\,\theta_{64}$	((133))	$\theta_2\,\theta_8\,\theta_{10}\,\theta_{32}\,\theta_{34}\,\theta_{40}\,\theta_{42}\,\theta_{64}$
((37))	$-\theta_4\,\theta_7\,\theta_{28}\,\theta_{31}\,\theta_{32}\,\theta_{35}\,\theta_{56}\,\theta_{59}$	((89))	$-\theta_2\,\theta_3\,\theta_4\,\theta_5\,\theta_{48}\,\theta_{49}\,\theta_{54}\,\theta_{55}$	((135))	$\theta_1\,\theta_2\,\theta_3\,\theta_{32}\,\theta_{33}\,\theta_{34}\,\theta_{35}\,\theta_{64}$

Table 2: Tsuyumine's maximal syzygetic sequences

Then Tsuyumine considers 34 Fs written as combinations of

- $\chi_{18} = \prod_{\theta_i \text{ even}} \theta_i$,
- a rational function of the 36 non-zero θ_i^4,
- the monomials $((i))$ defined in Table 2, and
- the squares of the gcd between two such $((i))$.

Using the map from modular forms to invariants of binary octics introduced by Igusa [Igu67], he proves the following result.

Theorem 2.4 (Tsuyumine [Tsu86, Sec. 20][1]). *The graded algebra $\mathbf{R}_3(\Gamma_3)$ is generated by the 34 modular forms defined in Table 3. Its Hilbert–Poincaré series is generated by the rational function*

$$\frac{(1+T^2)\,N(T)}{(1-T^4)\,(1-T^{12})^2\,(1-T^{14})\,(1-T^{18})\,(1-T^{20})\,(1-T^{30})}\,, \tag{2.5}$$

where

$$\begin{aligned} N(T) = {} & 1 - T^2 + T^4 + T^{10} + 3\,T^{16} - T^{18} + 3\,T^{20} + 2\,T^{22} + 2\,T^{24} + 3\,T^{26} + 4\,T^{28} + 2\,T^{30} + 7\,T^{32} + 3\,T^{34} \\ & + 7\,T^{36} + 5\,T^{38} + 9\,T^{40} + 6\,T^{42} + 10\,T^{44} + 8\,T^{46} + 10\,T^{48} + 9\,T^{50} + 12\,T^{52} + 7\,T^{54} + 14\,T^{56} \\ & + 7\,T^{58} + 12\,T^{60} + 9\,T^{62} + 10\,T^{64} + 8\,T^{66} + 10\,T^{68} + 6\,T^{70} + 9\,T^{72} + 5\,T^{74} + 7\,T^{76} + 3\,T^{78} + 7\,T^{80} \\ & + 2\,T^{82} + 4\,T^{84} + 3\,T^{86} + 2\,T^{88} + 2\,T^{90} + 3\,T^{92} - T^{94} + 3\,T^{96} + T^{102} + T^{108} - T^{110} + T^{112}\,. \end{aligned}$$

The modular forms f defined in Table 3 are all polynomials in the theta constants whose primitive part has all its coefficients equal ± 1 and whose content is

$$c(f) = \frac{\#\Gamma_3/\Gamma_3(2)}{\#\{\text{summands of } f\}} = \frac{2^9 \cdot 3^4 \cdot 5 \cdot 7}{\#\{\text{summands of } f\}} \in \mathbb{Z}\,.$$

In order to get simpler expressions when restricting to the hyperelliptic locus or on the decomposable one, Tsuyumine multiplies each f by an additional normalization constant (2nd column

[1] See [Tsu89, p. 44] for the $(1-T^{12})$ misprint in the denominator of Equation (2.5) in [Tsu86].

Name	[Tsu86] Coeff.	$F \in \mathbf{R}_3(\Gamma(2))$	#sum.
χ_{18}	$1/(2^9 \cdot 3^4 \cdot 5 \cdot 7)$	$\prod_{\theta_i \text{ even}} \theta_i$	1
χ_{28}	$1/(2^{10} \cdot 3^2 \cdot 5 \cdot 7)$	$\chi_{18}^2 \,/\, ((131))^2$	135
α_4	$1/(2^{13} \cdot 3 \cdot 7)$	$\gcd(\,((131)), ((132))\,)^2$	945
α_6	$1/(2^6 \cdot 3 \cdot 7)$	$\theta_{64}^4 \cdot ((131))$	1080
α_{10}	$-1/(2^4 \cdot 3^2 \cdot 5 \cdot 11)$	$(\theta_{16}\,\theta_{20}\,\theta_{32}\,\theta_{34}\,\theta_{48}\,\theta_{54})^2 \cdot ((131))$	30240
α_{12}	$1/(2^8 \cdot 3^5 \cdot 5)$	$(\theta_2\,\theta_{21}\,\theta_{24}\,\theta_{49}\,\theta_{62}\,\theta_{64})^4$	336
α'_{12}	$3/2^8$	$((85))^2 \cdot ((119))^2 \,/\, (\,\theta_1\,\theta_{64}\,)^4$	945
α_{16}	$-3^2/2^9$	$((85))^2 \cdot ((119)) \cdot ((131))$	3780
α_{18}	$-3^2/2^5$	$\theta_{64}^4\,((85))^2 \cdot ((119)) \cdot ((131))$	7560
α_{20}	$3/(2^9 \cdot 5)$	$(((85))^2 \cdot ((119))^2 \cdot ((131))^2 \,/\, (\theta_1\,\theta_{64})^4$	63
α_{24}	$3^2/2^3$	$\theta_{64}^4\,((85))^2 \cdot ((119))^2 \cdot ((131))^2 \,/\, \theta_1^4$	1260
α_{30}	$3^4/(2^8 \cdot 5)$	$(((85))^3 \cdot ((119))^3 \cdot ((131))^3 \,/\, (\theta_1\,\theta_{64}^2)^4$	1260
β_{14}	$1/(2^5 \cdot 3 \cdot 7)$	$\theta_{31}^8\,\chi_{18}/(\,((5)) \cdot ((54))\,)$	4320
β_{16}	$1/(2^6 \cdot 3)$	$((31)) \cdot ((43)) \cdot ((47)) \cdot ((51))$	7560
β_{22}	$-1/(2^4 \cdot 3)$	$(\theta_{27}\,\theta_{31}\,\theta_{54}\,\theta_{55}\,\theta_{59}\,\theta_{62})^4\,\chi_{18} \,/\, (\,((2)) \cdot ((54))\,)$	30240
β'_{22}	2^4	$\chi_{18}\,((119))^2 \cdot ((133))^2 \,/\, (\,\theta_{34}^4\theta_{64}^4\,((18)) \cdot ((34))\,)$	90720
β_{26}	$-1/2^2$	$((32)) \cdot ((36)) \cdot ((37)) \cdot ((45)) \cdot ((90)) \cdot ((111)) \cdot ((135)) \,/\, \theta_{64}^4$	362880
β_{28}	$-1/2^2$	$((32)) \cdot ((36)) \cdot ((37)) \cdot ((45)) \cdot ((90)) \cdot ((111)) \cdot ((135))$	362880
β_{32}	$1/2^2$	$\chi_{18}\,((85))^2 \cdot ((89))^2 \cdot ((90)) \cdot ((111)) \cdot ((135)) \,/\, (\,\theta_{48}^4\,\theta_{49}^4\,\theta_{64}^4((4)) \cdot ((99))\,)$	362880
β_{34}	$1/(2^3 \cdot 3)$	$\theta_{31}^8\,\chi_{18}\,((90))^2 \cdot ((111))^2 \cdot ((135))^2 \,/\, (\,\theta_{64}^4\,\theta_1^4\,((3)) \cdot ((31))\,)$	120960
γ_{20}	$1/(2^7 \cdot 3)$	$\theta_{31}^4\,\chi_{18}\,((135)) \,/\, ((1))$	7560
γ_{24}	$1/2^7$	$\theta_{31}^8\,\chi_{18}^2 \,/\, (\,((4)) \cdot ((5)) \cdot ((47)) \cdot ((54))\,)$	11340
γ_{26}	$1/2^6$	$(\theta_{31}\,\theta_{28})^4\,\chi_{18}\,((38)) \cdot ((135)) \,/\, ((1))$	22680
γ_{32}	$1/(2^3 \cdot 3)$	$(\theta_{16}\,\theta_{20}\,\theta_{31}\,\theta_{49}\,\theta_{54}\,\theta_{56}\,\theta_{59})^4\,\chi_{18}\,((135)) \,/\, ((1))$	120960
c'_{32}	$-1/2^3$	$\theta_{33}^4\,\chi_{18}\,((90))^2 \cdot ((111))^2 \cdot ((135)) \,/\, (\,(\theta_1\,\theta_{64})^4\,((1))\,)$	30240
γ_{36}	$-1/2^4$	$(\theta_{28}\,\theta_{31})^4\,\chi_{18}\,((38)) \cdot ((90)) \cdot ((111)) \cdot ((135))^2 \,/\, (\,\theta_1^4\,((1))\,)$	181440
γ_{38}	$1/2^4$	$\theta_{31}^{16}\,\chi_{18}^2\,((31)) \cdot ((39)) \cdot ((43)) \,/\, (\,\theta_7^4\,((4)) \cdot ((5)) \cdot ((47)) \cdot ((54))\,)$	90720
c'_{38}	$1/2^2$	$\theta_{31}^4\,\chi_{18}\,((38))^2 \cdot ((90)) \cdot ((111)) \cdot ((135))^2 \,/\, (\,\theta_1^4\,((1))\,)$	362880
γ_{42}	$1/2^3$	$\chi_{18}\,(((38)) \cdot ((85))^2 \cdot ((90)) \cdot ((111)) \cdot ((119))^2 \cdot ((135)) \,/\, (\,(\theta_1\,\theta_{64})^4\,((1))\,)$	181440
γ_{44}	$1/2^4$	$\chi_{18}^2\,\theta_{31}^8\,((45))^2 \cdot ((55))^2 \cdot ((103))^2 \,/\, (\,(\theta_{24}\,\theta_{64})^4\,((4)) \cdot ((5)) \cdot ((47)) \cdot ((54))\,)$	90720
δ_{30}	$2^7/3$	$(\theta_{28}\,\theta_{31})^4\,\chi_{18}\,((47)) \cdot ((115)) \cdot ((118)) \,/\, ((1))$	90720
δ_{36}	$1/2^3$	$(\theta_{28}\,\theta_{31}\,\theta_{64})^4\,\chi_{18}\,((31)) \cdot ((38)) \cdot ((118)) \cdot ((135)) \,/\, ((1))$	181440
δ_{46}	$-1/2$	$(\theta_{28}\,\theta_{31})^4\,\chi_{18}\,((31)) \cdot ((38)) \cdot ((90)) \cdot ((111)) \cdot ((118)) \cdot ((135))^2 \,/\, ((1))$	725760
c_{48}	$1/2$	$\theta_{28}^4\,\chi_{18}\,((31))^2 \cdot ((38)) \cdot ((90)) \cdot ((111)) \cdot ((118)) \cdot ((135))^2 \,/\, ((1))$	725760

