



HAL
open science

Horn inequalities for nonzero Kronecker coefficients

Nicolas Ressayre

► **To cite this version:**

Nicolas Ressayre. Horn inequalities for nonzero Kronecker coefficients. *Advances in Mathematics*, 2019, 356, pp.106809. 10.1016/j.aim.2019.106809 . hal-02186377

HAL Id: hal-02186377

<https://hal.science/hal-02186377v1>

Submitted on 17 Jul 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Horn inequalities for nonzero Kronecker coefficients

N. Ressayre

July 17, 2019

Abstract

The Kronecker coefficients $g_{\alpha\beta\gamma}$ and the Littlewood-Richardson coefficients $c_{\alpha\beta}^{\gamma}$ are nonnegative integers depending on three partitions α , β , and γ . By definition, $g_{\alpha\beta\gamma}$ (resp. $c_{\alpha\beta}^{\gamma}$) are the multiplicities of the tensor product decomposition of two irreducible representations of symmetric groups (resp. linear groups). By a classical Littlewood-Murnaghan's result the Kronecker coefficients extend the Littlewood-Richardson ones.

The nonvanishing of the Littlewood-Richardson coefficient $c_{\alpha\beta}^{\gamma}$ implies that (α, β, γ) satisfies some linear inequalities called Horn inequalities. In this paper, we extend the essential Horn inequalities to the triples of partitions corresponding to a nonzero Kronecker coefficient.

Along the way, we describe the set of tripless (α, β, γ) of partitions such that $c_{\alpha\beta}^{\gamma} \neq 0$ and $l(\alpha) \leq e$, $l(\beta) \leq f$ and $l(\gamma) \leq e + f$, for some given positive integers e and f . This set is the natural analogue of the classical Horn semigroup when one thinks about $c_{\alpha\beta}^{\gamma}$ as the branching multiplicities for the subgroup $\mathrm{GL}_e \times \mathrm{GL}_f$ of GL_{e+f} .

1 Introduction

If $\alpha = (\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_e \geq 0)$ is a partition, we set $|\alpha| = \sum_i \alpha_i$ in such a way α is a partition of $|\alpha|$. Consider the symmetric group S_n on n letters. The irreducible representations of S_n are parametrized by the partitions of n , see *e.g.* [Mac95, I. 7]. Let $[\alpha]$ denote the representation of $S_{|\alpha|}$ corresponding to α . The Kronecker coefficients $g_{\alpha\beta\gamma}$, depending on three partitions α , β , and γ of the same integer n , are defined by

$$[\alpha] \otimes [\beta] = \sum_{\gamma} g_{\alpha\beta\gamma} [\gamma]. \quad (1)$$

The length $l(\alpha)$ of the partition α is the number of nonzero parts α_i . Let V be a complex vector space of dimension d . If $l(\alpha) \leq d$ then $S^\alpha V$ denotes the Schur power (see *e.g.* [FH91]): it is an irreducible polynomial representation of the linear group $\mathrm{GL}(V)$. Let β be a second partition such that $l(\beta) \leq d$. Then the Littlewood-Richardson coefficients $c_{\alpha\beta}^\gamma$ are defined by

$$S^\alpha V \otimes S^\beta V = \sum_{\gamma} c_{\alpha\beta}^\gamma S^\gamma V. \quad (2)$$

The partition obtained by suppressing the first part of α is denoted by $\bar{\alpha} = (\alpha_2 \geq \alpha_3 \dots)$. Observe that $\bar{\alpha}_1 = \alpha_2$. We state a classical result due to Littlewood and Murnaghan (see for example [JK81]).

Proposition 1 *Let α , β and γ be three partitions of the same integer n .*

(i) *If $g_{\alpha\beta\gamma} \neq 0$ then*

$$(n - \alpha_1) + (n - \beta_1) \geq n - \gamma_1. \quad (3)$$

(ii) *If $(n - \alpha_1) + (n - \beta_1) = n - \gamma_1$ then*

$$g_{\alpha\beta\gamma} = c_{\bar{\alpha}\bar{\beta}}^{\bar{\gamma}}. \quad (4)$$

In this paper, we prove many other inequalities similar to the identity (3), that are consequences of the nonvanishing of $g_{\alpha\beta\gamma}$. For the partitions (α, β, γ) satisfying equality in such an inequality, we prove a reduction rule for $g_{\alpha\beta\gamma}$ similar to the identity (4).

Observe that the formula (4) shows that the Kronecker coefficients extend the Littlewood-Richardson ones. Indeed, given $\bar{\alpha}$, $\bar{\beta}$ and $\bar{\gamma}$, one can find $\alpha = (\alpha_1, \bar{\alpha})$, $\beta = (\beta_1, \bar{\beta})$ and $\gamma = (\gamma_1, \bar{\gamma})$ such that $|\alpha| = |\beta| = |\gamma| = n$, $(n - \alpha_1) + (n - \beta_1) = n - \gamma_1$. Then $c_{\bar{\alpha}\bar{\beta}}^{\bar{\gamma}} = g_{\alpha\beta\gamma}$ is a Kronecker coefficient. If $c_{\bar{\alpha}\bar{\beta}}^{\bar{\gamma}} \neq 0$ then $(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ satisfy the Horn inequalities (see *e.g.* [Ful00] or below for details). If $g_{\alpha\beta\gamma} \neq 0$, our inequalities for (α, β, γ) extend some Horn inequalities. Fix such an inequality $\varphi(\bar{\alpha}, \bar{\beta}, \bar{\gamma}) \geq 0$. We want to find an inequality $\tilde{\varphi}(\alpha, \beta, \gamma) \geq 0$ such that

(i) If $g_{\alpha\beta\gamma} \neq 0$ then $\tilde{\varphi}(\alpha, \beta, \gamma) \geq 0$;

(ii) If $(n - \alpha_1) + (n - \beta_1) = n - \gamma_1$ then $\tilde{\varphi}(\alpha, \beta, \gamma) = \varphi(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$.

For example, a Weyl's theorem [Wey12] asserts that if $c_{\bar{\alpha}\bar{\beta}}^{\bar{\gamma}} \neq 0$ then

$$\bar{\gamma}_{e+j-1} \leq \bar{\beta}_{j-1}, \quad (5)$$

whenever $l(\bar{\alpha}) \leq e$ and $j \geq 2$.

Before stating our extension of Weyl's theorem, we introduce some notation. Let $\mathcal{S}(r, d)$ denote the set of subsets of $\{1, \dots, d\}$ with r elements. Given $I = \{i_1 < \dots < i_r\} \in \mathcal{S}(r, d)$ and $\alpha = (\alpha_1 \geq \dots \geq \alpha_d)$ a partition of length at most d , we set $\alpha_I = (\alpha_{i_1} \geq \dots \geq \alpha_{i_r})$. Observe that $\bar{\alpha}_I = (\alpha_{i_1+1} \geq \dots \geq \alpha_{i_r+1})$.

Theorem 1 *Let e and f be two positive integers. Let α , β , and γ be three partitions of the same integer n such that*

$$l(\alpha) \leq e + 1, \quad l(\beta) \leq f + 1, \quad \text{and} \quad l(\gamma) \leq e + f + 1. \quad (6)$$

Let $j \in \{2, \dots, f + 1\}$.

(i) *If $g_{\alpha\beta\gamma} \neq 0$ then*

$$n + \gamma_1 + \gamma_{e+j} \leq \alpha_1 + \beta_1 + \beta_j \quad (7)$$

(ii) *Set $J = \{1, \dots, f\} - \{j - 1\}$ and $K = \{1, \dots, e + f\} - \{e + j - 1\}$. If $n + \gamma_1 + \gamma_{e+j} = \alpha_1 + \beta_1 + \beta_j$ then*

$$g(\alpha, \beta, \gamma) = \sum_{l(x) \leq 2e, l(y) \leq 2} c(x, \bar{\beta}_J; \bar{\gamma}_K) \cdot c(\gamma_1 \geq \gamma_j, \mathbf{y}; \beta_1 \geq \beta_j) \cdot g(\bar{\alpha}, x, y).$$

Remark. In the statement of Theorem 1 (and sometimes below) we denote $c_{\alpha\beta}^{\gamma}$ and $g_{\alpha\beta\gamma}$ respectively by $c(\alpha, \beta; \gamma)$ and $g(\alpha, \beta, \gamma)$.

Theorem 1 extends Weyl's theorem in the sense that if $(n - \alpha_1) + (n - \beta_1) = n - \gamma_1$ then the inequality (7) is equivalent to $\gamma_{e+j} \leq \beta_j$, that is to the inequality (5).

For $I \in \mathcal{S}(r, d)$, consider the partition

$$\tau^I = (d - r + 1 - i_1 \geq d - r + 2 - i_2 \geq \dots \geq d - i_r).$$

Set $|\alpha_I| := \sum_{i \in I} \alpha_i$. Observe that $|\bar{\alpha}_I| := \sum_{i \in I} \alpha_{i+1}$. We can now state our main result.

Theorem 2 Let α , β , and γ be three partitions of the same integer n satisfying the conditions (6).

Assume that $g_{\alpha\beta\gamma} \neq 0$. Then

$$n + |\bar{\alpha}_I| - \alpha_1 + |\bar{\beta}_J| - \beta_1 \geq |\bar{\gamma}_K| - \gamma_1, \quad (8)$$

for any $0 < r < e$, $0 < s < f$, $I \in \mathcal{S}(r, e)$, $J \in \mathcal{S}(s, f)$ and $K \in \mathcal{S}(r+s, e+f)$ such that

$$c_{\tau I \tau J}^{\tau K} = 1. \quad (9)$$

If $(n - \alpha_1) + (n - \beta_1) = n - \gamma_1$ then the inequality (8) is equivalent to

$$|\bar{\alpha}_I| + |\bar{\beta}_J| \geq |\bar{\gamma}_K|, \quad (10)$$

which is a Horn inequality (see [Ful00] or Section 4).

