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THE SMALL QUANTUM COHOMOLOGY OF THE CAYLEY GRASSMANNIAN

VLADIMIRO BENEDETTI AND LAURENT MANIVEL

ABSTRACT. We compute the small cohomology ring of the Cayley Grassmannian, that parametrizes four-dimensional subalgebras of the complexified octonions. We show that all the Gromov-Witten invariants in the multiplication table of the Schubert classes are non negative and deduce Golyshev's conjecture \mathcal{O} holds true for this variety. We also check that the quantum cohomology is semisimple and that there exists, as predicted by Dubrovin's conjecture, an exceptional collection of maximal length in the derived category.

1. THE SETUP

The Cayley Grassmannian CG is a closed subvariety of the complex Grassmannian $G(3, 7) \simeq G(4, 7)$, which can be described as follows:

- (1) either as the subvariety of $G(3, 7)$ parametrizing (the imaginary parts of) the four-dimensional subalgebras of the complexified octonions,
- (2) or as the zero-locus of a general section of the vector bundle $\wedge^3 T^*$, where T denotes the tautological bundle on $G(4, 7)$.

The equivalence of these two descriptions comes from the facts that a global section of $\wedge^3 T^*$ is a skew-symmetric three-form in seven variables, and that the stabilizer of a general such form is a copy of G_2 , the automorphism group of the octonions.

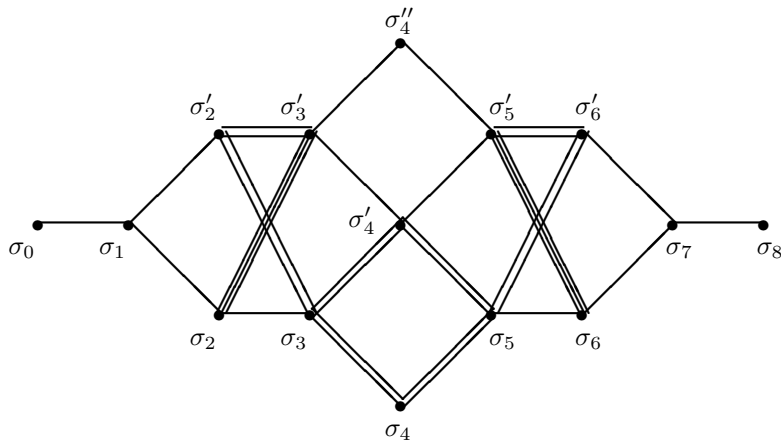
The reader will find in [Ma] more details on the geometry of the Cayley Grassmannian; let us just recall that it is a Fano eightfold of Picard number one and index four. Its equivariant cohomology ring (with respect to a maximal torus in G_2) has been computed in [Ma, 4.2] with the help of the classical localization techniques. These computations imply that the ordinary cohomology ring is generated (over the rational numbers) by the hyperplane class σ_1 and a codimension two class σ_2 . The even Betti numbers are 1, 1, 2, 2, 3, 2, 2, 1, 1 and there is an integral basis of *Schubert classes* for which we keep the notations of [Ma]. The so called Chevalley formula for the product by the hyperplane class is encoded in the graph below, where the number of edges between two classes is the coefficient of the larger degree class in the hyperplane product of the other one. Moreover the rational cohomology ring is defined by two relations in degree five and six:

Proposition 1.1. *The rational cohomology ring of CG is*

$$H^*(CG, \mathbb{Q}) = \mathbb{Q}[\sigma_1, \sigma_2] / \langle \sigma_1^5 - 5\sigma_1^3\sigma_2 + 6\sigma_1\sigma_2^2, 16\sigma_2^3 - 27\sigma_1^2\sigma_2^2 + 9\sigma_1^4\sigma_2 \rangle.$$

We will denote these two relations by R_5 and R_6 . The full multiplication table is given in [Ma, 4.3].

Beware! There is a typo in this table, a coefficient 3 in $\sigma_5'\sigma_2 = 3\sigma_7$ is missing.

The Bruhat graph of CG

2. HOW TO COMPUTE THE QUANTUM COHOMOLOGY RING

2.1. Deforming the cohomology ring. The small quantum cohomology ring of the Cayley Grassmannian, which we intend to determine, is a deformation of its ordinary cohomology ring. The quantum parameter q has degree four, the Fano index of CG . The quantum products of two Schubert classes are of the form

$$\sigma\sigma' = \sigma \cup \sigma' + \sum_{d>0} q^d \sum_{\tau} I_d(\sigma, \sigma', \tau) \tau^{\vee},$$

where the sum is over the Schubert classes τ whose Poincaré dual class has degree $\deg(\tau^{\vee}) = \deg(\sigma) + \deg(\sigma') - 4d$ (recall from [Ma] that the basis of Schubert classes is self-dual, up to order). Moreover $I_d(\sigma, \sigma', \tau)$ is the three-points degree d Gromov-Witten invariant associated to the three Schubert classes σ, σ', τ . Note that, since CG has dimension eight, d does never exceed four.

A first useful observation is that, by the results of [TS], the small quantum cohomology ring has a presentation of the form

$$QH^*(CG, \mathbb{Q}) = \mathbb{Q}[\sigma_1, \sigma_2, q] / \langle R_5(q), R_6(q) \rangle,$$

where the relations $R_5(q)$ and $R_6(q)$ are deformations of R_5 and R_6 : in fact we can just consider the latter relations, and evaluate them in the quantum cohomology ring rather than in ordinary cohomology. For degree reasons we will obtain deformed relations of the form

$$R_5(q) = R_5 + qQ_1, \quad R_6(q) = R_6 + qQ_2.$$

Since there is no term with degree in q bigger than one, these relations are completely determined by the Gromov-Witten invariants of degree one. In fact, as we are going to see, this turns out to be true for the full quantum product.

2.2. Degree one Gromov-Witten invariants are enough. In order to determine the full quantum products, the natural strategy would be to:

- (1) find the quantum relations $R_5(q), R_6(q)$;
- (2) express the Schubert classes as polynomials in σ_1, σ_2 and q : these are the quantum Giambelli formulas;

- (3) compute the products of the Schubert classes by σ_1 and σ_2 : these are the quantum Pieri classes.

The quantum products by σ_1 are given by the quantum Chevalley formulas. Because of the symmetries of the Gromov-Witten invariants, these products must be of the following form:

$$\begin{aligned}
 \sigma_1\sigma_1 &= \sigma_2 + \sigma_2' \\
 \sigma_2\sigma_1 &= \sigma_3 + 3\sigma_3' \\
 \sigma_2'\sigma_1 &= 2\sigma_3 + 2\sigma_3' \\
 \sigma_3\sigma_1 &= 2\sigma_4 + 2\sigma_4' + a_3q \\
 \sigma_3'\sigma_1 &= \sigma_4' + \sigma_4'' + a_3'q \\
 \sigma_4\sigma_1 &= 2\sigma_5 + a_4q\sigma_1 \\
 \sigma_4'\sigma_1 &= 2\sigma_5 + \sigma_5' + a_4'q\sigma_1 \\
 \sigma_4''\sigma_1 &= \sigma_5' + a_4''q\sigma_1 \\
 \sigma_5\sigma_1 &= \sigma_6 + 2\sigma_6' + a_5q\sigma_2 + b_5q\sigma_2' \\
 \sigma_5'\sigma_1 &= 3\sigma_6 + 2\sigma_6' + a_5'q\sigma_2 + b_5'q\sigma_2' \\
 \sigma_6\sigma_1 &= \sigma_7 + a_5q\sigma_3 + a_5'q\sigma_3' \\
 \sigma_6'\sigma_1 &= \sigma_7 + b_5q\sigma_3 + b_5'q\sigma_3' \\
 \sigma_7\sigma_1 &= \sigma_8 + a_4q\sigma_4 + a_4'q\sigma_4' + a_4''q\sigma_4'' + a_7q^2 \\
 \sigma_8\sigma_1 &= a_3q\sigma_5 + a_3'q\sigma_5' + a_7q^2\sigma_1.
 \end{aligned}$$

There are ten unknowns to compute; all of them are degree one Gromov-Witten invariants, except a_7 . Suppose we have computed them; then we can almost deduce the quantum Giambelli formulas, because most Schubert classes belong to the image of the multiplication map by σ_1 . To be precise, in ordinary cohomology the image of this multiplication map has codimension two; in order to generate the full cohomology ring, we just need to add for example σ_2 and σ_4 , or σ_2 and σ_2^2 . In particular, from quantum Chevalley and the quantum σ_2^2 , we will be able to deduce quantum Giambelli. Note that the computation of σ_2^2 only involves a degree one Gromov-Witten invariant.

Once we have quantum Giambelli, we can compute inductively the products by σ_2 , just using the fact that if a class σ satisfies $\sigma = \tau\sigma_1$ for some other class τ , then $\sigma\sigma_2$ can obviously be deduced from $\tau\sigma_2$ and quantum Chevalley. The missing ingredient is a formula for $\sigma_4\sigma_2$, or equivalently σ_2^3 . Again the computation of the quantum product $\sigma_4\sigma_2$ only involves degree one Gromov-Witten invariants.

Finally, the only invariant we used previously which is not of degree one is a_7 . From the quantum Giambelli formula in degree at most seven and the expression of $\sigma_7\sigma_1$ above, we can at this point express σ_8 in terms of $\sigma_1, \sigma_2, q, a_7$. But then we can deduce $\sigma_8\sigma_1$ as a linear combination of Schubert classes, and comparing with the formula above, this yields a non trivial equation in a_7 . So a_7 is also determined by the other coefficients. (In fact we will check that $a_7 = 0$.) We have proved:

Lemma 2.1. *The quantum cohomology ring $QH^*(CG, \mathbb{Q})$ is determined by:*

- (1) *the quantum Chevalley formula, up to degree seven;*
- (2) *the quantum products σ_2^2 and $\sigma_4\sigma_2$.*

In particular it is completely determined by degree one Gromov-Witten invariants.

