



**HAL**  
open science

# Normal forms for rank two linear irregular differential equations and moduli spaces

Karamoko Diarra, Frank Loray

► **To cite this version:**

Karamoko Diarra, Frank Loray. Normal forms for rank two linear irregular differential equations and moduli spaces. 2019. hal-02183978v1

**HAL Id: hal-02183978**

**<https://hal.science/hal-02183978v1>**

Preprint submitted on 15 Jul 2019 (v1), last revised 28 Jul 2020 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# NORMAL FORMS FOR RANK TWO LINEAR IRREGULAR DIFFERENTIAL EQUATIONS AND MODULI SPACES.

KARAMOKO DIARRA AND FRANK LORAY

ABSTRACT. We provide a unique normal form for rank two irregular connections on the Riemann sphere. In fact, we provide a birational model where we introduce apparent singular points and where the bundle has a fixed Birkhoff-Grothendieck decomposition. The essential poles and the apparent poles provide two parabolic structures. The first one only depend on the formal type of the singular points. The latter one determine the connection (accessory parameters). As a consequence, an open set of the corresponding moduli space of connections is canonically identified with an open set of some Hilbert scheme of points on the explicit blow-up of some Hirzebruch surface. This generalizes to the irregular case a description due to Oblezin in the logarithmic case.

## CONTENTS

|   |    |
|---|----|
| 1. Introduction   | 1  |
| 2. Local normal form and local formal data              | 3  |
| 3. Global normal form and explicit definition of $\Phi$ | 5  |
| 4. Birationality of $\Phi$                              | 7  |
| 5. Symplectic structure and Lagrangian fibration        | 9  |
| 6. Link with scalar equation                            | 9  |
| References  | 10 |

## 1. INTRODUCTION

In this paper, we deal with rank 2 meromorphic connections on the Riemann sphere  $\mathbb{P}^1 = \mathbb{P}^1(\mathbb{C})$ , which consist in the data  $(E, \nabla)$  of a rank 2 holomorphic vector bundle  $E$ , and a meromorphic connection  $\nabla : E \rightarrow E \otimes \Omega^1(D)$

- $D$  is the effective divisor  $D = \sum_{i=1}^{\nu} n_i \cdot [t_i]$  of poles on  $\mathbb{P}^1$ , i.e.  $t_i$  are distinct points and  $n_i \in \mathbb{Z}_{>0}$  are multiplicities;
- $\nabla$  is a  $\mathbb{C}$ -linear map satisfying Leibniz rule: for any open set  $U \subset \mathbb{P}^1$ , we have

$$\nabla(f \cdot s) = df \otimes s + f \cdot \nabla(s)$$

for any holomorphic function  $f$  on  $U$  and section  $s : U \rightarrow E$ .

---

*Date:* **July 15, 2019.**

*1991 Mathematics Subject Classification.* 34M55, 34M56, 34M03.

*Key words and phrases.* Ordinary differential equations, Normal forms.

We thank CNRS, Université de Rennes 1, Henri Lebesgue Center and ANR-16-CE40-0008 project “*Foliage*” for financial support.

On any strict open set  $U \subset \mathbb{P}^1$ , the vector bundle  $E$  is trivial, i.e. admits global coordinates  $Y \in \mathbb{C}^2$ , and in such trivialization, the connection writes

$$\nabla : Y \mapsto dY + \Omega \cdot Y,$$

where  $\Omega$  is the matrix-connection, a two-by-two matrix of meromorphic 1-forms, sections of  $\Omega^1(D)$ . In a coordinate  $x : U \rightarrow \mathbb{C}$ , we have

$$\Omega = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix} \frac{dx}{P(x)}$$

where  $P = \prod_{i=1}^{\nu} (x - t_i)^{n_i}$  is the monic polynomial with divisor  $\text{div}(P) = D|_U$  and  $a, b, c, d$  are holomorphic functions on  $U$ .

Fix local formal data  $\Lambda$  (see section 2) including the polar locus  $D$ . It is the local data of a global meromorphic connection  $(E, \nabla)$  provided that it satisfies Fuchs relation: the sum of residual eigenvalues equals  $\text{deg}(E)$ , and must therefore be an integer. The moduli space of those  $\Lambda$ -connections up to bundle isomorphisms can be constructed by Geometric Invariant Theory. After fixing convenient weights  $\mu$ , the moduli space  $\mathcal{M}_{\mu}^{\Lambda}$  of  $\mu$ -stable parabolic  $\Lambda$ -connections  $(E, \nabla, \mathbf{l})$  forms an irreducible quasi-projective variety of dimension  $2N$  where  $N = \text{deg}(D) - 3$  (see [4, 5, 6]). In fact, the role of parabolic data is mainly to avoid bad singularities in the quotient, and are useful only for special choices of  $\Lambda$ ; for generic  $\Lambda$ , the parabolic structure is automatically determined by  $(E, \nabla)$  and can therefore be omitted, so that  $\mathcal{M}_{\mu}^{\Lambda}$  can be viewed as the moduli space of  $\Lambda$ -connections  $(E, \nabla)$ . In this note, we would like to describe a natural open set of this moduli space in an explicit way, together with an explicit universal family. Natural operations on parabolic connections like twisting by rank one meromorphic connections, or applying convenient birational bundle modifications provide isomorphisms between moduli spaces, usually called canonical transformations. In particular, this has the effect to shift  $\text{deg}(E)$  by arbitrary integers, and we can assume  $\text{deg}(E) = 1$  without loss of generality.

