



HAL
open science

Numerical reconstruction based on Carleman estimates of a source term in a reaction-diffusion equation. *

Muriel Boulakia, Maya de Buhan, Erica Schwindt

► To cite this version:

Muriel Boulakia, Maya de Buhan, Erica Schwindt. Numerical reconstruction based on Carleman estimates of a source term in a reaction-diffusion equation. *. 2019. hal-02185889v1

HAL Id: hal-02185889

<https://hal.science/hal-02185889v1>

Preprint submitted on 16 Jul 2019 (v1), last revised 27 Jan 2021 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Numerical reconstruction based on Carleman estimates of a source term in a reaction-diffusion equation. *

Muriel Boulakia^{1,†} Maya de Buhan^{2,‡}

Erica L. Schwindt[§]

¹ Sorbonne Université, CNRS, Université de Paris, Inria, Laboratoire Jacques-Louis Lions, équipe COMMEDIA, F-75005 Paris

² Université de Paris, MAP5, CNRS, F-75006 Paris, France

³ Inria Nancy Grand-Est and CNRS UMR 7503 LORIA, Villers-lès-Nancy, France

July 16, 2019

Abstract

In this article, we consider a reaction-diffusion equation where the reaction term is given by a cubic function and we are interested in the numerical reconstruction of the time-independent part of the source term from measurements of the solution. For this identification problem, we present an iterative algorithm based on Carleman estimates which consists of minimizing at each iteration strongly convex cost functionals. Despite the nonlinear nature of the problem, we prove that our algorithm globally converges and the convergence speed evaluated in weighted norm is linear. In the last part of the paper, we illustrate the effectiveness of our algorithm with several numerical reconstructions in dimension one or two.

Keywords: inverse problems, nonlinear parabolic equations, Carleman estimates, numerical reconstruction.

AMS subject classifications: 35R30, 35K55, 35K57, 93B07.

1 Introduction

Let Ω be a C^2 bounded domain of \mathbb{R}^d for $d = 1, 2$ or 3 and $T > 0$. We consider the following reaction-diffusion equation

$$\begin{cases} \partial_t u(t, x) - \Delta u(t, x) + u^3(t, x) = \sigma(x)h(t, x), & (t, x) \in (0, T) \times \Omega, \\ u(t, x) = g(t, x), & (t, x) \in (0, T) \times \partial\Omega, \\ u(0, x) = u_0(x), & x \in \Omega, \end{cases} \quad (1.1)$$

*This work was partially funded by the DGA 2014-91-00-79 project. The second author was partially supported by the Project “Analysis and simulation of optimal shapes - application to life sciences” of the Paris City Hall.

[†]e-mail: boulakia@ljl11.math.upmc.fr

[‡]e-mail: maya.de-buhan@parisdescartes.fr

[§]e-mail: schwindt@math.cnrs.fr

where g is the Dirichlet boundary data and u_o is the initial condition. In the right hand side of the first equation, we assume that the time-dependent function h is known and we focus on the reconstruction of σ which is assumed to depend only on the spatial variable. To identify this unknown, we have two kinds of measurements, the flux of the solution on a part of a boundary and the solution in the whole domain at a given time:

$$\begin{cases} m(t, x) := \nabla u(t, x) \cdot n(x), & (t, x) \in (0, T) \times \Gamma, \\ r(x) := u(T_0, x), & x \in \Omega, \end{cases} \quad (1.2)$$

where $\Gamma \subset \partial\Omega$, $T_0 \in (0, T)$ and n is the outward-pointing unit normal vector defined on $\partial\Omega$. Regarding the applications, this model can represent the evolution of a pollutant in the atmosphere. The source in the right hand side corresponds to a spill of pollutant and we want to localise it. This model can also be viewed as a simplified model to represent the evolution of the electrical potential in the heart (we refer to [7] for a detailed presentation of this application domain and more precisely to [7, Subsection 2.9.7] for cubic-like reaction models). In our model, the natural propagation of the potential is initiated by the initial condition and the source in the right hand side may correspond to a secondary undesirable source that we want to identify.

Let $\sigma_{max} > 0$ be a fixed constant. We assume that the source term σ that we want to reconstruct belongs to $L^\infty(\Omega)$ and satisfies the following *a priori* bound:

$$\|\sigma\|_{L^\infty(\Omega)} \leq \sigma_{max}. \quad (1.3)$$

For this problem, according to Bukhgeim-Klibanov method, σ is uniquely determined by the measurements and a Lipschitz stability estimate holds under appropriate assumptions on the data (the precise result is stated in Proposition 2.6). Bukhgeim-Klibanov method [4] is a classical theoretical method to prove the uniqueness and stability for parameter identification problems. For a presentation of this method which relies on Carleman estimates and for a survey on its applications, we refer to [16] and in particular to section 3.3 on parabolic equations. For the inverse problem of coefficients identification in nonlinear parabolic equations, let us in particular mention that [3] and [8] deal with the theoretical stability of the reaction term in a semi-linear PDE.

In this paper, our aim is to tackle the numerical reconstruction of σ and to propose for this nonlinear problem a globally convergent algorithm. Our work is drawn from a numerical algorithm presented in [1] for the identification of a potential in a wave equation. The method strongly relies on Carleman inequalities and it consists of an iterative algorithm minimizing at each iteration a cost functional involving Carleman weights. The main strength of this numerical method is that it globally converges to the exact solution *i.e.* it converges independently of the initialization. In particular, contrary to classical minimization techniques like Tikhonov methods [14], it is not necessary to add a priori knowledge on the source term through the data of a background state to convexify the cost functional.

As pointed out in the introduction of [2], this method induces several numerical challenges. In particular, the classical Carleman weights have very strong variations due to the presence of a double exponential involving large coefficients. That is why, as in [2] for the wave equations, we need to construct new Carleman weights for the heat equation which involve single exponentials (these weights are given by (2.3) and (2.4)).

The presence of a nonlinearity in our PDE leads to additional difficulties in the study of the algorithm and in the numerical methods. In particular, the strong convexity properties of the cost functional are restricted to bounded spaces. Moreover, the operator appearing in the cost functional has to be modified by adding truncation operators in the nonlinear terms to tackle these terms in the proof of the convergence of the algorithm. At last, contrary to [2] where the PDE is linear, introducing a conjugate variable $e^{s\varphi}z$ does not allow to overcome the fact that, even with single exponentials, the minimization of the cost functional is challenging.

Let us mention that we could have considered general cubic functions of the form

$$a_3u^3 + a_2u^2 + a_1u$$

with $a_3 > 0$ instead of a simple cubic monomial in this semi-linear parabolic partial differential equations (1.1). By this way, the study includes the bistable equation or Allen-Cahn equation. At last, we refer to Remark 2.4 for some remarks on the case of other boundary data (Neumann boundary conditions instead of the second equation in (1.1) and boundary measurements on the solution itself).

For numerical studies applying Carleman estimates to controllability problems, we refer to [6] for the numerical controllability of the wave equation and [11] for the numerical controllability of the heat, Stokes and Navier-Stokes equation. In [17], the authors are interested by the reconstruction of a coefficient in a parabolic equation and present a gradient method applied to a strictly convex cost functional involving Carleman weights.

The paper is organized as follows. Sections 2 give some preliminary results. First, in Section 2.1, we present a Carleman estimate for the heat operator with Dirichlet boundary conditions. In this estimate, we consider two kinds of Carleman weights: the classical weights for the heat equation with a double exponential and new weights involving single exponentials which are introduced for numerical purposes. Then, in Section 2.2, we state a regularity result satisfied by the solution of equation (1.1). The proofs of the Carleman estimate and the regularity result are presented in Annexes A and B respectively. At last, in Section 2.3, we state the stability inequality associated to our inverse problem.

Section 3.1 is the core of the paper and presents the reconstruction algorithm of the source term. The latter is an iterative process which requires at each iteration the minimization of a functional based on the Carleman estimate. This section also states the well-posedness of the algorithm (Lemma 3.2) as well as its global convergence (Theorem 3.3). The proofs of these results are given in Section 3.3 and 3.4 respectively.

In Section 4, we give a simplified version of the algorithm that is able to reconstruct the source term in a faster way but only in a restricted case. The iterative process is much simpler because it is no more necessary to minimize a functional at each step. For this algorithm, the convergence result (Proposition 4.1) is limited to the case of small data.

Finally, Section 5 is devoted to the implementation of the algorithm and the numerical results obtained for various 1d and 2d test cases. In particular, we show examples where the simplified algorithm fails whereas our initial algorithm identifies accurately the source.

2 Preliminary results

2.1 Carleman inequality for the heat equation

Without loss of generality, from now on, we assume that $T_0 = \frac{T}{2}$.

In this section, we state a Carleman inequality for the heat equation in two cases. The first case corresponds to the classical weights with a double exponential while, in the second case, the weights only involve single exponentials as for instance in [22, Section 3]. Let us specify these two cases :

- **Case 1:** For $\lambda > 0$, we define θ and φ by: for all $(t, x) \in (0, T) \times \Omega$

$$\theta(t, x) = \frac{e^{\lambda(2\|\eta_0\|_\infty + \eta_0(x))}}{t(T-t)} \quad \varphi(t, x) = \frac{e^{\lambda(2\|\eta_0\|_\infty + \eta_0(x))} - e^{4\lambda\|\eta_0\|_\infty}}{t(T-t)} \quad (2.1)$$

where η_0 satisfies the following properties:

$$\eta_0 > 0 \text{ in } \Omega, \quad |\nabla \eta_0| \geq C > 0 \text{ in } \bar{\Omega} \quad \text{and} \quad \eta_0 = 0 \text{ on } \partial\Omega \setminus \Gamma. \quad (2.2)$$

- **Case 2:** For all $(t, x) \in (0, T) \times \Omega$, we define

$$\theta(t) = \frac{1}{t(T-t)} - \frac{1-\rho}{T_0^2} \quad (2.3)$$

and

$$\varphi(t, x) = \psi(x)\theta(t) \quad \text{with} \quad \psi(x) = |x - x_0|^2 - 2 \sup_{x \in \bar{\Omega}} |x - x_0|^2, \quad (2.4)$$

where x_0 is an arbitrary point in $\mathbb{R}^d \setminus \bar{\Omega}$ and ρ is a constant satisfying $0 < \rho < 1$. We notice that $\theta > 0$ and $\psi < 0$. In this case, we assume in addition that x_0 and Γ are such that

$$\{x \in \partial\Omega \mid (x - x_0) \cdot n(x) > 0\} \subset \Gamma. \quad (2.5)$$

This last condition in **Case 2** is unusual for the heat equation and is linked to this new choice of weights. With these weights, we have less flexibility in the computations and we need an extra condition on the measurement domain compared to the classical weights corresponding to **Case 1**. On the other hand, if we take the classical weights, the presence of a double exponential in the functional to minimize (see (3.11)) is prohibitive to address numerical applications (we refer to Remark 2.3 for additional comments). In all our numerical tests presented in Section 5.2, we have considered the weights given by **Case 2**.

Let us now formulate the Carleman inequality in **Case 1** and **Case 2**.

Theorem 2.1. *We assume that θ and φ are given by (2.1) where λ is fixed and large enough or by (2.4). In this last case, we assume that Γ is such that (2.5) holds. Then, there exists $s_0 > 0$ and $C > 0$ such that, for all $s \geq s_0$:*

$$\begin{aligned} & \int_0^T \int_\Omega e^{2s\varphi} \left(\frac{1}{s\theta} |\partial_t z|^2 + \frac{1}{s\theta} |\Delta z|^2 + s\theta |\nabla z|^2 + s^3 \theta^3 |z|^2 \right) dx dt \\ & \leq C \int_0^T \int_\Omega e^{2s\varphi} |\partial_t z - \Delta z|^2 dx dt + Cs \int_0^T \int_\Gamma e^{2s\varphi} \theta |\nabla z \cdot n|^2 d\gamma dt, \end{aligned} \quad (2.6)$$

for all $z \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega))$.

Here and in all the paper, we denote by C a positive constant which depends on T and Ω , λ in **Case 1** and ρ in **Case 2**, unless specified otherwise where appropriated. The proof of this theorem is given in Appendix A. A consequence of Theorem 2.1 is the following lemma:

Lemma 2.2. *Under the same assumptions as Theorem 2.1, there exist $s_0 > 0$ and $C > 0$ such that, for all $s \geq s_0$:*

$$s \int_{\Omega} e^{2s\varphi(T_0)} |z(T_0)|^2 dx \leq C \int_0^T \int_{\Omega} e^{2s\varphi} |\partial_t z - \Delta z|^2 dx dt + Cs \int_0^T \int_{\Gamma} e^{2s\varphi} \theta |\nabla z \cdot n|^2 d\gamma dt, \quad (2.7)$$

for all $z \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^2(\Omega)) \cap H_0^1(\Omega)$.

Proof. We have

$$\begin{aligned} \int_{\Omega} e^{2s\varphi(T_0)} |z(T_0)|^2 dx &= \int_0^{T_0} \frac{d}{dt} \left(\int_{\Omega} e^{2s\varphi} |z|^2 dx \right) dt = \int_0^{T_0} \int_{\Omega} \partial_t (e^{2s\varphi} |z|^2) dx dt \\ &= \int_0^{T_0} \int_{\Omega} e^{2s\varphi} \left(2z \left(s\theta^{1/2} \frac{1}{s\theta^{1/2}} \right) \partial_t z + 2s\partial_t \varphi |z|^2 \right) dx dt \\ &\leq \int_0^{T_0} \int_{\Omega} e^{2s\varphi} \frac{1}{s^2\theta} |\partial_t z|^2 dx dt + \int_0^{T_0} \int_{\Omega} e^{2s\varphi} (s^2\theta + 2s|\partial_t \varphi|) |z|^2 dx dt \\ &\leq \frac{C}{s} \left[\int_0^{T_0} \int_{\Omega} e^{2s\varphi} \frac{1}{s\theta} |\partial_t z|^2 dx dt + \int_0^{T_0} \int_{\Omega} e^{2s\varphi} s^3\theta^2 |z|^2 dx dt \right] \end{aligned}$$

where we have used that $|\partial_t \varphi| \leq C\theta^2$. Thus, the result follows from (2.6). \square

Remark 2.3. *To better design Carleman weights for numerical purposes, it would be interesting to make a comprehensive comparison between different possible choices of Carleman weights for the heat equation. In particular, in such a study which is beyond the scope of our paper, it would be necessary to spell the lower bound on s in the associated Carleman inequality.*

Remark 2.4. *We could have considered other kinds of boundary data by completing the first equation of (1.1) with Neumann conditions instead of Dirichlet conditions and by replacing the first measurement in (1.2) by a measurement on a part of the boundary of u itself. In this case, following [12] and [10], we still have a Carleman inequality with the classical weights (2.1) and we can still prove the global convergence of the algorithm. For the numerical tests, it would be interesting to see if we can get a Carleman inequality with weights similar to the ones of Case 2.*

2.2 Regularity result

Let us give a regularity result for problem (1.1). The proof of this result is presented in Appendix B.