Remark. Since inequalities (3), (7) and (8) are linear in (α, β, γ) , the condition $g_{\alpha\beta\gamma} \neq 0$ in Proposition 1 and Theorem 1 and 2 can be replaced by the weaker condition $g_{k\alpha k\beta k\gamma} \neq 0$ for some positive k .

We get a reduction formula for the coefficients $g_{\alpha\beta\gamma}$ if the inequality (8) is saturated. If $I \in \mathcal{S}(r, d)$, we denote by $I_- \in \mathcal{S}(d-r, d)$ the complement of I in $\{1, \dots, d\}$. By symmetry we also set $I_+ = I$.

Theorem 3 Let α , β , and γ be three partitions of the same integer n satisfying the conditions (6).

Let (I, J, K) be a triple that appears in Theorem 2 (in particular satisfying the condition (9)). We assume that

$$n + |\bar{\alpha}_I| - \alpha_1 + |\bar{\beta}_J| - \beta_1 = |\bar{\gamma}_K| - \gamma_1. \quad (11)$$

Then $g(\alpha, \beta, \gamma)$ is equal to

$$\sum_{a,b,x,y,u,v} c(\bar{\alpha}_{I_-}, \bar{\beta}_{J_-}; y) \cdot c(x, y; \bar{\gamma}_{K_+}) \cdot c(u, v; \bar{\gamma}_{K_-}) \cdot c(a, u; \bar{\alpha}_I) \cdot c(b, v; \bar{\beta}_J) \cdot g(a, b, x), \quad (12)$$

where the sum runs over the partitions a, b, x, y, u, v satisfying

$$\begin{aligned} l(x) &\leq (e-r)(f-s), & l(a) &\leq e-r, & l(u) &\leq e-r, \\ l(y) &\leq r+s, & l(b) &\leq f-s, & l(v) &\leq f-s. \end{aligned} \quad (13)$$

Note that in Theorem 3, we needn't assume that $g_{\alpha\beta\gamma} \neq 0$.

Let $\text{Kron}(e+1, f+1, e+f+1)$ denote the set of triples (α, β, γ) of partitions such that $|\alpha| = |\beta| = |\gamma|$, $g_{\alpha\beta\gamma} \neq 0$ and $l(\alpha) \leq e+1$, $l(\beta) \leq f+1$, $l(\gamma) \leq e+f+1$. Then $\text{Kron}(e+1, f+1, e+f+1)$ is a finitely generated semigroup in $\mathbb{Z}_{\geq 0}^{2e+2f+3}$. In particular, the cone $\mathbb{Q}_{\geq 0} \text{Kron}(e+1, f+1, e+f+1)$ generated by $\text{Kron}(e+1, f+1, e+f+1)$ is a closed convex polyhedral cone.

Theorem 4 *The inequalities (7) in Theorem 1 and the inequalities (8) in Theorem 2 are essential, that is correspond to codimension one faces of $\mathbb{Q}_{\geq 0} \text{Kron}(e+1, f+1, e+f+1)$.*

One can guess to describe the complete minimal list \mathcal{L} of inequalities characterizing $\mathbb{Q}_{\geq 0} \text{Kron}(e+1, f+1, e+f+1)$. Such a list is known for the Littlewood-Richardson coefficients (see Theorem 7 below for details). In principle, [Res10] gives \mathcal{L} . Nevertheless, it is known to be untractable to make this description very explicit. Indeed, one first need to describe the so-called adapted one-parameter subgroups by describing the collection of hyperplanes spanned by subsets of a given set: a tricky combinatorial problem. And secondly one need to understand an unknown Schubert problem. In this paper we describe a natural subset of \mathcal{L} related with the Horn cone.

Inequality (3) defines a codimension one face \mathcal{F}_{LM} of $\mathbb{Q}_{\geq 0} \text{Kron}(e+1, f+1, e+f+1)$. Here “LM” stands for Littlewood-Murnaghan. Each Horn inequality (10) or Weyl inequality (5) define a face \mathcal{F} of codimension two contained in \mathcal{F}_{LM} . By convex geometry \mathcal{F} has to be contained in a second codimension one face \mathcal{F}' of $\mathbb{Q}_{\geq 0} \text{Kron}(e+1, f+1, e+f+1)$. Basically, Theorem 1 and 2 describe this face \mathcal{F}' .

Comparison between Theorems 1 and 2. With I, J and K respectively equal to $\{1, \dots, e\}$, $\{1, \dots, f\} - \{j-1\}$, and $\{1, \dots, e+f\} - \{e+j-1\}$ (where $j \in \{2, \dots, f+1\}$), we have $c_{\tau_I \tau_J}^{\tau_K} = 1$. The inequality (8) gives

$$2n + 2\gamma_1 + \gamma_j \geq 2\alpha_1 + 2\beta_1 + \beta_j. \quad (14)$$

This inequality is satisfied if $g_{\alpha\beta\gamma} \neq 0$. But the corresponding face has codimension 2 in $\mathbb{Q}_{\geq 0} \text{Kron}(e+1, f+1, e+f+1)$. Hence the inequality (14) is not essential. More precisely, it is a consequence of inequalities (3) and (7).

In Section 2, we define and compare several semigroups. In Section 3, we recall some results from [Res10] that allows to describe some cones generated

by these semigroups. In Section 4, we describe the support of the LR-coefficients $c_{\alpha\beta}^\gamma$ for partitions satisfying $l(\alpha) \leq e$, $l(\beta) \leq f$ and $l(\gamma) \leq e + f$, for fixed positive integers e and f . Note that these assumptions are natural if one thinks about the LR-coefficients as multiplicities for the branching from $\mathrm{GL}_e \times \mathrm{GL}_f$ to GL_{e+f} . It is a variation of the classical Horn problem. The next sections contain the proofs of the statements of the introduction.

Acknowledgements. The author is partially supported by the French National Agency (Project GeoLie ANR-15-CE40-0012) and the Institut Universitaire de France (IUF).

2 Semigroups

2.1 Definitions

2.1.1 Kronecker semigroups

We extend the definition of $g_{\alpha\beta\gamma}$ to any triple (α, β, γ) of partitions by setting $g_{\alpha\beta\gamma} = 0$ if the condition $|\alpha| = |\beta| = |\gamma|$ does not hold. Let e , f , and g be three positive integers. We define $\mathrm{Kron}(e, f, g)$ to be the set of triples (α, β, γ) of partitions such that $g_{\alpha\beta\gamma} \neq 0$ and $l(\alpha) \leq e$, $l(\beta) \leq f$, $l(\gamma) \leq g$. It is well known that $\mathrm{Kron}(e, f, g)$ is a finitely generated semigroup of $\mathbb{Z}_{\geq 0}^{e+f+g}$.

2.1.2 Littlewood-Richardson semigroups

We define $\mathrm{LR}(e, f, g)$ to be the set of triples (α, β, γ) of partitions such that $c_{\alpha\beta}^\gamma \neq 0$ and $l(\alpha) \leq e$, $l(\beta) \leq f$, $l(\gamma) \leq g$. It is well known that $\mathrm{LR}(e, f, g)$ is a finitely generated semigroup of $\mathbb{Z}_{\geq 0}^{e+f+g}$.

2.1.3 Branching semigroups

Let G be a connected reductive subgroup of a complex connected reductive group \hat{G} . Fix maximal tori $T \subset \hat{T}$ and Borel subgroups $B \supset T$ and $\hat{B} \supset \hat{T}$ of G and \hat{G} . Let $X(T)$ denote the group of characters of T and let $X(T)^+$ denote the set of dominant characters. The irreducible representation of highest weight $\nu \in X(T)^+$ is denoted by V_ν . Similarly, we use the notation $X(\hat{T})$, $X(\hat{T})^+$, $V_{\hat{\nu}}$ relatively to \hat{G} . The subspace of G -fixed vectors of the

G -module V is denoted by V^G . Set

$$c_{\nu\hat{\nu}} = \dim(V_{\nu}^* \otimes V_{\hat{\nu}})^G. \quad (15)$$

The branching problem is equivalent to the knowledge of these coefficients since

$$V_{\hat{\nu}} = \sum_{\nu \in X(T)^+} c_{\nu\hat{\nu}} V_{\nu}, \quad (16)$$

as a G -module. Consider the set

$$\text{LR}(G, \hat{G}) = \{(\nu, \hat{\nu}) \in X(T)^+ \times X(\hat{T})^+ : c_{\nu\hat{\nu}} \neq 0\}.$$

By a result of Brion and Knop (see [É92]), $\text{LR}(G, \hat{G})$ is a finitely generated semigroup.

2.1.4 GIT semigroups

Let G be a complex reductive group acting on an irreducible projective variety X . Let $\text{Pic}^G(X)$ denote the group of G -linearized line bundles on X . The space $H^0(X, \mathcal{L})$ of regular sections of \mathcal{L} is a G -module. Consider the set

$$\text{LR}(G, X) = \{\mathcal{L} \in \text{Pic}^G(X) : H^0(X, \mathcal{L})^G \neq \{0\}\}. \quad (17)$$

Since X is irreducible, the product of two nonzero G -invariant sections is a nonzero G -invariant section and $\text{LR}(G, X)$ is a semigroup.