Table 3: Tsuyumine's generators (the index is their weight), Tsuyumine's normalization constant, the form F and the number of summands of the polynomial in the theta constants

of Table 3). For instance, as defined by Tsuyumine,

$$\chi_{28} := 2^{-10} \cdot 3^{-2} \cdot 5^{-1} \cdot 7^{-1} \sum_{M \in \Gamma_g/\Gamma_g(2)} M.(\chi_{18}^2 \,/\, ((131))^2)\,,$$

and, so, the 135 summands are each a (monic) monomial in the theta constants time $\pm(2^{-10} \cdot 3^{-2} \cdot 5^{-1} \cdot 7^{-1}) \cdot c(\chi_{28}) = \pm 1/30$.

Having in mind possible applications of our results to fields of positive characteristics, we replace the multiplication by Tsuyumine's constant by a multiplication by $1/c(f)$. In this way,

f is a sum of (monic) monomials in the theta constants with coefficients ± 1. To avoid confusion with Tsuyumine's notation, our modular forms will be denoted with bold font. Typically, $\boldsymbol{\chi}_{28} := 30\,\chi_{28}$, $\boldsymbol{\alpha}_4 := 112\,\alpha_4$, $\boldsymbol{\alpha}_6 := \alpha_6$, $\boldsymbol{\alpha}_{10} := 165\,\alpha_{10}$, etc.

Still driven by the link with the hyperelliptic locus, Tsuyumine adds to c'_{32} (resp. c'_{38} and c_{48}) some polynomials in modular forms of smaller weights and denote the result γ_{32} (resp. γ_{38} and δ_{48}). Theorem 2.4 as stated in [Tsu86] considers modular forms γ_{32}, γ_{38} and δ_{48}, instead of c'_{32}, c'_{38} and c_{48}. The two theorems are obviously equivalent. Here, we choose instead to define $\boldsymbol{\gamma}_{32} := c'_{32}/6$, $\boldsymbol{\gamma}_{38} := c'_{38}$ and $\boldsymbol{\delta}_{48} := c_{48}$.

Remark 2.5. Some of the modular forms in Table 3 have a large number of summands. If it would be cumbersome to store them, evaluating them is relatively quick as it basically consists in permuting theta constants up to some eighth roots of unity according to Eq. (2.3). Following Tsuyumine, the sum is computed in two steps. Let Θ be the conjugate subgroup of $\Gamma_3(1,2)$ that stabilizes θ_{61} ($\Gamma_3(1,2)$ stabilizes θ_{64}). Tsuyumine gives explicit coset representatives for Γ_3/Θ (36 elements) and $\Theta/\Gamma_3(2)$ (8! elements) and splits the sum in Eq. (2.4) as

$$f = \sum_{M' \in \Gamma_3/\Theta} M' \,.\, \sum_{M'' \in \Theta/\Gamma_3(2)} M''.F$$

We use this approach in order to perform the computation of the summands[2], provided the precomputation of the eighth roots of unity $\zeta_{M'}$ and $\zeta_{M''}$ on a fixed chosen matrix in $\mathbb{H}_3$.

3. A minimal set of generators for the algebra of modular forms of degree 3

3.1. **Fundamental set of modular forms.** Since we know the dimensions of each $\mathbf{R}_{3,h}(\Gamma_3)$ from the generating functions of Theorem 2.4, it is a matter of linear algebra to check that a given subset of Tsuyumine's generators is enough for generating the full algebra. However, it is difficult to perform these linear computations on the formal expressions in terms of the theta constants, since there exist numerous algebraic relations between the later. Therefore we favour an interpolation/evaluation strategy.