Those connections  $(E, \nabla)$  with  $E \simeq \mathcal{O} \oplus \mathcal{O}(1)$  such that the destabilizing subsheaf  $\mathcal{O}(1) \subset E$  is not  $\nabla$ -invariant form a Zariski open subset  $U \subset \mathcal{M}_{\mu}^{\Lambda}$ . We can define the apparent map as follows. Consider the sequence of maps

$$(1) \quad \begin{array}{ccccccc} \mathcal{O}(1) \hookrightarrow & E & \xrightarrow{\nabla} & E \otimes \Omega^1(D) & \xrightarrow{\text{quotient}} & (E/\mathcal{O}(1)) \otimes \Omega^1(D) & \simeq \mathcal{O}(n-2). \\ & & & & & \searrow \varphi_{\nabla} & \\ & & & & & & \end{array}$$

The composition map  $\varphi_{\nabla} : \mathcal{O}(1) \rightarrow \mathcal{O}(n-2)$  is a morphism (i.e. is  $\mathcal{O}$ -linear). By assumption on  $(E, \nabla)$ ,  $\varphi_{\nabla}$  is not the zero morphism and has therefore  $n-3$  zeroes counted with multiplicity. It thus defines an element of the linear system  $|\mathcal{O}(n-3)|$  and we get a well-defined map

$$(2) \quad \text{App} : U \rightarrow |\mathcal{O}(n-3)| \simeq \mathbb{P}^{(n-3)} ; (E, \nabla) \mapsto \text{div}(\varphi_{\nabla})$$

which is called *apparent map* (see [14, 11, 12]).

In this paper, we generalize to the irregular setting a construction appearing in a paper of Oblezin [14] allowing to define a birational model for  $\mathcal{M}_{\mu}^{\Lambda}$ :

**Theorem 1.** *There is a natural birational map*

$$(3) \quad \Phi : \mathcal{M}_{\mu}^{\Lambda} \dashrightarrow \text{Sym}^{(n-3)} \text{Tot}(\Omega^1(D))$$

where  $\text{Tot}$  stands for the total space, and  $\text{Sym}^{(n-3)}$  for the  $(n-3)^{\text{th}}$  symmetric product.

Precisely, consider the Zariski open subset  $V \subset U \subset \mathcal{M}_\mu^\Lambda$  formed by those connections  $(E = \mathcal{O} \oplus \mathcal{O}(1), \nabla)$  for which the corresponding map  $\varphi_\nabla$  defined as above has  $(n - 3)$  distinct zeroes  $q_1, \dots, q_{(n-3)}$ , all of them distinct from the poles  $t_1, \dots, t_\nu$  of the connection. Then, for such a connection, we can define a birational bundle transformation

$$(4) \quad \phi_\nabla := \text{id} \oplus \varphi_\nabla : \mathcal{O} \oplus \mathcal{O}(1) \dashrightarrow \mathcal{O} \oplus \Omega^1(D)$$

and consider the pushed-forward connection  $\nabla_0 := (\phi_\nabla)_* \nabla$  on  $E_0 = \mathcal{O} \oplus \Omega^1(D)$ . The new connection  $(E_0, \nabla_0)$  has simple poles at  $q_1, \dots, q_{(n-3)}$  with residual eigenvalues 0 and  $-1$  at each pole. One can use automorphisms of  $E_0$  to normalize  $\nabla_0$  in such a way that its restriction to the first factor  $\mathcal{O} \subset E_0$  is trivial; this normalization is unique up to scalar automorphisms. Once this has been done, the Hukuhara-Levelt-Turrittin splitting subsheaves for each singular point of  $\nabla_0$  are determined modulo  $(x - t_i)^{n_i}$  by the local formal data  $\Lambda$ . The 0-eigendirection of  $\nabla_0$  at  $q_j$  corresponds to a point  $\mathbf{p}_j \in \mathbb{P}(E_0)$ . On the other hand,  $\mathbb{P}(E_0)$  can be viewed as the fiber-compactification of  $\Omega^1(D)$  by identifying a section  $s \in \Gamma(U, \Omega^1(D))$  with the section  $\begin{pmatrix} 1 \\ s \end{pmatrix} \in \Gamma(U, E_0)$ ; we denote by  $\text{Tot}(\Omega^1(D)) \subset \mathbb{P}(E_0)$  this natural embedding. As the  $(-1)$ -eigendirection is contained in the second factor of  $E_0$ , we deduce that  $\mathbf{p}_j \in \text{Tot}(\Omega^1(D))$  and we define the map (3) by setting

$$(5) \quad \Phi(E_0, \nabla_0) = \{\mathbf{p}_1, \dots, \mathbf{p}_{(n-3)}\} \in \text{Sym}^{(n-3)} \text{Tot}(\Omega^1(D)).$$

The natural projection  $\pi : \text{Tot}(\Omega^1(D)) \rightarrow \mathbb{P}^1$ ;  $\mathbf{p}_j \mapsto q_j$  induces a morphism

$$\Pi : \text{Sym}^{(n-3)} \text{Tot}(\Omega^1(D)) \rightarrow \text{Sym}^{(n-3)} \mathbb{P}^1 \simeq \mathbb{P}^{(n-3)}$$

which is nothing but the apparent map (2), i.e. we have  $\text{App} = \Pi \circ \Phi$ . We prove that  $\Phi$ , defined as above, induces a 1-to-1 map between the open set  $V \subset \mathcal{M}_\mu^\Lambda$  and the open set  $\Pi^{-1}(W)$ , where  $W$  is the open set formed by those  $\{q_1, \dots, q_{(n-3)}\}$  for which  $q_j \neq t_i$  and  $q_j \neq q_k$  for all  $i, j, k, j \neq k$ . This is a consequence of our normal form (9).