2.2 Relations between these semigroups

2.2.1 Kronecker semigroups as branching semigroups.

Let E and F be two complex vector spaces of dimension e and f . Consider the group $G = \text{GL}(E) \times \text{GL}(F)$. Using Schur-Weyl duality, the Kronecker coefficient $g_{\alpha\beta\gamma}$ can be interpreted in terms of representations of G . Namely (see for example [Mac95, FH91]) $g_{\alpha\beta\gamma}$ is the multiplicity of $S^{\alpha}E \otimes S^{\beta}F$ in $S^{\gamma}(E \otimes F)$. More precisely, let γ be a partition such that $l(\gamma) \leq ef$. Then the simple $\text{GL}(E \otimes F)$ -module $S^{\gamma}(E \otimes F)$ decomposes as a sum of simple G -modules as follows

$$S^{\gamma}(E \otimes F) = \sum_{\substack{\text{partitions } \alpha, \beta \text{ s.t.} \\ l(\alpha) \leq e, l(\beta) \leq f}} g_{\alpha\beta\gamma} S^{\alpha}E \otimes S^{\beta}F. \quad (18)$$

As a consequence

$$\text{Kron}(e, f, ef) = \text{LR}(\text{GL}(E) \times \text{GL}(F), \text{GL}(E \otimes F)) \cap (\mathbb{Z}^e \times \mathbb{Z}^f \times (\mathbb{Z}_{\geq 0})^{ef}). \quad (19)$$

2.2.2 Littlewood-Richardson semigroups as branching semigroups

Since the Littlewood-Richardson coefficients are multiplicities for the tensor product decomposition of GL_n , we have

$$\mathrm{LR}(e, e, e) = \mathrm{LR}(\mathrm{GL}_e, \mathrm{GL}_e \times \mathrm{GL}_e) \cap ((\mathbb{Z}_{\geq 0})^e)^3. \quad (20)$$

The Littlewood-Richardson coefficients have another interpretation in terms of representations of linear groups. Consider the embedding of $\mathrm{GL}(E) \times \mathrm{GL}(F)$ in $\mathrm{GL}(E \oplus F)$ as a Levi subgroup by its natural action on $E \oplus F$. Then (see[Mac95, Chapter I, 5.9])

$$S^\gamma(E \oplus F) = \sum_{\substack{\text{partitions } \alpha, \beta \text{ s.t.} \\ l(\alpha) \leq e, l(\beta) \leq f}} c_{\alpha\beta}^\gamma S^\alpha E \otimes S^\beta F. \quad (21)$$

In particular

$$\mathrm{LR}(e, f, e + f) = \mathrm{LR}(\mathrm{GL}_e \times \mathrm{GL}_f, \mathrm{GL}_{e+f}) \cap (\mathbb{Z}^e \times \mathbb{Z}^f \times (\mathbb{Z}_{\geq 0})^{e+f}). \quad (22)$$

2.2.3 Branching semigroups as GIT semigroups

We use notation of Section 2.1.3 and we assume that G and \hat{G} are semisimple simply connected. Consider the diagonal action of G on $X = G/B \times \hat{G}/\hat{B}$. Note that $\mathrm{Pic}^G(X)$ identifies with $X(T) \times X(\hat{T})$. Then Borel-Weyl's theorem implies that $\mathrm{LR}(G, \hat{G}) = \mathrm{LR}(G, X)$.

2.2.4 Kronecker semigroups as GIT semigroups

If V is a complex finite dimensional vector space, let $\mathcal{F}l(V)$ denote the variety of complete flags of V . Given integers a_i such that $1 \leq a_1 < \dots < a_s \leq \dim(V) - 1$, we denote by $\mathcal{F}l(a_1, \dots, a_s; V)$ the variety of flags $V_1 \subset \dots \subset V_s \subset V$ such that $\dim(V_i) = a_i$ for any i . If α is a partition with at most $\dim(V)$ parts then \mathcal{L}_α (resp. \mathcal{L}^α) denotes the $\mathrm{GL}(V)$ -linearized line bundle on $\mathcal{F}l(V)$ such that the space $H^0(\mathcal{F}l(V), \mathcal{L}_\alpha)$ (resp. $H^0(\mathcal{F}l(V), \mathcal{L}^\alpha)$) is isomorphic to $S^\alpha V^*$ (resp. $S^\alpha V$) as a $\mathrm{GL}(V)$ -module.

Assume that E and F are two linear spaces of dimension $e + 1$ and $f + 1$. Set $G = \mathrm{GL}(E) \times \mathrm{GL}(F)$. Consider the variety

$$X = \mathcal{F}l(E) \times \mathcal{F}l(F) \times \mathcal{F}l(1, \dots, e + f + 1; E \otimes F)$$

endowed with its natural G -action. Let α , β , and γ be three partitions such that $l(\alpha) \leq e + 1$, $l(\beta) \leq f + 1$, and $l(\gamma) \leq e + f + 1$. Consider the $\mathrm{GL}(E)$ -linearized line bundle \mathcal{L}^α on $\mathcal{F}l(E)$, and respectively \mathcal{L}^β on $\mathcal{F}l(F)$. Since $l(\gamma) \leq e + f + 1$, the line bundle \mathcal{L}_γ on $\mathcal{F}l(E \otimes F)$ is the pullback of a line bundle (still denoted by \mathcal{L}_γ) on $\mathcal{F}l(1, \dots, e + f + 1; E \otimes F)$. Consider the line bundle $\mathcal{L} = \mathcal{L}^\alpha \otimes \mathcal{L}^\beta \otimes \mathcal{L}_\gamma$ on X endowed with its natural G -action. Then

$$H^0(X, \mathcal{L}) \simeq S^\alpha E \otimes S^\beta F \otimes S^\gamma(E \otimes F)^*,$$

and, by the formula (18),

$$g_{\alpha\beta\gamma} = \dim(H^0(X, \mathcal{L})^G). \quad (23)$$

The map $(\alpha, \beta, \gamma) \mapsto \mathcal{L}^\alpha \otimes \mathcal{L}^\beta \otimes \mathcal{L}_\gamma$ extends to a linear isomorphism from $\mathbb{Z}^{2e+2f+3}$ onto $\mathrm{Pic}^G(X)$. This isomorphism allows to identify $\mathrm{LR}(G, X)$ with a subset of $\mathbb{Z}^{2e+2f+3}$. The equality (23) implies that

$$\mathrm{Kron}(e + 1, f + 1, e + f + 1) = (\mathbb{Z}_{\geq 0})^{2e+2f+3} \cap \mathrm{LR}(G, X).$$

3 Descriptions of branching and GIT cones

3.1 GIT cones

Assume that the connected reductive group G acts on the smooth projective variety X and that $\mathrm{Pic}^G(X)$ has finite rank. Consider the cone $\mathbb{Q}_{\geq 0} \mathrm{LR}(G, X)$ generated in $\mathrm{Pic}^G(X) \otimes \mathbb{Q}$ by the points of $\mathrm{LR}(G, X)$. The G -linearized ample line bundles on X generated an open convex cone $\mathrm{Pic}^G(X)_{\mathbb{Q}}^+$ in $\mathrm{Pic}^G(X) \otimes \mathbb{Q}$. In this section, we recall from [Res10] a description of the faces of $\mathbb{Q}_{\geq 0} \mathrm{LR}(G, X)$ that intersect $\mathrm{Pic}^G(X)_{\mathbb{Q}}^+$.

Let \mathcal{L} be a G -linearized line bundle on X . Consider the associated set of semistable points

$$X^{\mathrm{ss}}(\mathcal{L}) = \{x \in X : \exists k > 0 \text{ and } \sigma \in H^0(X, \mathcal{L}^{\otimes k})^G \quad \sigma(x) \neq 0\}.$$

Assume that $X^{\mathrm{ss}}(\mathcal{L})$ is nonempty. Then the projective variety $\mathrm{Proj}(\bigoplus_{k \geq 0} H^0(X, \mathcal{L}^{\otimes k})^G)$ is denoted by $X^{\mathrm{ss}}(\mathcal{L})//G$. For later use, observe that $\dim(H^0(X, \mathcal{L}^{\otimes k})^G)$ is $O(k^{\dim(X^{\mathrm{ss}}(\mathcal{L})//G)})$. If moreover \mathcal{L} is ample, $X^{\mathrm{ss}}(\mathcal{L})//G$ is the categorical quotient of $X^{\mathrm{ss}}(\mathcal{L})$ by G . In general, there is a canonical G -invariant regular map

$$\pi : X^{\mathrm{ss}}(\mathcal{L}) \longrightarrow X^{\mathrm{ss}}(\mathcal{L})//G. \quad (24)$$

Let λ be a one parameter subgroup of G . The set

$$P(\lambda) = \{g \in G : \lim_{t \rightarrow 0} \lambda(t)g\lambda(t^{-1}) \text{ exists in } G\}$$

is a parabolic subgroup of G . Consider an irreducible component C of the fixed point set X^λ of λ in X . Set

$$C^+ = \{x \in X : \lim_{t \rightarrow 0} \lambda(t)x \in C\}.$$

By Białynicki-Birula's theorem, C^+ is an irreducible smooth locally closed subvariety of X . Moreover it is stable by the action of $P(\lambda)$. Consider on $G \times C^+$ the following action of the group $P(\lambda)$:

$$p.(g, x) = (gp^{-1}, px).$$

There exists a quotient variety denoted by $G \times_{P(\lambda)} C^+$. We denote by $[g : x]$, the class of $(g, x) \in G \times C^+$. The following formula

$$h.[g : x] = [hg : x] \quad \forall h \in G,$$

endows $G \times_{P(\lambda)} C^+$ with a G -action. Consider the G -equivariant morphism

$$\begin{aligned} \eta : G \times_{P(\lambda)} C^+ &\longrightarrow X \\ [g : x] &\longmapsto gx. \end{aligned}$$

The pair (C, λ) is said to be *well covering* if there exists a $P(\lambda)$ -stable open subset Ω of C^+ such that

- (i) the restriction of η to $G \times_{P(\lambda)} \Omega$ is an open immersion;
- (ii) Ω intersects C .