2.3. Enumerativity. Homogeneous varieties have the nice property that their Gromov-Witten invariants are enumerative: they can be effectively computed as numbers of rational curves touching suitable collections of Schubert varieties in general position. This is definitely no longer the case for non homogeneous varieties,

where certain Gromov-Witten invariants can be negative (and even certain classical intersection numbers). Fortunately, enumerativity can be preserved in certain specific situations. We will mainly use the following result from [GPPS] (Theorem 3.3):

Proposition 2.2. *Let X be a smooth projective variety, with an action of a reductive group G that has only finitely many orbits. Let $\gamma_1, \dots, \gamma_k$ be cohomology classes of degree bigger than one, represented by subvarieties Y_1, \dots, Y_k of X that are transverse to the orbits.*

Suppose moreover that the moduli space $M_{d,k}(X)$ of stable maps of genus zero and degree d with k marked points is irreducible, and such that the general such curve intersects the open orbit of X .

Then the Gromov-Witten invariant $I_d(\gamma_1, \dots, \gamma_k)$ can be computed as the number of stable curves of degree d that intersect general G -translates of Y_1, \dots, Y_k .

In the sequel we will only apply this statement to $d = 1$ and $k = 2$ or $k = 3$. Of course the Schubert varieties themselves are in general not transverse to the orbits of CG , so we need to be careful. We will show in the next section that general lines and general planes through a general point of CG satisfy this hypothesis.

Alternatively, we can use Schubert varieties on the ambient Grassmannian $G(4, 7)$, since the homogeneity of the latter allows to put their general PGL_7 -translates in general position with any finite collection of subvarieties, in particular with the orbits in CG (and CG itself). This means that the intersections with CG of general Schubert varieties in $G(4, 7)$ will have the required properties.

The cohomology classes of these intersections are given by the restrictions of Schubert classes, which were computed in [Ma, Proposition 4.7]. For future use we recall the results. Let us fix a complete flag $0 = V_0 \subset V_1 \subset \dots \subset V_7 \cong \mathbb{C}^7$. For $\lambda = (\lambda_1 \geq \dots \geq \lambda_4)$ an integer sequence with $\lambda_1 \leq 3$ and $\lambda_4 \geq 0$, we have the usual Schubert variety

$$X_\lambda = \{W \in G(4, 7) \text{ s.t. } \dim(W \cap V_{3-\lambda_j+j}) \geq j \text{ for } 1 \leq j \leq 4\},$$

of codimension $\sum_i \lambda_i$ inside $G(4, 7)$. We denote its cohomology class by τ_λ . In particular τ_1 is the hyperplane class, whose restriction to CG is just σ_1 . The remaining pull-backs, up to degree six, are as follows.

Proposition 2.3. *Let $\iota : CG \hookrightarrow G(4, 7)$ be the natural embedding. Then:*

$$\begin{aligned} \iota^* \tau_2 &= \sigma_2, & \iota^* \tau_{11} &= \sigma'_2, \\ \iota^* \tau_3 &= \sigma'_3, & \iota^* \tau_{111} &= \sigma_3, \\ \iota^* \tau_{1111} &= \sigma_4, & \iota^* \tau_{211} &= \sigma_4 + 2\sigma'_4, \\ \iota^* \tau_{22} &= \sigma_4 + \sigma'_4 + \sigma''_4, & \iota^* \tau_{31} &= \sigma'_4 + \sigma''_4, \\ \iota^* \tau_{2111} &= 2\sigma_5, & \iota^* \tau_{221} &= 3\sigma_5 + \sigma'_5, & \iota^* \tau_{311} &= \iota^* \tau_{32} = \sigma_5 + \sigma'_5, \\ \iota^* \tau_{2211} &= \sigma_6 + 3\sigma'_6, & \iota^* \tau_{222} &= 2\sigma_6 + 2\sigma'_6, \\ \iota^* \tau_{321} &= 3\sigma_6 + 3\sigma'_6, & \iota^* \tau_{33} &= \iota^* \tau_{3111} = \sigma_6 + \sigma'_6. \end{aligned}$$

Beware! In [Ma, Proposition 4.7], $\iota^* \tau_2$ and $\iota^* \tau_{11}$ have been interchanged.

3. A BIT OF GEOMETRY

In this section we study lines and planes on CG . We denote by Ω a general three-form on the seven dimensional vector space V_7 . One can then find a basis of

V_7 (a basis of eigenvectors for a maximal torus of the copy of G_2 that stabilizes Ω) for which Ω can be written as

$$\Omega = v_0 \wedge v_\alpha \wedge v_{-\alpha} + v_0 \wedge v_\beta \wedge v_{-\beta} + v_0 \wedge v_\gamma \wedge v_{-\gamma} + v_\alpha \wedge v_\beta \wedge v_\gamma + v_{-\alpha} \wedge v_{-\beta} \wedge v_{-\gamma}.$$

Moreover there is a G_2 -invariant quadratic form on V_7 , that one can write as

$$q = v_0^2 + v_\alpha v_{-\alpha} + v_\beta v_{-\beta} + v_\gamma v_{-\gamma}.$$

See [Ma, section 2.1] for more details.

3.1. Lines in the Cayley Grassmannian. Recall that the variety $F_1(G)$ of lines in the Grassmannian G is the flag variety $F(3, 5, 7)$ parametrizing flags of subspaces $V_3 \subset V_5 \subset V_7$. Its two projections p_3 and p_5 onto $G(3, 7)$ and $G(5, 7)$ are locally trivial, with Grassmannians $G(2, 4)$ and $G(3, 5)$ as respective fibers.

Proposition 3.1. *The variety $F_1(CG)$ of lines in CG is a smooth subvariety of $F_1(G)$, of the expected dimension 9.*

Proof. Let us denote by T_3 and T_5 the tautological bundles of rank 3 and 5 on $F_1(G)$. By restriction, the vector bundle $\wedge^2 T_3^* \otimes T_5^*$ maps to $\wedge^2 T_3^* \otimes T_3^*$. Let E denote the pre-image of $\wedge^3 T_3^* \subset \wedge^2 T_3^* \otimes T_3^*$. There is an exact sequence

$$0 \rightarrow \wedge^2 T_3^* \otimes (T_5/T_3)^* \rightarrow E \rightarrow \wedge^3 T_3^* \rightarrow 0.$$

By the Borel-Weil theorem $H^0(F_1(G), \wedge^2 T_3^* \otimes (T_5/T_3)^*) = 0$ and $H^0(F_1(G), \wedge^3 T_3^*) = \wedge^3 V_7^*$. Therefore $H^0(F_1(G), E) = \wedge^3 V_7^*$. Our three-form Ω thus defines a general section of E , whose zero-locus is exactly $F_1(CG)$. The statement follows from generic smoothness since E is globally generated. Indeed, let us choose a basis e_1, \dots, e_7 of V_7 such that e_1, \dots, e_3 is a basis of T_3 and e_1, \dots, e_5 a basis of T_5 . By choosing arbitrarily the coefficients Ψ_{123} and Ψ_{ijk} for $i, j \leq 3 < k \leq 5$ of a three-form Ψ , which we can do freely, we generate the whole fiber of E . \square

Now let us consider the set of lines in the Cayley Grassmannian passing through a given point $p \in CG$, corresponding to $A_4 \subset V_7$. Such a line is given by a pair (V_3, V_5) such that $V_3 \subset A_4 \subset V_5$ and $\Omega(V_3, V_3, V_5/A_4) = 0$ (note that $\Omega(V_3, V_3, A_4) = 0$ since Ω vanishes on A_4). Focusing on V_5 , consider the map

$$\theta : V_5/A_4 \longrightarrow \wedge^2 A_4^*$$

induced by Ω . Note that θ cannot be zero, since otherwise Ω would vanish on V_5 . This condition would define a codimension ten subvariety of $G(5, V_7)$, stable under G_2 , hence a collection of fixed points: but there is none. So for a suitable V_3 to exist, we need the image of θ to be generated by a rank two form; then V_3 must contain the kernel of this rank two form, hence varies in a \mathbb{P}^1 .

One can compute explicitly θ at three points p representing the three orbits in CG . The conclusion is that the locus in $\mathbb{P}(V_7/A_4)$ defined by the condition that θ drops rank is either:

- (1) a smooth conic if p belongs to the open orbit,
- (2) a reducible conic if p belongs to the codimension one orbit,
- (3) the whole plane if p belongs to the closed orbit.

From this we deduce:

Proposition 3.2.

- (1) *The variety of lines passing through a general point $p = [A_4]$ of CG is a copy of $\mathbb{P}^1 \times \mathbb{P}^1$, embedded inside $\mathbb{P}(A_4^*) \times \mathbb{P}(V_7/A_4)$ by a linear system of type $|\mathcal{O}(1, 1)| \otimes |\mathcal{O}(2, 0)|$.*
- (2) *The variety $F_1(CG)$ is irreducible.*

Proof. The first assertion is the result of a direct computation. For the second assertion, consider the point-line incidence variety $I \subset CG \times F_1(CG)$. On the one hand, its projection to $F_1(CG)$ is a \mathbb{P}^1 -bundle, so I is smooth of dimension 10, and irreducible if and only if $F_1(CG)$ is irreducible. On the other hand, the fibers of its projection to CG are smooth quadratic surfaces over the open orbit, and surfaces or threefolds over the other points. This implies that the preimage in I of the open orbit is irreducible, and what remains is too small to generate another dimension ten component. \square

Remark. Recall from [Ma] that the stabilizer of a general point in CG is isomorphic to $SL_2 \times SL_2$. This stabilizer acts transitively on the quadratic surface that parametrizes the lines through this point. As a consequence, G_2 has an open orbit in $F_1(CG)$. More precisely, G_2 acts transitively on the space of lines meeting the open orbit of CG .