Suppose that we want to prove that a given form f of weight h can be obtained from a given set $\{f_1, \ldots, f_m\}$. This set produces $F_1, \ldots, F_n$, homogeneous polynomials in the f_i of weight h. If $n < d = \dim \mathbf{R}_{3,h}(\Gamma_3)$, then all forms of weight h cannot be obtained. Assume that $n \geq d$. Then, if we can find $(\tau_i)_{i=1,\ldots,d} \in \mathbb{H}_g^d$ such that the matrix $(F_i(\tau_j))_{1\leq i,j\leq d}$ is of rank d, we know that f can be written in terms of the f_i, and even find such a relation. Equivalently, we will actually find a polynomial relation between f/θ_{64}^h and the $f_i/\theta_{64}^{w_i}$ where w_i denotes the weight of f_i.

By Remark 2.5, the evaluation of a form $f(\tau)/\theta_{64}^h(\tau)$ boils down to the computation of quotients $(\theta_i/\theta_{64})(\tau)$. A naive approach would be to use an arbitrary matrix $\tau \in \mathbb{H}_3$. But then the theta constants would in general be transcendental complex numbers which would make the computations much more costly and the final result hard to certify. We therefore prefer to consider a matrix τ coming from a complex torus $\operatorname{Jac} C$ attached to a smooth plane quartic C given by 7 general lines in $\mathbb{P}^2$. Indeed (see for instance [Web76, Rit04, NR17]), we can consider these 7 lines as an Aronhold system of 7 bitangents for a (unique) plane quartic C. Then, one can easily recover the equations of the 21 other bitangents and give an expression of the quotients

[2]There are two small typos in [Tsu86, pp. 842–846], the $(3,6)$-th coefficients of "$M_{①}$" must be -1 instead of 1, and the $(2,2)$-th coefficients of "$M_{㉗}$" must be 1 instead of 0. This modification makes $M_{①}$ and $M_{㉗}$ symplectic.

$(\theta_i/\theta_{64})^4(\tau')$ in terms of the coefficients of the linear forms defining the bitangents. Note that the Riemann matrix τ' is not explicitly known here (it is not even our final τ yet) and depends not only on C but also on the choice of a symplectic basis for $H_1(C,\mathbb{Z})$. But when each of the bitangents in the Aronhold system is defined over $\mathbb{Q}$, all computations can be performed over $\mathbb{Q}$ and $(\theta_i/\theta_{64})^4(\tau')$ is a rational number.

To remove the fourth root of unity ambiguity that remains, we compute independently an approximation of a Riemann matrix τ for the curve C over $\mathbb{C}$. We need to do it only at very low precision (a typical choice is 20 decimal digits) and this can be done efficiently either in MAPLE (package ALGCURVES by Deconinck et al. [DvH01]) or in MAGMA (package RIEMANN SURFACES by Neurohr [Neu18]). Then, we can calculate an approximation of the theta constants at τ. We note that [NR17, Theorem 3.1] shows that the set $\left\{\theta_j^8/\theta_i^8\right\}$ running through every even theta constants θ_i,θ_j depends only on C and not on the Riemann matrix. Indeed, the dependence on this matrix relies only on the quadratic form q_0 (in the notation of *loc. cit.*) whose contribution disappears in the eighth power. Therefore, there exist an integer i_0 and a permutation σ such that

$$\frac{\theta_{\sigma(i)}(\tau')^{\,8}}{\theta_{i_0}(\tau')^{\,8}}=\frac{\theta_i(\tau)^{\,8}}{\theta_{64}(\tau)^{\,8}}.$$

Since we know $\theta_{\sigma(i)}(\tau)/\theta_{64}(\tau)$ with small precision and its eighth power exactly, it is possible to give the exact value of $\theta_i(\tau)/\theta_{64}(\tau)$.

Using extensively this method leads to a set of generators for $\mathbf{R}_3(\Gamma_3)$. Moreover it is easy to prove, by the same algorithms, that this set is *fundamental*, *i.e.* one cannot remove any element and still generates the algebra $\mathbf{R}_3(\Gamma_3)$.

Theorem 3.1. *The* 19 *Siegel modular forms* $\boldsymbol{\alpha}_4$, $\boldsymbol{\alpha}_6$, $\boldsymbol{\alpha}_{10}$, $\boldsymbol{\alpha}_{12}$, $\boldsymbol{\alpha}'_{12}$, $\boldsymbol{\beta}_{14}$, $\boldsymbol{\alpha}_{16}$, $\boldsymbol{\beta}_{16}$, $\boldsymbol{\chi}_{18}$, $\boldsymbol{\alpha}_{18}$, $\boldsymbol{\alpha}_{20}$, $\boldsymbol{\gamma}_{20}$, $\boldsymbol{\beta}_{22}$, $\boldsymbol{\beta}'_{22}$, $\boldsymbol{\alpha}_{24}$, $\boldsymbol{\gamma}_{24}$, $\boldsymbol{\gamma}_{26}$, $\boldsymbol{\chi}_{28}$ *and* $\boldsymbol{\alpha}_{30}$ *define a fundamental set of generators for* $\mathbf{R}_3(\Gamma_3)$.

Remark 3.2. Note that [Run95] proved that $\mathbf{R}_3(\Gamma_3(2))$ has a fundamental set of generators of 30 elements.

A word on the complexity. The proof mainly consists in checking for all the even weight h between 4 and 48 that there exists an evaluation matrix of rank $\dim \mathbf{R}_{3,h}(\Gamma_3)$ for this set of 19 modular forms. It is a matter of few hours for the largest weight to perform this calculation in MAGMA. Most of the time is spent on the evaluation of the 19 forms f_i at a matrix τ_j, which takes about 1 mn on a laptop.

Additionally, we find the expressions of the remaining 15 modular forms given in Table 3. The first ones are