After pushing the Liouville form on  $\text{Tot}(\Omega^1(D))$  via the natural map  $\Omega^1 \rightarrow \Omega^1(D)$ , we get a rational 2-form on  $\text{Tot}(\Omega^1(D))$  with polar divisor  $\pi^*D$ , and which is non degenerate outside  $|\pi^*D|$ . This induces a holomorphic symplectic structure on the open set  $\Pi^{-1}(W)$  and we expect that it coincides with the natural symplectic structure on  $\mathcal{M}_\mu^\Lambda$  (see [8, 1, 5, 6]). This is checked in the Fuchsian case in section 6.

We think this approach to the moduli space of connections could be useful to describe the total moduli space of connections by studying what is happening when conditions  $q_j \neq t_i$  and  $q_j \neq q_k$  are violated. They could be also useful to better understand coalescence of poles  $t_i \rightarrow t_j$  and degenerescence diagram for Garnier systems. Finally, we hope they can be useful for better understanding isomonodromic deformations, and in particular how to compute general degenerate Garnier systems: so far, we only know the explicit form in one variable case (Painlevé equations) and two variables case (listed in [8, 7]).

## 2. LOCAL NORMAL FORM AND LOCAL FORMAL DATA

Let  $z$  be a local coordinate at a pole of  $\nabla$ , and  $Y$  a local trivialization of  $E$ . Then we can write  $\nabla : Y \mapsto dY + \Omega \cdot Y$  with

$$\Omega = \left( \underbrace{\frac{A_0}{z^{k+1}} + \frac{A_1}{z^k} + \dots + \frac{A_k}{z}}_{\Omega^{<0}} + A(z) \right) dz$$

where  $n := k + 1$  is the order of pole,  $A_0, A_1, \dots, A_k \in \mathfrak{gl}_n(\mathbb{C})$  are constant matrices,  $A_0 \neq 0$ , and  $A \in \mathfrak{gl}_n(\mathbb{C}\{z\})$  is holomorphic. We note  $[\Omega]_{z=0}^{<0}$  the negative (or principal) part of  $\Omega$  at  $z = 0$ . We can assume that  $A_0$  is not a scalar matrix, i.e.  $A_0 \neq c \cdot I$ , otherwise, tensoring by the rank 1 connection  $d - c \frac{dz}{z^{k+1}}$  would kill the matrix coefficient of  $\frac{dz}{z^{k+1}}$ , and therefore decrease the pole order, what we have excluded from the beginning.

Given  $M \in \mathfrak{gl}_n(\mathbb{C}[[z]])$ , the formal change of variable  $Y = M\tilde{Y}$  gives the new matrix connection

$$\nabla : \tilde{Y} \mapsto d\tilde{Y} + \underbrace{(M^{-1}\Omega M + M^{-1}dM)}_{\tilde{\Omega}} \tilde{Y}.$$

We note that the new negative part only depends on the negative part of  $M$ :

$$[\tilde{\Omega}]_{z=0}^{<0} = [M^{-1}[\Omega]_{z=0}^{<0} M]_{z=0}^{<0}.$$

**Logarithmic case**  $n = 1$  (and  $k = 0$ ). Clearly, the eigenvalues  $\theta_1, \theta_2 \in \mathbb{C}$  of  $A_0$  are invariant under formal gauge transformation  $M$  as above, and it is known that there exists a holomorphic  $M$  such that

$$(6) \quad \tilde{\Omega} = \begin{pmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{pmatrix} \frac{dz}{z} \quad \text{or} \quad \begin{pmatrix} \theta + n & x^n \\ 0 & \theta \end{pmatrix} \frac{dz}{z}$$

in the resonant case  $\{\theta_1, \theta_2\} = \{\theta, \theta + n\}$ , with  $n \in \mathbb{Z}_{\geq 0}$ .

**Irregular  $n > 1$  and unramified case.** Assume that the leading coefficient  $A_0$  is semi-simple: it has distinct eigenvalues (recall  $A_0$  cannot be scalar). Then, there exists a formal  $M$  such that

$$(7) \quad \tilde{\Omega} = \begin{pmatrix} \lambda^+ & 0 \\ 0 & \lambda^- \end{pmatrix} dz$$

where

$$\lambda^\pm = \frac{\theta_0^\pm}{z^{k+1}} + \frac{\theta_1^\pm}{z^k} + \dots + \frac{\theta_k^\pm}{z}.$$

Moreover,  $M$  is unique up to right multiplication by a constant diagonal or anti-diagonal matrix. We note that the formal invariants  $\lambda^\pm$  can be determined from the negative part  $[\Omega]_{z=0}^{<0}$  of the initial matrix: it suffices to solve

$$\det \left( [\Omega]_{z=0}^{<0} - \lambda I \right) = \lambda^2 - \text{tr} \left( [\Omega]_{z=0}^{<0} \right) \lambda + \det \left( [\Omega]_{z=0}^{<0} \right) = 0$$

up to the  $k + 1$  first coefficients.

An easy consequence of this normal form is that the connection admits a unique formal decomposition as product of rank 1 connections. Indeed, one can write  $E = L^+ \oplus L^-$  with formal line bundles  $L^\pm \subset E$  that are invariant by  $\nabla$ , and  $\nabla|_{L^\pm} = d + \lambda^\pm$ . This is the Hukuhara-Levelt-Turrittin splitting in its simplest form.