Proposition 2.5. *Assume that $u_o \in H^3(\Omega)$, $\sigma \in L^\infty(\Omega)$, $h \in H^1(0, T; L^2(\Omega))$ and $g \in H^1(0, T; H^{3/2}(\partial\Omega)) \cap H^2(0, T; H^{1/2}(\partial\Omega))$. Moreover, we assume that $h(0, \cdot) = 0$ in Ω .*

Then the solution u of (1.1) belongs to

$$u \in C^1(0, T; H^1(\Omega)) \cap H^2(0, T; L^2(\Omega)) \cap H^1(0, T; H^2(\Omega))$$

with the estimate

$$\begin{aligned}
& \|u\|_{C^1(0,T;H^1(\Omega))} + \|u\|_{H^2(0,T;L^2(\Omega))} + \|u\|_{H^1(0,T;H^2(\Omega))} \\
& \leq C \left(\|\sigma\|_{L^\infty(\Omega)} + \|\sigma\|_{L^\infty(\Omega)}^p \right) \left(\|h\|_{H^1(0,T;L^2(\Omega))} + \|h\|_{H^1(0,T;L^2(\Omega))}^p \right) + C \left(\|u_\circ\|_{H^3(\Omega)} + \|u_\circ\|_{H^3(\Omega)}^p \right) \\
& \quad + C \left(\|g\|_{H^1(0,T;H^{3/2}(\partial\Omega)) \cap H^2(0,T;L^2(\partial\Omega))} + \|g\|_{H^1(0,T;H^{3/2}(\partial\Omega)) \cap H^2(0,T;H^{1/2}(\partial\Omega))}^p \right)
\end{aligned} \tag{2.8}$$

where the power p is a fixed positive integer and C only depends on T and Ω .

Let us note that, in the above proposition, the regularity assumed for g is not optimal, it would indeed be sufficient to assume that $g \in H^1(0, T; H^{3/2}(\partial\Omega)) \cap H^2(0, T; H^\kappa(\partial\Omega))$ with $\kappa > 0$ (see [19, Chapter 1, Subsection 9.2]). In this result, if we do not make the assumption that $h(0, \cdot) = 0$ in Ω , it is necessary to assume that σ belongs to $H^1(\Omega)$ (since we need an initial condition in $H^1(\Omega)$ for the problem satisfied by $\partial_t u$). But this additional regularity assumption on σ leads to difficulties in the construction of the iterations in Algorithm 1.

2.3 Stability inequality

In this paragraph, we state a Lipschitz stability inequality for our inverse problem. This result asserts in particular that the unknown σ is identifiable from the measurements given by (1.2). It is obtained thanks to a direct application of Bukhgeim-Klibanov method [4] and relies on the Carleman inequality given by Theorem 2.1 and the regularity result given by Proposition 2.5. We do not give the proof here and refer to [15] and [13] for a closely related result.

Proposition 2.6. *We assume that $u_\circ \in H^3(\Omega)$, $g \in H^1(0, T; H^{3/2}(\partial\Omega)) \cap H^2(0, T; H^{1/2}(\partial\Omega))$ and $h \in H^1(0, T; L^\infty(\Omega))$ is such that $h(0, \cdot) = 0$ in Ω and $|h(T_0, \cdot)| \geq \beta > 0$ in Ω . We consider σ_1 and σ_2 in $L^\infty(\Omega)$ which satisfy (1.3). Then, for $i = 1, 2$, if we denote by u_i the solution of (1.1) associated to σ_i , we have the following inequality: there exists $C > 0$ such that*

$$\|\sigma_1 - \sigma_2\|_{L^2(\Omega)} \leq C(\|u_1(T_0) - u_2(T_0)\|_{H^2(\Omega)} + \|\nabla(u_1 - u_2) \cdot n\|_{H^1(0,T;L^2(\Gamma))}).$$

3 Reconstruction algorithm and theoretical study

3.1 Presentation of the algorithm and convergence

In this subsection, we construct a sequence $(\sigma_k)_{k \in \mathbb{N}}$ which approximates the unknown σ and we state the convergence of this sequence. We make the following assumptions:

Hypotheses 3.1. • $u_\circ \in H^3(\Omega)$ and $g \in H^1(0, T; H^{3/2}(\partial\Omega)) \cap H^2(0, T; H^{1/2}(\partial\Omega))$.

- $\sigma \in L^\infty(\Omega)$ satisfies (1.3).
- h satisfies

$$h \in H^1(0, T; L^\infty(\Omega)), \quad h(0, \cdot) = 0 \text{ in } \Omega \tag{3.1}$$

and

$$|h(T_0, \cdot)| \geq \beta > 0 \quad \text{in } \Omega. \tag{3.2}$$

- The weights θ and φ are given by **Case 1** or **Case 2** described at the beginning of paragraph 2.1. In **Case 1**, the parameter λ is fixed and large enough.

In our paper, we denote by M an arbitrary constant which only depends on T , Ω , σ_{max} , $\|u_o\|_{H^3(\Omega)}$, $\|h\|_{H^1(0,T;L^\infty(\Omega))}$ and $\|g\|_{H^1(0,T;H^{3/2}(\partial\Omega))\cap H^2(0,T;H^{1/2}(\partial\Omega))}$.

First, we initialize the algorithm with $\sigma_0 = 0$ (or any guess such that $\|\sigma_0\|_{L^\infty(\Omega)} \leq \sigma_{max}$). Now, let us assume that we are at step k and that we have constructed σ_k which satisfies

$$\|\sigma_k\|_{L^\infty(\Omega)} \leq \sigma_{max}. \quad (3.3)$$

We denote by u_k the solution of (1.1) associated to σ_k and by u_σ the solution of (1.1) associated to the unknown σ . Moreover, we set $v_k = u_\sigma - u_k$.

We then use Proposition 2.5 and we denote by $\bar{M} > 0$ a fixed constant depending on T , Ω , σ_{max} , $\|u_o\|_{H^3(\Omega)}$, $\|h\|_{H^1(0,T;L^2(\Omega))}$ and $\|g\|_{H^1(0,T;H^{3/2}(\partial\Omega))\cap H^2(0,T;H^{1/2}(\partial\Omega))}$ such that

$$\|v_k\|_{C([0,T]\times\bar{\Omega})} + \|v_k\|_{C^1(0,T;H^1(\Omega))} + \|v_k\|_{H^2(0,T;L^2(\Omega))} + \|v_k\|_{H^1(0,T;H^2(\Omega))} \leq \bar{M}. \quad (3.4)$$

The function v_k is solution of

$$\begin{cases} \partial_t v_k(t, x) - \Delta v_k(t, x) + v_k(t, x)q[v_k, u_k](t, x) = (\sigma(x) - \sigma_k(x))h(t, x), & (t, x) \in (0, T) \times \Omega, \\ v_k(t, x) = 0, & (t, x) \in (0, T) \times \partial\Omega, \\ v_k(0, x) = 0, & x \in \Omega, \end{cases} \quad (3.5)$$

where we have set $q[v, u] = 3u^2 + 3uv + v^2$. Let us differentiate the equation with respect to time. We introduce $w_k = \partial_t v_k$ which satisfies:

$$\begin{cases} \partial_t w_k(t, x) - \Delta w_k(t, x) + w_k(t, x)q[v_k, u_k](t, x) + v_k(t, x)\partial_t q[v_k, u_k](t, x) = f_k(t, x), & (t, x) \in (0, T) \times \Omega, \\ w_k(t, x) = 0, & (t, x) \in (0, T) \times \partial\Omega, \\ w_k(0, x) = 0, & x \in \Omega, \end{cases} \quad (3.6)$$

where, for all $(t, x) \in (0, T) \times \Omega$,

$$f_k(t, x) = (\sigma(x) - \sigma_k(x))\partial_t h(t, x). \quad (3.7)$$

We notice that

$$w_k(T_0, x) = \partial_t v_k(T_0, x) = \Delta v_k(T_0, x) - v_k(T_0, x)q[v_k, u_k](T_0, x) + (\sigma(x) - \sigma_k(x))h(T_0, x), \quad x \in \Omega. \quad (3.8)$$

Hence, thanks to hypothesis (3.2), knowing $w_k(T_0, \cdot)$ gives access to $\sigma - \sigma_k$, the other terms being given observations thanks to (1.2).

For the constant $\bar{M} > 0$ introduced in estimate (3.4), we consider the following function:

$$\begin{aligned} T_{\bar{M}} : \quad \mathbb{R} &\longrightarrow \mathbb{R} \\ X &\longmapsto X\Phi\left(\frac{X}{\bar{M}}\right), \end{aligned} \quad (3.9)$$

where $\Phi \in C_0^1(\mathbb{R})$ is such that $0 \leq \Phi \leq 1$ and

$$\Phi(X) = \begin{cases} 1, & \text{if } |X| \leq 1, \\ 0, & \text{if } |X| \geq 2. \end{cases} \quad (3.10)$$

The properties satisfied by $T_{\overline{M}}$ are given in section 3.2. For any μ in $L^2((0, T) \times \Gamma)$, we introduce the functional $J_0[\mu]$ by

$$J_0[\mu](z) = \frac{1}{2} \int_0^T \int_{\Omega} e^{2s\varphi} |P_k z|^2 dx dt + \frac{s}{2} \int_0^T \int_{\Gamma} e^{2s\varphi} \theta |\nabla z \cdot n - \mu|^2 d\gamma dt, \quad (3.11)$$

with

$$P_k z = \partial_t z - \Delta z + 3(u_k)^2 z + 6\partial_t u_k u_k T_{\overline{M}}(y) + 3\partial_t u_k T_{\overline{M}}(y)^2 + 6u_k z T_{\overline{M}}(y) + 3z T_{\overline{M}}(y)^2 \quad (3.12)$$

where

$$y(t, x) = v_k(T_0, x) + \int_{T_0}^t z(t', x) dt', \quad (t, x) \in (0, T) \times \Omega.$$

By this way, since v_k satisfies (3.4), $T_{\overline{M}}(v_k) = v_k$ and $P_k(w_k)$ corresponds to the left hand side of the first equation of (3.6).

We consider the functional $J_0[\mu]$ on the function space

$$\tilde{E} = \left\{ z : e^{s\varphi}(\partial_t z - \Delta z) \in L^2((0, T) \times \Omega), e^{s\varphi} \theta^{1/2} \nabla z \cdot n \in L^2((0, T) \times \Gamma), \right. \\ \left. e^{s\varphi} \theta^{3/2} z \in L^2((0, T) \times \Omega), e^{s\varphi} \theta^{-1/2} z \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega)) \right\} \quad (3.13)$$

endowed with its natural norm.

The next iteration σ_{k+1} is defined by following four steps:

Algorithm 1. Iteration: From k to $k+1$

- *Step 1* - We set $\mu_k = \partial_t(m - \nabla u_k \cdot n)$ on $(0, T) \times \Gamma$, where m is the measurement defined in (1.2).
- *Step 2* - According to Lemma 3.2, $J_0[\mu_k]$ (defined in (3.11)) admits a global minimizer in \tilde{E} . We denote it by Z_k (it depends on s but we drop this dependence to simplify the notations).
- *Step 3* - We set

$$\tilde{\sigma}_{k+1}(x) = \sigma_k(x) + \frac{Z_k(T_0, x) - \Delta v_k(T_0, x) + v_k(T_0, x)q[v_k, u_k](T_0, x)}{h(T_0, x)}, \quad x \in \Omega. \quad (3.14)$$

This is well-defined because h satisfies the positivity condition (3.2) and $h(T_0, \cdot) \in L^2(\Omega)$. Moreover, in this expression, $v_k(T_0, \cdot)$ is known and given by $v_k(T_0, \cdot) = r - u_k(T_0, \cdot)$ where r is the measurement defined in (1.2). Since $u_k(T_0)$ and $v_k(T_0)$ belong to $H^2(\Omega)$ and $Z_k(T_0)$ belongs to $L^2(\Omega)$, $\tilde{\sigma}_{k+1}$ belongs to $L^2(\Omega)$.

- *Step 4* - At last, we define

$$\sigma_{k+1} = \Pi_{\sigma_{max}}(\tilde{\sigma}_{k+1}),$$

where $\Pi_{\sigma_{max}}$ is given by

$$\Pi_{\sigma_{max}}(\sigma) = \begin{cases} \sigma, & \text{if } |\sigma| \leq \sigma_{max}, \\ \text{sign}(\sigma)\sigma_{max}, & \text{otherwise.} \end{cases}$$

By this way, σ_{k+1} satisfies (3.3) at step $k+1$.

The following lemma ensures the existence of Z_k at *Step 2* of Algorithm 1.

Lemma 3.2. *Let μ be given in $L^2((0, T) \times \Gamma)$ and assume that Hypotheses 3.1 hold. There exists $s_0 > 0$ which depends on T , Ω , σ_{max} , $\|u_\circ\|_{H^3(\Omega)}$, $\|h\|_{H^1(0, T; L^2(\Omega))}$, and $\|g\|_{H^1(0, T; H^{3/2}(\partial\Omega)) \cap H^2(0, T; H^{1/2}(\partial\Omega))}$ such that for all $s \geq s_0$, $J_0[\mu]$ admits a global minimizer in \tilde{E} . Moreover, if we define, for any $C > 0$*

$$E_C := \{z \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega)), \|z\|_{L^2(0, T; H^2(\Omega))} \leq C\}, \quad (3.15)$$

then, for s_0 large enough, $J_0[\mu]$ is strongly convex and admits a unique minimizer in E_C for any $s \geq s_0$.

This lemma will be proved in Subsection 3.3. Due to the nonlinearities in our equation, the strong convexity of $J_0[\mu]$ can only be stated in E_C where the bound in $L^2(0, T; H^2(\Omega))$ allows to deal with the nonlinearities. Contrary to the wave equations where the weights stay far from 0 (see [1, Section 4]), our weights, as usual for the heat equation, vanish at 0 and T and it is not clear that a minimizer of $J_0[\mu]$ in \tilde{E} will belong to E_C for some $C > 0$. Therefore, it is not possible to deduce in a direct way the uniqueness in \tilde{E} from the strong convexity in E_C .

Now we state the main theoretical result which gives the global linear convergence in the weighted L^2 -norm of the sequence $(\sigma_k)_{k \in \mathbb{N}}$:

Theorem 3.3. *Under Hypotheses 3.1, there exist $s_0 > 0$ and $M > 0$ such that for all $s \geq s_0$, for all $k \in \mathbb{N}$*

$$\int_{\Omega} e^{2s\varphi(T_0)} |\sigma_{k+1} - \sigma|^2 dx \leq \frac{M}{s} \int_{\Omega} e^{2s\varphi(T_0)} |\sigma_k - \sigma|^2 dx. \quad (3.16)$$

Thus, for s large enough, $(\sigma_k)_{k \in \mathbb{N}}$ tends to σ when k goes to $+\infty$.

This theorem will be proved in Subsection 3.4.

3.2 Properties satisfied by the function $T_{\overline{M}}$

Proposition 3.4. *The function $T_{\overline{M}}$ defined by (3.9) satisfies the following properties:*

a) For all $X \in \mathbb{R}$,

$$|T_{\overline{M}}(X)| \leq 2\overline{M}. \quad (3.17)$$

b) $T_{\overline{M}} \in C_0^1(\mathbb{R})$ and there exists $L > 0$ such that

$$|T_{\overline{M}}'(X)| \leq L\chi_{[-2\overline{M}, 2\overline{M}]}(X), \quad \forall X \in \mathbb{R}, \quad (3.18)$$

where χ_A is the characteristic function of a set A .

c) For all $X_1, X_2 \in \mathbb{R}$,

$$|T_{\overline{M}}(X_1) - T_{\overline{M}}(X_2)| \leq L|X_1 - X_2|, \quad (3.19)$$

which implies in particular that $T_{\overline{M}}$ is a Lipschitz operator.