For any $\mathcal{L} \in \text{Pic}^G(X)$, there exists an integer $\mu^{\mathcal{L}}(C, \lambda)$ such that

$$\lambda(t)\tilde{z} = t^{-\mu^{\mathcal{L}}(C, \lambda)}\tilde{z},$$

for any $t \in \mathbb{C}^*$, $z \in C$ and \tilde{z} in the fiber \mathcal{L}_z over z in \mathcal{L} .

Theorem 5 (see [Res10])

- (i) For any well covering pair (C, λ) and any $\mathcal{L} \in \text{LR}(G, X)$, we have $\mu^{\mathcal{L}}(C, \lambda) \leq 0$.
- (ii) For any face \mathcal{F} of $\mathbb{Q}_{\geq 0} \text{LR}(G, X)$ intersecting $\text{Pic}^G(X)_{\mathbb{Q}}^+$ there exists a well covering pair (C, λ) such that $(\mathcal{L} \otimes 1) \in \mathcal{F}$ if and only if $\mu^{\mathcal{L}}(C, \lambda) = 0$, for any ample \mathcal{L} in $\mathbb{Q}_{\geq 0} \text{LR}(G, X)$.
- (iii) Let (C, λ) be a well covering pair and \mathcal{L} be ample in $\text{LR}(G, X)$. Then $\mu^{\mathcal{L}}(C, \lambda) = 0$ if and only if $X^{\text{ss}}(\mathcal{L}) \cap C$ is not empty.

3.2 Branching cones

With notation of Section 2.1.3, we want to describe the cone $\mathbb{Q}_{\geq 0} \text{LR}(G, \hat{G})$ generated by $\text{LR}(G, \hat{G})$. We assume that no nonzero ideal of the Lie algebra $\text{Lie}(G)$ of G is an ideal of that $\text{Lie}(\hat{G})$ of \hat{G} : this assumption implies that the cone $\mathbb{Q}_{\geq 0} \text{LR}(G, \hat{G})$ has nonempty interior in $(X(T) \times X(\hat{T})) \otimes \mathbb{Q}$.

Consider the natural pairing $\langle \cdot, \cdot \rangle$ between the one parameter subgroups and the characters of tori T or \hat{T} . Let W (resp. \hat{W}) denote the Weyl group of T (resp. \hat{T}). If λ is a one parameter subgroup of T (and thus of \hat{T}), we denote by W_λ (resp. \hat{W}_λ) the stabilizer of λ for the natural action of the Weyl group.

The cohomology group $H^*(G/P(\lambda), \mathbb{Z})$ is freely generated by the Schubert classes σ_w parameterized by the elements $w \in W/W_\lambda$. Assume that λ is dominant. Let w_0 be the longest element of W . If $w \in W/W_\lambda$, we denote by $w^\vee \in W/W_\lambda$ the class of $w_0 w$. By this way σ_{w^\vee} and σ_w are Poincaré dual. We consider $\hat{G}/\hat{P}(\lambda)$, $\sigma_{\hat{w}}$ as above but with \hat{G} in place of G . Consider also the canonical G -equivariant immersion $\iota : G/P(\lambda) \rightarrow \hat{G}/\hat{P}(\lambda)$; and the corresponding morphism ι^* in cohomology.

Recall from [RR11], the definition of Levi-movability for the pair $(\sigma_w, \sigma_{\hat{w}})$. For the purpose of this paper it is only useful to know that if $(\sigma_w, \sigma_{\hat{w}})$ is Levi-movable then $\iota^*(\sigma_{\hat{w}}) \cdot \sigma_w$ is a nonzero multiple of the class $[pt]$ of the point. Moreover the converse is true if $\hat{G}/\hat{P}(\lambda)$ is minuscule.

Consider the set $\text{Wt}_T(\text{Lie}(\hat{G})/\text{Lie}(G))$ of nontrivial weights of T in $\text{Lie}(\hat{G})/\text{Lie}(G)$ and the set of hyperplanes H of $X(T) \otimes \mathbb{Q}$ spanned by some elements of $\text{Wt}_T(\text{Lie}(\hat{G})/\text{Lie}(G))$. For each such hyperplane H there exist exactly two opposite indivisible one parameter subgroups $\pm \lambda_H$ which are orthogonal (for the pairing $\langle \cdot, \cdot \rangle$) to H . The so obtained one parameter subgroups are called *admissible* and form a W -stable set.

Theorem 6 (see [Res10])

Recall that no nonzero ideal of $\text{Lie}(G)$ is an ideal of $\text{Lie}(\hat{G})$. Then, the cone $\mathbb{Q}_{\geq 0} \text{LR}(G, \hat{G})$ has nonempty interior in $X(T \times \hat{T}) \otimes \mathbb{Q}$.

A dominant weight $(\nu, \hat{\nu})$ belongs to $\mathbb{Q}_{\geq 0} \text{LR}(G, \hat{G})$ if and only if

$$\langle \hat{w}\lambda, \hat{\nu} \rangle \leq \langle w\lambda, \nu \rangle \quad (25)$$

for any dominant admissible one parameter subgroup λ of T and for any pair $(w, \hat{w}) \in W/W_\lambda \times \hat{W}/\hat{W}_\lambda$ such that

(i) $\iota^(\sigma_{\hat{w}}) \cdot \sigma_{w^\vee} = [pt] \in H^*(G/P(\lambda), \mathbb{Z})$, and*

(ii) the pair $(\sigma_{w^\vee}, \sigma_{\hat{w}})$ is Levi-movable.

Moreover, the inequalities (25) are pairwise distinct and no one can be omitted.

4 Description of $\text{LR}(e, f, e + f)$

4.1 The statement

Theorem 7 *Let α , β , and γ be three partitions such that $l(\alpha) \leq e$, $l(\beta) \leq f$ and $l(\gamma) \leq e + f$.*

Then $c_{\alpha\beta}^\gamma \neq 0$ if and only if

$$|\alpha| + |\beta| = |\gamma|, \quad (26)$$

and

$$\gamma_{f+i} \leq \alpha_i \leq \gamma_i, \quad \gamma_{e+j} \leq \beta_j \leq \gamma_j, \quad (27)$$

for any $i \in \{1, \dots, e\}$ and $j \in \{1, \dots, f\}$, and

$$|\gamma_K| \leq |\alpha_I| + |\beta_J|, \quad (28)$$

for any $0 < r < e$ and $0 < s < f$, for any $I \in \mathcal{S}(r, e)$, $J \in \mathcal{S}(s, f)$ and $K \in \mathcal{S}(r + s, e + f)$ such that

$$c_{\tau_I \tau_J}^{\tau_K} = 1. \quad (29)$$

Moreover, the inequalities (27) or (28) are pairwise distinct and no one can be omitted.

The partitions α and β in the statement of Theorem 7 are also partitions of length at most $e + f$. Hence the nonvanishing of $c_{\alpha\beta}^\gamma$ is equivalent to $(\alpha, \beta, \gamma) \in \mathbb{Q}_{\geq 0} \text{LR}(e + f, e + f, e + f)$. But, by the classical Horn conjecture (see *e.g.* [Ful00]), this cone is characterized by the inequalities

$$|\gamma_{K'}| \geq |\alpha_{I'}| + |\beta_{J'}| \quad (30)$$

where $\#I' = \#J' = \#K'$ and

$$c_{\tau_{I'} \tau_{J'}}^{\tau_{K'}} = 1. \quad (31)$$

In some sense, Theorem 7 selects among the inequalities (30) those that remain essential when one imposes $l(\alpha) \leq e$ and $l(\beta) \leq f$.

Each inequality (28) has to be consequence of at least one Horn inequality (30). Indeed, by setting $\tilde{I} = I_- \cup \{e + s + 1, \dots, e + f\}$ and $\tilde{J} = J_- \cup \{f + r + 1, \dots, e + f\}$, one can check that, under the assumptions of Theorem 7 and modulo the equality (26), the inequality (28) is equivalent to

$$|\gamma_{K_-}| \geq |\alpha_{\tilde{I}}| + |\beta_{\tilde{J}}|. \quad (32)$$

But $\#\tilde{I} = \#\tilde{J} = \#K_- = e + f - r - s$. One can check that

$$c_{\tau^I \tau^J}^{\tau^{K_-}} = c_{\tau^{\tilde{I}} \tau^{\tilde{J}}}^{\tau^{K_-}}.$$

Hence the assumption (29) implies that the condition (32) is an Horn inequality (30) for the cone $\mathbb{Q}_{\geq 0} \text{LR}(e + f, e + f, e + f)$.

For the proof of Theorem 7, we need to recall some notations and results on Schubert calculus on Grassmannians.