**Remark 2.** By a change  $z = \varphi(\tilde{z})$  of coordinate, one can change the normal form, and reduce the difference

$$\lambda^+ - \lambda^- = \frac{d\tilde{z}}{\tilde{z}^{k+1}} + (\theta_k^+ - \theta_k^-) \frac{d\tilde{z}}{\tilde{z}}.$$

However, in this paper, the global coordinate  $x$  is fixed, and relatively to it, we have  $2k + 2$  formal invariants. Isomonodromic deformations appear when we deform the local coordinate (see [10, 3]).

**Irregular  $n > 1$  and ramified case.** The leading coefficient  $A_0$  is a non trivial Jordan block: it has a single eigenvalue with multiplicity 2 but is not scalar. In that case, there exists a formal  $M$  such that

$$(8) \quad \tilde{\Omega} = \begin{pmatrix} \alpha & \beta \\ z\beta & \alpha - \frac{1}{2z} \end{pmatrix} dz$$

where

$$\begin{cases} \alpha = \frac{a_0}{z^{k+1}} + \frac{a_1}{z^k} + \dots + \frac{a_k}{z} \\ \beta = \frac{b_0}{z^{k+1}} + \frac{b_1}{z^k} + \dots + \frac{b_{k-1}}{z^2} \end{cases}$$

We have one less formal invariant, namely  $2k + 1$  instead of  $2k + 2$ . After ramification  $z = \zeta^2$ , and applying an elementary transformation

$$\tilde{Y} = \begin{pmatrix} 1 & 1 \\ \zeta & -\zeta \end{pmatrix} \bar{Y},$$

we get an unramified irregular singular point with matrix connection

$$\bar{\Omega} = \begin{pmatrix} \lambda^+ & 0 \\ 0 & \lambda^- \end{pmatrix}$$

where

$$\lambda^\pm = 2\zeta\alpha(\zeta^2) \pm 2\zeta^2\beta(\zeta^2).$$

Therefore, again we have a unique formal decomposition  $E = L^+ \oplus L^-$ , but in a ramified variable  $\zeta = \sqrt{z}$ , i.e.  $L^\pm$  is generated by the “section”

$$\zeta \mapsto \begin{pmatrix} 1 \\ \pm\zeta \end{pmatrix}.$$

### 3. GLOBAL NORMAL FORM AND EXPLICIT DEFINITION OF $\Phi$

We now consider global rank 2 connections  $(E, \nabla)$  on  $\mathbb{P}^1$ . Denote by  $x$  a coordinate on an affine chart of  $\mathbb{P}^1$  and by  $x = \infty$  the deleted point. The divisor of poles of  $\nabla$  can be written into the form

$$D = \sum_{i=1}^{\nu} n_i \cdot [t_i] + n_\infty \cdot [\infty] \quad \text{and} \quad n := \deg(D) = \sum_{i=1}^{\nu} n_i + n_\infty.$$

Let us first make more explicit the construction of  $\varphi_\nabla$  defined by (1) when  $E = \mathcal{O} \oplus \mathcal{O}(1)$ .

In matrix form, the vector bundle  $\mathcal{O} \oplus \mathcal{O}(k)$  is defined by two charts, one with coordinates  $(x, Y) \in \mathbb{C} \times \mathbb{C}^2$ , and another one with coordinates  $(\tilde{x}, \tilde{Y})$  satisfying

$$x = 1/\tilde{x}, \quad Y = \begin{pmatrix} 1 & 0 \\ 0 & x^k \end{pmatrix} \tilde{Y}.$$

A connection on  $\mathcal{O} \oplus \mathcal{O}(k)$  takes the form  $d + \Omega$  and  $d + \tilde{\Omega}$  in the respective charts with

$$\Omega = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix} dx \quad \text{and} \quad \tilde{\Omega} = \begin{pmatrix} -\tilde{x}^{-2}a(\tilde{x}^{-1}) & -\tilde{x}^{-k-2}b(\tilde{x}^{-1}) \\ -\tilde{x}^{k-2}c(\tilde{x}^{-1}) & -\tilde{x}^{-2}d(\tilde{x}^{-1}) - k\tilde{x}^{-1} \end{pmatrix} d\tilde{x}.$$

If  $E = \mathcal{O} \oplus \mathcal{O}(1)$ , then the matrix connection for  $\nabla$  takes the form

$$\Omega = \begin{pmatrix} \frac{A(x)}{P(x)} & \frac{B(x)}{P(x)} \\ \frac{C(x)}{P(x)} & \frac{D(x)}{P(x)} \end{pmatrix} dx$$

where  $P(x) = \prod_{i=1}^{\nu} (x - t_i)^{n_i}$  and  $A, B, C, D$  are polynomials of degree

$$\begin{cases} \deg(B) & \leq n - 3 \\ \deg(A), \deg(D) & \leq n - 2 \\ \deg(C) & \leq n - 1 \end{cases}$$

with at least one inequality which is an equality. Then the map  $\varphi_{\nabla}$  defined by (1) is here defined by  $B(x)$ . From now on, we assume that  $B$  has degree  $n - 3$ , with simple roots  $q_1, \dots, q_{(n-3)}$ , all of them distinct from the poles  $t_i$ 's. Now, define the birational bundle transformation

$$\phi : \mathcal{O} \oplus \mathcal{O}(1) \dashrightarrow \mathcal{O} \oplus \mathcal{O}(n - 2)$$

in matrix form by setting

$$Y_1 = MY \quad \text{with} \quad M = \begin{pmatrix} 1 & 0 \\ 0 & B(x) \end{pmatrix}.$$