Proof. a) For $X \in \mathbb{R}$, we have

$$|T_{\overline{M}}(X)| = \left| X \Phi\left(\frac{X}{\overline{M}}\right) \right| = \begin{cases} \leq |X|, & \text{if } |X| \leq 2\overline{M} \\ = 0, & \text{if } |X| > 2\overline{M} \end{cases} \leq 2\overline{M}.$$

b) By definition (3.9), $T_{\overline{M}} \in C_0^1(\mathbb{R})$ and $T'_{\overline{M}}(X) = 0$ for all $|X| \geq 2\overline{M}$. Moreover, for all $|X| \leq 2\overline{M}$, we have

$$\begin{aligned} |T'_{\overline{M}}(X)| &= \left| \Phi\left(\frac{X}{\overline{M}}\right) + \frac{X}{\overline{M}}\Phi'\left(\frac{X}{\overline{M}}\right) \right| \\ &\leq 1 + 2\|\Phi'\|_{C_0(\mathbb{R})}. \end{aligned}$$

c) This is a direct consequence of (3.18) and the mean value inequality. \square

3.3 Proof of Lemma 3.2

According to Proposition 2.5, there exists a constant $M > 0$ such that

$$\|u_k\|_{C^1(0,T;H^1(\Omega))} + \|u_k\|_{H^2(0,T;L^2(\Omega))} + \|u_k\|_{H^1(0,T;H^2(\Omega))} \leq M. \quad (3.20)$$

Using this estimate and (3.19), we have the continuity of $J_0[\mu]$ in \widetilde{E} . Moreover, since $J_0[\mu]$ is positive, it admits an infimum in \widetilde{E} and we can introduce a sequence $(z_n)_{n \in \mathbb{N}}$ such that

$$J_0[\mu](z_n) \xrightarrow{n \rightarrow +\infty} \inf_{z \in \widetilde{E}} J_0[\mu](z).$$

Let us study the convergence properties of the sequence $(z_n)_{n \in \mathbb{N}}$. Using inequality (3.20), we have

$$\begin{aligned} J_0[\mu](z_n) &\geq \frac{1}{4} \int_0^T \int_{\Omega} e^{2s\varphi} |\partial_t z_n - \Delta z_n|^2 dx dt + \frac{s}{4} \int_0^T \int_{\Gamma} e^{2s\varphi} \theta |\nabla z_n \cdot n|^2 d\gamma dt \\ &\quad - \frac{s}{2} \int_0^T \int_{\Gamma} e^{2s\varphi} \theta \mu^2 d\gamma dt - M - M \int_0^T \int_{\Omega} e^{2s\varphi} |z_n|^2 dx dt. \end{aligned}$$

Thus, according to the Carleman inequality given by (2.6) and using the fact that $se^{2s\varphi}\theta \leq C$ in $(0, T) \times \Omega$ for the third term in the right hand side, we get that, for s large enough,

$$\int_0^T \int_{\Omega} e^{2s\varphi} \left(\frac{1}{s\theta} |\partial_t z_n|^2 + \frac{1}{s\theta} |\Delta z_n|^2 + s\theta |\nabla z_n|^2 + s^3 \theta^3 |z_n|^2 \right) dx dt \leq J_0[\mu](z_n) + M + C \|\mu\|_{L^2((0,T) \times \Gamma)}^2. \quad (3.21)$$

By construction of $(z_n)_{n \in \mathbb{N}}$, the sequence $(J_0[\mu](z_n))_{n \in \mathbb{N}}$ is bounded and thus the left hand side of this last inequality is bounded. According to the definitions of θ and φ which are given by (2.1) or by (2.4), we have in $(0, T) \times \Omega$

$$|\partial_t \theta| + |\partial_t \varphi| \leq C\theta^2 \quad \text{and} \quad |\nabla \theta| + |\nabla \varphi| + |D^2 \theta| + |D^2 \varphi| \leq C\theta$$

and thus $(e^{s\varphi}\theta^{-1/2}z_n)_{n \in \mathbb{N}}$ is bounded in $H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^2(\Omega))$. We deduce that, $(e^{s\varphi}\theta^{-1/2}z_n)_{n \in \mathbb{N}}$ weakly converges to some element in $H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^2(\Omega))$ that we denote $e^{s\varphi}\theta^{-1/2}\tilde{z}$ (all the convergence results given in this proof are valid up to a subsequence but we do not specify it in order to lighten the writing). Moreover, since $H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^2(\Omega))$ is compactly embedded in $L^2((0, T) \times \Omega)$,

$$e^{s\varphi}\theta^{-1/2}z_n \rightarrow e^{s\varphi}\theta^{-1/2}\tilde{z} \quad \text{in } L^2((0, T) \times \Omega) \quad (3.22)$$

and, by identification of the limit, since θ^{-1} belongs to $L^\infty((0, T) \times \Omega)$, we also have

$$e^{s\varphi}\theta^{3/2}z_n \rightharpoonup e^{s\varphi}\theta^{3/2}\tilde{z} \text{ weakly in } L^2((0, T) \times \Omega)$$

and

$$e^{s\varphi}\theta^{1/2}\nabla z_n \rightharpoonup e^{s\varphi}\theta^{1/2}\nabla\tilde{z} \text{ weakly in } L^2(0, T; H^1(\Omega)). \quad (3.23)$$

Let us now prove that $\lim_{n \rightarrow +\infty} J_0[\mu](z_n) = J_0[\mu](\tilde{z})$ which will imply that \tilde{z} minimizes $J_0[\mu]$. Since $(J_0[\mu](z_n))_{n \in \mathbb{N}}$ is bounded, $(e^{s\varphi}\theta^{1/2}(\nabla z_n \cdot n))_{n \in \mathbb{N}}$ weakly converges in $L^2((0, T) \times \Gamma)$ and $(e^{s\varphi}P_k z_n)_{n \in \mathbb{N}}$ weakly converges in $L^2((0, T) \times \Omega)$ and it is sufficient to identify their weak limits. The fact that

$$e^{s\varphi}\theta^{1/2}(\nabla z_n \cdot n) \rightharpoonup e^{s\varphi}\theta^{1/2}(\nabla\tilde{z} \cdot n) \text{ weakly in } L^2((0, T) \times \Gamma)$$

directly comes from (3.23). To identify the limit of $(e^{s\varphi}P_k z_n)_{n \in \mathbb{N}}$, we will prove that

$$e^{s\varphi}\theta^{-1/2}P_k z_n \rightharpoonup e^{s\varphi}\theta^{-1/2}P_k\tilde{z} \text{ weakly in } L^2((0, T) \times \Omega). \quad (3.24)$$

We first consider in the definition (3.12) of P_k the three first terms which correspond to the linear part. The weak convergence of $(e^{s\varphi}\theta^{-1/2}z_n)_{n \in \mathbb{N}}$ to $e^{s\varphi}\theta^{-1/2}\tilde{z}$ in $H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^2(\Omega))$ implies that

$$e^{s\varphi}\theta^{-1/2}(\partial_t z_n - \Delta z_n + 3(u_k)^2 z_n) \rightharpoonup e^{s\varphi}\theta^{-1/2}(\partial_t \tilde{z} - \Delta \tilde{z} + 3(u_k)^2 \tilde{z}) \text{ weakly in } L^2((0, T) \times \Omega). \quad (3.25)$$

Now we define, for all $t \in (0, T)$

$$y_n(t) = v_k(T_0) + \int_{T_0}^t z_n(t') dt' \quad \text{and} \quad \tilde{y}(t) = v_k(T_0) + \int_{T_0}^t \tilde{z}(t') dt'.$$

For the other terms in the operator P_k , let us first prove that $(e^{s\varphi}\theta^{-1/2}T_{\overline{M}}(y_n))_{n \in \mathbb{N}}$ strongly converges to $e^{s\varphi}\theta^{-1/2}T_{\overline{M}}(\tilde{y})$ in $L^\infty(0, T; L^2(\Omega))$. To do so, we observe that

$$\begin{aligned} \int_{\Omega} e^{2s\varphi}\theta^{-1/2}|y_n - \tilde{y}|^2 dx &= \int_{\Omega} e^{2s\varphi}\theta^{-1/2} \left| \int_{T_0}^t (z_n - \tilde{z})(t', x) dt' \right|^2 dx \\ &\leq C \int_{\Omega} e^{2s\varphi}\theta^{-1/2} \left| \int_{T_0}^t |z_n - \tilde{z}|^2(t', x) dt' \right| dx. \end{aligned}$$

By definition (2.4) of φ and θ and since $T_0 = \frac{T}{2}$, we have, for all t' between T_0 and t , for all $x \in \Omega$

$$\varphi(t, x) \leq \varphi(t', x) \text{ and } \theta(t, x) \geq \theta(t', x). \quad (3.26)$$

This implies that

$$\|e^{s\varphi}\theta^{-1/2}(y_n - \tilde{y})\|_{L^\infty(0, T; L^2(\Omega))}^2 \leq C \int_0^T \int_{\Omega} e^{2s\varphi}\theta^{-1/2}|z_n - \tilde{z}|^2 dx dt.$$

Thus, according to (3.22), $(e^{s\varphi}\theta^{-1/2}y_n)_{n \in \mathbb{N}}$ strongly converges to $e^{s\varphi}\theta^{-1/2}\tilde{y}$ in $L^\infty(0, T; L^2(\Omega))$ and since $T_{\overline{M}}$ satisfies (3.19), this implies that

$$e^{s\varphi}\theta^{-1/2}T_{\overline{M}}(y_n) \rightarrow e^{s\varphi}\theta^{-1/2}T_{\overline{M}}(\tilde{y}) \text{ in } L^\infty(0, T; L^2(\Omega)). \quad (3.27)$$

We can now study the limit of the remaining terms of $e^{s\varphi}P_k z_n$ when n tends to $+\infty$: using (3.17), (3.20) and (3.27), we have

$$e^{s\varphi}\theta^{-1/2}\partial_t u_k u_k T_{\overline{M}}(y_n) \rightarrow e^{s\varphi}\theta^{-1/2}\partial_t u_k u_k T_{\overline{M}}(\tilde{y}) \quad \text{in } L^2((0, T) \times \Omega). \quad (3.28)$$

and

$$e^{s\varphi}\theta^{-1/2}\partial_t u_k T_{\overline{M}}(y_n)^2 \rightarrow e^{s\varphi}\theta^{-1/2}\partial_t u_k T_{\overline{M}}(\tilde{y})^2 \quad \text{in } L^2((0, T) \times \Omega). \quad (3.29)$$

Let us now prove that

$$e^{s\varphi}\theta^{-1/2}u_k z_n T_{\overline{M}}(y_n) \rightarrow e^{s\varphi}\theta^{-1/2}u_k \tilde{z} T_{\overline{M}}(\tilde{y}) \quad \text{in } L^2((0, T) \times \Omega). \quad (3.30)$$

The strong convergence (3.22) of $(e^{s\varphi}\theta^{-1/2}z_n)_{n \in \mathbb{N}}$ implies the almost everywhere convergence of $(z_n)_{n \in \mathbb{N}}$ to \tilde{z} and the existence of a function z_b in $L^2((0, T) \times \Omega)$ such that, for all $n \in \mathbb{N}$

$$|e^{s\varphi}\theta^{-1/2}z_n| \leq z_b.$$

Moreover, the strong convergence of $(e^{s\varphi}\theta^{-1/2}y_n)_{n \in \mathbb{N}}$ in $L^2((0, T) \times \Omega)$ implies the almost everywhere convergence of $(y_n)_{n \in \mathbb{N}}$ to \tilde{y} . Thus, we deduce that

$$e^{s\varphi}\theta^{-1/2}u_k z_n T_{\overline{M}}(y_n) \rightarrow e^{s\varphi}\theta^{-1/2}u_k \tilde{z} T_{\overline{M}}(\tilde{y}) \quad \text{a.e.}$$

and

$$|e^{s\varphi}\theta^{-1/2}u_k z_n T_{\overline{M}}(y_n)| \leq M z_b.$$

And these two properties imply (3.30) according to Lebesgue's dominated convergence theorem. At last, we use the same arguments to prove that

$$e^{s\varphi}\theta^{-1/2}z_n T_{\overline{M}}(y_n)^2 \rightarrow e^{s\varphi}\theta^{-1/2}u_k \tilde{z} T_{\overline{M}}(\tilde{y}) \quad \text{in } L^2((0, T) \times \Omega). \quad (3.31)$$

Finally, gathering (3.25) and (3.28) to (3.31), we obtain (3.24) and we conclude that \tilde{z} is a minimizer of $J_0[\mu]$.

Let us now prove the end of Lemma 3.2. On \tilde{E} , we consider the norm $\|\cdot\|_s$ defined by

$$\|z\|_s^2 = \int_0^T \int_{\Omega} e^{2s\varphi} \left(\frac{1}{s\theta} |\partial_t z|^2 + \frac{1}{s\theta} |\Delta z|^2 + s\theta |\nabla z|^2 + s^3 \theta^3 |z|^2 \right) dx dt.$$

For any fixed $C > 0$, the set E_C defined by (3.15) is convex and closed in $(\tilde{E}, \|\cdot\|_s)$ and we can prove that, for s large enough, $J_0[\mu]$ is strongly convex in E_C i.e. that there exists $\delta > 0$ such that, for all z_1 and z_2 in E_C

$$DJ_0[\mu](z_1)(z_1 - z_2) - DJ_0[\mu](z_2)(z_1 - z_2) \geq \delta \int_0^T \int_{\Omega} e^{2s\varphi} |z_1 - z_2|^2 dx dt.$$

We do not detail the proof of this inequality because it follows exactly the same steps as in the next paragraph to deal with the terms in the left hand side of (3.35) (with respectively z_1 and z_2 replaced by w_k and Z_k) and we have chosen to give these details in the next paragraph to present a complete proof for the convergence of the algorithm. At last, the strong convexity of $J_0[\mu]$ implies that it admits a unique minimizer in E_C .

3.4 Proof of the convergence of the algorithm given by Theorem 3.3

For $\mu_k = \partial_t(m - \nabla u_k \cdot n)$ on $(0, T) \times \Gamma$, we define the functional

$$J[\mu_k](z) = \frac{1}{2} \int_0^T \int_{\Omega} e^{2s\varphi} |P_k z - f_k|^2 dxdt + \frac{s}{2} \int_0^T \int_{\Gamma} e^{2s\varphi} \theta |\nabla z \cdot n - \mu_k|^2 d\gamma dt,$$

where P_k is given by (3.12) and f_k is defined by (3.7). We notice that w_k , solution of the equation (3.6) is the unique function in \tilde{E} which minimizes $J[\mu_k]$. Indeed, according to (3.4), $T_{\overline{M}}(v_k) = v_k$ and this implies that $J[\mu_k](w_k) = 0$.

Let us now compute the Gâteaux derivative of P_k at point w , for any $w \in \tilde{E}$. Let $z \in \tilde{E}$,

$$\begin{aligned} DP_k(w)(z) &= \lim_{\epsilon \rightarrow 0} \frac{P_k(w + \epsilon z) - P_k(w)}{\epsilon} \\ &= \partial_t z - \Delta z + 3z \left((u_k)^2 + 2u_k T_{\overline{M}}(v) + T_{\overline{M}}(v)^2 \right) \\ &\quad + 6T'_{\overline{M}}(v) \bar{y} \left(\partial_t u_k u_k + \partial_t u_k T_{\overline{M}}(v) + u_k w + w T_{\overline{M}}(v) \right), \end{aligned} \quad (3.32)$$

where $v(t) = v_k(T_0) + \int_{T_0}^t w(t') dt'$, $\bar{y}(t) = \int_{T_0}^t z(t') dt'$.