3.2. Planes in the Cayley Grassmannian. In order to simplify the computations of certain Gromov-Witten invariants, it will be useful to understand the planes in the Cayley Grassmannian. Indeed the two degree six Schubert classes σ_6 and σ'_6 both represent planes contained in CG .

The Grassmannian $G = G(4, 7)$ contains two different kind of planes, α -planes parametrized by $F(3, 6, 7)$ and β -planes parametrized by $F(2, 5, 7)$. A β -plane is made of spaces contained in a codimension two subspace of V_7 , so it cannot meet a class $\iota^* \tau_2 = \sigma_2$. Since according to [Ma] $\sigma_6 \sigma_2 \neq 0$ but $\sigma'_6 \sigma_2 = 0$, we deduce:

Lemma 3.3. *The class of an α -plane in CG is σ_6 , the class of a β -plane is σ'_6 .*

Let us discuss these planes separately.

Proposition 3.4. *The family of α -planes in CG is parametrized by the quadric \mathbb{Q}^5 . There is no α -plane through the general point of CG .*

Proof. An α -plane is defined by a pair $(V_3 \subset V_6)$ of subspaces of V_7 . It is contained in CG if and only if $\Omega(V_3, V_3, V_3) = 0$ and $\Omega(V_3, V_3, V_6) = 0$. If we denote by ω the restriction of Ω to V_6 , the latter condition means that ω belongs to $\wedge^2 V_3^\perp \wedge V_6^*$, which is the tangent space to the Grassmannian $G(3, V_6^*)$ at V_3^\perp . In particular ω is not a generic three-form.

There are only two G_2 -orbits of hyperplanes in V_7 : the orthogonal line (with respect to the invariant quadratic form) can be isotropic or not. Let us choose representatives of these orbits.

A non isotropic vector is e_0 , its orthogonal being $V_6 = \langle e_\alpha, e_\beta, e_\gamma, e_{-\alpha}, e_{-\beta}, e_{-\gamma} \rangle$. The restriction of Ω to this hyperplane is $\omega = v_\alpha \wedge v_\beta \wedge v_\gamma + v_{-\alpha} \wedge v_{-\beta} \wedge v_{-\gamma}$, a generic three-form.

An isotropic vector is e_α , its orthogonal being $V_6 = \langle e_0, e_\alpha, e_\beta, e_\gamma, e_{-\beta}, e_{-\gamma} \rangle$. The restriction of Ω to this hyperplane is $\omega = v_0 \wedge v_\beta \wedge v_{-\beta} + v_0 \wedge v_\gamma \wedge v_{-\gamma} + v_\alpha \wedge v_\beta \wedge v_\gamma$, which belongs to $\wedge^2 V_3^\perp \wedge V_6^*$ only for $V_3^\perp = \langle v_0, v_\beta, v_\gamma \rangle$. Note that V_3 contains e_α , so that any point A in the α -plane defined by V_6 verifies $V_6^\perp \subset A \subset V_6$.

As a consequence, the restriction of the invariant quadratic form on A must be degenerate; equivalently, by [Ma, Proposition 3.1], A does not belong to the open orbit in CG . \square

Proposition 3.5. *The family of β -planes in CG has dimension seven. There is a conic of β -planes passing through the general point of CG . Moreover, a generic β -plane is transverse to the orbit stratification.*

Proof. A β -plane is defined by a pair $(V_2 \subset V_5)$ of subspaces of V_7 . It is contained in CG if and only if $\Omega(V_2, V_2, V_5) = 0$ and $\Omega(V_2, V_5, V_5) = 0$.

Let us describe the β -planes passing through a general point p of CG , which we choose to be the point defined by $A = \langle e_\alpha, e_\beta, e_{-\alpha}, e_{-\beta} \rangle$. Suppose that V_5 is generated by A and $a = xe_0 + ye_\gamma + ze_{-\gamma}$. The conditions on $V_2 \subset A$ now restrict to $\Omega(V_2, A, a) = 0$, which means that V_2 must be contained in the kernel of the four linear forms $\Omega(a, e_\alpha, \bullet) = xv_{-\alpha} + yv_\beta$, $\Omega(a, e_\beta, \bullet) = xv_{-\beta} - yv_\alpha$, $\Omega(a, e_{-\alpha}, \bullet) = -xv_\alpha + zv_{-\beta}$, $\Omega(a, e_{-\beta}, \bullet) = -xv_\beta - zv_{-\alpha}$. This systems of linear forms has rank four in general, and rank two if $x^2 = yz$. There is therefore a conic of β -planes through a general point of CG , from which one can deduce that there is a seven dimensional family of β -planes on CG .

Let us choose for example $V_5 = \langle e_\alpha, e_\beta, e_{-\alpha}, e_{-\beta}, e_\gamma \rangle$. Then $V_2 = \langle e_{-\alpha}, e_{-\beta} \rangle$ and our plane $V_2 \subset A \subset V_5$ can be described by $A^\perp = \langle e_0, e_\gamma, ue_{-\alpha} + ve_{-\beta} + we_{-\gamma} \rangle$. The rank of the invariant quadratic form on such a three-space is 3 for $w \neq 0$ and 1 for $w = 0$. In particular it is transverse to the orbit stratification. \square

4. SOME GROMOV-WITTEN INVARIANTS

In this section we compute explicitly the Gromov-Witten invariants that we need. By Lemma 2.1, these will only be degree one invariants. By Propositions 3.1 and 3.2, the variety of lines in the Cayley Grassmannian is smooth and irreducible of the expected dimension, so we can apply Proposition 2.2 if we use classes of varieties that are transverse to the orbit stratification. We will use either restrictions of Schubert classes from the ambient Grassmannian $G(4, 7)$, or when convenient, the classes σ_8 , σ_7 and σ'_6 of points, lines and β -planes in CG , which are in general transverse to the orbit stratification (this is obvious for lines, and for planes this is Proposition 3.5).

4.1. The quantum Chevalley formula. The degree one Gromov-Witten invariants that appear in the quantum Chevalley formula are of type $I_1(\sigma_1, \sigma_k, \sigma_\ell)$ for $k + \ell = 11$. By the divisor axiom this reduces to the two-points Gromov-Witten invariant $I_1(\sigma_k, \sigma_\ell)$.

4.1.1. $I_1(\sigma_3, \sigma_8)$. This invariant is equal to $I_1(\iota^* \tau_{111}, \sigma_8)$, and can thus be computed as the number of lines passing through a general point A , and containing a point B representing a four-space that meets a generic $U_5 \subset V_7$ in dimension at least three. The base of the line is a hyperplane A_3 of A , that meets U_5 in dimension at least two, since it is also a hyperplane in B . But $A_2 = A \cap U_5$ has dimension two, so we need that $A_2 \subset A_3 \subset A$. The existence of B is then guaranteed by the fact that the induced map $\wedge^2 A_3 \rightarrow (U_5/A_2)^*$ does not have maximal rank. Generically the rank can drop only by one and B is then uniquely determined. Hence

$$I_1(\sigma'_3, \sigma_8) = c_1(\wedge^2 A_3^*) = 2,$$

where here A_3 is considered as vector bundle on $\mathbb{P}(A/A_2) = \mathbb{P}^1$.

4.1.2. $I_1(\sigma'_3, \sigma_8)$. This invariant is equal to $I_1(\iota^* \tau_3, \sigma_8)$ and therefore can be computed as the number of lines passing through a general point A , and containing a point B representing a four-space that contains a generic $U_1 \subset V_7$. The base of the line is a hyperplane A_3 of A such that $B = A_3 \oplus U_1$ is in CG . The condition for this is that the induced map $\wedge^2 A_3 \rightarrow U_1^*$ is zero. As a consequence

$$I_1(\sigma_3, \sigma_8) = c_3(\wedge^2 A_3^*) = 0,$$

where A_3 is considered as vector bundle on $\mathbb{P}(A) = \mathbb{P}^3$. The latter Chern class is zero because it is the number of isotropic hyperplanes for a non-degenerate two-forms in four variables.

4.1.3. $I_1(\iota^* \tau_{1111}, \sigma_7)$. Consider a general line d in CG defined by a pair $V_3 \subset V_5$ representing σ_7 (the same notation will be used for the computations of the next two invariants involving σ_7). This invariant is the number of lines in CG meeting d , say at A , and passing through some B which is contained in a general hyperplane V_6 . The axis of the line is $A_3 \subset V_5 \cap V_6$. Moreover A_3 meets V_3 in codimension one, so necessarily along $V_3 \cap V_6$. Finally, for A_3 to be contained in some $B \subset V_6$ belonging to CG we need that the map $\wedge^2 A_3 \rightarrow (V_6/A_3)^*$ does not have maximal rank. Therefore

$$I_1(\iota^* \tau_{1111}, \sigma_7) = c_1(\wedge^2 A_3^*) - c_1((V_6/A_3)^*) = 1,$$

where here A_3 is considered as vector bundle on $\mathbb{P}(V_5 \cap V_6/V_3 \cap V_6) = \mathbb{P}^1$.