We obtain the new matrix connection

$$\Omega_1 = \left( \begin{array}{cc} \frac{A(x)}{P(x)} & \frac{1}{P(x)} \\ \frac{Q(x)C(x)}{P(x)} & \frac{D(x)}{P(x)} - \sum_{i=1}^{n-3} \frac{1}{x-q_i} \end{array} \right) dx$$

and the change of chart is now given by

$$\tilde{x} = 1/x, \quad \tilde{Y}_1 = \begin{pmatrix} 1 & 0 \\ 0 & x^{n-2} \end{pmatrix} Y_1.$$

We note that  $\phi$  is obtained by applying a positive elementary transformation directed by  $\mathcal{O}(1)$  over each point  $x = q_i$ . Finally, we normalize by applying a holomorphic bundle automorphism. Automorphisms take the form

$$Y_2 = MY_1 \quad \text{with} \quad M = \begin{pmatrix} u & 0 \\ F & v \end{pmatrix}$$

where  $u, v \in \mathbb{C}^*$  are constants and  $F(x)$  is a polynomial of degree  $\leq n - 2$ . By choosing  $F = -A$ , we can kill the  $(1, 1)$ -coefficient of the matrix

$$\Omega_2 = \begin{pmatrix} 0 & \frac{1}{P(x)} \\ * & * \end{pmatrix} dx.$$

More geometrically, this means that *we can deform the subbundle  $\mathcal{O}$  in a unique way such that  $\nabla|_{\mathcal{O}}$  is the trivial connection!* The remaining freedom is given by scalar automorphisms (with  $u = v$  and  $F = 0$ ), which commute with everything. Taking into account that

$$\deg(c_2) = n + n_{\infty} - 4 \quad \text{and} \quad \deg(d_2) = n_{\infty} - 2,$$

we arrive at the final global normal form

$$(9) \quad \Omega_2 = \begin{pmatrix} 0 & \frac{1}{P(x)} \\ c_2(x) & d_2(x) \end{pmatrix} dx \quad \text{with} \quad \begin{cases} c_2(x) = \sum_{i=1}^{\nu} \frac{C_i(x)}{(x-t_i)^{n_i}} + \sum_{j=1}^{n-3} \frac{p_j}{x-q_j} + \tilde{C}(x) + x^{n-3}C_{\infty}(x) \\ d_2(x) = \sum_{i=1}^{\nu} \frac{D_i(x)}{(x-t_i)^{n_i}} + \sum_{j=1}^{n-3} \frac{-1}{x-q_j} + D_{\infty}(x) \end{cases}$$

where

- $\deg(C_i), \deg(D_i) \leq n_i - 1$  for  $i = 1, \dots, \nu$ ,
- $\deg(C_{\infty}) \leq n_{\infty} - 1$  and  $\deg(D_{\infty}) \leq n_{\infty} - 2$ ,
- $\deg(\tilde{C}) \leq n - 4$ .

The 0-eigendirection of  $\nabla_2$  over  $x = q_j$  is defined by  $\begin{pmatrix} 1 \\ p_j \end{pmatrix}$  so that we have canonically associated the collection of points  $\{\mathbf{p}_1, \dots, \mathbf{p}_{(n-3)}\}$  where  $\mathbf{p}_j = (q_j, p_j)$  stands for the point of  $\text{Tot}(\Omega^1(D))$  defined by the section  $p_j \cdot \frac{dx}{P(x)}$  at  $x = q_j$ . The map  $\Phi$  of the introduction is therefore explicitly defined on the open set  $V \subset \mathcal{M}_\mu^\Lambda$  of those connections  $(E, \nabla)$  with  $E = \mathcal{O} \oplus \mathcal{O}(1)$  with  $q_j \neq t_i$  and  $q_j \neq q_k, j \neq k$ . Indeed, we set  $\Phi(E, \nabla) = \{\mathbf{p}_1, \dots, \mathbf{p}_{(n-3)}\} \in \text{Sym}^{(n-3)}\text{Tot}(\Omega^1(D))$ .

#### 4. BIRATIONALITY OF $\Phi$

In this section, we want to inverse the map  $\Phi$  on a Zariski open set. Namely, we now want to prove that, knowing local formal data  $\Lambda$ , i.e. those  $\lambda^\pm$  defined at each singular point like in section 2, and knowing the position  $\{\mathbf{p}_1, \dots, \mathbf{p}_{(n-3)}\}$  of 0-eigendirections of apparent singular points, we can uniquely determine the matrix connection  $\Omega_2$ , i.e. the connection  $(E_2, \nabla_2)$ . If true, we can recover  $(E, \nabla)$  up to isomorphism by applying an elementary transformation at the 0-eigendirection of each apparent pole  $q_j$ .