$$\begin{aligned}
2^5\cdot 3^4\cdot 5\cdot 7^2\cdot 11\ \boldsymbol{\beta}_{26} &= 7\,\boldsymbol{\alpha}_6\,\boldsymbol{\alpha}_{10}^2-3080\,\boldsymbol{\alpha}_6^2\,\boldsymbol{\beta}_{14}-145530\,\boldsymbol{\alpha}_{12}\,\boldsymbol{\beta}_{14}+194040\,\boldsymbol{\alpha}'_{12}\,\boldsymbol{\beta}_{14}-11760\,\boldsymbol{\alpha}_{10}\,\boldsymbol{\alpha}_{16}\\
&\quad-7040\,\boldsymbol{\alpha}_4\,\boldsymbol{\alpha}_6\,\boldsymbol{\beta}_{16}+16660\,\boldsymbol{\alpha}_{10}\,\boldsymbol{\beta}_{16}-20824320\,\boldsymbol{\alpha}_4^2\,\boldsymbol{\chi}_{18}-4435200\,\boldsymbol{\alpha}_6\,\boldsymbol{\alpha}_{20}\\
&\quad+2822512\,\boldsymbol{\alpha}_6\,\boldsymbol{\gamma}_{20}-55440\,\boldsymbol{\alpha}_4\,\boldsymbol{\beta}_{22}+36960\,\boldsymbol{\alpha}_4\,\boldsymbol{\beta}'_{22}-105557760\,\boldsymbol{\gamma}_{26}\,,\\[2ex]
2^8\cdot 3^4\cdot 5^2\cdot 7^4\ \boldsymbol{\beta}_{28} &= -105\,\boldsymbol{\alpha}_4^2\,\boldsymbol{\alpha}_{10}^2-42000\,\boldsymbol{\alpha}_4^2\,\boldsymbol{\alpha}_6\,\boldsymbol{\beta}_{14}+66885\,\boldsymbol{\alpha}_4\,\boldsymbol{\alpha}_{10}\,\boldsymbol{\beta}_{14}+129654\,\boldsymbol{\beta}_{14}^2-96000\,\boldsymbol{\alpha}_4^3\,\boldsymbol{\beta}_{16}\\
&\quad+77792400\,\boldsymbol{\alpha}_{12}\,\boldsymbol{\beta}_{16}+207446400\,\boldsymbol{\alpha}'_{12}\,\boldsymbol{\beta}_{16}+5399533440\,\boldsymbol{\alpha}_4\,\boldsymbol{\alpha}_6\,\boldsymbol{\chi}_{18}-9996323400\,\boldsymbol{\alpha}_{10}\,\boldsymbol{\chi}_{18}\\
&\quad-4321800\,\boldsymbol{\alpha}_{10}\,\boldsymbol{\alpha}_{18}+320544000\,\boldsymbol{\alpha}_4^2\,\boldsymbol{\alpha}_{20}+82576256\,\boldsymbol{\alpha}_4^2\,\boldsymbol{\gamma}_{20}-12965400\,\boldsymbol{\alpha}_6\,\boldsymbol{\beta}_{22}\\
&\quad-17287200\,\boldsymbol{\alpha}_6\,\boldsymbol{\beta}'_{22}-666792000\,\boldsymbol{\alpha}_4\,\boldsymbol{\alpha}_{24}-700378560\,\boldsymbol{\alpha}_4\,\boldsymbol{\gamma}_{24}-442172001600\,\boldsymbol{\chi}_{28}\,,\\[2ex]
2^3\cdot 3^4\cdot 5\cdot 7^4\ \boldsymbol{\delta}_{30} &= -37044\,\boldsymbol{\beta}_{14}\,\boldsymbol{\beta}_{16}+23040\,\boldsymbol{\alpha}_4^3\,\boldsymbol{\chi}_{18}+987840\,\boldsymbol{\alpha}_6^2\,\boldsymbol{\chi}_{18}+47508930\,\boldsymbol{\alpha}_{12}\,\boldsymbol{\chi}_{18}\\
&\quad+133358400\,\boldsymbol{\alpha}'_{12}\,\boldsymbol{\chi}_{18}-1568\,\boldsymbol{\alpha}_4\,\boldsymbol{\alpha}_6\,\boldsymbol{\gamma}_{20}+46305\,\boldsymbol{\alpha}_{10}\,\boldsymbol{\gamma}_{20}-246960\,\boldsymbol{\alpha}_6\,\boldsymbol{\gamma}_{24}+282240\,\boldsymbol{\alpha}_4\,\boldsymbol{\gamma}_{26}\,,
\end{aligned}$$

$$2\cdot 3\cdot 5\cdot 7\ \gamma'_{32} = \chi_{18}\,(\alpha_4\,\alpha_{10} - 252\,\beta_{14})\,.$$

The last ones, for instance γ_{44}, δ_{46} and δ_{48}, tend to be heavily altered with the relations that exist between these 19 modular forms, and have huge coefficients (thousands of digits).

3.2. **Module of relations between the generators.** We now quickly deal with the relations defining the algebra $\mathbf{R}_3(\Gamma_3)$. With the same techniques, involving modular forms up to weight 70 (see Remark 4.4 for speeding up the computations), we find a (possibly incomplete) list of 55 relations for our generators of $\mathbf{R}_3(\Gamma_3)$ given by weighted polynomials of degree 32 to 58 (*cf.* Table 4).

Weight	32	34	36	38	40	42	44	46	48	50	52	54	56	58
Number	1	1	2	4	5	5	7	6	8	6	5	2	2	1

Table 4: number of relations of a given weight in $\mathbf{R}_3(\Gamma_3)$

The relations of weight 32 and 34 are relatively small,

$$\begin{aligned}
0 = {} & -25226544365568\,\beta_{16}^2 + 50854572195840\,\beta_{16}\,\alpha_{16} - 25092716544000\,\alpha_{16}^2 + 13916002383360\,\alpha_{18}\,\beta_{14} \\
& - 18410871153185280\,\chi_{18}\,\beta_{14} + 1109304189987840\,\gamma_{20}\,\alpha_{12p} - 1951854879744000\,\alpha_{20}\,\alpha_{12p} \\
& - 413549252645760\,\gamma_{20}\,\alpha_{12} + 1463891159808000\,\alpha_{20}\,\alpha_{12} + 474409172160\,\beta_{22p}\,\alpha_{10} + 355806879120\,\beta_{22}\,\alpha_{10} \\
& + 8471592360\,\alpha_{12p}\,\alpha_{10}^2 - 3882813165\,\alpha_{12}\,\alpha_{10}^2 + 14993672601600\,\gamma_{26}\,\alpha_6 - 1800579432960\,\beta_{14}\,\alpha_{12p}\,\alpha_6 \\
& + 559752621120\,\beta_{14}\,\alpha_{12}\,\alpha_6 + 14755739264\,\beta_{16}\,\alpha_{10}\,\alpha_6 - 25299240960\,\alpha_{16}\,\alpha_{10}\,\alpha_6 - 4775514472960\,\gamma_{20}\,\alpha_6^2 \\
& + 10174277836800\,\alpha_{20}\,\alpha_6^2 - 43285228\,\alpha_{10}^2\,\alpha_6^2 + 7065470720\,\beta_{14}\,\alpha_6^3 + 779296133468160\,\chi_{28}\,\alpha_4 \\
& - 530133424128\,\beta_{14}^2\,\alpha_4 - 2857212610560\,\beta_{16}\,\alpha_{12p}\,\alpha_4 + 1510363895040\,\beta_{16}\,\alpha_{12}\,\alpha_4 - 5020202880\,\alpha_{18}\,\alpha_{10}\,\alpha_4 \\
& + 59052646477440\,\chi_{18}\,\alpha_{10}\,\alpha_4 - 104866460160\,\beta_{22p}\,\alpha_6\,\alpha_4 - 38488222080\,\beta_{22}\,\alpha_6\,\alpha_4 + 16149647360\,\beta_{16}\,\alpha_6^2\,\alpha_4 \\
& + 642585968640\,\gamma_{24}\,\alpha_4^2 + 516363724800\,\alpha_{24}\,\alpha_4^2 + 1529966592\,\beta_{14}\,\alpha_{10}\,\alpha_4^2 - 5609877504000\,\chi_{18}\,\alpha_6\,\alpha_4^2 \\
& - 130817347584\,\gamma_{20}\,\alpha_4^3 - 154557849600\,\alpha_{20}\,\alpha_4^3 - 1036728\,\alpha_{10}^2\,\alpha_4^3 + 97574400\,\beta_{14}\,\alpha_6\,\alpha_4^3 + 223027200\,\beta_{16}\,\alpha_4^4\,,
\end{aligned}$$