4.2 Schubert Calculus

Let $\mathbb{G}(r, n)$ be the Grassmann variety of r -dimensional linear subspaces of $V = \mathbb{C}^n$. Let $F_\bullet: \{0\} = F_0 \subset F_1 \subset F_2 \subset \dots \subset F_n = V$ be a complete flag of V . Let $I = \{i_1 < \dots < i_r\} \in \mathcal{S}(r, n)$. The Schubert variety $X_I(F_\bullet)$ in $\mathbb{G}(r, n)$ is defined to be

$$X_I(F_\bullet) = \{L \in \mathbb{G}(r, n) : \dim(L \cap F_{i_j}) \geq j \text{ for } 1 \leq j \leq r\}.$$

The Poincaré dual of the homology class of $X_I(F_\bullet)$ is denoted by σ_I . It does not depend on F_\bullet . The classes σ_I form a \mathbb{Z} -basis of the cohomology ring of $\mathbb{G}(r, n)$. Recall from the introduction the definition of the partition τ^I . Then σ_I has degree $2|\tau^I|$. A first cohomological interpretation of the Littlewood-Richardson coefficients is given by the formula (see *e.g.* [Man01])

$$\sigma_I \cdot \sigma_J = \sum_{K \in \mathcal{S}(r, n)} c_{\tau^I \tau^J}^{\tau^K} \sigma_K, \quad (33)$$

for any I, J in $\mathcal{S}(r, n)$.

Let r and s be two integers such that $0 < r < e$ and $0 < s < f$. Fix an identification $\mathbb{C}^{e+f} = \mathbb{C}^e \oplus \mathbb{C}^f$ and consider the morphism

$$\begin{aligned} \phi_{r,s} : \mathbb{G}(r, e) \times \mathbb{G}(s, f) &\longrightarrow \mathbb{G}(r+s, e+f) \\ (F, G) &\longmapsto F \oplus G. \end{aligned}$$

The associated comorphism in cohomology is

$$\phi_{r,s}^* : \mathbb{H}^*(\mathbb{G}(r+s, e+f), \mathbb{Z}) \longrightarrow \mathbb{H}^*(\mathbb{G}(r, e) \times \mathbb{G}(s, f), \mathbb{Z}).$$

By Kuneth's formula, the family $(\sigma_I \otimes \sigma_J)_{(I,J) \in \mathcal{S}(r,e) \times \mathcal{S}(s,f)}$ is a basis of $\mathbb{H}^*(\mathbb{G}(r, e) \times \mathbb{G}(s, f), \mathbb{Z})$. A second cohomological interpretation of the Littlewood-Richardson coefficients is given by the formula

$$\phi_{r,s}^*(\sigma_K) = \sum_{(I,J) \in \mathcal{S}(r,e) \times \mathcal{S}(s,f)} c_{\tau^I \tau^J}^{\tau^K} (\sigma_I \otimes \sigma_J), \quad (34)$$

for any $K \in \mathcal{S}(r+s, e+f)$.

4.3 Proof of Theorem 7

By the Knutson-Tao theorem of saturation (see [KT99]), $c_{\alpha\beta}^\gamma \neq 0$ if and only if (α, β, γ) belongs to the cone $\mathbb{Q}_{\geq 0} \text{LR}(e, f, e+f)$. It remains to prove that the inequalities (27) and (28) characterize the cone $\mathbb{Q}_{\geq 0} \text{LR}(e, f, e+f)$ in a minimal way.

Let us fix bases for the two vector spaces E and F of dimension e and f . Consider the group $\hat{G} = \text{SL}(E \oplus F)$, its subgroup $G = S(\text{GL}(E) \times \text{GL}(F))$ and on $E \oplus F$ the basis obtained by concatenating the bases of E and F . Let \hat{T} be the maximal torus of \hat{G} consisting in diagonal matrices. It is contained in G ; set $T = \hat{T}$. Let \hat{B} be the Borel subgroup of \hat{G} consisting in upper triangular matrices. Set $B = \hat{B} \cap G$. Let ε_i be the character of \hat{T} mapping a matrix in \hat{T} to its i^{th} diagonal entry. Since $\sum_i \varepsilon_i = 0$, $(\varepsilon_1, \dots, \varepsilon_{e+f-1})$ is a \mathbb{Z} -basis of $X(\hat{T})$.

Let α, β , and γ be three partitions of length less or equal to e, f , and $e+f$. The highest weight of the \hat{G} -module $S^\gamma(E \oplus F)$ is $\tilde{\gamma} = (\gamma_1 - \gamma_{e+f})\varepsilon_1 + \dots + (\gamma_{e+f-1} - \gamma_{e+f})\varepsilon_{e+f-1}$. The highest weight of the G -module $S^\alpha E \otimes S^\beta F$ is $(\alpha, \beta) = (\alpha_1 - \beta_f)\varepsilon_1 + \dots + (\alpha_e - \beta_f)\varepsilon_e + (\beta_1 - \beta_f)\varepsilon_{e+1} + \dots + (\beta_{f-1} - \beta_f)\varepsilon_{e+f-1}$. Then, by the formula (21)

$$\begin{aligned} (\alpha, \beta, \gamma) \in \text{LR}(e, f, e+f) &\iff (S^\alpha E^* \otimes S^\beta F^* \otimes S^\gamma(E \oplus F))^{\text{GL}(E) \times \text{GL}(F)} \neq 0, \\ &\iff |\alpha| + |\beta| = |\gamma| \\ &\text{and } (S^\alpha E^* \otimes S^\beta F^* \otimes S^\gamma(E \oplus F))^G \neq 0, \\ &\iff |\alpha| + |\beta| = |\gamma| \\ &\text{and } ((\alpha, \beta), \tilde{\gamma}) \in \text{LR}(G, \hat{G}). \end{aligned}$$

In particular, to determine the inequalities for the cone $\mathbb{Q}_{\geq 0} \text{LR}(e, f, e+f)$, it is sufficient to describe $\mathbb{Q}_{\geq 0} \text{LR}(G, \hat{G})$. We do this using Theorem 6.

The set of weights of T acting on $\text{Lie}(\hat{G})/\text{Lie}(G)$ is the set of weights of T acting on $F^* \otimes E$ and their opposite. Explicitly $\text{Wt}_T(\text{Lie}(\hat{G})/\text{Lie}(G)) = \pm\{\varepsilon_i - \varepsilon_{e+j} \mid 1 \leq i \leq e \text{ and } 1 \leq j \leq f\}$. Let $(a_1, \dots, a_e, b_1, \dots, b_f) \in \mathbb{Z}^{e+f}$ be the exponents of a one parameter subgroup λ of T ; they satisfy $\sum_i a_i + \sum_j b_j = 0$. Then $\langle \lambda, \varepsilon_i - \varepsilon_{e+j} \rangle = 0$ if and only if $a_i = b_j$. It follows that if λ is admissible then the integers a_i and b_j take at most two values. If moreover λ is dominant then there exist integers r, s , and $c > d$ such that $a_1 = \dots = a_r = b_1 = \dots = b_s = c$ and $a_{r+1} = \dots = a_e = b_{s+1} = \dots = b_f = d$. If moreover λ is indivisible, $c = \frac{e+f-r-s}{(r+s)\wedge(e+f)}$ and $d = \frac{-r-s}{(r+s)\wedge(e+f)}$, where \wedge denotes the gcd. Let $\lambda_{r,s}$ denote the so obtained one-parameter subgroup of T . Conversely, one easily checks that $\lambda_{r,s}$ is an admissible dominant one-parameter subgroup of T , if $0 < r < e$ and $0 < s < f$ or if the pair (r, s) is one of the four exceptional ones $\{(1, 0), (0, 1), (e-1, f), (e, f-1)\}$.

The inclusions $G/P(\lambda_{r,s}) \subset \hat{G}/\hat{P}(\lambda_{r,s})$ associated to the four exceptional cases are $\mathbb{P}(E) \subset \mathbb{P}(E \oplus F)$, $\mathbb{P}(F) \subset \mathbb{P}(E \oplus F)$, $\mathbb{P}(E^*) \subset \mathbb{P}(E^* \oplus F^*)$ and $\mathbb{P}(F^*) \subset \mathbb{P}(E^* \oplus F^*)$. Consider $\mathbb{P}(E) \subset \mathbb{P}(E \oplus F)$. The restriction of $\sigma_{\{f+i\}} \in \mathbb{H}^*(\mathbb{P}(E \oplus F), \mathbb{Z})$ in $\mathbb{H}^*(\mathbb{P}(E), \mathbb{Z})$ is $\sigma_{\{i\}}$. Then Theorem 6 implies that

$$(e+f)\alpha_i - |\alpha| - |\beta| \geq (e+f)\gamma_{f+i} - |\gamma|.$$

Modulo the identity (26), this is equivalent to $\gamma_{f+i} \leq \alpha_i$. Similarly, we get the three other inequalities (27).

Fix now $0 < r < e$ and $0 < s < f$. The inclusion $G/P(\lambda_{r,s}) \subset \hat{G}/\hat{P}(\lambda_{r,s})$ is the morphism $\phi_{r,s}$ defined in Section 4.2. Consider $\sigma_I \otimes \sigma_J \in \mathbb{H}^*(\mathbb{G}(r, e) \times \mathbb{G}(s, f), \mathbb{Z})$ and $\sigma_K \in \mathbb{H}^*(\mathbb{G}(r+s, e+f), \mathbb{Z})$ such that $\phi_{r,s}^*(\sigma_K) \cdot (\sigma_I \otimes \sigma_J)^\vee = [pt]$. Here the Levi movability is automatic since $\hat{G}/\hat{P}(\lambda_{r,s})$ is cominusculé. Modulo (26), the inequality (25) of Theorem 6 corresponding to $\sigma_I \otimes \sigma_J$ and σ_K is the inequality (28). Then the theorem follows from Theorem 6. \square

4.4 Complement on stretched Littlewood-Richardson coefficients

Lemma 1 *Let α, β , and γ be three partitions such that $l(\alpha) \leq e$, $l(\beta) \leq f$ and $l(\gamma) \leq e+f$.*

Then, the map $n \mapsto c_{n\alpha n\beta}^{n\gamma}$ is polynomial of degree not greater than

$$\binom{e}{2} + \binom{f}{2} + \binom{e+f}{2} - e^2 - f^2 + 1,$$

where $\binom{e}{2} = \frac{e(e-1)}{2}$.