**Lemma 3.** *Polynomials  $C_i, D_i$  are determined by the local formal invariants  $\lambda^\pm$  at  $t_i$  for  $i = 1, \dots, \nu, \infty$ .*

*Proof.* For singular points  $t_i, i \neq \infty$ , having local normal forms (6) non resonant or (7)

$$(10) \quad \tilde{\Omega} = \begin{pmatrix} \lambda^+ & 0 \\ 0 & \lambda^- \end{pmatrix} dz, \quad \text{with} \quad \lambda^\pm = \frac{\theta_0^\pm}{z^{k+1}} + \frac{\theta_1^\pm}{z^k} + \dots + \frac{\theta_k^\pm}{z} =: \frac{\Lambda^\pm}{z^{k+1}}$$

in variable  $z = x - t_i, k + 1 = n_i$ , we pass from this local normal form to the global one (9) by a formal holomorphic gauge transformation  $\tilde{Y} = MY_2$ . Therefore, the negative part is given by

$$(11) \quad [\Omega_2]_{x=t_i}^{<0} = \left[ \begin{pmatrix} 0 & \frac{1}{P(x)} \\ \frac{C_i(x)}{(x-t_i)^{n_i}} & \frac{D_i(x)}{(x-t_i)^{n_i}} \end{pmatrix} dx \right]_{x=t_i}^{<0} = [M^{-1}\tilde{\Omega}M]_{x=t_i}^{<0}$$

as  $M^{-1}dM$  is holomorphic at  $x = t_i$ , and therefore

$$\left[ \text{trace}(\Omega_2) \right]_{x=t_i}^{<0} = \left[ \text{trace}(\tilde{\Omega}) \right]_{x=t_i}^{<0} \quad \text{and} \quad \left[ (x-t_i)^{n_i} \det(\Omega_2) \right]_{x=t_i}^{<0} = \left[ (x-t_i)^{n_i} \det(\tilde{\Omega}) \right]_{x=t_i}^{<0}.$$

We deduce that polynomials  $C_i$  and  $D_i$  are determined by

$$(12) \quad \begin{cases} C_i(x) = -J^{n_i-1} \left( \Lambda^+ \cdot \Lambda^- \cdot \prod_{j \neq i} (x-t_j)^{n_j} \right) \\ D_i(x) = \Lambda^+ + \Lambda^- \end{cases}$$

where  $J^k$  denotes the jet of order  $k$  of the power-series at  $x = t_i$ , and  $\Lambda^\pm$  defined in (10).

Consider now a ramified singular points  $t_i, i \neq \infty$ , having local normal form (8)

$$\tilde{\Omega} = \begin{pmatrix} \frac{A_i(z)}{z^{n_i}} & \frac{B_i(z)}{z^{n_i}} \\ \frac{\tilde{B}_i(z)}{z^{n_i-1}} & \frac{A_i(z)}{z^{n_i}} - \frac{1}{2z} \end{pmatrix} dz, \quad \text{with} \quad \begin{cases} \deg(A_i) = n_i - 1 \\ \deg(B_i) = n_i - 2 \end{cases}$$

in variable  $z = x - t_i$ . A similar discussion shows that the negative part of  $\Omega_2$  is given by (11) with

$$(13) \quad \begin{cases} C_i(x) = -J^{n_i-1} \left( \left( A_i^2 - \frac{(x-t_i)^{n_i-1}}{2} A_i - (x-t_i) B_i^2 \right) \cdot \prod_{j \neq i} (x-t_j)^{n_j} \right) \\ D_i(x) = 2A_i - \frac{(x-t_i)^{n_i-1}}{2} \end{cases}$$



therefore determining  $C_i$  and  $D_i$  in terms of formal invariants  $A_i, B_i$ .

Finally, the negative part at  $x = \infty$  ( $\Leftrightarrow \zeta = 0$ ) is given by

$$[\tilde{\Omega}_2]_{\zeta=0}^{<0} = - \left( \begin{array}{cc} 0 & \left[ \frac{1}{\zeta P(\frac{1}{\zeta})} \right]_{\zeta=0}^{<0} \\ \frac{1}{\zeta} C_\infty \left( \frac{1}{\zeta} \right) & \frac{1}{\zeta^2} D_\infty \left( \frac{1}{\zeta} \right) + \frac{1 + \sum_{i=1}^{\nu} \tau_i}{\zeta} \end{array} \right) d\zeta$$

where  $\tau_i = \text{Res}_{x=t_i} \frac{D_i(x)}{(x-t_i)^{n_i}} dx$  is the residue of the (2,2)-coefficient at  $t_i$ . Note that the trace of residue at  $x = \infty$  has to be  $1 + \sum_{i=1}^{\nu} \tau_i$  by Fuchs relation for the initial connection  $\nabla$  on  $E = \mathcal{O} \oplus \mathcal{O}(1)$ . Again, we can determine  $C_\infty, D_\infty$  in terms of local formal invariants  $\lambda^\pm$  by a formula similar to (12) or (12).  $\square$

**Remark 4.** *We have just proved that the principal part of the matrix connection at essential poles  $t_i$ 's depends only on local formal data  $\Lambda$ . It is independant of the choice of the connection in the moduli space  $\mathcal{M}_\mu^\Lambda$ . Moreover, the local Hukuhara-Levelt-Turritti decomposition at poles  $t_i$ 's is fixed independly of the connection up to order  $n_i - 1$  as proved in the following.*

**Corollary 5.** *The local (possibly ramified) formal decomposition  $E = L^+ \oplus L^-$  by  $\nabla_2$ -invariant subbundles  $L^\pm$  at each singular point  $t_i$  is determined up to order  $n_i - 1$  by the following equation*

$$J^{n_i-1} \left( L^\pm \right) = \left( J^{n_i-1}(y) \right) \quad \text{where} \quad y^2 = \left( \prod_{j \neq i} (x - t_j)^{n_j} \right) \left( C_i(x) + y D_i(x) \right).$$

*In particular, it depends only on formal invariants, not of accessory parameters.*

*Proof.* Straightforward computation with the negative part of (9) which is given by (11).  $\square$

**Lemma 6.** *Coefficients of the polynomial  $\tilde{C}$  are determined by the fact that singular points  $x = q_j$  are not logarithmic (they are apparent).*