$$\begin{aligned}
0 = {} & -113265734400\,\alpha_{18}\,\beta_{16} - 107036119008000\,\chi_{18}\,\beta_{16} + 130691232000\,\alpha_{18}\,\alpha_{16} + 123503214240000\,\chi_{18}\,\alpha_{16} \\
& + 711613758240\,\gamma_{20}\,\beta_{14} + 242595599400\,\beta_{22}\,\alpha'_{12} - 121297799700\,\beta'_{22}\,\alpha_{12} - 107820266400\,\gamma_{24}\,\alpha_{10} \\
& + 670881657600\,\chi_{28}\,\alpha_6 - 399334320\,\beta_{14}^2\,\alpha_6 + 2662228800\,\beta_{16}\,\alpha'_{12}\,\alpha_6 + 3993343200\,\beta_{16}\,\alpha_{12}\,\alpha_6 \\
& + 80673600\,\alpha_{18}\,\alpha_{10}\,\alpha_6 + 699198091200\,\chi_{18}\,\alpha_{10}\,\alpha_6 - 221852400\,\beta'_{22}\,\alpha_6^2 - 665557200\,\beta_{22}\,\alpha_6^2 \\
& + 657308736\,\beta_{16}\,\beta_{14}\,\alpha_4 - 4978713600\,\alpha_{16}\,\beta_{14}\,\alpha_4 + 37811907302400\,\chi_{18}\,\alpha'_{12}\,\alpha_4 + 16298463535200\,\chi_{18}\,\alpha_{12}\,\alpha_4 \\
& + 5427686880\,\gamma_{20}\,\alpha_{10}\,\alpha_4 + 21254365440\,\gamma_{24}\,\alpha_6\,\alpha_4 - 2545060\,\beta_{14}\,\alpha_{10}\,\alpha_6\,\alpha_4 - 194295615360\,\chi_{18}\,\alpha_6^2\,\alpha_4 \\
& - 2123573760\,\gamma_{26}\,\alpha_4^2 + 27165600\,\beta_{14}\,\alpha'_{12}\,\alpha_4^2 - 50935500\,\beta_{14}\,\alpha_{12}\,\alpha_4^2 - 7299040\,\beta_{16}\,\alpha_{10}\,\alpha_4^2 + 9466800\,\alpha_{16}\,\alpha_{10}\,\alpha_4^2 \\
& - 1339238208\,\gamma_{20}\,\alpha_6\,\alpha_4^2 + 5145\,\alpha_{10}^2\,\alpha_6\,\alpha_4^2 + 5174400\,\beta'_{22}\,\alpha_4^3 - 19404000\,\beta_{22}\,\alpha_4^3 - 4011279360\,\chi_{18}\,\alpha_4^4\,.
\end{aligned}$$

Runge [Run93, Cor.6.3] shows that $\mathbf{R}_3(\Gamma_3)$ is a Cohen-Macaulay algebra. There exists a strong link between a minimal free resolution of a Cohen-Macaulay algebra and its Hilbert series. Let us rewrite Equation (2.5) as a rational fraction with denominator $\prod_{d_i}(1-T^{d_i})$ where the degrees d_i run through the weights of the fundamental set of generators. We obtain a numerator with 140 non-zero coefficients, the first and last ones of which are

$$\begin{aligned}
& 1 - T^{32} - T^{34} - 2\,T^{36} - 4\,T^{38} - 5\,T^{40} - 5\,T^{42} - 7\,T^{44} - 6\,T^{46} - 8\,T^{48} - 5\,T^{50} - 4\,T^{52} \\
& \quad + 4\,T^{56} + 9\,T^{58} + 15\,T^{60} + 22\,T^{62} + 27\,T^{64} + 32\,T^{66} + 36\,T^{68} + 39\,T^{70} + 36\,T^{72} + 34\,T^{74} + 26\,T^{76} + \dots \\
& \qquad \dots - 5\,T^{296} - 8\,T^{298} - 6\,T^{300} - 7\,T^{302} - 5\,T^{304} - 5\,T^{306} - 4\,T^{308} - 2\,T^{310} - T^{312} - T^{314} + T^{346}.
\end{aligned}$$

The coefficients of the numerator give information on the weights and numbers of relations. They are consistent with Table 4 up to weight 48. The drop from 6 (relations) to a coefficient 5 in weight 50 indicates that there is a first syzygy (*i.e.* a relation between the relations) of weight 50.

3.3. **A homogeneous system of parameters.** Having these relations, one can also try to work out a homogeneous system of parameters (HSOP) for $\mathbf{R}_3(\Gamma_3)$. Recall that this is a set of elements $(f_i)_{1\leq i\leq m}$ of the algebra, which are algebraically independent, and such that $\mathbf{R}_3(\Gamma_3)$ is a $\mathbb{C}[f_1,\ldots,f_m]$-module of finite type. Equation (2.5) suggests that a HSOP of weight 4, 12, 12, 14, 18, 20 and 30 may exist. An easy Gröbner basis computation made in MAGMA with the lexicographic order $\boldsymbol{\alpha}_6 < \boldsymbol{\alpha}_{10} < \ldots < \boldsymbol{\gamma}_{26} < \boldsymbol{\chi}_{28}$ shows that when we set to zero $\boldsymbol{\alpha}_4$, $\boldsymbol{\alpha}_{12}$, $\boldsymbol{\alpha}'_{12}$, $\boldsymbol{\beta}_{14}$, $\boldsymbol{\chi}_{18}$, $\boldsymbol{\alpha}_{20}$ and $\boldsymbol{\alpha}_{30}$ in the 55 relations of Table 4, the remaining 12 Siegel modular forms of the generator set of Theorem 3.1 must be zero as well. As it is well known that the dimension of $\mathrm{Proj}(\mathbf{R}_3(\Gamma_3))$ is 6, this yields the following theorem.

Theorem 3.3. *A homogeneous system of parameters for* $\mathbf{R}_3(\Gamma_3)$ *is given by the* 7 *forms* $\boldsymbol{\alpha}_4$, $\boldsymbol{\alpha}_{12}$, $\boldsymbol{\alpha}'_{12}$, $\boldsymbol{\beta}_{14}$, $\boldsymbol{\chi}_{18}$, $\boldsymbol{\alpha}_{20}$ *and* $\boldsymbol{\alpha}_{30}$.

4. A DICTIONARY BETWEEN MODULAR FORMS AND INVARIANTS OF TERNARY QUARTICS

In [Dix87], Dixmier gives a homogeneous system of parameters for the graded $\mathbb{C}$-algebra $\mathbf{I}_3$ of invariants of ternary quartic forms under the action of $\mathrm{SL}_3(\mathbb{C})$. They are denoted I_3, I_6, I_9, I_{12}, I_{15}, I_{18} and I_{27}. This list is completed by Ohno with six invariants, J_9, J_{12}, J_{15}, J_{18} and J_{21}, into a list of 13 generators for $\mathbf{I}_3$, the so-called *Dixmier-Ohno invariants* [Ohn07, Els15]. Note that $2^{40}\cdot I_{27} = D_{27}$ where D_{27} denotes the normalized discriminant of plane quartics in the sense of [GKZ94, p.426] or [Dem12, Prop.11].

Using the morphism ρ_3 defined in [Igu67], Tsuyumine in [Tsu86, pp. 847–864] relates each of the Siegel modular forms given in Table 3 with an invariant for the graded ring of binary octics under the action of $\mathrm{SL}_2(\mathbb{C})$. He uses this key argument to prove Theorem 2.4. More generally, there is a way to canonically associate to a modular form an invariant. After briefly recalling the way to do so when $g=3$, we establish a complete dictionary between $\mathbf{R}_3(\Gamma_3)$ and $\mathbf{I}_3$.