Then, w_k satisfies the first order optimality condition given by

$$\int_0^T \int_{\Omega} e^{2s\varphi} (P_k w_k - f_k) DP_k(w_k)(z) dxdt + s \int_0^T \int_{\Gamma} e^{2s\varphi} \theta (\nabla w_k \cdot n - \mu_k) (\nabla z \cdot n) d\gamma dt = 0, \quad \forall z \in \tilde{E}. \quad (3.33)$$

Similarly, Z_k satisfies the first order optimality condition

$$\int_0^T \int_{\Omega} e^{2s\varphi} (P_k Z_k) DP_k(Z_k)(z) dxdt + s \int_0^T \int_{\Gamma} e^{2s\varphi} \theta (\nabla Z_k \cdot n - \mu_k) (\nabla z \cdot n) d\gamma dt = 0, \quad \forall z \in \tilde{E}. \quad (3.34)$$

Let us define $z_k = w_k - Z_k$. We compute the difference between (3.33) and (3.34) and take $z = z_k$. We get

$$\begin{aligned} &\int_0^T \int_{\Omega} e^{2s\varphi} (P_k w_k DP_k(w_k)(z_k) - P_k Z_k DP_k(Z_k)(z_k)) dxdt + s \int_0^T \int_{\Gamma} e^{2s\varphi} \theta |\nabla z_k \cdot n|^2 d\gamma dt \\ &= \int_0^T \int_{\Omega} e^{2s\varphi} f_k DP_k(w_k)(z_k) dxdt. \end{aligned} \quad (3.35)$$

For the first term in the left hand-side, we have:

$$P_k w_k DP_k(w_k)(z_k) - P_k Z_k DP_k(Z_k)(z_k) = (P_k w_k - P_k Z_k) DP_k(Z_k)(z_k) + P_k w_k (DP_k(w_k)(z_k) - DP_k(Z_k)(z_k)).$$

Thus, (3.35) implies

$$\begin{aligned} &\int_0^T \int_{\Omega} e^{2s\varphi} (P_k w_k - P_k Z_k) DP_k(Z_k)(z_k) dxdt + s \int_0^T \int_{\Gamma} e^{2s\varphi} \theta |\nabla z_k \cdot n|^2 dxdt \leq \int_0^T \int_{\Omega} e^{2s\varphi} |f_k|^2 d\gamma dt \\ &+ \frac{1}{4} \int_0^T \int_{\Omega} e^{2s\varphi} |DP_k(w_k)(z_k)|^2 dxdt + \int_0^T \int_{\Omega} e^{2s\varphi} |P_k w_k| |DP_k(w_k)(z_k) - DP_k(Z_k)(z_k)| dxdt. \end{aligned} \quad (3.36)$$

We will estimate separately the different terms of this inequality. For what follows, we define, for all $t \in (0, T)$

$$Y_k(t, \cdot) = v_k(T_0, \cdot) + \int_{T_0}^t Z_k(t', \cdot) dt' \quad \text{and} \quad \bar{y}_k(t, \cdot) = v_k(t, \cdot) - Y_k(t, \cdot) = \int_{T_0}^t z_k(t', \cdot) dt'.$$

- Let us first find a lower bound for the first term in the left-hand side.

$$\begin{aligned} P_k w_k - P_k Z_k &= \partial_t z_k - \Delta z_k + 3(u_k)^2 z_k + 6\partial_t u_k u_k (T_{\bar{M}}(v_k) - T_{\bar{M}}(Y_k)) \\ &\quad + 3\partial_t u_k (T_{\bar{M}}(v_k)^2 - T_{\bar{M}}(Y_k)^2) + 6u_k (T_{\bar{M}}(v_k) w_k - T_{\bar{M}}(Y_k) Z_k) \\ &\quad + 3(T_{\bar{M}}(v_k)^2 w_k - T_{\bar{M}}(Y_k)^2 Z_k) \\ &= \partial_t z_k - \Delta z_k + 3(u_k)^2 z_k + 6\partial_t u_k u_k (T_{\bar{M}}(v_k) - T_{\bar{M}}(Y_k)) \\ &\quad + 3\partial_t u_k (T_{\bar{M}}(v_k) - T_{\bar{M}}(Y_k))(T_{\bar{M}}(v_k) + T_{\bar{M}}(Y_k)) + 6u_k z_k T_{\bar{M}}(Y_k) \\ &\quad + 6u_k (T_{\bar{M}}(v_k) - T_{\bar{M}}(Y_k)) w_k + 3z_k T_{\bar{M}}(Y_k)^2 \\ &\quad + 3(T_{\bar{M}}(v_k) + T_{\bar{M}}(Y_k))(T_{\bar{M}}(v_k) - T_{\bar{M}}(Y_k)) w_k \\ &:= \partial_t z_k - \Delta z_k + R_{1,k}. \end{aligned}$$

Using (3.17), (3.19) and (3.20), we can estimate $R_{1,k}$

$$|R_{1,k}| \leq M|z_k| + M|\bar{y}_k|(|\partial_t u_k| + |w_k|).$$

Moreover, from (3.32), we can write $DP_k(Z_k)(z_k) = \partial_t z_k - \Delta z_k + R_{2,k}$ where, according to (3.17), (3.18) and (3.20)

$$|R_{2,k}| \leq M|z_k| + M|T'_{\bar{M}}(Y_k)| |\bar{y}_k| (|\partial_t u_k| + |Z_k|) \leq M|z_k| + M\chi_{[-2\bar{M}, 2\bar{M}]}(Y_k) |\bar{y}_k| (|\partial_t u_k| + |Z_k|).$$

We deduce from these inequalities that

$$\begin{aligned} \int_0^T \int_{\Omega} e^{2s\varphi} (P_k w_k - P_k Z_k) DP_k(Z_k)(z_k) dx dt &\geq \frac{3}{4} \int_0^T \int_{\Omega} e^{2s\varphi} |\partial_t z_k - \Delta z_k|^2 dx dt \\ &\quad - M \int_0^T \int_{\Omega} e^{2s\varphi} |z_k|^2 dx dt - M \int_0^T \int_{\Omega} e^{2s\varphi} |\bar{y}_k|^2 (|\partial_t u_k|^2 + |w_k|^2 + \chi_{[-2\bar{M}, 2\bar{M}]}(Y_k) |Z_k|^2) dx dt. \end{aligned} \quad (3.37)$$

For the last term, we use that $Z_k = w_k - z_k$ and that $\chi_{[-2\bar{M}, 2\bar{M}]}(Y_k) \leq \chi_{[-3\bar{M}, 3\bar{M}]}(\bar{y}_k)$, according to (3.4). Thus, we get

$$\begin{aligned} \int_0^T \int_{\Omega} e^{2s\varphi} |\bar{y}_k|^2 (|\partial_t u_k|^2 + |w_k|^2 + \chi_{[-2\bar{M}, 2\bar{M}]}(Y_k) |Z_k|^2) dx dt \\ \leq C \int_0^T \int_{\Omega} e^{2s\varphi} |\bar{y}_k|^2 (|\partial_t u_k|^2 + |w_k|^2 + \chi_{[-3\bar{M}, 3\bar{M}]}(\bar{y}_k) |z_k|^2) dx dt. \end{aligned} \quad (3.38)$$

Let us estimate this last integral. We first notice that

$$\int_0^T \int_{\Omega} e^{2s\varphi} |\bar{y}_k|^2 \chi_{[-3\bar{M}, 3\bar{M}]}(\bar{y}_k) |z_k|^2 dx dt \leq M \int_0^T \int_{\Omega} e^{2s\varphi} |z_k|^2 dx dt. \quad (3.39)$$

Next, according to (3.20) and (3.4), we have

$$\begin{aligned} \int_0^T \int_{\Omega} e^{2s\varphi} |\bar{y}_k|^2 (|\partial_t u_k|^2 + |w_k|^2) dx dt &\leq \|e^{s\varphi} \bar{y}_k\|_{L^\infty(0,T;L^2(\Omega))}^2 (\|\partial_t u_k\|_{L^2(0,T;L^\infty(\Omega))}^2 + \|w_k\|_{L^2(0,T;L^\infty(\Omega))}^2) \\ &\leq M \|e^{s\varphi} \bar{y}_k\|_{L^\infty(0,T;L^2(\Omega))}^2. \end{aligned}$$

We have, for all $t \in (0, T)$

$$\begin{aligned} \int_{\Omega} e^{2s\varphi(t,x)} |\bar{y}_k(t,x)|^2 dx &= \int_{\Omega} e^{2s\varphi(t,x)} \left| \int_{T_0}^t z_k(t',x) dt' \right|^2 dx \\ &\leq C \int_{\Omega} e^{2s\varphi(t,x)} \left| \int_{T_0}^t |z_k(t',x)|^2 dt' \right| dx \leq C \int_0^T \int_{\Omega} e^{2s\varphi} |z_k|^2 dx dt' \end{aligned}$$

using inequality (3.26) for φ . By this way, we deduce that

$$\int_0^T \int_{\Omega} e^{2s\varphi} |\bar{y}_k|^2 (|\partial_t u_k|^2 + |w_k|^2) dx dt \leq M \int_0^T \int_{\Omega} e^{2s\varphi} |z_k|^2 dx dt. \quad (3.40)$$

Using, (3.38), (3.39) and this last inequality, (3.37) becomes

$$\begin{aligned} \int_0^T \int_{\Omega} e^{2s\varphi} (P_k w_k - P_k Z_k) DP_k(Z_k)(z_k) dx dt &\geq \frac{3}{4} \int_0^T \int_{\Omega} e^{2s\varphi} |\partial_t z_k - \Delta z_k|^2 dx dt \\ &\quad - M \int_0^T \int_{\Omega} e^{2s\varphi} |z_k|^2 dx dt. \end{aligned} \quad (3.41)$$

- To bound the second term in the right-hand side of (3.36), by definition (3.32) of DP_k , we have, according to (3.18) and (3.20)

$$|DP_k(w_k)(z_k)|^2 \leq 2|\partial_t z_k - \Delta z_k|^2 + M|z_k|^2 + M|\bar{y}_k|^2 (|\partial_t u_k|^2 + |w_k|^2).$$

Thus, using again (3.40), we get

$$\frac{1}{4} \int_0^T \int_{\Omega} e^{2s\varphi} |DP_k(w_k)(z_k)|^2 dx dt \leq \frac{1}{2} \int_0^T \int_{\Omega} e^{2s\varphi} |\partial_t z_k - \Delta z_k|^2 dx dt + M \int_0^T \int_{\Omega} e^{2s\varphi} |z_k|^2 dx dt. \quad (3.42)$$

- To bound the last term of (3.36), we notice that

$$\begin{aligned} &DP_k(w_k)(z_k) - DP_k(Z_k)(z_k) = 3z_k (2u_k T'_M(v_k) + T'_M(v_k)^2) - 3z_k (2u_k T'_M(Y_k) + T'_M(Y_k)^2) \\ &\quad + 6T'_M(v_k) \bar{y}_k (\partial_t u_k u_k + \partial_t u_k T'_M(v_k) + u_k w_k + w_k T'_M(v_k)) \\ &\quad - 6T'_M(Y_k) \bar{y}_k (\partial_t u_k u_k + \partial_t u_k T'_M(Y_k) + u_k Z_k + Z_k T'_M(Y_k)) \\ &= 6z_k u_k (T'_M(v_k) - T'_M(Y_k)) + 3z_k (T'_M(v_k)^2 - T'_M(Y_k)^2) + 6\partial_t u_k u_k \bar{y}_k (T'_M(v_k) - T'_M(Y_k)) \\ &\quad + 6\partial_t u_k \bar{y}_k (T'_M(v_k) - T'_M(Y_k)) T'_M(v_k) + 6\partial_t u_k T'_M(Y_k) \bar{y}_k (T'_M(v_k) - T'_M(Y_k)) \\ &\quad + 6u_k \bar{y}_k (T'_M(v_k) - T'_M(Y_k)) w_k + 6u_k T'_M(Y_k) \bar{y}_k z_k + 6\bar{y}_k (T'_M(v_k) - T'_M(Y_k)) w_k T'_M(v_k) \\ &\quad + 6T'_M(Y_k) \bar{y}_k w_k (T'_M(v_k) - T'_M(Y_k)) + 6T'_M(Y_k) \bar{y}_k z_k T'_M(Y_k). \end{aligned}$$

Hence, using that $|T'_M(Y_k)| \leq L\chi_{[-2M, 2M]}(Y_k) \leq L\chi_{[-3M, 3M]}(\bar{y}_k)$

$$|DP_k(w_k)(z_k) - DP_k(Z_k)(z_k)| \leq M|z_k| + M|\bar{y}_k|(|\partial_t u_k| + |w_k| + |z_k|\chi_{[-3M, 3M]}(\bar{y}_k)).$$

This implies that

$$\begin{aligned} & \int_0^T \int_{\Omega} e^{2s\varphi} |P_k w_k| |DP_k(w_k)(z_k) - DP_k(Z_k)(z_k)| dx dt \\ & \leq \frac{1}{2} \int_0^T \int_{\Omega} e^{2s\varphi} |P_k w_k|^2 dx dt + \frac{1}{2} \int_0^T \int_{\Omega} e^{2s\varphi} |DP_k(w_k)(z_k) - DP_k(Z_k)(z_k)|^2 dx dt \\ & \leq \frac{1}{2} \int_0^T \int_{\Omega} e^{2s\varphi} |f_k|^2 dx dt + M \int_0^T \int_{\Omega} e^{2s\varphi} |z_k|^2 dx dt \end{aligned} \quad (3.43)$$

according to (3.39) and (3.40).

Using (3.41), (3.42) and (3.43), inequality (3.36) becomes:

$$\begin{aligned} & \frac{1}{4} \int_0^T \int_{\Omega} e^{2s\varphi} |\partial_t z_k - \Delta z_k|^2 dx dt + s \int_0^T \int_{\Gamma} e^{2s\varphi} \theta |\nabla z_k \cdot n|^2 d\gamma dt \\ & \leq \frac{3}{2} \int_0^T \int_{\Omega} e^{2s\varphi} |f_k|^2 dx dt + M \int_0^T \int_{\Omega} e^{2s\varphi} |z_k|^2 dx dt. \end{aligned} \quad (3.44)$$

Using Theorem 2.1, we can eliminate the last term in the right hand-side of (3.44) for s larger than some constant s_0 . Thus, using inequality (2.7), we get the following bound on $z(T_0)$:

$$s \int_{\Omega} e^{2s\varphi(T_0)} |z_k(T_0)|^2 dx \leq M \int_0^T \int_{\Omega} e^{2s\varphi} |f_k|^2 dx dt. \quad (3.45)$$

In the left hand-side of this inequality, we have $z_k(T_0, x) = w_k(T_0, x) - Z_k(T_0, x)$ for $x \in \Omega$ and, using (3.8) and (3.14), we get that

$$z_k(T_0, x) = -h(T_0, x)(\tilde{\sigma}_{k+1}(x) - \sigma(x)), \quad x \in \Omega.$$

In the right hand-side of (3.45), since $f_k = (\sigma_k - \sigma)\partial_t h$ and h is assumed to be bounded in $H^1(0, T; L^\infty(\Omega))$, we have

$$\int_0^T \int_{\Omega} e^{2s\varphi} |f_k|^2 dx dt \leq M \int_{\Omega} e^{2s\varphi(T_0)} |\sigma_k - \sigma|^2 dx.$$

Using (3.2), we get that

$$s \int_{\Omega} e^{2s\varphi(T_0)} |\tilde{\sigma}_{k+1} - \sigma|^2 dx \leq M \int_{\Omega} e^{2s\varphi(T_0)} |\sigma_k - \sigma|^2 dx.$$

Now, to estimate $\sigma_{k+1} = \Pi_{\sigma_{max}}(\tilde{\sigma}_{k+1})$, we notice that, since σ satisfies (1.3), we have

$$|\sigma_{k+1} - \sigma| \leq |\tilde{\sigma}_{k+1} - \sigma| \text{ in } \Omega. \quad (3.46)$$

Thus, we get (3.16) and, applying iteratively this estimate, we obtain that

$$\int_{\Omega} e^{2s\varphi(T_0)} |\sigma_{k+1} - \sigma|^2 dx \leq \left(\frac{M}{s}\right)_{k+1} \int_{\Omega} e^{2s\varphi(T_0)} |\sigma_0 - \sigma|^2 dx.$$

Thus, for s large enough we deduce the convergence of the algorithm.