*Proof.* For each  $x = q_j$ , we can write

$$\Omega_2 = \left( \begin{array}{cc} 0 & \frac{1}{P(x)} \\ \frac{p_j}{x-q_j} + \tilde{C}(x) + \hat{c}_j(x) & -\frac{1}{x-q_j} + \hat{d}_j(x) \end{array} \right) dx$$

where  $\hat{c}_j, \hat{d}_j$  are holomorphic at  $x = q_j$ . The singular point  $x = q_j$  is apparent if, and only if, it disappears after elementary transformation  $Y_2 = M\tilde{Y}_2$  with

$$M = \begin{pmatrix} 1 & 0 \\ p_j & x - q_j \end{pmatrix}.$$

We can check that the new matrix connection is holomorphic if, and only if,

$$\tilde{C}(q_j) + \hat{c}_j(q_j) + p_j \hat{d}_j(q_j) - \frac{p_j^2}{P(q_j)} = 0.$$

Running over the  $n - 3$  distinct points  $q_j$ , we get  $n - 3$  independant relations on  $\tilde{C}(x)$  which is therefore uniquely determined.  $\square$

5. SYMPLECTIC STRUCTURE AND LAGRANGIAN FIBRATION

We have identified the open set  $V \subset \mathcal{M}_\mu^\Lambda$  of those connections  $(E, \nabla)$  with  $E = \mathcal{O} \oplus \mathcal{O}(1)$  with  $q_j \neq t_i$  and  $q_j \neq q_k, j \neq k$ , with the open set  $\Pi^{-1}(W)$  of  $\text{Sym}^{(n-3)}\text{Tot}(\Omega^1(D))$ , also defined by the same restrictions on  $q_j$  (see introduction).

On the other hand, this latter space has dimension  $2(n-3)$  and can be equipped with a meromorphic 2-form  $\omega$  inducing a symplectic structure on a large open set including  $\Pi^{-1}(W)$ . Indeed, the natural map which to a (local holomorphic) section  $y(x) \cdot \frac{dx}{P(x)}$  of  $\Omega^1(D)$  associates the meromorphic section  $\frac{y(x)}{P(x)} \cdot dx$  of  $\Omega^1$  induces a rational map  $\psi : \text{Tot}(\Omega^1(D)) \dashrightarrow \text{Tot}(\Omega^1)$ . The Liouville symplectic form  $\omega_{Liouv}$  on  $\text{Tot}(\Omega^1)$  induces by pull-back a 2-form  $\omega := \psi^* \omega_{Liouv}$ . In local coordinates  $(x, y)$  on  $\text{Tot}(\Omega^1(D))$  where  $y$  stands for the section  $y \cdot \frac{dx}{P(x)}$ , we have  $\omega = dy \wedge \frac{dx}{P(x)}$ . The 2-form  $\sum_{j=1}^{n-3} dy_j \wedge \frac{dx_j}{P(x_j)}$  on the product  $\text{Tot}(\Omega^1(D))^{(n-3)}$  is obviously invariant under permutation of factors and defines a rational 2-form on the quotient  $\text{Sym}^{(n-3)}\text{Tot}(\Omega^1(D))$  with poles along hypersurfaces  $t_i = q_j$ . Finally, the symplectic form induced on  $V \subset \mathcal{M}_\mu^\Lambda$  via  $\Phi$  has the explicit expression

$$(14) \quad \omega = \sum_{j=1}^{n-3} dp_j \wedge \frac{dq_j}{P(q_j)}$$

We expect that this symplectic structure coincides with the natural one constructed for instance in [1]. We check this in the Fuchsian case where everything is known to be explicit.

6. LINK WITH SCALAR EQUATION

Our normal form (9) almost looks like a companion matrix. After setting

$$Y_2 = MY_3 \quad \text{with} \quad M = \begin{pmatrix} 1 & 0 \\ 0 & -P(x) \end{pmatrix},$$

we obtain the matrix connection in companion form

$$\Omega_3 = \begin{pmatrix} 0 & -1 \\ -\frac{c_2}{P} & d_2 + \frac{P'}{P} \end{pmatrix}$$

that corresponds to the scalar differential equation

$$(15) \quad u'' + \left(d_2 + \frac{P'}{P}\right) u' + \left(-\frac{c_2}{P}\right) u = 0.$$

Starting from Okamoto's Fuchsian equation for Garnier system (see [8, formula (0.8)])

$$\begin{aligned} & u'' + \left( \frac{1-\kappa_0}{x} + \frac{1-\kappa_1}{x-1} + \sum_{i=1}^{n-3} \frac{1-\theta_i}{x-t_i} - \sum_{j=1}^{n-3} \frac{1}{x-q_j} \right) u' \\ & + \left( \frac{\kappa}{x(x-1)} - \sum_{i=1}^{n-3} \frac{t_i(t_i-1)K_i}{x(x-1)(x-t_i)} + \sum_{j=1}^{n-3} \frac{q_j(q_j-1)\mu_j}{x(x-1)(x-q_j)} \right) u = 0 \end{aligned}$$

with  $K_i$  determined by [8, formula (0.9)], we get the normal form

$$\Omega_2 = \begin{pmatrix} 0 & \frac{1}{P(x)} \\ c_2(x) & d_2(x) \end{pmatrix} dx$$

with

$$\begin{cases} c_2(x) = -\sum_{j=1}^{n-3} \frac{P(q_j)\mu_j}{x-q_j} + C(x), \\ d_2(x) = -\frac{\kappa_0}{x} - \frac{\kappa_1}{x-1} - \sum_{i=1}^{n-3} \frac{\theta_i}{x-t_i} - \sum_{j=1}^{n-3} \frac{1}{x-q_j} \end{cases}$$

where  $C(x)$  is a polynomial of degree  $\leq n-3$  and  $P(x) = x(x-1)\prod_{i=1}^{n-3}(x-t_i)$ . The symplectic form  $\omega_0 = \sum_{j=1}^{n-3} d\mu_j \wedge d\lambda_j$  in [8, page 47] (here  $\lambda_j = q_j$ ) coincide up to a sign with the natural symplectic form of our normal form  $\omega = \sum_{j=1}^{n-3} dp_j \wedge \frac{dq_j}{P(q_j)}$ .