Proof. Since $c_{\alpha\beta}^\gamma = c_{\beta\alpha}^\gamma$, we may assume that $e \leq f$. By [DW02], the function $\mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}$, $n \mapsto c_{n\alpha n\beta}^{n\gamma}$ is polynomial.

Recall that E and F are complex vector spaces of dimension e and f . Set $G = \mathrm{GL}(E) \times \mathrm{GL}(F)$ and $X = \mathcal{F}l(E) \times \mathcal{F}l(F) \times \mathcal{F}l(E \oplus F)$. Consider on X the line bundle $\mathcal{L} = \mathcal{L}^\alpha \otimes \mathcal{L}^\beta \otimes \mathcal{L}^\gamma$. Since $c_{n\alpha n\beta}^{n\gamma} = \dim(\mathrm{H}^0(X, \mathcal{L}^{\otimes n})^G)$, the degree of $c_{n\alpha n\beta}^{n\gamma}$ is equal to the dimension of $X^{\mathrm{ss}}(\mathcal{L})//G$.

Consider the map π defined in (24). By Chevalley Theorem, since π is dominant, for any general $y \in X^{\mathrm{ss}}(\mathcal{L})$, one has

$$\dim \pi^{-1}(\pi(y)) = \dim(X^{\mathrm{ss}}(\mathcal{L})) - \dim(X^{\mathrm{ss}}(\mathcal{L})//G).$$

But, π is G -invariant and $\pi^{-1}(\pi(y))$ contains $G.y$. Then

$$\dim \pi^{-1}(\pi(y)) \geq \dim(G.y) = \dim(G) - \dim(G_y),$$

where G_y is the stabilizer of y in G . But, for any $x \in X$, we have $\dim(G.x) \leq \dim(G.y)$ and

$$\dim(X^{\mathrm{ss}}(\mathcal{L})//G) \leq \dim(X) - \dim(G) + \dim(G_x).$$

We now claim that there exists x such that $\dim(G_x) = 1$. Then the lemma follows.

We now prove the claim by constructing explicitly x , that is, defining complete flags of E , F and $E \oplus F$. Fix bases (η_1, \dots, η_e) and $(\zeta_1, \dots, \zeta_f)$ of E and F . On E and F , we consider the two standard flags F_\bullet^E and F_\bullet^F in these bases. Consider on $E \oplus F$, the following base

$$(\eta_e + \zeta_f, \eta_e + \eta_{e-1} + \zeta_{f-1}, \dots, \eta_e + \dots + \eta_1 + \zeta_{f-e+1}, \eta_1 + \zeta_{f-e}, \dots, \eta_1 + \zeta_1)$$

and the associated flag $F_\bullet^{E \oplus F}$. One easily checks that $x = (F_\bullet^E, F_\bullet^F, F_\bullet^{E \oplus F})$ works. \square

5 Faces of $\mathbb{Q}_{\geq 0} \mathrm{Kron}(e+1, f+1, e+f+1)$

5.1 Murnaghan's face

The cone $\mathbb{Q}_{\geq 0} \mathrm{Kron}(e+1, f+1, e+f+1)$ is contained in the linear subspace of points $(\alpha, \beta, \gamma) \in \mathbb{Q}^{e+1} \times \mathbb{Q}^{f+1} \times \mathbb{Q}^{e+f+1}$ that satisfy $|\alpha| = |\beta| = |\gamma|$. In particular its dimension is at most $2e + 2f + 1$.

Recall that $\bar{\alpha} = (\alpha_2 \geq \alpha_3 \cdots)$, if $\alpha = (\alpha_1 \geq \alpha_2 \cdots)$. By Proposition 1, the points (α, β, γ) in $\mathbb{Q}_{\geq 0} \text{Kron}(e+1, f+1, e+f+1)$ satisfy

$$|\bar{\alpha}| + |\bar{\beta}| \geq |\bar{\gamma}|. \quad (35)$$

The set of points of $\mathbb{Q}_{\geq 0} \text{Kron}(e+1, f+1, e+f+1)$ such that equality holds in the inequality (35) is a face \mathcal{F}^M (M stands for Murnaghan) of the cone $\mathbb{Q}_{\geq 0} \text{Kron}(e+1, f+1, e+f+1)$. Consider the linear map

$$\begin{aligned} \pi : \mathbb{Q}^{2e+2f+3} &\longrightarrow \mathbb{Q}^{2e+2f} \\ (\alpha, \beta, \gamma) &\longmapsto (\bar{\alpha}, \bar{\beta}, \bar{\gamma}). \end{aligned}$$

Lemma 2 *The face \mathcal{F}^M maps by π to $\mathbb{Q}_{\geq 0} \text{LR}(e, f, e+f)$. Moreover each fiber of π over $\mathbb{Q}_{\geq 0} \text{LR}(e, f, e+f)$ contains an unbounded interval.*

The cone $\mathbb{Q}_{> 0} \text{Kron}(e+1, f+1, e+f+1)$ has dimension $2e+2f+1$ and the face \mathcal{F}^M has dimension $2e+2f$.

Proof. Assume that equality holds in the formula (35). Assume also that the coordinates of α , β , and γ are nonnegative integers. Then

$$g_{\alpha\beta\gamma} = c_{\bar{\alpha}\bar{\beta}}^{\bar{\gamma}}. \quad (36)$$

Thus the face \mathcal{F}^M maps by π on $\mathbb{Q}_{\geq 0} \text{LR}(e, f, e+f)$. Conversely let $(\lambda, \mu, \nu) \in \text{LR}(e, f, e+f)$. Let a be an integer and set $b = a + |\lambda| - |\mu|$ and $c = a + |\lambda| - |\nu|$. If a is big enough then $a \geq \lambda_1$, $b \geq \mu_1$ and $c \geq \nu_1$. Therefore $\alpha := (a, \lambda)$, $\beta = (b, \mu)$ and $\gamma = (c, \nu)$ are three partitions of the same integer such that equality holds in the inequality (35). Thus the equality (36) holds and (α, β, γ) belongs to \mathcal{F}^M . In particular the fiber $\pi^{-1}(\lambda, \mu, \nu)$ contains an unbounded segment.

Since $\mathbb{Q}_{\geq 0} \text{LR}(e, f, e+f)$ has dimension $2e+2f-1$ and the fibers of π have dimension at least one, the cone \mathcal{F}^M has dimension at least $2e+2f$. We had already noticed that $\mathbb{Q}_{\geq 0} \text{Kron}(e+1, f+1, e+f+1)$ has dimension at most $2e+2f+1$. These two inequalities (and the fact that \mathcal{F}^M is a strict face of the cone $\mathbb{Q}_{\geq 0} \text{Kron}(e+1, f+1, e+f+1)$) imply the lemma. \square

5.2 Proof of Theorem 2

Let r, s, I, J , and K be like in Theorem 2. To such a triple (I, J, K) , Theorem 7 associates a codimension one face of $\text{LR}(e, f, e+f)$. Using Lemma 2, this face corresponds to a face \mathcal{F}_{IJK} of $\mathbb{Q}_{\geq 0} \text{Kron}(e+1, f+1, e+f+1)$

of codimension two. Explicitly, \mathcal{F}_{IJK} is the set of $(\alpha, \beta, \gamma) \in \mathbb{Q}_{\geq 0} \text{Kron}(e + 1, f + 1, e + f + 1)$ such that

$$\begin{cases} |\bar{\gamma}| = |\bar{\alpha}| + |\bar{\beta}|, \\ |\bar{\gamma}_K| = |\bar{\alpha}_I| + |\bar{\beta}_J|. \end{cases} \quad (37)$$

This face \mathcal{F}_{IJK} is contained in two codimension one faces, \mathcal{F}^M and another one \mathcal{F}_{IJK}^M that we want to determine.

Let φ_τ denote the linear form defined by

$$\varphi_\tau(\alpha, \beta, \gamma) = \tau(|\bar{\alpha}| + |\bar{\beta}| - |\bar{\gamma}|) + (|\bar{\alpha}_I| + |\bar{\beta}_J| - |\bar{\gamma}_K|),$$

where τ is any rational number. Set also

$$\varphi_\infty(\alpha, \beta, \gamma) = |\bar{\alpha}| + |\bar{\beta}| - |\bar{\gamma}|.$$

By the theory of convex polyhedral cones, there exists τ_0 such that for any $\tau > \tau_0$, φ_τ is nonnegative on the cone and the associated face is \mathcal{F}_{IJK} , and, φ_{τ_0} corresponds to \mathcal{F}_{IJK}^M .

Here, E and F are two linear spaces of dimension $e + 1$ and $f + 1$ and $G = \text{GL}(E) \times \text{GL}(F)$. Consider the variety

$$X = \mathcal{F}l(E) \times \mathcal{F}l(F) \times \mathcal{F}l(1, \dots, e + f + 1; E \otimes F).$$

We identify $\text{Pic}^G(X)$ with $\mathbb{Z}^{2e+2f+3}$ like in Section 2.2.4.

Geometric description of φ_∞ . The inequality corresponding to \mathcal{F}^M is $\varphi_\infty \geq 0$. By Section 2.2.4, \mathcal{F}^M generates a face of $\mathbb{Q}_{\geq 0} \text{LR}(G, X)$. Theorem 5 shows that there exists a well covering pair $(C_\infty, \lambda_\infty)$ of X such that $\varphi_\infty(\alpha, \beta, \gamma) = -\mu^{\mathcal{L}^\alpha \otimes \mathcal{L}^\beta \otimes \mathcal{L}^\gamma}(C_\infty, \lambda_\infty)$.