4.1.4. $I_1(\iota^* \tau_{211}, \sigma_7)$. This invariant is the number of lines in CG meeting d at some A , and passing through some B such that $\dim(B \cap U_2) \geq 1$ and $\dim(B \cap U_5) \geq 3$, for $U_2 \subset U_5$ generic. The axis of the line is $A_3 \subset V_5$, and necessarily $\dim(A_3 \cap U_5) \geq 2$. We also have $\dim(A_3 \cap V_3) \geq 2$, and therefore $\dim(A_3 \cap U_5 \cap V_3) \geq 1$, which means that A_3 contains the line $U_1 = U_5 \cap V_3$. Our parameter space for A_3 is therefore $\mathbb{P}(V_3/U_1) \times \mathbb{P}(U_5 \cap V_5/U_1) \simeq \mathbb{P}^1 \times \mathbb{P}^1$. The condition for the existence of B is that the induced map $\wedge^2 A_3 \rightarrow U_2^*$ does not have full rank. By the Thom-Porteous formula we deduce that

$$I_1(\iota^* \tau_{211}, \sigma_7) = c_2(\wedge^2 A_3^*) = 3.$$

Indeed, if h and h' are the hyperplane classes on the two copies of \mathbb{P}^1 , then $c(A_3^*) = (1+h)(1+h')$ and $c(\wedge^2 A_3^*) = (1+h)(1+h')(1+h+h')$, hence $c_2(\wedge^2 A_3^*) = hh' + (h+h')^2 = 3hh'$.

4.1.5. $I_1(\iota^* \tau_{22}, \sigma_7)$. This is the number of lines in CG meeting d at some A , and passing through some B such that $\dim(B \cap U_3) \geq 2$ for $U_3 \subset V_7$ generic. The axis of the line is $A_3 \subset V_5$, and necessarily $\dim(A_3 \cap U_3) \geq 1$, so A_3 contains the line $U_1 = U_3 \cap V_5$. Our parameter space for A_3 is therefore $\mathbb{P}(V_3^*) \simeq \mathbb{P}^2$. The condition for the existence of B is that the induced map $\wedge^2 A_3 \rightarrow (U_3/U_1)^*$ does not have full rank. By the Thom-Porteous formula again we deduce that

$$I_1(\iota^* \tau_{22}, \sigma_7) = c_2(\wedge^2 A_3^*) = 2.$$

By the restriction formulas 2.3, $\iota^* \tau_{1111} = \sigma_4$, $\iota^* \tau_{211} = \sigma_4 + 2\sigma'_4$ and $\iota^* \tau_{22} = \sigma_4 + \sigma'_4 + \sigma''_4$, so we deduce that

$$I_1(\sigma_4, \sigma_7) = I_1(\sigma'_4, \sigma_7) = 1, \quad I_1(\sigma''_4, \sigma_7) = 0.$$

4.1.6. $I_1(\iota^* \tau_{2111}, \sigma'_6)$. This is the number of lines $d = (U_3, U_5)$ in CG meeting

- (1) a general β -plane $P(V_2, V_5)$ at a point A , such that $V_2 \subset A \subset V_5$,
- (2) a general Schubert cycle $\tau_{2111}(W_2, W_6)$ at a point B , such that $B \cap W_2 \neq 0$ and $B \subset W_6$.

Since U_3 must be a hyperplane in both A and B , we need that $U_3 \subset V_4 := V_5 \cap W_6$. Moreover $U_3 \cap V_2$ must contain a certain one dimensional subspace V_1 . But $U_3 \cap V_2 \subset W_6 \cap V_2$, so this must be equal to V_1 . Once U_3 is fixed, since $U_3 \cap W_2 \subset V_5 \cap W_2 = 0$, there must exist a unique line $L_1 \subset W_2$ such that $B = U_3 + L_1$. Since $A = U_3 + V_2$, the line d is then determined. The set of lines to be considered is thus isomorphic to $\mathbb{P}(W_2) \times \mathbb{P}(V_4/V_1)^* \simeq \mathbb{P}^1 \times \mathbb{P}^2$.

The condition for d to be contained in CG is that $\Omega(U_3, U_3, L_1) = 0$, which can be interpreted as the vanishing of a general section of the bundle $L_1^* \otimes \wedge^2 U_3^*$. Let us denote by $h_1 = c_1(L_1^*)$ and h_2 the hyperplane classes of our two projective spaces. The Chern roots of U_3^* are $0, a, b$, with $a + b = h_2$ and $ab = h_2^2$. Then the Chern roots of $L_1^* \otimes \wedge^2 U_3^*$ are $h_1 + a, h_1 + b, h_1 + a + b$ and therefore the invariant we are looking for is

$$c_3(L_1^* \otimes \wedge^2 U_3^*) = (h_1 + a)(h_1 + b)(h_1 + a + b) = 2h_1 h_2^2 = 2.$$

4.1.7. $I_1(\iota^* \tau_{221}, \sigma'_6)$. This is the number of lines $d = (U_3, U_5)$ in CG meeting

- (1) a general β -plane $P(V_2, V_5)$ at a point A , such that $V_2 \subset A \subset V_5$,
- (2) a general Schubert cycle $\tau_{221}(W_3, W_5)$ at a point B , such that $\dim(B \cap W_3) \geq 2$ and $\dim(B \subset W_5) \geq 3$.

Since U_3 must be a hyperplane in both A and B , we first deduce that $\dim(U_3 \cap V_2) \geq 1$ and $\dim(U_3 \cap W_5) \geq 2$. Since $V_2 \cap W_5 = 0$, this implies that $U_3 = L_1 + P_2$ for a line $L_1 \subset V_2$ and a plane $P_2 \subset V_3 := V_5 \cap W_5$. Moreover $P_2 \cap W_3 \subset W_1 := V_5 \cap W_3$ and since $U_3 \cap W_3$ must be non zero, we need the equality $P_2 \cap W_3 = W_1$. Once U_3 is fixed, we need that $B = U_3 + B_2$ with $W_1 \subset B_2 \subset W_3$ (because of the condition $\dim(B \subset W_3) \geq 2$), and $A = U_3 + V_2$, so the line d is determined.

The set of lines to be considered is thus isomorphic to $\mathbb{P}(V_2) \times \mathbb{P}(V_3/W_1) \times \mathbb{P}(W_3/W_1) \simeq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Let us denote by h_1, h_2, h_3 the hyperplane classes of our three projective lines. The condition for d to be contained in CG is that $\Omega(U_3, U_3, B_2/W_1) = 0$, which we interpret again as the vanishing of a general section of the bundle $(B_2/W_1)^* \otimes \wedge^2 U_3^*$. The Chern roots of U_3^* are $0, h_1, h_2$, hence the invariant we are looking for is

$$c_3((B_2/W_1)^* \otimes \wedge^2 U_3^*) = (h_1 + h_3)(h_2 + h_3)(h_1 + h_2 + h_3) = 3h_1 h_2 h_3 = 3.$$

By 2.3, $\iota^* \tau_{2111} = 2\sigma_5$, and $\iota^* \tau_{221} = 3\sigma_5 + \sigma'_5$, so we deduce that

$$I_1(\sigma_5, \sigma'_6) = 1 \quad \text{and} \quad I_1(\sigma'_5, \sigma'_6) = 0.$$

4.1.8. $I_1(\iota^* \tau_{2111}, \iota^* \tau_{2211})$. This is the number of lines in CG joining A and B such that

- (1) $A \subset U_6$ and $\dim(A \cap U_2) \geq 1$ for some generic $U_2 \subset U_6$,
- (2) $B \subset V_6$ and $\dim(B \cap V_3) \geq 2$ for some generic $V_3 \subset V_6$.

The axis D_3 of such a line must be contained in $W_5 = U_6 \cap V_6$ and meet V_3 non trivially, along a line $D_1 \subset W_2 = U_6 \cap V_3$. The parameter space P for the pair $D_1 \subset D_3$ is the quadric bundle $G(2, W_5/D_1)$ over $\mathbb{P}(W_2) \simeq \mathbb{P}^1$. We need that the three-form Ω vanish on D_3 , a codimension one condition. If D_3 does not contain W_2 or $W_1 = U_2 \cap V_6$, for A and B to exist we need that the induced maps $\wedge^2 D_3 \rightarrow U_2^*$

and $\wedge^2 D_3 \rightarrow (V_3/D_1)^*$ do not have maximal ranks, each of which is a codimension two condition. The number of points satisfying these conditions is given by:

$$c_1(D_3^*)c_2(\wedge^2 D_3^*)c_2(\wedge^2 D_3^* - (V_3/D_1)) = 7.$$

Indeed, this intersection product has to be taken in P , whose cohomology ring is generated by the hyperplane class $h = c_1(D_1^*)$ pulled-back from \mathbb{P}^1 , and the Chern classes a_1, a_2 of the dual tautological vector bundle $(D_3/D_1)^*$. We leave the details of the computation to the reader.

However, the important point to notice is that among the seven points p_1, \dots, p_7 at which our previous conditions are satisfied, some may not correspond to an actual line inside CG passing through A and B . Indeed, it could happen that $W_2 \subset D_3$ or $W_1 \subset D_3$. Let us study these two cases separately.

Consider the case where $W_2 \subset D_3$. The parameter space for D_3 is then $\mathbb{P}(W_5/W_2)$. Among p_1, \dots, p_7 , the points that satisfy $W_2 \subset D_3$ are those for which $\omega|_{D_3} = 0$ (a codimension one condition) and the induced map $\wedge^2 D_3 \rightarrow U_2^*$ does not have maximal rank (a codimension two condition). Indeed, notice that $W_2 \subset D_3$ implies automatically that $\wedge^2 D_3 \rightarrow (V_3/D_1)^*$ does not have maximal rank. As $\dim(\mathbb{P}(W_5/W_2)) = 2$, these conditions will not be satisfied generically.