Starting from Kimura's most degenerate scalar equation (see [8, L(9/2;2) page 37])

$$u'' + \left( -\sum_{j=1}^2 \frac{1}{x-q_j} \right) u' + \left( -9x^5 - 9t_1x^3 - 3t_2x^2 - 3K_2x - 3K_1 + \sum_{j=1}^2 \frac{\mu_j}{(x-q_j)} \right) u = 0$$

with  $K_1, K_2$  determined in [8, H(9/2) page 40], we get the normal form (9) with

$$\begin{cases} c_2(x) = 9x^5 + 9t_1x^3 + 3t_2x^2 + 3K_2x + 3K_1 - \sum_{j=1}^2 \frac{\mu_j}{(x-q_j)}, \\ d_2(x) = -\sum_{j=1}^2 \frac{1}{x-q_j} \end{cases}$$

and again we find the same symplectic structure up to a sign:  $\omega = \sum_{j=1}^2 dp_j \wedge dq_j$ ,  $p_j = -\mu_j$ .

#### REFERENCES

1. P. BOALCH, *Symplectic manifolds and isomonodromic deformations*. Adv. Math. **163** (2001) 137-205.
2. A. A. BOLIBRUCH, S. MALEK, C. MITSCHI, *On the generalized Riemann-Hilbert problem with irregular singularities*. Expo. Math. **24** (2006) 235-272.
3. V. HEU, *Universal isomonodromic deformations of meromorphic rank 2 connections on curves*. Ann. Inst. Fourier **60** (2010) 515-549.
4. M. INABA, K. IWASAKI AND M.-H. SAITO, *Dynamics of the sixth Painlevé equation*. Théories asymptotiques et équations de Painlevé, 103-167, Sémin. Congr., **14**, Soc. Math. France, Paris, 2006.
5. M. INABA, M.-H. SAITO, *Moduli of unramified irregular singular parabolic connections on a smooth projective curve*. Kyoto J. Math. **53** (2013) 433-482.
6. M. INABA, *Moduli space of irregular singular parabolic connections of generic ramified type on a smooth projective curve*. arXiv:1606.02369
7. H. KAWAMUKO, *On the Garnier system of half-integer type in two variables*. Funkcial. Ekvac. **52** (2009) 181-201.
8. H. KIMURA, *The degeneration of the two-dimensional Garnier system and the polynomial Hamiltonian structure*. Ann. Mat. Pura Appl. **155** (1989) 25-74.
9. A. KOMYO, M.-H. SAITO, *Explicit description of jumping phenomena on moduli spaces of parabolic connections and Hilbert schemes of points on surfaces*. arXiv:math/1611.00971 (accepted in Kyoto J. of Math.)
10. I. M. KRICHEVER, *Isomonodromy equations on algebraic curves, canonical transformations and Whitham equations*. Dedicated to Yuri I. Manin on the occasion of his 65th birthday. Mosc. Math. J. **2** (2002) 717-752.
11. F. LORAY, M.-H. SAITO, C. SIMPSON, *Foliations on the moduli space of rank two connections on the projective line minus four points*. Geometric and differential Galois theories, 117-170, Sémin. Congr., **27**, Soc. Math. France, Paris, 2013.
12. F. LORAY, M.-H. SAITO, *Lagrangian fibrations in duality on moduli spaces of rank 2 logarithmic connections over the projective line*. Int. Math. Res. Not. IMRN 2015, no. 4, 995-1043.
13. J. MARTINET, J.-P. RAMIS, *Théorie de Galois différentielle et resommation*. Computer algebra and differential equations, 117-214, Comput. Math. Appl., Academic Press, London, 1990.
14. S. OBLEZIN, *Isomonodromic deformations of  $sl(2)$ -Fuchsian systems on the Riemann sphere*. Mosc. Math. J. **5** (2005) 415-441.
15. K. OKAMOTO, *Isomonodromic deformation and Painlevé equations, and the Garnier system*. J. Fac. Sci. Univ. Tokyo Sect. IA Math. **33** (1986) 575-618.
16. M. VAN DER PUT, M.-H. SAITO, *Moduli spaces for linear differential equations and the Painlevé equations*. Ann. Inst. Fourier (Grenoble) **59** (2009) 2611-2667.

DER DE MATHÉMATIQUES ET D'INFORMATIQUE, FAST, UNIVERSITÉ DES SCIENCES, DES TECHNIQUES ET DES TECHNOLOGIES DE BAMAKO, BP: E 3206 MALI.

*E-mail address:* `karamoko.diara2005@yahoo.fr`

UNIV RENNES, CNRS, IRMAR - UMR 6625, F-35000 RENNES, FRANCE.

*E-mail address:* `frank.loray@univ-rennes1.fr`