To describe such a pair $(C_\infty, \lambda_\infty)$, fix decompositions $E = \bar{E} \oplus l$ and $F = \bar{F} \oplus m$, where \bar{E} and \bar{F} are hyperplanes and l and m are lines. Let λ_∞ be the one-parameter subgroup of G acting with weight 1 on \bar{E} and \bar{F} , and with weight 0 on l and m . Let C_∞ be the set of points in X such that

- the hyperplanes of the complete flags of E and F are respectively \bar{E} and \bar{F} ,
- the line of the partial flag of $E \otimes F$ is $l \otimes m$,
- the $(e + f + 1)$ -dimensional subspace of the partial flag of $E \otimes F$ is $(l \otimes m) \oplus (\bar{E} \otimes m) \oplus (l \otimes \bar{F})$.

One can check that $(C_\infty, \lambda_\infty)$ works (see [Res11c] for details).

Geometric description of φ_τ . Fix decompositions $\bar{E} = E_+ \oplus E_-$ and $\bar{F} = F_+ \oplus F_-$, where E_+ and F_+ have dimension r and s . Assume that $\tau > 1$ and write $\tau = \frac{p}{q}$ with two integers p and q satisfying $p \wedge q = 1$ and $q > 0$. Let λ_τ be the one parameter subgroup of G acting with weight $q + p$ on E_+ and F_+ , with weight p on E_- and F_- and with weight 0 on l and m . The weight spaces of the action of λ_τ on $E \otimes F$ are

Space	$E_+ \otimes F_+$	$E_+ \otimes F_- \oplus E_- \otimes F_+$	$E_- \otimes F_-$	$E_+ \oplus F_+$	$E_- \oplus F_-$	$l \otimes m$
Weight	$2p + 2q$	$2p + q$	$2p$	$p + q$	p	0

where some “ $\otimes m$ ” and “ $l \otimes$ ” have been forgotten.

To $I^\vee = \{e + 1 - i : i \in I\}$ is associated an embedding ι_{I^\vee} of $\mathcal{F}l(E_+) \times \mathcal{F}l(E_-)$ in $\mathcal{F}l(\bar{E})$. Explicitly

$$\begin{aligned} \iota_{I^\vee} : \mathcal{F}l(E_+) \times \mathcal{F}l(E_-) &\longrightarrow \mathcal{F}l(\bar{E}) \\ ((V_i), (W_j)) &\longmapsto (V_{\#I^\vee \cap [1, k]} \oplus W_{k - \#I^\vee \cap [1, k]})_{1 \leq k \leq e}. \end{aligned}$$

Similarly we consider ι_{J^\vee} and ι_K . Observe that C_∞ is canonically isomorphic to $\mathcal{F}l(\bar{E}) \times \mathcal{F}l(\bar{F}) \times \mathcal{F}l(\bar{E} \oplus \bar{F})$. Consider the embedding $(\iota_{I^\vee}, \iota_{J^\vee}, \iota_K)$ of

$$\mathcal{F}l(E_+) \times \mathcal{F}l(E_-) \times \mathcal{F}l(F_+) \times \mathcal{F}l(F_-) \times \mathcal{F}l(E_+ \oplus F_+) \times \mathcal{F}l(E_- \oplus F_-)$$

in C_∞ . Denote by C_τ its image. Using for example [Res11b, Proposition 1 and Theorem 1], one can check that, for τ big enough, (C_τ, λ_τ) is a well covering pair. Moreover, $\varphi_\tau(\alpha, \beta, \gamma) = -q\mu^{\mathcal{L}^\alpha \otimes \mathcal{L}^\beta \otimes \mathcal{L}^\gamma}(C_\tau, \lambda_\tau)$.

For any $\tau > 1$, C_τ is an irreducible component of λ_τ . Moreover, C_τ^+ and $P(\lambda_\tau)$ do not depend on $\tau > 1$. In Particular, (C_τ, λ_τ) is a well covering pair for any $\tau > 1$.

Theorem 5 shows that the face determined by the inequality φ_τ only depends on C_τ , and so does not depend on $\tau > 1$: it is \mathcal{F}_{IJK} . This implies that $\varphi_1 \geq 0$ on $\mathbb{Q}_{\geq 0} \text{Kron}(e + 1, f + 1, e + f + 1)$. Theorem 2 follows. \square

Remark. Let \mathcal{F}_1 denote the face associated to φ_1 . Up to now, we have not proved that \mathcal{F}_1 has codimension 1 or equivalently that $\mathcal{F}_1 = \mathcal{F}_{IJK}^M$. This is the aim of Section 7.

6 Proof of Theorem 3

Keeping the notation of Section 5.2, we give a geometric description of φ_1 . The weight spaces of the action of λ_1 on $E \otimes F$ are

$E_+ \otimes F_+$	$E_+ \otimes F_- \oplus E_- \otimes F_+$	$E_- \otimes F_- \oplus E_+ \oplus F_+$	$E_- \oplus F_-$	$l \otimes m$
4	3	2	1	0

The irreducible component C_1 of X^{λ_1} containing C_τ (for $\tau > 1$) is isomorphic to

$$\begin{aligned} & \mathcal{F}l(E_+) \times \mathcal{F}l(E_-) \times \mathcal{F}l(F_+) \times \mathcal{F}l(F_-) \\ & \times \mathcal{F}l(1, \dots, r+s; E_- \otimes F_- \oplus E_+ \oplus F_+) \times \mathcal{F}l(E_- \oplus F_-). \end{aligned}$$

Moreover, $C_1^+ = C_\tau^+$ and $P(\lambda_1) = P(\lambda_\tau)$. In particular the pair (C_1, λ_1) is well covering.

Let G^{λ_1} denote the centralizer of λ_1 in G . Note that $G^{\lambda_1} = \mathrm{GL}(E_+) \times \mathrm{GL}(E_-) \times \mathbb{C}^* \times \mathrm{GL}(F_+) \times \mathrm{GL}(F_-) \times \mathbb{C}^*$. By [Res11c, Theorem 2], $g_{\alpha\beta\gamma}$ is the dimension of

$$H^0(C_1, (\mathcal{L}^\alpha \otimes \mathcal{L}^\beta \otimes \mathcal{L}^\gamma)|_{C_1})^{G^{\lambda_1}}.$$

We have to determine the restriction $(\mathcal{L}^\alpha \otimes \mathcal{L}^\beta \otimes \mathcal{L}^\gamma)|_{C_1}$ via the identification of C_1 with a product of flag varieties. Fix a basis of \bar{E} starting with a basis of E_+ followed by a basis of E_- . For the group $\mathrm{GL}(\bar{E})$ we consider standard maximal tori and Borel subgroups in this basis. Similarly, we choose subgroups of $\mathrm{GL}(\bar{F})$.

The maximal torus of $\mathrm{GL}(E)$ acts on the fiber in \mathcal{L}^α over the base point of $\mathcal{F}l(E)$ with weight $(\alpha_{e+1}, \dots, \alpha_1)$. The maximal torus \bar{T} of $\mathrm{GL}(\bar{E})$ acts on the fiber in \mathcal{L}^α over the base point of $\mathcal{F}l(\bar{E})$ (embedded in $\mathcal{F}l(E)$ like C_∞ is embedded in X) with weight $(\alpha_{e+1}, \dots, \alpha_2)$. Let w_{I^\vee} in the symmetric group S_e associated to I^\vee ($w_{I^\vee}(k)$ is the k^{th} elements of I^\vee and $w_{I^\vee}(r+k)$ is the k^{th} elements of I^\vee). Then ι_{I^\vee} maps the base point of $\mathcal{F}l(E_+) \times \mathcal{F}l(E_-)$ to the image by $w_{I^\vee}^{-1}$ of the base point of $\mathcal{F}l(\bar{E})$. It follows that \bar{T} acts on the fiber in $\iota_{I^\vee}^*(\mathcal{L}_{C_\infty}^\alpha)$ by the weight $w_{I^\vee}^{-1}(\alpha_{e+1}, \dots, \alpha_2)$. After computation this gives

$$H^0(C_1, (\mathcal{L}^\alpha \otimes \mathcal{L}^0 \otimes \mathcal{L}_0)|_{C_1}) = S^{\bar{\alpha}_{I^+}} E_+ \otimes S^{\bar{\alpha}_{I^-}} E_-.$$

Similarly

$$H^0(C_1, (\mathcal{L}^0 \otimes \mathcal{L}^\beta \otimes \mathcal{L}_0)|_{C_1}) = S^{\bar{\beta}_{J^+}} F_+ \otimes S^{\bar{\beta}_{J^-}} F_-,$$

and

$$H^0(C_1, (\mathcal{L}^0 \otimes \mathcal{L}^0 \otimes \mathcal{L}^\gamma)|_{C_1}) = S^{\bar{\gamma}_{K^+}} (E_- \otimes F_- \oplus E_+ \oplus F_+)^* \otimes S^{\bar{\gamma}_{K^-}} (E_- \oplus F_-)^*.$$

We deduce that $g_{\alpha\beta\gamma}$ is the multiplicity of the $\mathrm{GL}(E_+) \times \mathrm{GL}(E_-) \times \mathrm{GL}(F_+) \times \mathrm{GL}(F_-)$ -simple module

$$S^{\bar{\alpha}_{I_+}} E_+ \otimes S^{\bar{\alpha}_{I_-}} E_- \otimes S^{\bar{\beta}_{J_+}} F_+ \otimes S^{\bar{\beta}_{J_-}} F_-$$

in the module

$$S^{\bar{\gamma}_{K_+}} (E_- \otimes F_- \oplus E_+ \oplus F_+) \otimes S^{\bar{\gamma}_{K_-}} (E_- \oplus F_-). \quad (38)$$

Now the theorem is obtained by using repeatedly the formulas (2), (18) and (21) to decompose the module (38).