4.1. **Modular forms in terms of invariants.** Let us recall from [LRZ10, 2.2] how to associate to $f\in\mathbf{R}_{3,h}(\Gamma_3)$ an element of $\mathbf{I}_3$. This morphism only depends on the choice of a universal basis of regular differentials ω which can be fixed "canonically" for smooth plane quartics (in the sense that it is a basis over $\mathbb{Z}$). Let $Q\in\mathbb{C}[x_1,x_2,x_3]$ be a ternary quartic form such that $C:Q=0$ is a smooth genus 3 curve. Let $\Omega=\left[\begin{smallmatrix}\Omega_1\\ \Omega_2\end{smallmatrix}\right]$ be the 6×3 period matrix of C defined by integrating ω_C with respect to an arbitrary symplectic basis of $H_1(C,\mathbb{Z})$. We have $\tau=\Omega_2^{-1}\Omega_1\in\mathbb{H}_3$. The function

$$Q\mapsto \Phi_3(f)(Q)=\left(\frac{(2i\pi)^3}{\det\Omega_2}\right)^h\cdot f(\tau) \tag{4.1}$$

is a homogeneous element of $\mathbf{I}_3$ of degree $3h$ (confusing the polynomial with its polynomial function).

Remark 4.1. A similar construction can be worked out with invariants of binary octics (see [IKL+19]). Up to a normalization constant, this is actually the same morphism as defined by [Igu67].

Chai's expansion principle [Cha86] shows that if the Fourier expansion of f has coefficients in a ring $R\subset\mathbb{C}$, then $\Phi_3(f)$ is defined over R as well. When f is given by a polynomial in the theta constants with coefficients in $\mathbb{Z}$, we can take $R=\mathbb{Z}$. A particular case is given by the

modular form $\boldsymbol{\chi}_{18}$ which is the product of the 36 theta constants. In [LRZ10] (see also [Ich18b]) one shows the following precise form of Klein's formula [Kle90, Eq. 118, p. 462],

$$\Phi_3(\boldsymbol{\chi}_{18}) = -2^{28} \cdot D_{27}^2 = -2^{28} \cdot (\, 2^{40}\, I_{27}\,)^2\,. \tag{4.2}$$

Remark 4.2. The map (4.1) is obtained by pulling back *geometric modular forms* to invariants as described in [LRZ10]. Within this background, it is for instance possible to speak about the reduction modulo a prime of modular forms and to consider the algebra that they generate. In small characteristics, one still encounters similar accidents as in the case of invariants. We will not study this question further here, but for instance, our 19 generators are not linearly independent modulo 11 since

$$\boldsymbol{\beta}_{16} + 9\,\boldsymbol{\alpha}_{16} + 3\,\boldsymbol{\alpha}_{10}\,\boldsymbol{\alpha}_6 = 0 \bmod 11.$$

We have seen in Section 3 that we have an evaluation/interpolation strategy to handle quotient of modular forms by a power of θ_{64}. This strategy can also be used to find the relations with invariants. But now, we also need to take care of the transcendental factor $\mu := (2i\pi)^3/\det \Omega_2$. This is done in the following way.

(i) Assume that a relation $\Phi_3(f_0) = I_0$ is known for a modular form f_0 of weight h_0. This is the case for $\boldsymbol{\chi}_{18}$ (*cf.* Eq. (4.2)) and we will start with this one, but switch to lower weights one (*i.e.* 4 with $\boldsymbol{\alpha}_4$ or even 2 with $\boldsymbol{\chi}_{18}/\boldsymbol{\alpha}_4^4$) after a first round of the following steps (this simplifies the last step).

(ii) Let now f be one of the generators from Theorem 3.1 of weight h and compute a basis $j_1, \ldots, j_d$ of invariants of degree $3h$. We aim at finding $a_1, \ldots, a_d \in \mathbb{Q}$ such that $\Phi_3(f) = \sum a_i j_i$. This is done by evaluation/interpolation at Riemann models until one gets a system of d linearly independent equations. More precisely, for a given $Q = 0$ and an associate $\tau \in \mathbb{H}_3$:

(a) Compute the values of $(j_1, \ldots, j_d)$ at Q;

(b) Using the same procedure as in Section 3, compute $(f/\theta_{64}^{2h})(\tau)$ and $(f_0/\theta_{64}^{2h_0})(\tau)$;

(c) Let $p = \mathrm{lcm}(h_0, h)$. Since

$$\frac{(f/\theta_{64}^{2h})^{p/h}}{(f_0/\theta_{64}^{2h_0})^{p/h_0}} = \frac{(\mu^h f)^{p/h}}{(\mu^{h_0} f_0)^{p/h_0}} = \frac{\Phi_3(f)^{p/h}}{\Phi_3(f_0)^{p/h_0}}\,,$$

we get the value of $\Phi_3(f)^{p/h}$. An approximate computation at low precision can then give the exact value.

The above strategy provides explicit expressions for $\Phi_3(f)$ where f is any modular form in the fundamental set defined in Theorem 3.1.

Proposition 4.3. *Let f be one modular form of weight h from Theorem 3.1. There exists an explicit polynomial P_f of degree $3h$ in the Dixmier-Ohno invariants such that*

$$\Phi_3(f) = P_f(\, I_3, I_6, \ldots, I_{27}\,)\,.$$

The first ones[3] are

$$\begin{aligned}
\Phi_3(\boldsymbol{\alpha}_4) &= \ 2^{20}\cdot 3^3\cdot 7\ \big(486\, I_{12} - 155520\, I_6^2 - 423\, J_9\, I_3 + 117\, I_9\, I_3 + 14418\, I_6\, I_3^2 + 8\, I_3^4\big)\,,\\
5\cdot 7\ \Phi_3(\boldsymbol{\alpha}_6) &= -\,2^{28}\cdot 3^4\ \big(40415760\, J_{18} - 1224720\, I_{18} - 2664900\, J_9^2 - 8323560\, J_9\, I_9 + 2506140\, I_9^2 - 76982400\, J_{12}\, I_6\\
&\quad - 1143538560\, I_{12}\, I_6 + 135992908800\, I_6^3 - 40041540\, J_{15}\, I_3 + 2143260\, I_{15}\, I_3 + 247160160\, J_9\, I_6\, I_3\\
&\quad + 289325520\, I_9\, I_6\, I_3 + 400950\, J_{12}\, I_3^2 - 6206220\, I_{12}\, I_3^2 - 7357573440\, I_6^2\, I_3^2 + 1527453\, J_9\, I_3^3\\
&\quad - 266481\, I_9\, I_3^3 - 36764280\, I_6\, I_3^4 - 62720\, I_3^6\big)\,,\\
\Phi_3(\boldsymbol{\alpha}_{12}) &= \ 2^{75}\cdot 3\ \big(495\, I_{27}\, J_9 - 261\, I_{27}\, I_9 - 14580\, I_{27}\, I_6\, I_3 + 32\, I_{27}\, I_3^3\big)\,,\\
\Phi_3(\boldsymbol{\beta}_{14}) &= \ 2^{81}\cdot 3^4\ \big(-540\, I_{27}\, J_{15} - 4860\, I_{27}\, I_{15} + 285120\, I_{27}\, J_9\, I_6 - 45360\, I_{27}\, I_9\, I_6 + 810\, I_{27}\, J_{12}\, I_3\\
&\quad + 12204\, I_{27}\, I_{12}\, I_3 - 18057600\, I_{27}\, I_6^2\, I_3 - 8541\, I_{27}\, J_9\, I_3^2 + 2961\, I_{27}\, I_9\, I_3^2 + 213912\, I_{27}\, I_6\, I_3^3 - 128\, I_{27}\, I_3^5\big)\,,\\
7\ \Phi_3(\boldsymbol{\beta}_{22}) &= -\,2^{135}\cdot 3^5\ \big(540\, I_{27}^2\, J_{12} - 4590\, I_{27}^2\, I_{12} - 151200\, I_{27}^2\, I_6^2 + 4005\, I_{27}^2\, J_9\, I_3 - 1683\, I_{27}^2\, I_9\, I_3\\
&\quad - 143010\, I_{27}^2\, I_6\, I_3^2 + 56\, I_{27}^2\, I_3^4\big)\,.
\end{aligned}$$