Consider then the case $W_1 \subset D_3$. The parameter space for $D_1 \subset D_3$ is now the projective bundle $\mathbb{P}(W_5/(W_1 + D_1))$ over $\mathbb{P}(W_2)$. Among p_1, \dots, p_7 , the points that satisfy $W_1 \subset D_3$ are those for which $\omega|_{D_3} = 0$ (codimension one) and the induced map $\wedge^2 D_3 \rightarrow (V_3/D_1)^*$ does not have maximal rank (codimension two). Their number is $c_1(D_3^*)c_2(\wedge^2 D_3^* - (V_3/D_1)) = 1$.

As a consequence, we obtain:

$$I_1(\iota^* \tau_{2111}, \iota^* \tau_{2211}) = 2I_1(\sigma_5, \sigma_6) + 6I_1(\sigma_5, \sigma'_6) = 7 - 1 = 6.$$

4.1.9. $I_1(\iota^* \tau_{32}, \iota^* \tau_{2211})$. This is the number of lines in CG joining A and B such that

- (1) $U_1 \subset A$ and $\dim(A \cap U_3) \geq 2$ for some generic $U_1 \subset U_3$,
- (2) $B \subset V_6$ and $\dim(B \cap V_3) \geq 2$ for some generic $V_3 \subset V_6$.

The axis D_3 of such a line must be contained in V_6 ; moreover it must satisfy $D_3 \cap U_1 \subset U_1 \cap V_6 = 0$, $\dim(D_3 \cap U_3) \geq 1$ and $\dim(D_3 \cap V_3) \geq 1$. The last two conditions imply the existence of two lines D_1 and D'_1 inside D_3 which are contained respectively in $V_6 \cap U_3 = W_2$ and V_3 . The parameter space P for $D_1, D'_1 \subset D_3$ is the projective bundle $\mathbb{P}(V_6/(D_1 + D'_1))$ over $\mathbb{P}(W_2) \times \mathbb{P}(V_3)$. We need that the three-form ω vanishes on D_3 , a codimension one condition. If $D_3 \cap V_3 = D'_1$, for A and B to exist we need that the induced maps $\wedge^2 D_3 \rightarrow U_1^*$ and $\wedge^2 D_3 \rightarrow (V_3/D'_1)^*$ do not have maximal ranks, the first being a codimension three and the second a codimension two condition. The number of points satisfying these conditions is given by:

$$c_1(D_3^*)c_3(\wedge^2 D_3^*)c_2(\wedge^2 D_3^* - (V_3/D'_1)) = 4,$$

this intersection product being taken in P . One can easily verify that none of these four points satisfy $\dim(D_3 \cap V_3) \geq 2$. As a consequence, we obtain:

$$I_1(\iota^* \tau_{32}, \iota^* \tau_{2211}) = I_1(\sigma_5, \sigma_6) + 3I_1(\sigma_5, \sigma'_6) + I_1(\sigma'_5, \sigma_6) + 3I_1(\sigma'_5, \sigma'_6) = 4.$$

This is enough to deduce:

Proposition 4.1. *Up to degree three the quantum product with the hyperplane class is equal to the classical product. Up to degree seven, it is given by:*

$$\begin{aligned}
 \sigma_3\sigma_1 &= 2\sigma_4 + 2\sigma'_4 + 2q, \\
 \sigma'_3\sigma_1 &= \sigma'_4 + \sigma''_4, \\
 \sigma_4\sigma_1 &= 2\sigma_5 + q\sigma_1, \\
 \sigma'_4\sigma_1 &= 2\sigma_5 + \sigma'_5 + q\sigma_1, \\
 \sigma''_4\sigma_1 &= \sigma'_5, \\
 \sigma_5\sigma_1 &= \sigma_6 + 2\sigma'_6 + q\sigma'_2, \\
 \sigma'_5\sigma_1 &= 3\sigma_6 + 2\sigma'_6 + q\sigma_2, \\
 \sigma_6\sigma_1 &= \sigma_7 + q\sigma'_3, \\
 \sigma'_6\sigma_1 &= \sigma_7 + q\sigma_3.
 \end{aligned}$$

4.2. The missing invariants. Recall that in order to determine completely the quantum multiplication, we just need to determine the products σ_2^2 and $\sigma_4\sigma_2$. This requires the computation of three more Gromov-Witten invariants.

4.2.1. $I_1(\sigma_2, \sigma_2, \sigma_8)$. This invariant is $I_1(\iota^*\tau_2, \iota^*\tau_2, \sigma_8)$. It can therefore be computed as the number of lines $d = (D_3, D_5)$ in CG joining three points A, B, C such that A (resp. B) meets non trivially a general A_2 (resp. B_2), and C is a general point in CG . Since $C \cap B_2 = 0$, we must have $D_5 \subset C + B_2$, hence $A \cap A_2 \subset (C + B_2) \cap A_2$, which is a line A_1 , and there must be equality. Symmetrically $B \cap B_2 = B_1 := (C + A_2) \cap B_2$. The line d is therefore determined by $U_3 \subset C$, which gives $A = D_3 + A_1$ and $B = D_3 + B_1$.

The condition that d is contained in CG reduces to $\Omega(D_3, D_3, A_1) = 0$. So the invariant we are looking for is computed on $\mathbb{P}(C^*)$ as

$$I_1(\sigma_2, \sigma_2, \sigma_8) = c_3(\wedge^2 D_3^*) = 0.$$

4.2.2. $I_1(\sigma_2, \sigma_4, \sigma'_6)$. We compute this Gromov-Witten invariant as $I_1(\iota^*\tau_2, \iota^*\tau_{1111}, \sigma'_6)$, so as the number of lines in CG joining three points A, B, C such that

- (1) A meets non trivially a general U_2 ,
- (2) B is contained in a general hyperplane H_6 ,
- (3) C belongs to a general β -plane defined by a pair $V_2 \subset V_5$.

The axis D_3 of the line is contained in A, B, C , hence in $V_5 \cap H_6$. Moreover it must meet V_2 in dimension at least one, so necessarily along $V_2 \cap H_6$. Thus $V_2 \cap H_6 \subset D_3 \subset V_5 \cap H_6$ and D_3 is parametrized by a \mathbb{P}^2 . Then to get A we need the induced map $\wedge^2 D_3 \rightarrow U_2^*$ not to be of maximal rank. Necessarily $C = D_3 + V_2$. Since $D_3 \subset H_6$ there is a unique B on the line joining A to C which is contained in H_6 . It is automatically in CG since A and C are. We conclude that

$$I_1(\sigma_2, \sigma_4, \sigma'_6) = c_2(\wedge^2 D_3^*) = 2.$$

4.2.3. $I_1(\sigma_2, \sigma_4, \sigma_6)$. Let us compute $I_1(\iota^*\tau_{1111}, \iota^*\tau_2, \iota^*\tau_{33}) = I_1(\sigma_4, \sigma_2, \sigma_6 + \sigma'_6)$. This is the number of lines $d = (D_3, D_5)$ in CG joining three points A, B, C such that

- (1) A is contained in a general A_6 ,
- (2) B meets non trivially a general B_2 ,
- (3) C contains a general C_2 .

The axis D_3 must then be contained in A_6 , and meet C_2 along a line, necessarily $C_1 := C_2 \cap A_6$. Then C must be $D_3 + C_2$ and B must be generated by D_3 and a line B_1 of B_2 . (Beware that potentially the intersection $B_1 = B \cap B_2$ could

be contained in D_3 , which would impose $B_1 = A_6 \cap B_2$. But then the isotropy conditions would include $\Omega(B_1, C_2, C_2) = 0$, which is not possible.) We then get $D_5 = D_3 + C_2 + B_1$ and $A = D_5 \cap A_6$. We are thus led to consider a set of lines parametrized by $\mathbb{P}(B_2) \times G(2, A_6/C_1) = \mathbb{P}^1 \times G(2, 5)$.

The isotropy conditions are $\Omega(D_3, D_3, C_1) = 0$, $\Omega(D_3, D_3, C_2/C_1) = 0$ and $\Omega(D_3, D_3, B_1) = 0$. The bundle D_3/C_1 is just the tautological bundle on the Grassmannian, let us denote the Chern roots of its dual by a, b with $a + b = \tau_1$, $ab = \tau_{11}$ the usual Schubert classes. Let h be the hyperplane class on $\mathbb{P}(B_2)$. Our invariant is equal to $c_1(\wedge^2(D_3/C_1)^*)c_3(\wedge^2 D_3^*)c_3(B_1^* \otimes \wedge^2 D_3^*)$, that is,

$$\tau_1 ab(a+b)(a+h)(b+h)(a+b+h) = \tau_1^2 \tau_{11} (\tau_{11} + \tau_1^2) h = 3.$$

This gives three points that satisfy the conditions $\Omega(D_3, D_3, C_1) = 0$, $\Omega(D_3, D_3, C_2/C_1) = 0$ and $\Omega(D_3, D_3, B_1) = 0$ over $\mathbb{P}(B_2) \times G(2, A_6/C_1)$. Among them, we still need to remove those for which $B = C$. As we already know that $C_1 \subset D_3$, we have $B = C$ if $\dim(D_3 \cap W_3) \geq 2$, where $W_3 = A_6 \cap (B_2 + C_2)$. Let $C_1 \subset D_2 \subset (D_3 \cap W_3)$ with $\dim(D_2) = 2$. Then the parameter space of $D_2 \subset D_3$ is the projective bundle $\mathbb{P}(A_6/D_2)$ over $\mathbb{P}(W_3/C_1)$, and the conditions we need to impose are $\Omega(D_3, D_3, C_1) = 0$ and $\Omega(D_3, D_3, C_2/C_1) = 0$. Let l be the hyperplane in $\mathbb{P}(W_3/C_1) \cong \mathbb{P}^1$, and m the relative hyperplane class in $\mathbb{P}(A_6/D_2)$. The number of points that we need to remove is therefore

$$c_1(\wedge^3 D_3^*)c_3(\wedge^2 D_3^*) = (l+m)lm(l+m) = lm^3 = 1,$$

and we get

$$I_1(\sigma_4, \sigma_2, \sigma_6) = 3 - 1 - I_1(\sigma_2, \sigma_4, \sigma_6') = 0.$$

