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EXPONENTIAL CONVERGENCE TO EQUILIBRIUM IN SUPERCRITICAL KINETICALLY CONSTRAINED MODELS AT HIGH TEMPERATURE

LAURE MARÊCHÉ

Abstract: Kinetically constrained models (KCMs) were introduced by physicists to model the liquid-glass transition. They are interacting particle systems on $\mathbb{Z}^d$ in which each element of $\mathbb{Z}^d$ can be in state 0 or 1 and tries to update its state to 0 at rate q and to 1 at rate $1-q$, provided that a constraint is satisfied. In this article, we prove the first non-perturbative result of convergence to equilibrium for KCMs with general constraints: for any KCM in the class termed "supercritical" in dimension 1 and 2, when the initial configuration has product Bernoulli$(1-q')$ law with $q' \neq q$, the dynamics converges to equilibrium with exponential speed when q is close enough to 1, which corresponds to the high temperature regime.

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1. Introduction

Kinetically constrained models (KCMs) are interacting particle systems on $\mathbb{Z}^d$, in which each element (or *site*) of $\mathbb{Z}^d$ can be in state 0 or 1. Each site tries to update its state to 0 at rate q and to 1 at rate $1-q$, with $q \in [0,1]$ fixed, but an update is accepted if and only if a *constraint* is satisfied. This constraint is defined via an *update family* $\mathcal{U} = \{X_1, \dots, X_m\}$, where $m \in \mathbb{N}^*$ and the X_i, called *update rules*, are finite nonempty subsets of $\mathbb{Z}^d \setminus \{0\}$: the constraint is satisfied at a site x if and only if there exists $X \in \mathcal{U}$ such that all the sites in $x + X$ have state zero. Since the constraint at a site does not depend on the state of the site, it can be easily checked that the product Bernoulli$(1-q)$ measure, ν_q, satisfies the detailed balance with respect to the dynamics, hence is reversible and invariant. ν_q is the *equilibrium measure* of the dynamics.

KCMs were introduced in the physics literature by Fredrickson and Andersen [11] to model the liquid-glass transition, an important open problem in condensed matter physics (see [22, 13]). In addition to this physical interest, KCMs are also mathematically challenging, because the presence of the constraints make them very different from classical Glauber dynamics and prevents the use of most of the usual tools.

One of the most important features of KCMs is the existence of blocked configurations. These blocked configurations imply that the equilibrium measure ν_q is not the only invariant measure, which complicate a lot the study of the out-of equilibrium behavior of KCMs; even the basic question of their convergence to ν_q remains open in most cases.

Because of the blocked configurations, one cannot expect such a convergence to equilibrium for all initial laws. Initial measures particularly relevant for physicists are the $\nu_{q'}$ with $q' \neq q$ (see [17]). Indeed, q is a measure of the temperature of the system: the closer q is to 0, the lower the temperature is. Therefore, starting the dynamics with a configuration of law $\nu_{q'}$ means starting with a temperature

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different from the equilibrium temperature. In this case, KCMs are expected to converge to equilibrium with exponential speed as soon as no site is blocked for the dynamics in a configuration of law ν_q or $\nu_{q'}$. However, there have been few results in this direction so far (see [7, 3, 8, 21, 18]), and they have been restricted to particular update families or initial laws.

Furthermore, general update families have attracted a lot of attention in recent years. Indeed, there recently was a breakthrough in the study of a monotone deterministic counterpart of KCMs called bootstrap percolation. Bootstrap percolation is a discrete-time dynamics in which each site of $\mathbb{Z}^d$ can be *infected* or not; infected sites are the bootstrap percolation equivalent of sites at zero. To define it, we fix an update family $\mathcal{U}$ and choose a set A_0 of initially infected sites; then for any $t \in \mathbb{N}^*$, the set of sites that are infected at time t is

$$A_t = A_{t-1} \cup \{x \in \mathbb{Z}^d \,|\, \exists X \in \mathcal{U}, x + X \subset A_{t-1}\},$$

which means that the sites that were infected at time $t-1$ remain infected at time t and a site x that was not infected at time $t-1$ becomes infected at time t if and only if there exists $X \in \mathcal{U}$ such that all sites of $x+X$ are infected at time $t-1$. Until recently, bootstrap percolation had only been considered with particular update families, but the study of general update families was opened by Bollobás, Smith and Uzzell in [6]. Along with Balister, Bollobás, Przykucki and Smith [1], they proved that general update families satisfy the following universality result: in dimension 2, they can be sorted into three classes, *supercritical*, *critical* and *subcritical* (see definition 2), which display different behaviors (their result for the critical class was later refined by Bollobás, Duminil-Copin, Morris and Smith in [5]).

These works opened the study of KCMs with general update families. In [20, 19, 14, 15], Hartarsky, Martinelli, Morris, Toninelli and the author showed that the grouping of two-dimensional update families into supercritical, critical and subcritical is still relevant for KCMs, and established an even more precise classification. However, these results deal only with equilibrium dynamics. Until now, nothing had been shown on out-of-equilibrium KCMs with general update families, apart from a perturbative result in dimension 1 [7].