Beside Klein's formula $\Phi_3(\boldsymbol{\chi}_{18}) = -\,2^{108}\, I_{27}^2$, one finds a surprisingly compact expression for $\boldsymbol{\chi}_{28}$,

$$\Phi_3(\boldsymbol{\chi}_{28}) = -\,2^{171}\cdot 3^3\ I_{27}^3\, I_3\,.$$

If we do not not pay attention, the rational coefficients of these formulas tend to have prime factors greater than 7 in their denominators, especially for the forms of higher weight. We have eliminated all these "bad primes" using the relations that exist between the Dixmier-Ohno invariants. It is also a good way to reduce significantly the size of these expressions. All in all, we gain a factor of 3 in the amount of memory to store the results (*cf.* Table 5).

Form	Leading coeff.	Terms	Digits
$\boldsymbol{\alpha}_4$	$2^{20}\cdot 3^3\cdot 7$	6	6
$\boldsymbol{\alpha}_6$	$-2^{28}\cdot 3^4\cdot 5^{-1}\cdot 7^{-1}$	19	12
$\boldsymbol{\alpha}_{10}$	$-2^{44}\cdot 3^3\cdot 5^{-4}\cdot 7^{-2}$	98	23
$\boldsymbol{\alpha}_{12}$	$2^{75}\cdot 3$	4	5
$\boldsymbol{\alpha}_{12p}$	$2^{52}\cdot 3^2\cdot 5^{-4}\cdot 7^{-3}$	200	26
$\boldsymbol{\beta}_{14}$	$2^{81}\cdot 3^4$	11	8
$\boldsymbol{\alpha}_{16}$	$2^{66}\cdot 3^4\cdot 5^{-6}\cdot 7^{-3}$	703	35
$\boldsymbol{\beta}_{16}$	$2^{83}\cdot 3^4\cdot 5^{-2}\cdot 7^{-2}$	29	16
$\boldsymbol{\chi}_{18}$	-2^{108}	1	1
$\boldsymbol{\alpha}_{18}$	$-2^{80}\cdot 3^3\cdot 5^{-5}\cdot 7^{-2}$	813	36

Form	Leading coeff.	Terms	Digits
$\boldsymbol{\alpha}_{20}$	$2^{75}\cdot 3^2\cdot 5^{-13}\cdot 7^{-6}$	1941	52
$\boldsymbol{\gamma}_{20}$	$2^{122}\cdot 3^4$	2	3
$\boldsymbol{\beta}_{22}$	$-2^{135}\cdot 3^5\cdot 7^{-1}$	7	6
$\boldsymbol{\beta}_{22p}$	$2^{90}\cdot 3^4\cdot 5^{-14}\cdot 7^{-6}$	4000	56
$\boldsymbol{\alpha}_{24}$	$2^{89}\cdot 3^2\cdot 5^{-17}\cdot 7^{-7}$	6572	67
$\boldsymbol{\gamma}_{24}$	$2^{96}\cdot 3^3\cdot 5^{-14}\cdot 7^{-7}$	6585	62
$\boldsymbol{\gamma}_{26}$	$-2^{105}\cdot 3^5\cdot 5^{-17}\cdot 7^{-7}$	10750	67
$\boldsymbol{\chi}_{28}$	$2^{171}\cdot 3^3$	1	1
$\boldsymbol{\alpha}_{30}$	$2^{109}\cdot 3\cdot 5^{-21}\cdot 7^{-9}$	25630	86

Table 5: Polynomial expressions of the modular forms from Theorem 3.1 in terms of Dixmier-Ohno invariants: their content, their number of monomials, and the number of digits of the largest coefficient of their primitive part.

Remark 4.4. When we deal with the Jacobian of a curve with coefficients in $\mathbb{Q}$, what is a matter of few integer arithmetic operations to evaluate modular forms from invariants is a matter of high precision floating point arithmetic over the complex with analytic computations of Riemann matrices. In practical calculations, such as the computations in Section 3.2, it is thus much better to use the former, since a calculation that would take the order of the minute ultimately requires only a few milliseconds.

[3] We make available the list of these 19 polynomials at [LR19, file "`SiegelMFfromDO.txt`"].

4.2. **Invariants in terms of modular functions.** Conversely, we can look for expressions of a generator set of invariants in terms of modular forms. Using [Tsu86, LG19], one obtains such a result for invariants of binary octics. We focus here on the case of Dixmier-Ohno invariants.

Since the locus of plane quartic over $\mathbb{C}$ such that $I_{27} \neq 0$ corresponds to the locus of non-hyperelliptic curve of genus 3 and then to principally polarized abelian threefolds $\mathbb{C}^3/(\tau\mathbb{Z}^3+\mathbb{Z}^3)$ for which $\boldsymbol{\chi}_{18}(\tau) \neq 0$ [Igu67, Lem. 10, 11], we see that any invariant in $\mathbf{I}_3$ can be obtained as a quotient of a modular form by a power of I_{27}.

Proposition 4.5. *Let I be a Dixmier-Ohno invariant of degree $3k$. There exist a polynomial P_I in the modular forms from Theorem 3.1, of weight $28\,k$, such that*

$$I_{27}^{3k} \cdot I = \Phi_3(\, P_I(\, \boldsymbol{\alpha}_4,\, \boldsymbol{\alpha}_6, \ldots, \boldsymbol{\alpha}_{30}\,)\,)\,.$$