This concludes the proof of Theorem 3.3.

4 A simplified algorithm

In this section, we propose a simplified algorithm where we skip *Step 2* in the Algorithm 1 and choose $Z_k = 0$ instead. By this way, *Step 3* of the algorithm becomes

- *Step 3 bis* - We set

$$\tilde{\sigma}_{k+1}(x) = \sigma_k(x) - \frac{\Delta v_k(T_0, x) - v_k(T_0, x)q[v_k, u_k](T_0, x)}{h(T_0, x)}. \quad (4.1)$$

This new algorithm was motivated by numerical tests where we observed that the method described by Algorithm 1 reconstructs sometimes the source whereas the minimization problem is not solved.

The following proposition states the convergence of the sequence $(\sigma_k)_{k \in \mathbb{N}}$ defined by this simplified algorithm as soon as the data are small enough. The proof of this result is far more basic than the proof of the global convergence (Theorem 3.3) for the complete algorithm and essentially relies on regularity results. An interesting fact is that the convergence is proved for the L^∞ -norm.

Proposition 4.1. *We consider the sequence $(\sigma_k)_{k \in \mathbb{N}}$ defined by Algorithm 1 where we have replaced Step 3 by Step 3 bis. We assume that*

- h belongs to $H^2(0, T; L^2(\Omega))$, satisfies (3.2) and $h(0, \cdot) = 0$ in Ω ,
- $u_\circ \in H^3(\Omega)$, $\sigma \in L^\infty(\Omega)$ and $g \in H^1(0, T; H^{3/2}(\partial\Omega)) \cap H^2(0, T; H^{1/2}(\partial\Omega))$.

Then, there exists a constant $\epsilon > 0$ depending on T , Ω , σ_{max} and β such that, if

$$\|u_\circ\|_{H^3(\Omega)} + \|h\|_{H^2(0, T; L^2(\Omega))} + \|g\|_{H^1(0, T; H^{3/2}(\partial\Omega)) \cap H^2(0, T; H^{1/2}(\partial\Omega))} \leq \epsilon, \quad (4.2)$$

the sequence $(\sigma_k)_{k \in \mathbb{N}}$ linearly converges to σ in $L^\infty(\Omega)$.

Proof. According to equation (3.5) satisfied by v_k , $\tilde{\sigma}_{k+1}$ defined by (4.1) satisfies

$$\tilde{\sigma}_{k+1}(x) = \sigma(x) + \frac{\partial_t v_k(T_0, x)}{h(T_0, x)}$$

Thus, using (3.2), we have

$$\|\tilde{\sigma}_{k+1} - \sigma\|_{L^\infty(\Omega)} \leq \frac{1}{\beta} \|w_k\|_{C^0(0, T; L^\infty(\Omega))}. \quad (4.3)$$

According to Proposition 2.5, and since $\|\sigma_k\|_{L^\infty(\Omega)} \leq \sigma_{max}$,

$$\|u_k\|_{C^1(0, T; H^1(\Omega))} + \|u_k\|_{H^2(0, T; L^2(\Omega))} + \|u_k\|_{H^1(0, T; H^2(\Omega))} \leq M$$

and

$$\|u\|_{C^1(0, T; H^1(\Omega))} + \|u\|_{H^2(0, T; L^2(\Omega))} + \|u\|_{H^1(0, T; H^2(\Omega))} \leq M$$

where M has the following expression:

$$\begin{aligned} M = & C(\sigma_{max} + \sigma_{max}^p) \left(\|h\|_{H^1(0, T; L^2(\Omega))} + \|h\|_{H^1(0, T; L^2(\Omega))}^p \right) + C \left(\|u_\circ\|_{H^3(\Omega)} + \|u_\circ\|_{H^3(\Omega)}^p \right) \\ & + C \left(\|g\|_{H^1(0, T; H^{3/2}(\partial\Omega)) \cap H^2(0, T; L^2(\partial\Omega))} + \|g\|_{H^1(0, T; H^{3/2}(\partial\Omega)) \cap H^2(0, T; H^{1/2}(\partial\Omega))}^p \right) \end{aligned} \quad (4.4)$$

where the power p is a fixed positive integer.

Using these estimates and adapting the proof of Proposition 2.5 to the system (3.5) satisfied by v_k , we can prove in a similar way that

$$v_k \in C^1(0, T; H^1(\Omega)) \cap H^2(0, T; L^2(\Omega)) \cap H^1(0, T; H^2(\Omega))$$

with the estimate

$$\|v_k\|_{C^1(0, T; H^1(\Omega))} + \|v_k\|_{H^2(0, T; L^2(\Omega))} + \|v_k\|_{H^1(0, T; H^2(\Omega))} \leq M \|\sigma_k - \sigma\|_{L^\infty(\Omega)} \quad (4.5)$$

where M is also of the form (4.4). If we compute the energy estimate for the system satisfied by $\partial_t w_k$, we get that w_k belongs to $C^1(0, T; L^2(\Omega))$ with the estimate:

$$\|w_k\|_{C^1(0, T; L^2(\Omega))} \leq (M + \|h\|_{H^2(0, T; L^2(\Omega))}) \|\sigma_k - \sigma\|_{L^\infty(\Omega)}.$$

Looking at (3.6) as an elliptic problem, we deduce that

$$\|w_k\|_{C(0, T; H^2(\Omega))} \leq (M + \|h\|_{H^2(0, T; L^2(\Omega))}) \|\sigma_k - \sigma\|_{L^\infty(\Omega)}.$$

We then use this estimate in (4.3) to bound the right hand side and get:

$$\|\tilde{\sigma}_{k+1} - \sigma\|_{L^\infty(\Omega)} \leq \frac{1}{\beta} (M + \|h\|_{H^2(0, T; L^2(\Omega))}) \|\sigma_k - \sigma\|_{L^\infty(\Omega)}.$$

To conclude the proof, we use the same argument as in the proof of Theorem 3.3: since $\sigma_{k+1} = \Pi_{\sigma_{max}}(\tilde{\sigma}_{k+1})$ satisfies (3.46), we have

$$\|\sigma_{k+1} - \sigma\|_{L^\infty(\Omega)} \leq \frac{1}{\beta} (M + \|h\|_{H^2(0, T; L^2(\Omega))}) \|\sigma_k - \sigma\|_{L^\infty(\Omega)}.$$

Thus, according to the expression of M given by (4.4), if u_o , h and g satisfy (4.2) for $\epsilon > 0$ small enough with respect to T , Ω , σ_{max} and β , the sequence $(\sigma_k)_{k \in \mathbb{N}}$ linearly converges to σ in $L^\infty(\Omega)$. \square

5 Numerical issues

5.1 Numerical methods

In this subsection, we present the discretization procedure and the numerical methods used in our numerical simulations. To simplify the presentation, we explain the discretization scheme in the one-dimensional case and assume that $\Omega = (0, L)$ for $L > 0$ and $\Gamma = \{x = L\}$.

Generation of the data

In this article, we work with synthetic data. To discretize the reaction-diffusion equation (1.1) for the exact source σ , we use a finite differences scheme based on the three-point backward Euler scheme and a linearization of the cubic term. We denote by $N_x \in \mathbb{N}$ the number of discretization points in the interior of $[0, L]$ and by $N_t \in \mathbb{N}$ the number of discretization points in the interior of $[0, T]$. The

space and time steps are denoted by $\Delta x = \frac{L}{N_x + 1}$ and $\Delta t = \frac{T}{N_t + 1}$ respectively and we define, for $0 \leq j \leq N_x + 1$ and $0 \leq n \leq N_t + 1$, u_j^n a numerical approximation of the solution $u(t^n, x_j)$ with $t^n = n\Delta t$ and $x_j = j\Delta x$. The approximated solution is computed in the following way:

Initialize: $u_j^0 = u_o(x_j)$, $0 \leq j \leq N_x + 1$.

For $0 \leq n \leq N_t$, knowing u^n , compute u^{n+1} as the solution of the linear system:

$$\begin{cases} \frac{u_j^{n+1} - u_j^n}{\Delta t} - \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{\Delta x^2} + (u_j^n)^3 + 3(u_j^n)^2(u_j^{n+1} - u_j^n) = \sigma(x_j)h(t^n, x_j), \\ u_0^{n+1} = g(t^{n+1}, 0) \quad \text{and} \quad u_{N_x+1}^{n+1} = g(t^{n+1}, L), \quad 1 \leq j \leq N_x, \end{cases} \quad (5.1)$$

where the time implicit cubic term $(u_j^{n+1})^3$ has been approximated by its first order Taylor expansion $(u_j^n)^3 + 3(u_j^n)^2(u_j^{n+1} - u_j^n)$. Then, we compute the counterpart of the continuous measurements r and m given in (1.2) as follows:

$$m^n = \frac{u_{N_x+1}^n - u_{N_x}^n}{\Delta x}, \quad 0 \leq n \leq N_t + 1 \quad \text{and} \quad r_j = u_j^{n_0}, \quad 0 \leq j \leq N_x + 1,$$

with n_0 is the integer part of $N_t/2 + 1$.

On the computed data, we may add a Gaussian noise:

$$\begin{aligned} m^n &\leftarrow m^n + \alpha(\max_n m^n)\mathcal{N}(0, 1), & 0 \leq n \leq N_t + 1, \\ r_j &\leftarrow r_j + \alpha(\max_j r_j)\mathcal{N}(0, 1), & 0 \leq j \leq N_x + 1, \end{aligned} \quad (5.2)$$

where $\mathcal{N}(0, 1)$ satisfies a centered normal law with deviation 1 and α is the level of noise (*i.e.* $\alpha = 0.01$ corresponds to a noise of 1%).

Discrete algorithm

We present in this subsection the discrete version of Algorithm 1.

Algorithm 2. Initialisation : Start with $\bar{\sigma} = 0$.

Iteration : Until the convergence criteria is reached, do

- *Step 1* - Knowing $\bar{\sigma} \in \mathbb{R}^{N_x}$, solve

$$\begin{cases} \frac{\bar{u}_j^{n+1} - \bar{u}_j^n}{\Delta t} - \frac{\bar{u}_{j+1}^{n+1} - 2\bar{u}_j^{n+1} + \bar{u}_{j-1}^{n+1}}{\Delta x^2} + (\bar{u}_j^n)^3 = \bar{\sigma}_j h(t^n, x_j), \\ \bar{u}_0^{n+1} = g(t^{n+1}, 0) \quad \text{and} \quad \bar{u}_{N_x+1}^{n+1} = g(t^{n+1}, L), & 0 \leq n \leq N_t, \\ \bar{u}_j^0 = u_o(x_j), & 1 \leq j \leq N_x, \end{cases} \quad (5.3)$$

and set $v_j = r_j - \bar{u}_j^{n_0}$.

- *Step 2* - Define for $1 \leq n \leq N_t$,

$$\mu^n = \frac{\left(m - \frac{\bar{u}_{N_x+1} - \bar{u}_{N_x}}{\Delta x}\right)^{n+1} - \left(m - \frac{\bar{u}_{N_x+1} - \bar{u}_{N_x}}{\Delta x}\right)^{n-1}}{2\Delta t} \quad (5.4)$$

and discretize the functional (3.11) as follows:

$$J_0[\mu](z) = \frac{1}{2} \Delta t \Delta x \sum_{n=1}^{N_t} \sum_{j=1}^{N_x} e^{2s\varphi(t^n, x_j)} |(P_k z)_j^n|^2 + \frac{s}{2} \Delta t \sum_{n=1}^{N_t} e^{2s\varphi(t^n, L)} \theta(t^n) \left| \frac{-z_{N_x}^n}{\Delta x} - \mu^n \right|^2, \quad (5.5)$$

where

$$(P_k z)_j^n = \frac{z_j^{n+1} - z_j^{n-1}}{2\Delta t} - \frac{z_{j+1}^n - 2z_j^n + z_{j-1}^n}{\Delta x^2} + 3(\bar{u}_j^n)^2 z_j^n + 6 \frac{\bar{u}_j^{n+1} - \bar{u}_j^{n-1}}{2\Delta t} \bar{u}_j^n T_M(y_j^n) + 3 \frac{\bar{u}_j^{n+1} - \bar{u}_j^{n-1}}{2\Delta t} T_M(y_j^n)^2 + 6\bar{u}_j^n z_j^n T_M(y_j^n) + 3z_j^n T_M(y_j^n)^2, \quad (5.6)$$

with

$$\begin{cases} y_j^{n_0} = v_j, & 1 \leq j \leq N_x, \\ y_j^n = y_j^{n-1} + \Delta x z_j^n, & \text{if } n > n_0, \\ y_j^n = y_j^{n+1} - \Delta x z_j^n, & \text{if } n < n_0. \end{cases}$$

Minimize $J_0[\mu]$ by a Newton-Krylov method [18] using the gradient of $J_0[\mu]$ and obtain the minimum $Z = (Z_j^n)_{1 \leq j \leq N_x, 1 \leq n \leq N_t}$.

• *Step 3* - Update

$$\bar{\sigma}_j \leftarrow \bar{\sigma}_j + \frac{Z_j^{n_0} - \frac{v_{j+1} - 2v_j + v_{j-1}}{\Delta x^2} + v_j q[v_j, \bar{u}_j^{n_0}]}{h(t^{n_0}, x_j)}, \quad 1 \leq j \leq N_x. \quad (5.7)$$

• *Step 4* - At last, define

$$\bar{\sigma}_j \leftarrow \text{sign}(\bar{\sigma}_j) \min(\sigma_{max}, |\bar{\sigma}_j|).$$

The iterative loop is stopped when two consecutive $\bar{\sigma}$ are closer than a fixed relative tolerance ε or when the maximal number of iterations is reached. In the absence of knowledge of the exact solution σ , the quality of the converged solution is measured thanks to the following criteria

$$\text{err}_r = \frac{\|r - \bar{u}^{n_0}\|_2}{\|r\|_2} \quad \text{and} \quad \text{err}_m = \frac{\left\| m - \frac{\bar{u}_{N_x+1} - \bar{u}_{N_x}}{\Delta x} \right\|_2}{\|m\|_2}, \quad (5.8)$$

that should be of the order of the noise level on the observations. If the exact solution σ is known, we can also compute the relative error

$$\text{err}_\sigma = \frac{\|\sigma - \bar{\sigma}\|_2}{\|\sigma\|_2}.$$

Remark 5.1. *In order to avoid the inverse crime, we introduce a bias by taking different schemes for the direct and the inverse problems. Hence, we solve (5.1) associated to σ thanks to a linearized implicit scheme and we use an explicit scheme for the nonlinear term in equation (5.3) with $\bar{\sigma} = \sigma_k$.*

Numerical challenges

One of the main drawbacks of the method presented in Algorithm 1 is that we have to differentiate in time the observation m in (5.4) and to take the Laplacian of the observation r in (5.7). Thus, even a small perturbation (noise) on the observations may induce a large perturbation on its derivatives. In order to partially remedy this problem in the presence of noise, we first regularize the data (m, r) thanks to a 3-order low-pass Butterworth filter [5] associated to a cutoff frequency ω . We also replace the classical finite difference formulae in (5.4) and (5.7) that generate instabilities by a Savitzki-Golay formula [20] associated with a cubic polynomial and a window size of 5 points.

As already mentioned previously, a difficulty in our approach is the presence of the exponential weights in the functional that leads to severe numerical difficulties when performing the minimization for s large. In [2], this difficulty was solved by choosing a functional that only depended on the conjugate variable $e^{s\varphi}z$ and the corresponding conjugate operator. But this was possible because the considered operator was linear. Here, we managed to deal with this difficulty by introducing the new weight functions (2.4). In Figure 1, we plot $e^{s\varphi}$ in $(0, T) \times (0, L)$ for different s . Notice that even for s large, the function does not vanish at the observation time $T_0 = 0.5$ what allows a good reconstruction of the source term in the whole domain Ω .