7 Proof of Theorem 4

Recall that the aim is to prove that \mathcal{F}_1 has codimension one. Since \mathcal{F}_{IJK} has codimension two and it is contained in \mathcal{F}_1 , it remains to prove that $\mathcal{F}_{IJK} \neq \mathcal{F}_1$.

Assume now that (α, β, γ) belongs \mathcal{F}_{IJK} . Since \mathcal{F}_{IJK} is contained in \mathcal{F}^M , $g_{\alpha\beta\gamma} = c_{\bar{\alpha}\bar{\beta}}^{\bar{\gamma}}$. Then (see *eg* [DW11, Theorem 7.4]) $g_{\alpha\beta\gamma} = c_{\bar{\alpha}_I \bar{\beta}_J}^{\bar{\gamma}_{K_-}} \cdot c_{\bar{\alpha}_{I_-} \bar{\beta}_{J_-}}^{\bar{\gamma}_{K_-}}$. In particular, Lemma 1 shows that $g_{n\alpha n\beta n\gamma}$ is a polynomial function of n of degree at most

$$d_{max} = \binom{r}{2} + \binom{s}{2} + \binom{r+s}{2} + \binom{a}{2} + \binom{b}{2} + \binom{a+b}{2} - r^2 - s^2 - a^2 - b^2 + 2, \quad (39)$$

where $a = e - r$ and $b = f - s$.

Given an algebraic group Γ acting on an irreducible variety Y , we denote by $\mathrm{mod}(\Gamma, Y)$ the minimal codimension of the Γ -orbits. By [Res11a, Lemma 2], for any \mathcal{L} in the relative interior of $\mathbb{Q}_{\geq 0} \mathrm{LR}(G^{\lambda_1}, C_1)$, the dimension of $C_1^{\mathrm{ss}}(\mathcal{L})//G^{\lambda_1}$ is equal to $\mathrm{mod}(G^{\lambda_1}, C_1)$.

By [Res10, Theorem 4], there exists a line bundle \mathcal{M} on X such that $\mathcal{M}|_{C_1}$ belongs to the relative interior of $\mathbb{Q}_{\geq 0} \mathrm{LR}(G^{\lambda_1}, C_1)$. We may assume that $\mathcal{M} = \mathcal{L}^\alpha \otimes \mathcal{L}^\beta \otimes \mathcal{L}^\gamma$ for three partitions α , β and γ . But, by [Res10, Theorem 8], $X^{\mathrm{ss}}(\mathcal{M})//G \simeq C_1^{\mathrm{ss}}(\mathcal{M}|_{C_1})//G^{\lambda_1}$. It follows that \mathcal{M} is a point on \mathcal{F}_1 satisfying $\dim(X^{\mathrm{ss}}(\mathcal{M})//G) = \mathrm{mod}(G^{\lambda_1}, C_1)$. In particular,

$$n \mapsto g_{n\alpha n\beta n\gamma} \quad \text{cannot be a polynomial function} \\ \text{of degree less than } \mathrm{mod}(G^{\lambda_1}, C_1). \quad (40)$$

Regarding the assertions (39) and (40), to prove that $\mathcal{F}_{IJK} \neq \mathcal{F}_1$ it is sufficient to prove the claim: $\text{mod}(G^{\lambda_1}, C_1) > d_{max}$. The center of G^{λ_1} contains a dimension 3 torus acting trivially on C_1 . Hence

$$\text{mod}(G^{\lambda_1}, C_1) \geq \dim(C_1) - \dim(G^{\lambda_1}) + 3.$$

After simplification, we get

$$\text{mod}(G^{\lambda_1}, C_1) - d_{max} \geq ab(r + s) - 1.$$

Since $a, b, r,$ and s are positive integers, the claim follows.

8 Proof of Theorem 1

We keep notation of Section 5.2, but now $r = e$ and $I = \{1, \dots, e\}$. In particular E_- is trivial and the weight spaces of the action of λ_τ of $E \otimes F$ are

$\bar{E} \otimes F_+$	$\bar{E} \otimes F_-$	$\bar{E} \oplus F_+$	F_-	$l \otimes m$
$2p + 2q$	$2p + q$	$p + q$	p	0

Hence $C_\tau \simeq \mathcal{F}l(\bar{E}) \times \mathcal{F}l(F_+) \times \mathcal{F}l(\bar{E} \oplus F_+)$ for τ big enough. Then, for any $\tau > 0$, (C_τ, λ_τ) is a well covering pair. We conclude like in Section 5.2 that φ_0 is nonnegative on the Kronecker cone. This proves the first assertion of the theorem.

Consider now the limit case $\tau = 0$. The weight spaces are

$\bar{E} \otimes F_+$	$\bar{E} \otimes (F_- \oplus m) \oplus F_+$	$l \otimes (F_- \oplus m)$
2	1	0

Hence $C_0 \simeq \mathcal{F}l(\bar{E}) \times \mathcal{F}l(F_+) \times \mathbb{P}(F_- \oplus m) \times \mathcal{F}l(1, \dots, e + f - 1; \bar{E} \otimes (F_- \oplus m) \oplus F_+) \times \mathbb{P}(F_- \oplus m)$. Moreover $G^{\lambda_0} \simeq \text{GL}(\bar{E}) \times \mathbb{C}^* \times \text{GL}(F_+) \times \text{GL}(F_- \oplus m)$. By [Res11c, Theorem 2]

$$g_{\alpha\beta\gamma} = \dim(\text{H}^0(C_0, (\mathcal{L}^\alpha \otimes \mathcal{L}^\beta \otimes \mathcal{L}_\gamma)|_{C_0})^{G^{\lambda_0}}).$$

The computation of this dimension is made using the formulas (2), (18) and (21) like in Section 6.

9 A final inequality

All but two of the inequalities of Theorem 7 had been extended to the Kronecker coefficients by Theorems 1 and 2. The two exceptions are $\alpha_i \leq \gamma_i$ and $\beta_j \leq \gamma_j$. Consider the second one, up to permuting (α, e) and (β, f) . The extended inequality is

$$\alpha_1 + \beta_1 - \beta_j - n \leq \gamma_1 - \gamma_j,$$

for any $f + 1 \geq j \geq 2$.

This inequality is satisfied if $g_{\alpha\beta\gamma} \neq 0$. The proof is obtained by considering $I = \emptyset$ and $J = K = \{j - 1\}$ in Section 5.2.

References

- [DW02] Harm Derksen and Jerzy Weyman, *On the Littlewood-Richardson polynomials*, J. Algebra **255** (2002), no. 2, 247–257.
- [DW11] Harm Derksen and Jerzy Weyman, *The combinatorics of quiver representations*, Ann. Inst. Fourier **61** (2011), no. 3, 1061–1131.
- [É92] Alexander G. Élashvili, *Invariant algebras*, Lie groups, their discrete subgroups, and invariant theory, Adv. Soviet Math., vol. 8, Amer. Math. Soc., Providence, RI, 1992, pp. 57–64.
- [FH91] William Fulton and Joe Harris, *Representation theory*, Springer-Verlag, New York, 1991, A first course, Readings in Mathematics.
- [Ful00] William Fulton, *Eigenvalues, invariant factors, highest weights, and Schubert calculus*, Bull. Amer. Math. Soc. (N.S.) **37** (2000), no. 3, 209–249.
- [JK81] Gordon James and Adalbert Kerber, *The representation theory of the symmetric group*, Encyclopedia of Mathematics and its Applications, vol. 16, Addison-Wesley Publishing Co., Reading, Mass., 1981, With a foreword by P. M. Cohn, With an introduction by Gilbert de B. Robinson.
- [KT99] Allen Knutson and Terence Tao, *The honeycomb model of $GL_n(\mathbb{C})$ tensor products. I. Proof of the saturation conjecture*, J. Amer. Math. Soc. **12** (1999), no. 4, 1055–1090.

- [Mac95] Ian Grant Macdonald, *Symmetric functions and Hall polynomials*, second ed., Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1995, With contributions by A. Zelevinsky, Oxford Science Publications.
- [Man01] Laurent Manivel, *Symmetric functions, Schubert polynomials and degeneracy loci*, SMF/AMS Texts and Monographs, vol. 6, American Mathematical Society, Providence, RI, 2001, Translated from the 1998 French original by John R. Swallow, Cours Spécialisés [Specialized Courses], 3.
- [Res10] Nicolas Ressayre, *Geometric invariant theory and generalized eigenvalue problem*, Invent. Math. **180** (2010), 389–441.
- [Res11a] ———, *A cohomology free description of eigencones in type A, B and C*, Internat. Math. Res. Notices (to appear) (2011), 1–35.
- [Res11b] ———, *Multiplicative formulas in Schubert calculus and quiver representation*, Indag. Math. (N.S.) **22** (2011), no. 1-2, 87–102.
- [Res11c] Nicolas Ressayre, *Reductions for branching coefficients*, ArXiv e-prints **1102.0196** (2011), 1–13.
- [RR11] Nicolas Ressayre and Edward Richmond, *Branching Schubert calculus and the Belkale-Kumar product on cohomology*, Proc. Amer. Math. Soc. **139** (2011), 835–848.
- [Wey12] Hermann Weyl, *Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen (mit einer Anwendung auf die Theorie der Hohlraumstrahlung)*, Math. Ann. **71** (1912), no. 4, 441–479.

- \diamond -