The first ones[4] are

$$\begin{aligned}
2^{171} \cdot 3^3 \;\; I_{27}^3\, I_3 &= \Phi_3(\text{-}\boldsymbol{\chi}_{28})\,,\\
2^{144} \cdot 3^8 \cdot 5 \;\; I_{27}^6\, I_6 &= \Phi_3(\boldsymbol{\chi}_{28}^2 - 2^4 \cdot 3^2\, \boldsymbol{\chi}_{18}^2\, \boldsymbol{\gamma}_{20})\,,\\
2^{515} \cdot 3^{12} \cdot 5 \cdot 7^4 \;\; I_{27}^9\, I_9 &= \Phi_3(\,(-11735539200\, \boldsymbol{\alpha}_{12}' - 2920548960\, \boldsymbol{\alpha}_{12} - 86929920\, \boldsymbol{\alpha}_6^2 - 2027520\, \boldsymbol{\alpha}_4^3)\, \boldsymbol{\chi}_{18}^4\\
&\quad + (3259872\, \boldsymbol{\beta}_{16}\, \boldsymbol{\beta}_{14} - 4074840\, \boldsymbol{\gamma}_{20}\, \boldsymbol{\alpha}_{10} + 21732480\, \boldsymbol{\gamma}_{24}\, \boldsymbol{\alpha}_6 - 24837120\, \boldsymbol{\gamma}_{26}\, \boldsymbol{\alpha}_4\\
&\quad + 137984\, \boldsymbol{\gamma}_{20}\, \boldsymbol{\alpha}_6\, \boldsymbol{\alpha}_4)\, \boldsymbol{\chi}_{18}^3 + 153856080\, \boldsymbol{\chi}_{28}\, \boldsymbol{\gamma}_{20}\, \boldsymbol{\chi}_{18}^2 - 1764735\, \boldsymbol{\chi}_{28}^3\,)\,,\\
2^{515} \cdot 3^{12} \cdot 5^2 \cdot 7^4 \;\; I_{27}^9\, J_9 &= \Phi_3(\,(-30939148800\, \boldsymbol{\alpha}_{12}' - 2200413600\, \boldsymbol{\alpha}_{12} - 229178880\, \boldsymbol{\alpha}_6^2 - 5345280\, \boldsymbol{\alpha}_4^3)\, \boldsymbol{\chi}_{18}^4\\
&\quad + (8594208\, \boldsymbol{\beta}_{16}\, \boldsymbol{\beta}_{14} - 10742760\, \boldsymbol{\gamma}_{20}\, \boldsymbol{\alpha}_{10} + 57294720\, \boldsymbol{\gamma}_{24}\, \boldsymbol{\alpha}_6 - 65479680\, \boldsymbol{\gamma}_{26}\, \boldsymbol{\alpha}_4\\
&\quad + 363776\, \boldsymbol{\gamma}_{20}\, \boldsymbol{\alpha}_6\, \boldsymbol{\alpha}_4)\, \boldsymbol{\chi}_{18}^3 + 558376560\, \boldsymbol{\chi}_{28}\, \boldsymbol{\gamma}_{20}\, \boldsymbol{\chi}_{18}^2 - 5294205\, \boldsymbol{\chi}_{28}^3\,)\,.
\end{aligned}$$

In this setting, one can also write $I_{27}^{27}\, I_{27} = \Phi_3((2^{-108}\, \boldsymbol{\chi}_{18})^{14})$.

Unlike the previous computations, one cannot obtain the above ones by a direct application of the evaluation/interpolation strategy as the degrees (and weights) are sometimes too large. For the invariant I_{21}, for instance, one would potentially need to interpolate on a vector space of modular forms of weight 196, which is huge (its dimension is 869 945). The trick is to proceed by steps and first look for expressions of a small power of I_{27} by the desired invariant I, not only in terms of modular forms, but also in terms of invariants I_{3k} of smaller degrees. For instance in the case of I_{21},

$$\begin{aligned}
2^{63} \cdot 3^{21} \cdot 5^{21} \cdot 7^{10} \cdot 11 \;\; I_{27}\, I_{21} &= 2^{51} \cdot 3^{15} \cdot 5^{18} \cdot 7^9 \cdot 11 \;\; I_{27}\, (-16156800\, J_{12}\, J_9 + 5680595070\, I_{12}\, J_9 + 109296000\, J_{12}\, I_9\\
&\quad - 3076972650\, I_{12}\, I_9 - 216169581600\, J_{15}\, I_6 + 439538400\, I_{15}\, I_6\\
&\quad - 770217033600\, J_9\, I_6^2 + 2235454502400\, I_9\, I_6^2 + 8070768720\, J_{18}\, I_3\\
&\quad - 622051920\, I_{18}\, I_3 - 3928070295\, J_9^2\, I_3 + 1754339490\, J_9\, I_9\, I_3 - 182964375\, I_9^2\, I_3\\
&\quad + 70135124400\, J_{12}\, I_6\, I_3 - 611730004680\, I_{12}\, I_6\, I_3 - 18401013388800\, I_6^3\, I_3\\
&\quad - 8799659820\, J_{15}\, I_3^2 + 1352865780\, I_{15}\, I_3^2 + 237928085190\, J_9\, I_6\, I_3^2\\
&\quad - 56462733090\, I_9\, I_6\, I_3^2 + 294430290\, J_{12}\, I_3^3 - 1980696900\, I_{12}\, I_3^3\\
&\quad - 4995876680760\, I_6^2\, I_3^3 - 65637369\, J_9\, I_3^4 - 76264307\, I_9\, I_3^4 + 4016874680\, I_6\, I_3^5) + 2^{12}\\
&\quad \cdot 3^6 \cdot 5^3 \cdot 7\, \Phi_3(-19003712\, \boldsymbol{\beta}_{16} - 10671360\, \boldsymbol{\alpha}_{16} - 11116\, \boldsymbol{\alpha}_{10}\, \boldsymbol{\alpha}_6 - 1844513\, \boldsymbol{\alpha}_{12}\, \boldsymbol{\alpha}_4)\,.
\end{aligned}$$

Then, mechanically, through a sequence of substitutions of the invariants of smaller degrees by their expression in terms of the modular forms, we arrive to expressions for $I_{27}^{3k}\, I_{3k}$ purely in terms of modular forms. These formulas are very sparse, given their weight (see Table 5).

Remark 4.6. We are also able to eliminate the primes greater than 7 in the denominators of the coefficients in these formulas using the relations that exist between Siegel modular forms (*cf.* Section 3.2), with the notable exception of the primes 11 and 19 (*cf.* Table 5). We suspect that the reason behind this difficulty is that, similarly to the prime 11 (*cf.* Remark 4.2), one cannot

[4] We make available the list of these 13 polynomials at [LR19, file "`SiegelMFtoDO.txt`"].

DO inv.	Leading coeff.	Terms	Digits
I_3	$2^{-171} \cdot 3^{-3}$	1	1
I_6	$2^{-344} \cdot 3^{-8} \cdot 5$	2	3
I_9	$2^{-515} \cdot 3^{-12} \cdot 5 \cdot 7^{-4}$	11	11
J_9	$2^{-515} \cdot 3^{-12} \cdot 5^{-2} \cdot 7^{-4}$	11	11
I_{12}	$2^{-686} \cdot 3^{-16} \cdot 5^{-2} \cdot 7^{-4}$	13	13
J_{12}	$2^{-686} \cdot 3^{-16} \cdot 5 \cdot 7^{-3}$	14	13

DO inv.	Leading coeff.	Terms	Digits
I_{15}	$2^{-859} \cdot 3^{-20} \cdot 5^{-2} \cdot 7^{-4} \cdot 11^{-1}$	58	17
J_{15}	$2^{-859} \cdot 3^{-18} \cdot 5^{-3} \cdot 7^{-4} \cdot 11^{-1}$	58	17
I_{18}	$2^{-1030} \cdot 3^{-24} \cdot 5^{-2} \cdot 7^{-7} \cdot 11^{-2} \cdot 19^{-1}$	1321	237
J_{18}	$2^{-1030} \cdot 3^{-24} \cdot 5^{-3} \cdot 7^{-8} \cdot 11^{-3} \cdot 19^{-1}$	1321	238
I_{21}	$2^{-1202} \cdot 3^{-29} \cdot 5^{-6} \cdot 7^{-8} \cdot 11^{-3} \cdot 19^{-1}$	1382	242
J_{21}	$2^{-1201} \cdot 3^{-27} \cdot 5^{-6} \cdot 7^{-8} \cdot 11^{-3} \cdot 19^{-1}$	1382	242
I_{27}	2^{-1512}	1	1

Table 6: Polynomial expressions of the Dixmier-Ohno in terms of the 19 generators from Theorem 3.1: the content, the number of monomial, and the number of digits of the largest coefficient of the primitive parts.

extend Theorem 3.1 *mutatis mutandis* to characteristic 19. Although we do not go further on the topic, it is possible to work directly in these characteristics and find specific formulas valid there.

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