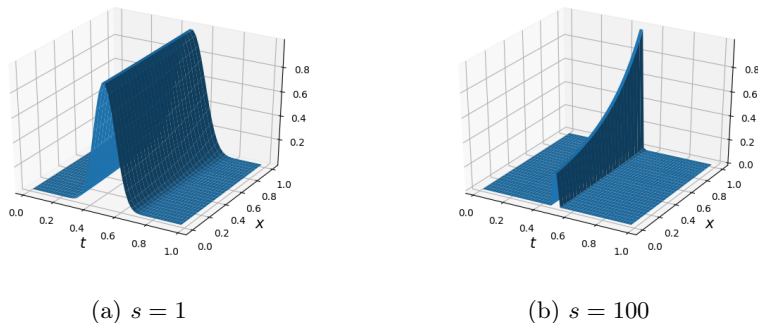


Figure 1: Carleman weight function $e^{s\varphi}$ defined in (2.4) for different values of s .

5.2 Numerical results

This subsection is devoted to the presentation of some numerical examples to illustrate the properties of the reconstruction algorithm and its efficiency. All simulations are executed with PYTHON. The source codes are available on request. Table 1 gathers the numerical values used for all the following examples, unless specified otherwise where appropriate. Moreover, we construct the function Φ introduced in

(3.10) in the form:

$$\Phi(X) = \begin{cases} 1, & \text{if } |X| \leq 1, \\ \frac{\int_1^{|X|} \exp\left(\frac{-1}{(x-1)(2-x)}\right) dx}{\int_1^2 \exp\left(\frac{-1}{(x-1)(2-x)}\right) dx}, & \text{if } 1 < |X| < 2, \\ 0, & \text{if } |X| \geq 2. \end{cases}$$

Figure 2 presents some examples of data generated by the direct problem. In all the figures presenting the numerical results, the exact source that we want to recover is plotted by a red line, whereas the numerical source recovered by the algorithm is represented by a dotted black line. The convergence informations (number of iterations, running time, convergence errors) are reported in Table 2.

L	T	N_x	N_t	g	u_o	σ_{max}
1	1	25	50	0	0	2
α	x_0	s	M	ρ	ε	ω
0	-0.3	100	10	10^{-3}	10^{-3}	0.15

Table 1: Numerical values for the variables.

Example	Number of iterations	Running time in seconds	err_m	err_r	err_σ
Figure 3 (a)	4	1	0.2%	0.2%	0.03%
Figure 4 (a)	3	117	0.1%	0.2%	0.02%
Figure 3 (b)	$N_{max} = 30$	6	> 100%	50%	> 100%
Figure 4 (b)	16	554	0.7%	0.1%	0.8%
Figure 6 (a)	3	87	1%	0.3%	2%
Figure 6 (b)	3	91	1%	0.3%	4%
Figure 6 (c)	3	97	3%	0.5%	9%
Figure 7 (b)	4	497	0.05%	0.1%	0.05%

Table 2: Convergence results of the test cases.

Simulations with the simplified algorithm

In Figure 3, we present the results obtained at each iteration of Algorithm 1 without the minimisation step for the case of the reconstruction of the source $\sigma(x) = \sin(\pi x)$. One can observe that the convergence depends on the choice for h . For this reason, this algorithm is not robust, even if it is sometimes very rapid and efficient (see Table 2).

Simulations from data without noise

In Figure 4, we present the successive results obtained at each iteration of Algorithm 1 in the case of the reconstruction of the source $\sigma(x) = \sin(\pi x)$ for two different choices of h . One can observe that

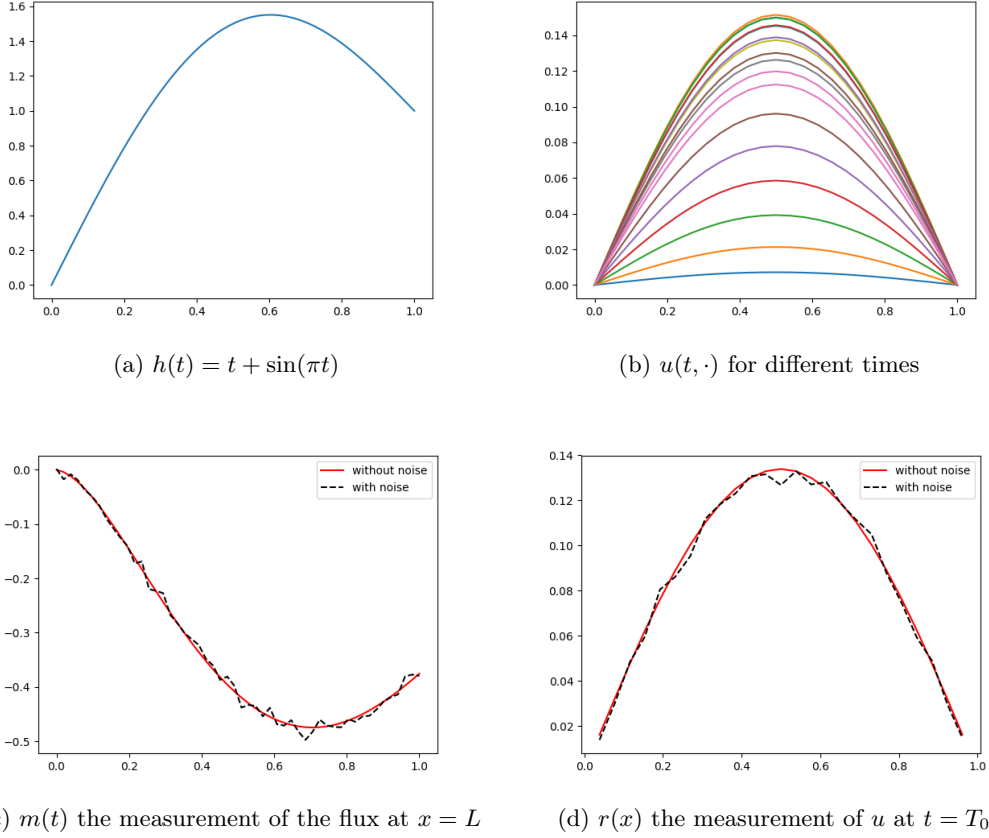


Figure 2: Examples of data used in the numerical examples for $\sigma(x) = \sin(\pi x)$ and $\alpha = 2\%$.

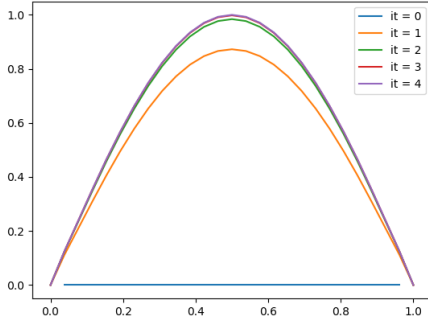
in both cases the convergence criteria (5.8) for ε is met in less than 20 iterations. In Figure 5, several results of reconstruction of sources obtained using Algorithm 1 in the absence of noise are given.

Simulations with several levels of noise

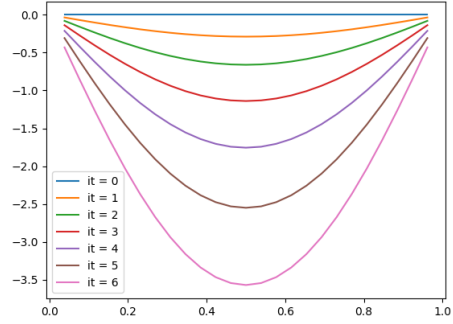
Figure 6 shows the results for $\sigma(x) = \sin(\pi x)$ with different levels of noise in the measurements ($\alpha = 1\%$, 2% and 5%). In Table 2, we report the corresponding errors on the reconstructed source. In fact, we observe that a noise of level α in the measurements gives rise to an error of order 2α in the recovered source.

Simulations in two dimensions

We also performed some reconstructions in two dimensions where $\Omega = (0, 1)^2$, $x_0 = (-0.3, -0.3)$ and $\Gamma = (\{0\} \times [0, 1]) \cup ([0, 1] \times \{0\})$. By this way, assumption (2.5) is satisfied. Figure 7 presents the results obtained for two different sources in the absence of additional noise. The gray scales are identical for the exact and the recovered graphics. The final error (reported in Table 2) is less than 0.1% what

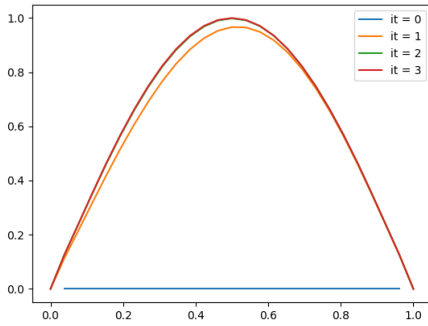


(a) $h(t) = t + \sin(\pi t) \rightarrow$ Convergence

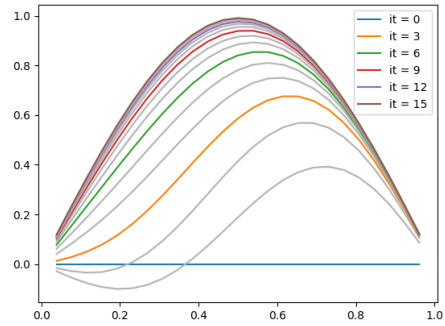


(b) $h(t) = t + \sin(7\pi t) \rightarrow$ Divergence

Figure 3: Reconstruction of $\sigma(x) = \sin(\pi x)$ without minimisation. We considered two different choices for h and represented the corresponding convergence/divergence history.



(a) $h(t) = t + \sin(\pi t) \rightarrow$ Convergence



(b) $h(t) = t + \sin(7\pi t) \rightarrow$ Convergence

Figure 4: Reconstruction of $\sigma(x) = \sin(\pi x)$ with minimisation. Different choices for h and the corresponding convergence history.

shows the effectiveness of the reconstruction obtained in a few minutes on a personal laptop.

Appendix

A Proof of the Carleman inequality given by Theorem 2.1

Proof. If we are in **Case 1** with the first choice of weight (2.1), this result is proved in an identical way as Lemma 1.2 in [12] which considers the case of intern measurements. Assume now that we are in **Case 2** where θ and ψ are given by (2.4).

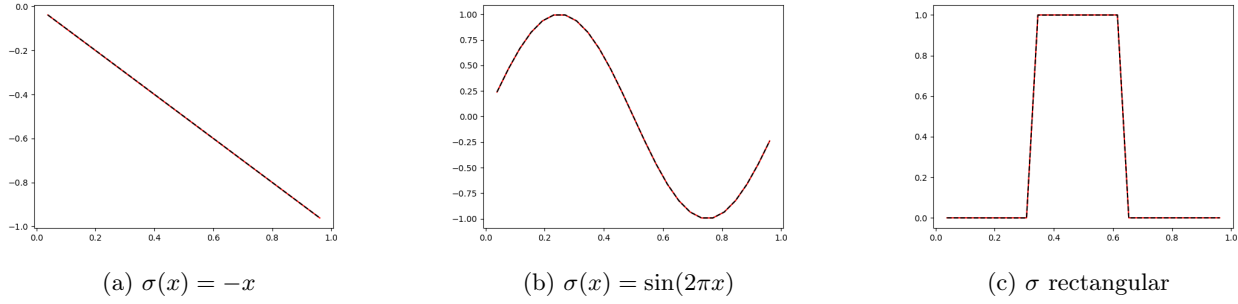


Figure 5: Different examples of reconstruction for $h(t) = t + \sin(\pi t)$.

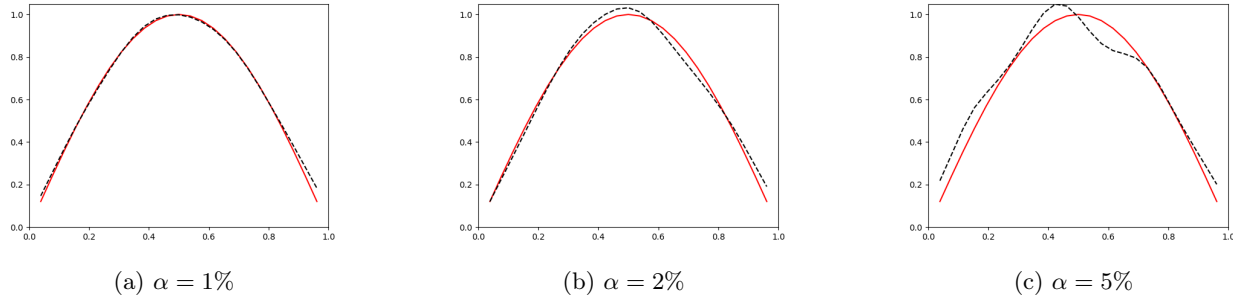


Figure 6: Reconstruction of the source $\sigma(x) = \sin(\pi x)$ for $h(t) = t + \sin(\pi t)$ in presence of noise in the data. The level of noise is denoted by α .

Let us give some properties on φ which will be useful in what follows:

$$\varphi(t, x) \leq \varphi(T_0, x), \forall (t, x) \in (0, T) \times \Omega, \quad \nabla^2 \varphi = 2\theta I_d, \quad (\text{A.1})$$

$$|\nabla \varphi| \leq C\theta, \quad |\partial_t \varphi| \leq C\theta^2, \quad |\partial_t \nabla \varphi| \leq C\theta^2, \quad |\partial_{tt} \varphi| \leq C\theta^3. \quad (\text{A.2})$$

In the proof, we assume that z belongs to $C^2([0, T] \times \bar{\Omega})$ and satisfies $z = 0$ on $(0, T) \times \partial\Omega$. A density argument allows to come back to the regularity hypotheses of the theorem.

For all $s > 0$, we set $w = e^{s\varphi} z$ and we introduce the conjugate operator Q defined by

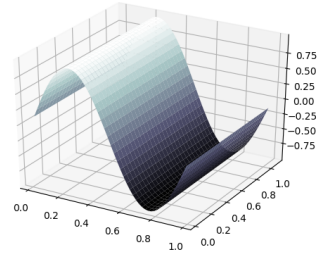
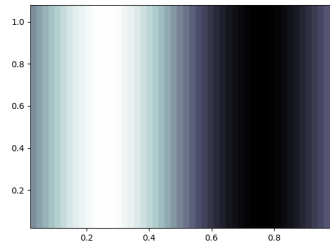
$$Qw = e^{s\varphi} (\partial_t - \Delta)(e^{-s\varphi} w). \quad (\text{A.3})$$

If we set $f = \partial_t z - \Delta z$, we have

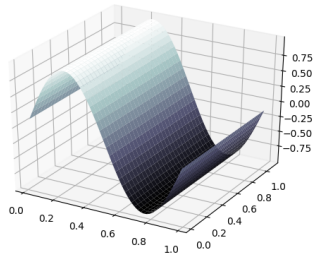
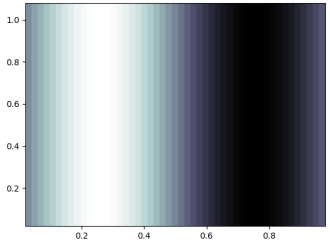
$$Qw = e^{s\varphi} f.$$

Some computations give

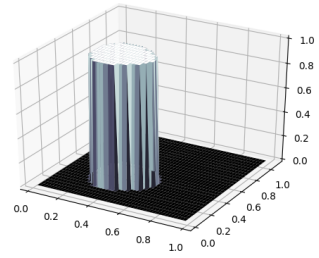
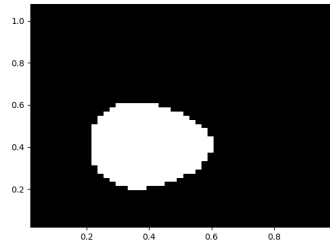
$$Qw = \partial_t w + 2s \nabla \varphi \cdot \nabla w + s \Delta \varphi w - \Delta w - (s^2 |\nabla \varphi|^2 + s \partial_{tt} \varphi) w = Q_+ w + Q_- w,$$



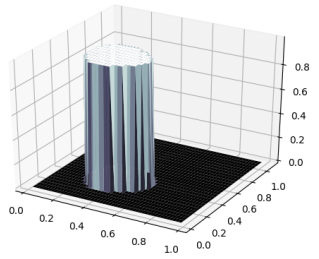
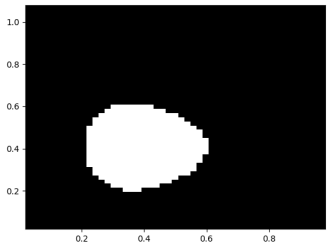
(a) Exact sources.