This finally yields the two missing products:

$$\sigma_2^2 = \sigma_4 + 2\sigma_4' + 2\sigma_4'', \quad \sigma_4\sigma_2 = \sigma_6 + \sigma_6' + 2q\sigma_2'.$$

4.3. A presentation of the quantum cohomology ring. We now have enough information to deduce a presentation of the quantum cohomology ring. We first use the relations we have obtained so far to express the Schubert classes in terms of the generators σ_1 and σ_2 . We could also have chosen the other degree two class $\sigma_2' = \sigma_1^2 - \sigma_2$ but the formulas would be slightly worse. In degree three there is no quantum corrections, we easily get

$$\sigma_3 = \frac{1}{4}(3\sigma_1^3 - 5\sigma_1\sigma_2), \quad \sigma_3' = \frac{1}{4}(3\sigma_1\sigma_2 - \sigma_1^3).$$

In degree four knowing $\sigma_3\sigma_1$, $\sigma_3'\sigma_1$, and σ_2^2 that we have just computed, we deduce:

$$\begin{aligned} \sigma_4 &= \sigma_2^2 - \frac{3}{2}\sigma_2\sigma_1^2 + \frac{1}{2}\sigma_1^4, \\ \sigma_4' &= -\sigma_2^2 + \frac{7}{8}\sigma_2\sigma_1^2 - \frac{1}{8}\sigma_1^4 - q, \\ \sigma_4'' &= \sigma_2^2 - \frac{1}{8}\sigma_2\sigma_1^2 - \frac{1}{8}\sigma_1^4 + q. \end{aligned}$$

The quantum products of these classes by the hyperplane class will give not only an expression of σ_5 and σ_5' in terms of the generators, but also a degree five relation

in the quantum cohomology ring. We get

$$\begin{aligned}\sigma_5 &= \frac{1}{2}\sigma_2\sigma_1^3 - \sigma_2^2\sigma_1 - \frac{3}{2}q\sigma_1, \\ \sigma'_5 &= -\frac{3}{4}\sigma_2\sigma_1^3 + \frac{7}{4}\sigma_2^2\sigma_1 + \frac{3}{2}q\sigma_1,\end{aligned}$$

plus the degree five relation $(\sigma'_4 - \sigma_4 - \sigma''_4)\sigma_1 = 0$. In degree six, we directly get σ_6 and σ'_6 from $\sigma_5\sigma_1$ and $\sigma'_5\sigma_1$:

$$\begin{aligned}\sigma_6 &= -\frac{5}{8}\sigma_2\sigma_1^4 + \frac{11}{8}\sigma_2^2\sigma_1^2 + q(2\sigma_1^2 - \sigma_2), \\ \sigma'_6 &= \frac{9}{16}\sigma_2\sigma_1^4 - \frac{19}{16}\sigma_2^2\sigma_1^2 + q(\sigma_2 - \frac{9}{4}\sigma_1^2).\end{aligned}$$

Our computation of $\sigma_4\sigma_2$ yields a degree 6 relation and we get:

Proposition 4.2. *The rational quantum cohomology ring of the Cayley Grassmannian is $QH^*(CG, \mathbb{Q}) = \mathbb{Q}[\sigma_1, \sigma_2, q]/\langle R_5(q), R_6(q) \rangle$, for the quantum relations*

$$\begin{aligned}R_5(q) &= \sigma_1^5 - 5\sigma_1^3\sigma_2 + 6\sigma_1\sigma_2^2 + 4q\sigma_1, \\ R_6(q) &= 16\sigma_2^3 - 27\sigma_1^2\sigma_2^2 + 9\sigma_1^4\sigma_2 + 32q\sigma_2 - 28q\sigma_1^2.\end{aligned}$$

A routine computation allows to check that for $q \neq 0$, these two equations define a reduced scheme.

Corollary 4.3. *The quantum cohomology ring $QH^*(CG, \mathbb{Q})|_{q=1}$ is semisimple.*

4.4. Completing the Chevalley and Giambelli formulas. We have enough information to complete the quantum Chevalley formula up to degree seven. The quantum product $\sigma_7\sigma_1$ yields σ_8 up to a potential term in q^2 . Plugging this into the product $\sigma_8\sigma_1$, we conclude that this term is in fact zero. This allows to complete the quantum Giambelli formula by the two equations

$$\begin{aligned}\sigma_7 &= \frac{1}{18}\sigma_2^3\sigma_1 + q\left(\frac{13}{36}\sigma_1\sigma_2 - \frac{17}{36}\sigma_1^3\right), \\ \sigma_8 &= \frac{1}{9}\sigma_2^4 + q\left(\frac{2}{9}\sigma_2^2 + \frac{29}{36}\sigma_1^2\sigma_2 - \frac{27}{36}\sigma_1^4\right) + q^2.\end{aligned}$$

Finally, the missing products in the quantum Chevalley formula are

$$\sigma_7\sigma_1 = \sigma_8 + q\sigma_4 + q\sigma'_4, \quad \sigma_8\sigma_1 = 2q\sigma_5.$$

From that, it is straightforward to deduce the full multiplication table, that we report in the Appendix. Remarkably, all the coefficients are non negative.

The mere fact that the quantum product by the hyperplane class has only non negative coefficients allows to ensure, following the approach of [CL], that conjecture \mathcal{O} from [GGI] is verified in this case:

Corollary 4.4. *The Cayley Grassmannian satisfies conjecture \mathcal{O} .*

The eigenvalues of the quantum product by the hyperplane class are the roots of the polynomial

$$p(t) = -t^{15} + 102t^{11}q - 317t^7q^2 + 2048t^3q^3 = t^3 f(t^4),$$

where $f(y) = -y^3 + 102y^2q - 317yq^2 + 2048q^3$. The equation $f(y) = 0$ has three distinct solutions, among which the one with maximal modulus is real, namely

$y_{max} \simeq 99.00713881372502$. As far as the anticanonical class $-K_{CG} = 4\sigma_1$ is concerned, we deduce that its spectral radius is

$$T(CG) = 4(y_{max})^{1/4} \simeq 12.6175960332 > \dim CG + 1,$$

in agreement with a conjecture of Galkin [Ga].

5. ON THE DERIVED CATEGORY OF CG

Denote by T and Q the ranks four and three tautological bundles of the Grassmannian G , and their restrictions to CG as well.

5.1. An exceptional collection. According to Dubrovin's conjecture, the semisimplicity of the cohomology ring of CG should imply that the bounded derived category of sheaves $D^b(CG)$ admits a full exceptional collection. The length of this collection should be equal to the rang of the Grothendieck group of vector bundles on CG , which is equal to 15. We have been able to find an exceptional collection of this length, as follows.

Consider the collections:

$$\begin{aligned} \mathcal{C}_0 &= \langle \mathfrak{sl}(Q), \mathcal{O}_G, Q, \wedge^2 T^\vee, \wedge^2 Q \rangle, \\ \mathcal{C}_1 &= \langle \mathcal{O}_G, Q, \wedge^2 T^\vee, \wedge^2 Q \rangle, \\ \mathcal{C}_2 &= \langle \mathcal{O}_G, Q, \wedge^2 Q \rangle, \\ \mathcal{C}_3 &= \mathcal{C}_2. \end{aligned}$$

Proposition 5.1. $\mathcal{C} = \langle \mathcal{C}_0, \mathcal{C}_1(1), \mathcal{C}_2(2), \mathcal{C}_3(3) \rangle$ is a Lefschetz exceptional collection inside the bounded category of sheaves on CG .

Here "Lefschetz" refers to the particular structure of the collection [Ku1]: we have $\mathcal{C}_0 \supset \mathcal{C}_1 \supset \mathcal{C}_2 \supset \mathcal{C}_3$ and $\mathcal{C}_i(i)$ is the result of twisting the objects in \mathcal{C}_i by $\mathcal{O}_{CG}(i)$.

Of course we expect \mathcal{C} to be full, in the sense that it should generate the full derived category. In order to prove such a statement, it can be useful to have a good birational model of the variety under consideration. We will end this paper by providing such a simple model.

5.2. A birationality. Consider a decomposition $V_7 = V_6 \oplus V_1$. For any k , there is an induced projection $\pi_k : G(k, V_7) \dashrightarrow G(k, V_6)$. The exceptional locus of this rational map is the sub-Grassmannian of k -planes containing V_1 , which is isomorphic to $G(k-1, V_6)$. Let $\tilde{G}(k, V_7)$ be the variety of pairs (V_k, U_k) of k -planes with $U_k \subset V_6$ and $V_k \subset U_k \oplus V_1 \subset V_7$. The two projections induce a diagram

$$\begin{array}{ccc} \tilde{G}(k, V_7) & \xrightarrow{\sim} & G(k, T \oplus V_1) \\ \swarrow p & & \downarrow q \\ G(k, V_7) & \dashrightarrow^{\pi_k} & G(k, V_6), \end{array}$$

where p is the blow-up of $G(k-1, V_6)$ and q is a relative Grassmannian.