(b) Sources recovered numerically.



(c) Exact sources.



(d) Sources recovered numerically.

Figure 7: Different examples of reconstruction in the 2d case.

where the operators Q_+ and Q_- are defined by

$$Q_+w = -\Delta w - (s^2|\nabla\varphi|^2 + s\partial_t\varphi)w, \quad (\text{A.4})$$

$$Q_-w = \partial_t w + 2s\nabla\varphi \cdot \nabla w + s\Delta\varphi w. \quad (\text{A.5})$$

In a classical way, we write that

$$\int_0^T \int_{\Omega} e^{2s\varphi} |f|^2 dxdt = \int_0^T \int_{\Omega} |Q_+w|^2 dxdt + \int_0^T \int_{\Omega} |Q_-w|^2 dxdt + 2 \int_0^T \int_{\Omega} Q_+w Q_-w dxdt. \quad (\text{A.6})$$

The main part of the proof consists of bounding from below the terms in the right hand side by positive and dominant terms and a negative observation term located in $(0, T) \times \Gamma$. For the sake of clarity, we divide the proof in several steps.

- *Step 1* - Explicit calculation of the cross-term.

We set

$$\int_0^T \int_{\Omega} Q_+w Q_-w dxdt = \sum_{1 \leq i \leq 2, 1 \leq k \leq 3} I_{i,k},$$

where $I_{i,k}$ is the integral of the product of the i th-term in Q_+w and the k th-term in Q_-w . Integrations by parts in time give easily

$$\begin{aligned} I_{11} &= \int_0^T \int_{\Omega} (-\Delta w) \partial_t w dxdt = \int_0^T \int_{\Omega} \nabla w \cdot \nabla \partial_t w dxdt - \int_0^T \int_{\partial\Omega} \nabla w \cdot n \partial_t w d\gamma dt \\ &= \frac{1}{2} \left[\int_{\Omega} |\nabla w|^2 dx \right]_0^T - \int_0^T \int_{\partial\Omega} \nabla w \cdot n \partial_t w d\gamma dt = 0 \end{aligned}$$

since $w(0) = w(T) = 0$ in Ω and $w = 0$ on $(0, T) \times \partial\Omega$. An integration by parts in time gives for I_{21}

$$I_{21} = - \int_0^T \int_{\Omega} (s^2|\nabla\varphi|^2 + s\partial_t\varphi)w \partial_t w dxdt = \frac{1}{2} \int_0^T \int_{\Omega} \partial_t (s^2|\nabla\varphi|^2 + s\partial_t\varphi) |w|^2 dxdt.$$

We compute in the same way, by integrating by parts in space

$$\begin{aligned} I_{12} &= - \int_0^T \int_{\Omega} \Delta w (2s\nabla\varphi \cdot \nabla w) dxdt \\ &= 2s \int_0^T \int_{\Omega} \nabla w \cdot \nabla (\nabla\varphi \cdot \nabla w) dxdt - 2s \int_0^T \int_{\partial\Omega} \nabla w \cdot n (\nabla\varphi \cdot \nabla w) d\gamma dt \\ &= 2s \int_0^T \int_{\Omega} (\nabla^2\varphi) \nabla w \cdot \nabla w dxdt + 2s \int_0^T \int_{\Omega} (\nabla^2 w) \nabla w \cdot \nabla\varphi dxdt - 2s \int_0^T \int_{\partial\Omega} \nabla w \cdot n (\nabla\varphi \cdot \nabla w) d\gamma dt \\ &= 2s \int_0^T \int_{\Omega} (\nabla^2\varphi) \nabla w \cdot \nabla w dxdt - s \int_0^T \int_{\Omega} |\nabla w|^2 \Delta\varphi dxdt + s \int_0^T \int_{\partial\Omega} |\nabla w|^2 \nabla\varphi \cdot n d\gamma dt \\ &\quad - 2s \int_0^T \int_{\partial\Omega} \nabla w \cdot n (\nabla\varphi \cdot \nabla w) d\gamma dt \end{aligned}$$

and

$$I_{22} = - \int_0^T \int_{\Omega} (s^2 |\nabla \varphi|^2 + s \partial_t \varphi) w (2s \nabla \varphi \cdot \nabla w) dx dt = \int_0^T \int_{\Omega} \nabla \cdot [(s^3 |\nabla \varphi|^2 + s^2 \partial_t \varphi) \nabla \varphi] |w|^2 dx dt.$$

Since $\Delta \varphi$ is independent of x , by integration by parts in space, we have

$$I_{13} = - \int_0^T \int_{\Omega} \Delta w (s \Delta \varphi w) dx dt = s \int_0^T \int_{\Omega} |\nabla w|^2 \Delta \varphi dx dt.$$

At last, we have

$$I_{23} = - \int_0^T \int_{\Omega} (s^2 |\nabla \varphi|^2 + s \partial_t \varphi) w (s \Delta \varphi w) dx dt = - \int_0^T \int_{\Omega} (s^3 |\nabla \varphi|^2 + s^2 \partial_t \varphi) \Delta \varphi |w|^2 dx dt.$$

Gathering all these computations and using the second property in (A.1), we get

$$\begin{aligned} & \int_0^T \int_{\Omega} Q_{+w} Q_{-w} dx dt = 4s \int_0^T \int_{\Omega} \theta |\nabla w|^2 dx dt \\ & + \int_0^T \int_{\Omega} \left(\frac{1}{2} \partial_t (s^2 |\nabla \varphi|^2 + s \partial_t \varphi) + \nabla \cdot [(s^3 |\nabla \varphi|^2 + s^2 \partial_t \varphi) \nabla \varphi] - (s^3 |\nabla \varphi|^2 + s^2 \partial_t \varphi) \Delta \varphi \right) |w|^2 dx dt \\ & + s \int_0^T \int_{\partial \Omega} |\nabla w|^2 \nabla \varphi \cdot n d\gamma dt - 2s \int_0^T \int_{\partial \Omega} \nabla w \cdot n (\nabla \varphi \cdot \nabla w) d\gamma dt. \end{aligned} \tag{A.7}$$

For the second term in the right hand side, we notice that the main part in s^3 is given by

$$s^3 (\nabla \cdot (|\nabla \varphi|^2 \nabla \varphi) - |\nabla \varphi|^2 \Delta \varphi) = s^3 \nabla (|\nabla \varphi|^2) \cdot \nabla \varphi = 8s^3 \theta^3 |x - x_0|^2 \geq Cs^3 \theta^3.$$

For the boundary terms in (A.7), we notice that, since $z = 0$ on $(0, T) \times \partial \Omega$, $\nabla w = e^{s\varphi} \nabla z$. In particular, $\nabla w \cdot \tau = 0$ on $(0, T) \times \partial \Omega$. Thus, we get

$$s \int_0^T \int_{\partial \Omega} |\nabla w|^2 \nabla \varphi \cdot n d\gamma dt - 2s \int_0^T \int_{\partial \Omega} \nabla w \cdot n (\nabla \varphi \cdot \nabla w) d\gamma dt = -s \int_0^T \int_{\partial \Omega} |\nabla w \cdot n|^2 \nabla \varphi \cdot n d\gamma dt.$$

We divide this last integral as follows

$$-s \int_0^T \int_{\partial \Omega} |\nabla w \cdot n|^2 \nabla \varphi \cdot n d\gamma dt = -s \int_0^T \int_{\Gamma} |\nabla w \cdot n|^2 \nabla \varphi \cdot n d\gamma dt - s \int_0^T \int_{\partial \Omega \setminus \Gamma} |\nabla w \cdot n|^2 \nabla \varphi \cdot n d\gamma dt.$$

According to (2.5), the second integral is positive and the first integral corresponds to an observation integral.

Gathering these estimates, (A.7) becomes, for s large enough

$$\begin{aligned} \int_0^T \int_{\Omega} Q_{+w} Q_{-w} dx dt & \geq 4s \int_0^T \int_{\Omega} \theta |\nabla w|^2 dx dt + Cs^3 \int_0^T \int_{\Omega} \theta^3 |w|^2 dx dt \\ & - s \int_0^T \int_{\Gamma} |\nabla w \cdot n|^2 \nabla \varphi \cdot n d\gamma dt. \end{aligned} \tag{A.8}$$

- *Step 2* - Bounds on Δw and $\partial_t w$.

From the definition of Q_- (A.5), we have

$$\begin{aligned} \frac{1}{2} \int_0^T \int_{\Omega} \frac{1}{s\theta} |\partial_t w|^2 dxdt &\leq \int_0^T \int_{\Omega} \frac{1}{s\theta} |Q_- w|^2 dxdt + \int_0^T \int_{\Omega} \frac{1}{s\theta} |2s\nabla\varphi \cdot \nabla w + s\Delta\varphi w|^2 dxdt \\ &\leq \int_0^T \int_{\Omega} |Q_- w|^2 dxdt + C \left(\int_0^T \int_{\Omega} s\theta |\nabla w|^2 dxdt + \int_0^T \int_{\Omega} s\theta |w|^2 dxdt \right). \end{aligned}$$

In the same way,

$$\begin{aligned} \frac{1}{2} \int_0^T \int_{\Omega} \frac{1}{s\theta} |\Delta w|^2 dxdt &\leq \int_0^T \int_{\Omega} \frac{1}{s\theta} |Q_+ w|^2 dxdt + \int_0^T \int_{\Omega} \frac{1}{s\theta} (s^2 |\nabla\varphi|^2 + s\partial_t\varphi)^2 |w|^2 dxdt \\ &\leq \int_0^T \int_{\Omega} |Q_+ w|^2 dxdt + C \int_0^T \int_{\Omega} s^3 \theta^3 |w|^2 dxdt. \end{aligned}$$

Thus, coming back to (A.6) and, gathering (A.8) and these last two estimates, we get, for s large enough

$$\begin{aligned} \int_0^T \int_{\Omega} \left(\frac{1}{s\theta} |\partial_t w|^2 + \frac{1}{s\theta} |\Delta w|^2 + s\theta |\nabla w|^2 + s^3 \theta^3 |w|^2 \right) dxdt \\ \leq C \int_0^T \int_{\Omega} e^{2s\varphi} |f|^2 dxdt + Cs \int_0^T \int_{\Gamma} |\nabla w \cdot n|^2 \nabla\varphi \cdot n d\gamma dt. \end{aligned} \tag{A.9}$$

- *Step 3* - Back to the variable z .

Since $z = e^{-s\varphi} w$ and according to (A.2), we have, in $\Omega \times (0, T)$

$$\begin{aligned} |\partial_t z|^2 &\leq C e^{-2s\varphi} (|\partial_t w|^2 + s^2 \theta^4 |w|^2), \quad |\nabla z|^2 \leq C e^{-2s\varphi} (|\nabla w|^2 + s^2 \theta^2 |w|^2), \\ |\Delta z|^2 &\leq C e^{-2s\varphi} (|\Delta w|^2 + s^2 \theta^2 |\nabla w|^2 + s^4 \theta^4 |w|^2). \end{aligned}$$

Thus, (A.9) gives inequality (2.6) for s large enough. □

B Proof of the regularity result given by Proposition 2.5

Proof. We split the proof in several steps.

- *Step 1* - A lifting of the boundary condition of (1.1).

First, we will use a lifting for the boundary condition. Since $g \in H^1(0, T; H^{3/2}(\partial\Omega)) \cap H^2(0, T; H^{1/2}(\partial\Omega))$, from trace theorem, we deduce that there exists a function $\tilde{u} \in H^1(0, T; H^2(\Omega)) \cap H^2(0, T; H^1(\Omega))$ such that $\tilde{u} = g$ on $(0, T) \times \partial\Omega$ and

$$\|\tilde{u}\|_{H^1(0, T; H^2(\Omega))} \leq C \|g\|_{H^1(0, T; H^{3/2}(\partial\Omega))}, \quad \|\tilde{u}\|_{H^2(0, T; H^1(\Omega))} \leq C \|g\|_{H^2(0, T; H^{1/2}(\partial\Omega))}. \tag{B.1}$$

The function $\bar{u} = u - \tilde{u}$ satisfies

$$\begin{cases} \partial_t \bar{u} - \Delta \bar{u} + \bar{u}^3 + 3\tilde{u}\bar{u}^2 + 3\tilde{u}^2\bar{u} = F, & \text{in } (0, T) \times \Omega, \\ \bar{u} = 0, & \text{on } (0, T) \times \partial\Omega, \\ \bar{u}(0, \cdot) = u_\circ - \tilde{u}(0, \cdot), & \text{in } \Omega, \end{cases} \quad (\text{B.2})$$

with F defined by $F = \sigma h - \partial_t \tilde{u} + \Delta \tilde{u} - \tilde{u}^3$.

Multiplying the main equation of (B.2) by $\phi \in H_0^1(\Omega)$ and integrating by parts, we obtain

$$\begin{aligned} \int_{\Omega} \partial_t \bar{u}(t, x) \phi(x) \, dx + \int_{\Omega} \nabla \bar{u}(t, x) \cdot \nabla \phi(x) \, dx + \int_{\Omega} \bar{u}^3(t, x) \phi(x) \, dx + 3 \int_{\Omega} \tilde{u} \bar{u}^2(t, x) \phi(x) \, dx \\ + 3 \int_{\Omega} \tilde{u}^2 \bar{u}(t, x) \phi(x) \, dx = \int_{\Omega} F(t, x) \phi(x) \, dx, \end{aligned} \quad (\text{B.3})$$

a.e. $t \in (0, T)$.

- *Step 2* - Finite-dimensional approximated solutions.

At this step, we use the Faedo-Galerkin method and introduce a family of functions $\{\phi_m\}_{m \geq 1}$ in $H_0^1(\Omega)$ which is an orthogonal basis in $H_0^1(\Omega)$ and an orthonormal basis in $L^2(\Omega)$.

A positive integer m being fixed, we look for an approximated solution of (B.3) $\bar{u}_m : [0, T] \rightarrow H_0^1(\Omega)$ under the form

$$\bar{u}_m(t) = \sum_{i=1}^m \alpha_{im}(t) \phi_i, \quad (\text{B.4})$$

where the coefficients $(\alpha_{im})_{1 \leq i \leq m}$ being to be determined by the conditions:

$$\begin{aligned} \int_{\Omega} \partial_t \bar{u}_m(t, x) \phi_i(x) \, dx + \int_{\Omega} \nabla \bar{u}_m(t, x) \cdot \nabla \phi_i(x) \, dx + \int_{\Omega} \bar{u}_m^3(t, x) \phi_i(x) \, dx + 3 \int_{\Omega} \tilde{u} \bar{u}_m^2(t, x) \phi_i(x) \, dx \\ + 3 \int_{\Omega} \tilde{u}^2 \bar{u}_m(t, x) \phi_i(x) \, dx = \int_{\Omega} F(t, x) \phi_i(x) \, dx, \quad \forall i = 1, \dots, m \end{aligned} \quad (\text{B.5})$$

along with

$$\alpha_{im}(0) = \int_{\Omega} \bar{u}(0, x) \phi_i(x) \, dx, \quad \forall i = 1, \dots, m. \quad (\text{B.6})$$

From Picard-Lindelöf theorem (see, for example [21]), the system (B.5)-(B.6) of nonlinear ordinary differential equations, admits a unique local solution $(\alpha_{im})_{1 \leq i \leq m}$ in C^1 defined on a maximal interval $(0, T_m)$.

- *Step 3* - A priori estimates.

Multiplying the equation (B.5) by α_{im} , summing over i and integrating on $(0, t)$, we deduce that

$$\begin{aligned} \|\bar{u}_m\|_{C^0(0,t;L^2(\Omega))} + \|\bar{u}_m\|_{L^2(0,t;H^1(\Omega))} + \|\bar{u}_m\|_{L^4((0,t) \times \Omega)}^2 \\ \leq C \left(\|F\|_{L^2((0,T) \times \Omega)} + \|\tilde{u}\|_{L^4((0,T) \times \Omega)}^2 + \|\bar{u}(0, \cdot)\|_{L^2(\Omega)} \right). \end{aligned}$$

Thus, the coefficients $(\alpha_{im})_{1 \leq i \leq m}$ stay bounded in $C^0(0, T_m)$ and this ensures that they are defined on the global interval $(0, T)$.

- *Step 4* - Passage to the limit $m \rightarrow \infty$.

Now, we multiply the equation (B.5) by α'_{im} , sum over i and integrate on $(0, T)$. We get that

$$\begin{aligned} & \|\bar{u}_m\|_{H^1(0, T; L^2(\Omega))} + \|\bar{u}_m\|_{L^\infty(0, T; H^1(\Omega))} \\ & \leq C \left(1 + \|\tilde{u}\|_{L^\infty((0, T) \times \Omega)}^2\right) \left(\|F\|_{L^2((0, T) \times \Omega)} + \|\tilde{u}\|_{L^4((0, T) \times \Omega)}^2 + \|\bar{u}(0, \cdot)\|_{H^1(\Omega)}\right). \end{aligned}$$

Thus we deduce that, up to a subsequence, $(\bar{u}_m)_m$ weakly converges in $H^1(0, T; L^2(\Omega)) \cap L^\infty(0, T; H_0^1(\Omega))$ and strongly converges in $L^2((0, T) \times \Omega)$. This convergence properties allow to deduce that the limit \bar{u} satisfies the weak formulation (B.3) and the estimate

$$\begin{aligned} & \|\bar{u}\|_{H^1(0, T; L^2(\Omega))} + \|\bar{u}\|_{L^\infty(0, T; H^1(\Omega))} + \|\bar{u}\|_{L^4((0, T) \times \Omega)}^2 \\ & \leq C \left(1 + \|\tilde{u}\|_{L^\infty((0, T) \times \Omega)}^2\right) \left(\|F\|_{L^2((0, T) \times \Omega)} + \|\tilde{u}\|_{L^4((0, T) \times \Omega)}^2 + \|\bar{u}(0, \cdot)\|_{H^1(\Omega)}\right). \end{aligned} \tag{B.7}$$

- *Step 5* - Higher regularity.

Looking at (B.2) as an elliptic problem by putting $\partial_t \bar{u}$ in the right hand side, the elliptic regularity implies that \bar{u} belongs to $L^2(0, T; H^2(\Omega))$ and

$$\begin{aligned} \|\bar{u}\|_{L^2(0, T; H^2(\Omega))} & \leq C \left(1 + \|\tilde{u}\|_{L^\infty((0, T) \times \Omega)}^2\right)^2 \left(\|F\|_{L^2((0, T) \times \Omega)} + \|\tilde{u}\|_{L^4((0, T) \times \Omega)}^2 + \|\bar{u}(0, \cdot)\|_{H^1(\Omega)}\right) \\ & \quad + C \left(1 + \|\tilde{u}\|_{L^\infty((0, T) \times \Omega)}^2\right)^3 \left(\|F\|_{L^2((0, T) \times \Omega)} + \|\tilde{u}\|_{L^4((0, T) \times \Omega)}^2 + \|\bar{u}(0, \cdot)\|_{H^1(\Omega)}\right)^3. \end{aligned}$$

Moreover, since \bar{u} belongs to $H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^2(\Omega))$, according to [9, Section 5.9, Theorem 4], we deduce that \bar{u} belongs to $C^0(0, T; H^1(\Omega))$.

- *Step 6* - Return to the variable u .

Coming back to $u = \bar{u} + \tilde{u}$ and using (B.1), we conclude that

$$u \in L^2(0, T; H^2(\Omega)) \cap C^0(0, T; H^1(\Omega)) \cap H^1(0, T; L^2(\Omega))$$

with the following estimate

$$\begin{aligned} & \|u\|_{L^2(0, T; H^2(\Omega))} + \|u\|_{C^0(0, T; H^1(\Omega))} + \|u\|_{H^1(0, T; L^2(\Omega))} \leq C \left(\|\sigma h\|_{L^2(0, T; L^2(\Omega))} + \|\sigma h\|_{L^2(0, T; L^2(\Omega))}^p \right. \\ & \quad \left. + \|g\|_{H^1(0, T; H^{3/2}(\partial\Omega))} + \|g\|_{H^1(0, T; H^{3/2}(\partial\Omega))}^p + \|u_\circ\|_{H^1(\Omega)} + \|u_\circ\|_{H^1(\Omega)}^p \right), \end{aligned} \tag{B.8}$$

where the power p is a positive integer that can change from line to line.

- *Step 7 - Improved regularity.*

Next, let us consider $w = \partial_t u$ which is, according to (1.1) and (3.1), formally the solution of

$$\begin{cases} \partial_t w - \Delta w + 3u^2 w = \sigma \partial_t h, & \text{in } (0, T) \times \Omega, \\ w = \partial_t g, & \text{on } (0, T) \times \partial\Omega, \\ w(0, \cdot) = \Delta u_\circ - (u_\circ)^3, & \text{in } \Omega. \end{cases} \quad (\text{B.9})$$

We use the same lifting as in *Step 1* and define the function $\bar{w} = w - \partial_t \tilde{u}$, which satisfies

$$\begin{cases} \partial_t \bar{w} - \Delta \bar{w} + 3u^2 \bar{w} = G, & \text{in } (0, T) \times \Omega, \\ \bar{w} = 0, & \text{on } (0, T) \times \partial\Omega, \\ \bar{w}(0, \cdot) = w(0, \cdot) - \partial_t \tilde{u}(0, \cdot), & \text{in } \Omega, \end{cases} \quad (\text{B.10})$$

with G defined by $G = \sigma \partial_t h - \partial_{tt} \tilde{u} + \Delta \partial_t \tilde{u} - 3u^2 \partial_t \tilde{u}$. For this system, we have a unique solution

$$\bar{w} \in C^0(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$$

which satisfies

$$\|\bar{w}\|_{C^0(0, T; L^2(\Omega))} + \|\bar{w}\|_{L^2(0, T; H^1(\Omega))} \leq C \left(\|G\|_{L^2((0, T) \times \Omega)} + \|\bar{w}(0, \cdot)\|_{L^2(\Omega)} \right).$$

For the first term in the right hand side, we have

$$\begin{aligned} \|G\|_{L^2((0, T) \times \Omega)} &\leq \|\sigma h\|_{H^1(0, T; L^2(\Omega))} + \|\tilde{u}\|_{H^2(0, T; L^2(\Omega)) \cap H^1(0, T; H^2(\Omega))} \\ &\quad + C \left(\|u\|_{C^0(0, T; H^1(\Omega))}^2 + \|\tilde{u}\|_{H^1(0, T; L^2(\Omega))} \right). \end{aligned}$$

Taking account (B.1) and (B.8), and going back to $w = \bar{w} + \partial_t \tilde{u}$, we conclude that

$$u \in C^1(0, T; L^2(\Omega)) \cap H^1(0, T; H^1(\Omega)),$$

with the following estimate

$$\begin{aligned} \|u\|_{C^1(0, T; L^2(\Omega))} + \|u\|_{H^1(0, T; H^1(\Omega))} &\leq C \left(\|\sigma h\|_{H^1(0, T; L^2(\Omega))} + \|\sigma h\|_{H^1(0, T; L^2(\Omega))}^p \right. \\ &\quad \left. + \|g\|_{H^1(0, T; H^{3/2}(\partial\Omega)) \cap H^2(0, T; L^2(\partial\Omega))} + \|g\|_{H^1(0, T; H^{3/2}(\partial\Omega)) \cap H^2(0, T; L^2(\partial\Omega))}^p + \|u_\circ\|_{H^2(\Omega)} + \|u_\circ\|_{H^2(\Omega)}^p \right). \end{aligned}$$

Thus, if look at (1.1) as an elliptic problem, we get that $u \in C^0(0, T; H^2(\Omega))$ and we have the estimate

$$\begin{aligned} &\|u\|_{C^0(0, T; H^2(\Omega))} + \|u\|_{C^1(0, T; L^2(\Omega))} + \|u\|_{H^1(0, T; H^1(\Omega))} \\ &\leq C \left(\|\sigma h\|_{H^1(0, T; L^2(\Omega))} + \|\sigma h\|_{H^1(0, T; L^2(\Omega))}^p + \|u_\circ\|_{H^2(\Omega)} + \|u_\circ\|_{H^2(\Omega)}^p \right. \\ &\quad \left. + \|g\|_{H^1(0, T; H^{3/2}(\partial\Omega)) \cap H^2(0, T; L^2(\partial\Omega))} + \|g\|_{H^1(0, T; H^{3/2}(\partial\Omega)) \cap H^2(0, T; L^2(\partial\Omega))}^p \right). \end{aligned} \quad (\text{B.11})$$

Let us note that, since $u_\circ \in H^3(\Omega)$, the initial condition

$$\bar{w}(0, \cdot) = \Delta u_\circ - (u_\circ)^3 - \partial_t \tilde{u}(0, \cdot)$$

belongs to $H^1(\Omega)$. Then, if we multiply the equation (B.10) by $\partial_t \bar{w}$ and integrate in $(0, T) \times \Omega$, we obtain that $\partial_t \bar{w} \in L^2(0, T; L^2(\Omega))$ with

$$\|\partial_t \bar{w}\|_{L^2(0, T; L^2(\Omega))} \leq C \left(\|G\|_{L^2((0, T) \times \Omega)} + \|\bar{w}(0)\|_{H^1(\Omega)} \right).$$

Hence, if we look at (B.10) as an elliptic problem, we deduce that \bar{w} belongs to $L^2(0, T; H^2(\Omega))$ with the following estimate

$$\|\bar{w}\|_{L^2(0, T; H^2(\Omega))} \leq C \left(\|G\|_{L^2((0, T) \times \Omega)} + \|u\|_{C^0(0, T; H^2(\Omega))}^2 \|\bar{w}\|_{L^2((0, T) \times \Omega)} + \|\bar{w}(0)\|_{H^1(\Omega)} \right).$$

Besides, since \bar{w} belongs to $H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^2(\Omega))$, we deduce that \bar{w} belongs to $C^0(0, T; H^1(\Omega))$.

Coming back to $\partial_t u = \bar{w} + \partial_t \tilde{u}$, we finally deduce that u belongs to $H^1(0, T; H^2(\Omega)) \cap C^1(0, T; H^1(\Omega)) \cap H^2(0, T; L^2(\Omega))$ along with the estimate (2.8).

□

References

- [1] L. Baudouin, M. de Buhan, and S. Ervedoza. Global Carleman estimates for waves and applications. *Comm. Partial Differential Equations*, 38(5):823–859, 2013.
- [2] L. Baudouin, M. de Buhan, and S. Ervedoza. Convergent algorithm based on Carleman estimates for the recovery of a potential in the wave equation. *SIAM J. Numer. Anal.*, 55(4):1578–1613, 2017.
- [3] M. Boulakia, C. Grandmont, and A. Osses. Some inverse stability results for the bistable reaction-diffusion equation using Carleman inequalities. *C. R. Math. Acad. Sci. Paris*, 347(11-12):619–622, 2009.
- [4] A. L. Bukhgeim and M. V. Klibanov. Uniqueness in the large of a class of multidimensional inverse problems. *Dokl. Akad. Nauk SSSR*, 260(2):269–272, 1981.
- [5] S. Butterworth. On the theory of filter amplifiers. *Wireless Engineer*, vol. 7, p. 536-541, 1930.
- [6] N. Cîndea, E. Fernández-Cara, and A. Münch. Numerical controllability of the wave equation through primal methods and Carleman estimates. *ESAIM Control Optim. Calc. Var.*, 19(4):1076–1108, 2013.
- [7] Piero Colli Franzone, Luca F. Pavarino, and Simone Scacchi. *Mathematical cardiac electrophysiology*, volume 13 of *MS&A. Modeling, Simulation and Applications*. Springer, Cham, 2014.
- [8] H. Egger, H. W. Engl, and M. V. Klibanov. Global uniqueness and Hölder stability for recovering a nonlinear source term in a parabolic equation. *Inverse Problems*, 21(1):271–290, 2005.
- [9] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010.

- [10] E. Fernández-Cara, M. González-Burgos, S. Guerrero, and J.-P. Puel. Exact controllability to the trajectories of the heat equation with Fourier boundary conditions: the semilinear case. *ESAIM Control Optim. Calc. Var.*, 12(3):466–483, 2006.
- [11] E. Fernández-Cara, A. Münch, and D. A. Souza. On the numerical controllability of the two-dimensional heat, Stokes and Navier-Stokes equations. *J. Sci. Comput.*, 70(2):819–858, 2017.
- [12] A. V. Fursikov and O. Yu. Imanuvilov. *Controllability of evolution equations*, volume 34 of *Lecture Notes Series*. Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1996.
- [13] Oleg Yu. Imanuvilov and Masahiro Yamamoto. Lipschitz stability in inverse parabolic problems by the Carleman estimate. *Inverse Problems*, 14(5):1229–1245, 1998.
- [14] K. Ito and B. Jin. *Inverse problems*, volume 22 of *Series on Applied Mathematics*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2015. Tikhonov theory and algorithms.
- [15] M. V. Klibanov. Global uniqueness of a multidimensional inverse problem for a nonlinear parabolic equation by a carleman estimate. *Inverse Problems*, 20(4):1003–1032, 2004.
- [16] M. V. Klibanov. Carleman estimates for global uniqueness, stability and numerical methods for coefficient inverse problems. *J. Inverse Ill-Posed Probl.*, 21(4):477–560, 2013.
- [17] M. V. Klibanov and V. G. Kamburg. Globally strictly convex cost functional for an inverse parabolic problem. *Math. Methods Appl. Sci.*, 39(4):930–940, 2016.
- [18] D. A. Knoll and D. E. Keyes. Jacobian-free newton-krylov methods: a survey of approaches and applications. *Journal of Computational Physics*. 193 (2): 357, 2004.
- [19] J. L. Lions and E. Magenes. *Non-homogeneous boundary value problems and applications*, volume 1. Springer Science & Business Media, 2012.
- [20] A. Savitzky and M. Golay. Smoothing and differentiation of data by simplified least squares procedures. *Analytical Chemistry*, vol. 8, no 36, p. 1627-1639, 1964.
- [21] K. Schmitt and R. Thompson. Nonlinear analysis and differential equations: An introduction. *Lecture Notes, University of Utah, Department of Mathematics*, 1998.
- [22] M. Yamamoto. Carleman estimates for parabolic equations and applications. *Inverse Problems*, 2009.