Recall that CG is defined by a three-form Ω on V_7 . We let ω denote its restriction to V_6 . Moreover, we can fix a generator e of V_1 , and get on V_6 the two-form $\alpha = \Omega(e, \bullet, \bullet)$. For a general choice of V_1 , this is a non degenerate two-form. The following result is Proposition 6.3 in [Ku3].

Proposition 5.2. *The projection π_2 sends the adjoint variety G_2^{ad} to its image $Y_2 \subset IG_\alpha(2, 6) \subset G(2, V_6)$.*

Proof. The intersection of G_2^{ad} with the exceptional locus of π_2 is the variety of planes of the form $V_2 = U_1 \oplus V_1$ for which $\Omega(v_2, v'_2, \bullet) = 0$ for any basis v_2, v'_2 . Choosing $v'_2 = e$, this amounts to $\alpha(v_2, \bullet) = 0$. Equivalently, v_2 would belong to the kernel of α ; but this kernel is zero, so the restriction of π_2 to G_2^{ad} is well-defined.

So any point of G_2^{ad} is of the form $V_2 = U_2^\phi = \{x + \phi(x), x \in U_2\}$ for some $U_2 \subset V_6$ and $\phi \in Hom(U_2, V_1)$. The condition for V_2 to belong to G_2^{ad} is that $U_2 = \langle u, u' \rangle$ is α -isotropic and

$$\omega(u, u', v) + \phi(u)\alpha(u', v) - \phi(u')\alpha(u, v) = 0 \quad \text{mod } U_2$$

for any $v \in V_6$. In particular, since the linear forms $\alpha(u, \bullet)$ and $\alpha(u', \bullet)$ are linearly independent, ϕ is uniquely determined. So G_2^{ad} is projected bijectively onto its image Y in $G(2, V_6)$. \square

By orthogonality with respect to α , $IG_\alpha(2, 6)$ is mapped isomorphically to the variety of coisotropic 4-planes, those 4-planes on which α has rank two. (in other words, they contain their orthogonal 2-plane). Moreover, the image of Y_2 in $G(4, V_6)$ can be interpreted as the variety Y_4 of pairs $(U_2 \subset U_4 = U_2^\perp)$ such that $\omega(U_2, U_2, U_4) = 0$.

Proposition 5.3. *The projection π_4 maps the Cayley Grassmannian CG birationally onto $G(4, V_6)$. This birational map is resolved by blowing up a hyperplane section of the Lagrangian Grassmannian $Z_3 = LG_\alpha(3, 6) \cap H_\omega$, which yields an isomorphism with the blowup of $G(4, V_6)$ along Y_4 .*

Proof. The intersection of CG with the exceptional locus of π_4 is the variety of 4-planes of the form $V_4 = U_3 \oplus V_1$ on which Ω vanishes identically. Equivalently, ω and α need to vanish on U_3 , or in other words U_3 has to belong to $Z_3 = LG_\alpha(3, 6) \cap H_\omega$. The preimage of such a V_4 in $\tilde{G}(4, V_7)$ is then isomorphic to the variety of 4-planes U_4 such that $U_3 \subset U_4 \subset V_6$, which yields a projective plane. Restricting the previous diagram, we get

$$\begin{array}{ccc} & \tilde{CG} & \\ p_{Z_3} \swarrow & & \searrow q \\ Z_3 \subset CG & \xrightarrow{\pi_4|_{CG}} & G(4, V_6), \end{array}$$

where p_{Z_3} is the blow-up of Z_3 .

Let us check that the projection π_4 restricted to CG is birational. The preimage of a 4-plane $U_4 \subset V_6$ is the set of 4-planes $V_4 \subset V_7$ of the form $V_4 = U_4^\phi$, for $\phi \in Hom(U_4, V_1)$. The condition that Ω vanishes on V_4 is equivalent to the condition that $\omega + \phi \wedge \alpha = 0$ on U_4 . When $\alpha|_{U_4}$ has rank four, the wedge product by this two-form yields an isomorphism from U_4^\vee to $\wedge^3 U_4^\vee$, so that ϕ is determined uniquely.

This also implies that q has non-trivial fibers only over the locus of those U_4 's on which α has rank two, which means that U_4 is coisotropic; let $U_2 = U_4^\perp$. Suppose that $V_4 \subset U_4 \oplus V_1$ belongs to CG . If $V_4 = U_3 \oplus V_1$, we need that ω and α vanish on U_3 , and the latter condition means that $U_2 \subset U_3 \subset U_4$. If V_4 is of the form U_4^ϕ with $\omega + \phi \wedge \alpha = 0$ on U_4 , then clearly $\omega(U_2, U_2, U_4) = 0$, and conversely this condition implies the existence of a suitable ϕ . So the exceptional locus Y of q^{-1} is isomorphic to G_2^{ad} . Moreover, ϕ is determined in V_4^\vee only up to U_2^\perp , which means

that the fiber contains an affine linear open subset of dimension two; so the fiber, being the closure of this affine linear subspace, is in fact a projective plane. This implies that the projection q is in fact the blowup of Y_4 in $G(4, V_6)$. \square

We get the following diagram

$$\begin{array}{ccc} & \tilde{C}G & \\ \swarrow^{Bl_{Z_3}} & & \searrow^{Bl_{Y_4}} \\ Z_3 \subset CG & \text{---} & G(4, V_6) \supset Y_4. \end{array}$$

Note that Z_3 and $Y_4 \simeq G_2^{ad}$ are two examples of five dimensional *minifolds* [GKMS]: their derived categories admit full exceptional collections of length 6.

Indeed Kuznetsov proved in [Ku1, 6.4] that the derived category of the adjoint variety G_2^{ad} has a Lefschetz decomposition

$$D(G_2^{ad}) = \langle \mathcal{Y}_0, \mathcal{Y}_1(1), \mathcal{Y}_2(2) \rangle,$$

where $\mathcal{Y}_0 = \mathcal{Y}_1 = \mathcal{Y}_2 = \langle \mathcal{O}, T^\vee \rangle$, the bundle T being the restriction of the rank two tautological bundle from the Grassmannian $G(2, 7)$.

Moreover, a completely similar statement holds for a smooth hyperplane section Z_3 of the Lagrangian Grassmannian $LG(3, 6)$: according to [Sa],

$$D(Z_3) = \langle \mathcal{Z}_0, \mathcal{Z}_1(1), \mathcal{Z}_2(2) \rangle,$$

where $\mathcal{Z}_0 = \mathcal{Z}_1 = \mathcal{Z}_2 = \langle \mathcal{O}, U^\vee \rangle$, the bundle U being the restriction of the rank three tautological bundle from the Grassmannian $G(3, 6)$.

For the derived category of the Grassmannian $G(2, 6)$, several full exceptional collections are known: the two Kapranov collections, and also the minimal Lefschetz collection found by Kuznetsov [Ku2]:

$$D(G(2, 6)) = \langle \mathcal{A}_0, \mathcal{A}_1(1), \mathcal{A}_2(2), \mathcal{A}_3(3), \mathcal{A}_4(4), \mathcal{A}_5(5) \rangle,$$

where $\mathcal{A}_0 = \mathcal{A}_1 = \mathcal{A}_2 = \langle \mathcal{O}, T^\vee, S^2 T^\vee \rangle$ and $\mathcal{A}_3 = \mathcal{A}_4 = \mathcal{A}_5 = \langle \mathcal{O}, T^\vee \rangle$. This Lefschetz collection induces a minimal Lefschetz collection on $IG(2, 6)$ [Ku2].

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APPENDIX: THE QUANTUM MULTIPLICATION TABLE

Here is the complete quantum multiplication table of the Cayley Grassmannian:

$$\begin{aligned}
\sigma_1\sigma_1 &= \sigma_2 + \sigma'_2, & \sigma_4''\sigma_1 &= \sigma'_5, \\
\sigma_2\sigma_1 &= \sigma_3 + 3\sigma'_3, & \sigma_5\sigma_1 &= \sigma_6 + 2\sigma'_6 + q\sigma'_2, \\
\sigma'_2\sigma_1 &= 2\sigma_3 + 2\sigma'_3, & \sigma'_5\sigma_1 &= 3\sigma_6 + 2\sigma'_6 + q\sigma_2, \\
\sigma_3\sigma_1 &= 2\sigma_4 + 2\sigma'_4 + 2q, & \sigma_6\sigma_1 &= \sigma_7 + q\sigma'_3, \\
\sigma'_3\sigma_1 &= \sigma'_4 + \sigma''_4, & \sigma'_6\sigma_1 &= \sigma_7 + q\sigma_3, \\
\sigma_4\sigma_1 &= 2\sigma_5 + q\sigma_1, & \sigma_7\sigma_1 &= \sigma_8 + q\sigma_4 + q\sigma'_4, \\
\sigma'_4\sigma_1 &= 2\sigma_5 + \sigma'_5 + q\sigma_1, & \sigma_8\sigma_1 &= 2q\sigma_5, \\
\\
\sigma_2\sigma_2 &= \sigma_4 + 2\sigma'_4 + 2\sigma''_4, & \sigma'_2\sigma'_2 &= 3\sigma_4 + 3\sigma'_4 + \sigma''_4 + 2q, \\
\sigma'_2\sigma_2 &= \sigma_4 + 3\sigma'_4 + \sigma''_4 + 2q, & \sigma'_2\sigma_3 &= 5\sigma_5 + \sigma'_5 + 3q\sigma_1, \\
\sigma_3\sigma_2 &= 3\sigma_5 + \sigma'_5 + 3q\sigma_1, & \sigma'_2\sigma'_3 &= \sigma_5 + \sigma'_5 + q\sigma_1, \\
\sigma'_3\sigma_2 &= \sigma_5 + \sigma'_5, & \sigma'_2\sigma_4 &= 2\sigma_6 + 3\sigma'_6 + q\sigma_2 + q\sigma'_2, \\
\sigma_4\sigma_2 &= \sigma_6 + \sigma'_6 + 2q\sigma'_2, & \sigma'_2\sigma'_4 &= 3\sigma_6 + 3\sigma'_6 + q\sigma_2 + 2q\sigma'_2, \\
\sigma'_4\sigma_2 &= 2\sigma_6 + 3\sigma'_6 + q\sigma_2 + q\sigma'_2, & \sigma'_2\sigma''_4 &= \sigma_6 + \sigma'_6 + q\sigma_2, \\
\sigma''_4\sigma_2 &= 2\sigma_6 + \sigma'_6, & \sigma'_2\sigma_5 &= 2\sigma_7 + 2q\sigma_3 + 2q\sigma'_3, \\
\sigma_5\sigma_2 &= \sigma_7 + 2q\sigma_3 + q\sigma'_3, & \sigma'_2\sigma'_5 &= 2\sigma_7 + 2q\sigma_3 + 4q\sigma'_3, \\
\sigma'_5\sigma_2 &= 3\sigma_7 + q\sigma_3 + 2q\sigma'_3, & \sigma'_2\sigma_6 &= q\sigma_4 + q\sigma'_4 + q\sigma''_4, \\
\sigma_6\sigma_2 &= \sigma_8 + q\sigma'_4, & \sigma'_2\sigma'_6 &= \sigma_8 + q\sigma_4 + 2q\sigma'_4 + q^2, \\
\sigma'_6\sigma_2 &= 2q\sigma_4 + q\sigma'_4 + q^2, & \sigma'_2\sigma_7 &= 3q\sigma_5 + q\sigma'_5 + q^2\sigma_1, \\
\sigma_7\sigma_2 &= 3q\sigma_5 + q^2\sigma_1, & \sigma'_2\sigma_8 &= 2q\sigma_6 + 2q\sigma'_6 + q^2\sigma'_2, \\
\sigma_8\sigma_2 &= 2q\sigma'_6 + q^2\sigma'_2, & \\
\\
\sigma_3\sigma_3 &= 3\sigma_6 + 5\sigma'_6 + q\sigma_2 + 3q\sigma'_2, & \sigma'_3\sigma'_3 &= \sigma_6 + \sigma'_6, \\
\sigma'_3\sigma_3 &= \sigma_6 + \sigma'_6 + q\sigma_2 + q\sigma'_2, & \sigma'_3\sigma_4 &= q\sigma_3 + q\sigma'_3, \\
\sigma_4\sigma_3 &= 2\sigma_7 + 2q\sigma_3 + 2q\sigma'_3, & \sigma'_3\sigma'_4 &= \sigma_7 + q\sigma_3 + q\sigma'_3, \\
\sigma'_4\sigma_3 &= 2\sigma_7 + 3q\sigma_3 + 4q\sigma'_3, & \sigma'_3\sigma''_4 &= \sigma_7, \\
\sigma''_4\sigma_3 &= q\sigma_3 + 2q\sigma'_3, & \sigma'_3\sigma_5 &= q\sigma_4 + q\sigma'_4 + q^2, \\
\sigma_5\sigma_3 &= \sigma_8 + 2q\sigma_4 + 3q\sigma'_4 + q\sigma''_4 + q^2, & \sigma'_3\sigma'_5 &= \sigma_8 + q\sigma_4 + q\sigma'_4, \\
\sigma'_5\sigma_3 &= 2q\sigma_4 + 4q\sigma'_4 + 2q\sigma''_4 + 2q^2, & \sigma'_3\sigma_6 &= q\sigma_5, \\
\sigma_6\sigma_3 &= q\sigma_5 + q\sigma'_5 + q^2\sigma_1, & \sigma'_3\sigma'_6 &= q\sigma_5 + q^2\sigma_1, \\
\sigma'_6\sigma_3 &= 3q\sigma_5 + q\sigma'_5 + q^2\sigma_1, & \sigma'_3\sigma_7 &= q\sigma'_6 + q^2\sigma'_2, \\
\sigma_7\sigma_3 &= 3q\sigma_6 + 3q\sigma'_6 + q^2\sigma_2 + q^2\sigma'_2, & \sigma'_3\sigma_8 &= q^2\sigma_3, \\
\sigma_8\sigma_3 &= 2q\sigma_7 + q^2\sigma_3 + 2q^2\sigma'_3, & &
\end{aligned}$$

$$\begin{aligned}
\sigma_4\sigma_4 &= \sigma_8 + 2q\sigma'_4 + q^2, & \sigma_4\sigma'_4 &= 2q\sigma_4 + 2q\sigma'_4 + q\sigma''_4 + q^2, \\
\sigma'_4\sigma'_4 &= \sigma_8 + 2q\sigma_4 + 3q\sigma'_4 + q\sigma''_4 + 2q^2, & \sigma_4\sigma''_4 &= q\sigma'_4 + q^2, \\
& & \sigma'_4\sigma''_4 &= q\sigma_4 + q\sigma'_4, \\
& & \sigma''_4\sigma''_4 &= \sigma_8, \\
\sigma_4\sigma_5 &= 2q\sigma_5 + q\sigma'_5 + q^2\sigma_1, & \sigma'_5\sigma_4 &= 2q\sigma_5 + q\sigma'_5 + 2q^2\sigma_1, \\
\sigma'_4\sigma_5 &= 3q\sigma_5 + q\sigma'_5 + 2q^2\sigma_1, & \sigma'_5\sigma'_4 &= 4q\sigma_5 + q\sigma'_5 + 2q^2\sigma_1, \\
& & \sigma'_5\sigma''_4 &= 2q\sigma_5, \\
& & \sigma''_4\sigma_5 &= q\sigma_5 + q^2\sigma_1, \\
\sigma_4\sigma_6 &= q\sigma'_6 + q^2\sigma_2, & \sigma'_6\sigma_4 &= 2q\sigma_6 + q\sigma'_6 + q^2\sigma'_2, \\
\sigma'_4\sigma_6 &= q\sigma_6 + q\sigma'_6 + q^2\sigma'_2, & \sigma'_6\sigma'_4 &= q\sigma_6 + 2q\sigma'_6 + q^2\sigma_2 + q^2\sigma'_2, \\
& & \sigma'_6\sigma''_4 &= q^2\sigma'_2, \\
& & \sigma''_4\sigma_6 &= q\sigma'_6, \\
\sigma_4\sigma_7 &= q\sigma_7 + q^2\sigma_3 + 2q^2\sigma'_3, & \sigma_8\sigma_4 &= q^2\sigma_4 + q^2\sigma'_4 + q^2\sigma''_4, \\
\sigma'_4\sigma_7 &= q\sigma_7 + 2q^2\sigma_3 + 2q^2\sigma'_3, & \sigma_8\sigma'_4 &= q^2\sigma_4 + 2q^2\sigma'_4 + q^3, \\
& & \sigma_8\sigma''_4 &= q^2\sigma_4 + q^3, \\
& & \sigma''_4\sigma_7 &= q^2\sigma_3, \\
\sigma_5\sigma_5 &= 2q\sigma_6 + 2q\sigma'_6 + q^2\sigma_2 + q^2\sigma'_2, & \sigma_6\sigma_6 &= q^2\sigma_4, \\
\sigma'_5\sigma_5 &= q\sigma_6 + 2q\sigma'_6 + q^2\sigma_2 + 2q^2\sigma'_2, & \sigma'_6\sigma_6 &= q^2\sigma'_4 + q^3, \\
\sigma'_5\sigma'_5 &= 2q\sigma_6 + 4q\sigma'_6 + 2q^2\sigma'_2, & \sigma'_6\sigma'_6 &= q^2\sigma_4 + q^2\sigma'_4 + q^2\sigma''_4, \\
& & \sigma'_5\sigma_6 &= q\sigma_7 + q^2\sigma_3, \\
& & \sigma'_5\sigma'_6 &= 2q^2\sigma_3 + 2q^2\sigma'_3, \\
& & \sigma'_6\sigma_5 &= q\sigma_7 + q^2\sigma_3 + 2q^2\sigma'_3, \\
\sigma_7\sigma_5 &= q^2\sigma_4 + 2q^2\sigma'_4 + q^2\sigma''_4 + q^3, & \sigma'_5\sigma_7 &= 2q^2\sigma_4 + 2q^2\sigma'_4 + 2q^3, \\
\sigma_8\sigma_5 &= q^2\sigma_5 + q^2\sigma'_5 + q^3\sigma_1, & \sigma'_5\sigma_8 &= 2q^2\sigma_5 + 2q^3\sigma_1, \\
& & \sigma'_6\sigma_7 &= q^2\sigma_5 + q^2\sigma'_5 + q^3\sigma_1, \\
& & \sigma'_6\sigma_8 &= q^2\sigma_6 + q^2\sigma'_6 + q^3\sigma_2, \\
& & \sigma_7\sigma_6 &= q^2\sigma_5 + q^3\sigma_1, \\
& & \sigma_8\sigma_6 &= q^3\sigma'_2, \\
\sigma_7\sigma_7 &= q^2\sigma_6 + q^2\sigma'_6 + q^3\sigma_2 + q^3\sigma'_2, & \sigma_7\sigma_8 &= q^3\sigma_3 + 2q^3\sigma'_3, \\
& & \sigma_8\sigma_8 &= q^3\sigma'_4 + q^3\sigma''_4 + q^4.
\end{aligned}$$

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