In this article, we prove that for all supercritical update families, for any initial law $\nu_{q'}$, $q' \in]0,1]$, when q is close enough to 1, the dynamics of the KCM converges to equilibrium with exponential speed. This result holds in dimension 2 and also in dimension 1 for a good definition of one-dimensional supercritical update families. It is the first non-perturbative result of convergence to equilibrium holding for a whole class of update families.

This result may help to gain a better understanding of the out-of-equilibrium behavior of supercritical KCMs. In particular, such results of convergence to equilibrium were key in proving "shape theorems" for specific one-dimensional constraints in [2, 12, 4].

2. Notations and result

Let $d \in \mathbb{N}^*$. We denote by $\|.\|_\infty$ the ℓ^∞-norm on $\mathbb{Z}^d$. For any set S, $|S|$ will denote the cardinal of S.

For any configuration $\eta \in \{0,1\}^{\mathbb{Z}^d}$, for any $x \in \mathbb{Z}^d$, we denote $\eta(x)$ the value of η at x. Moreover, for any $S \subset \mathbb{Z}^d$, we denote η_S the restriction of η to S, and 0_S (or just 0 when S is clear from the context) the configuration on $\{0,1\}^S$ that contains only zeroes.

We set an update family $\mathcal{U} = \{X_1, \dots, X_m\}$ with $m \in \mathbb{N}^*$ and the X_i finite nonempty subsets of $\mathbb{Z}^d \setminus \{0\}$. To describe the classification of update families, we need the concept of *stable directions*.

Definition 1. For $u \in S^{d-1}$, we denote $\mathbb{H}_u = \{x \in \mathbb{R}^d \,|\, \langle x, u\rangle < 0\}$ the half-space with boundary orthogonal to u. We say that u is a *stable direction* for the update family $\mathcal{U}$ if there does not exist $X \in \mathcal{U}$ such that $X \subset \mathbb{H}_u$; otherwise u is *unstable*. We denote by $\mathcal{S}$ the set of stable directions.

[6] gave a classification of two-dimensional update families into supercritical, critical or subcritical depending on their stable directions. Here is the generalization proposed for d-dimensional update families by [5] (definition 9.1 therein), where for any $\mathcal{E} \subset S^{d-1}$, $\mathrm{int}(\mathcal{E})$ is the interior of $\mathcal{E}$ in the usual topology on S^{d-1}.

Definition 2. A d-dimensional update family $\mathcal{U}$ is

- supercritical if there exists an open hemisphere $C \subset S^{d-1}$ that contains no stable direction;
- critical if every open hemisphere $C \subset S^{d-1}$ contains a stable direction, but there exists a hemisphere $C \subset S^{d-1}$ such that $\mathrm{int}(C \cap \mathcal{S}) = \emptyset$;
- subcritical if $\mathrm{int}(C \cap \mathcal{S}) \neq \emptyset$ for every hemisphere $C \subset S^{d-1}$.

Our result will be valid for supercritical update families.

The KCM process with update family $\mathcal{U}$ can be constructed as follows. We set $q \in [0,1]$. Independently for all $x \in \mathbb{Z}^d$, we define two independent Poisson point processes $\mathcal{P}_x^0$ and $\mathcal{P}_x^1$ on $[0,+\infty[$, with respective rates q and $1-q$. We call the elements of $\mathcal{P}_x^0 \cup \mathcal{P}_x^1$ *clock rings* and denote them by $t_{1,x} < t_{2,x} < \cdots$. The elements of $\mathcal{P}_x^0$ will be *0-clock rings* and the elements of $\mathcal{P}_x^1$ will be *1-clock rings*. For any intial configuration $\eta \in \{0,1\}^{\mathbb{Z}^d}$, we construct the KCM as the continuous-time process $(\eta_t)_{t\in[0,+\infty[}$ on $\{0,1\}^{\mathbb{Z}^d}$ defined thus: for any $x \in \mathbb{Z}^d$, $\eta_t(x) = \eta_0(x)$ for $t \in [0, t_{1,x}[$, and for any $k \in \mathbb{N}^*$,

- if there exists $X \in \mathcal{U}$ such that $(\eta_{t_{k,x}^-})_{x+X} = 0_{x+X}$, then $\eta_t(x) = \varepsilon$ for $t \in [t_{k,x}, t_{k+1,x}[$, where $t_{x,k}$ is a ε-clock ring, $\varepsilon \in \{0,1\}$;
- if such an X does not exist, $\eta_t(x) = \eta_{t_{k-1,x}}(x)$ for $t \in [t_{k,x}, t_{k+1,x}[$.

In other words, sites try to update themselves to 0 when there is a 0-clock ring, which happens at rate q, and to 1 when there is a 1-clock ring, which happens at rate $1-q$, but an update at x is successful if and only if there exists an update rule X such that all sites of $x+X$ are at zero. This construction is known as *Harris graphical construction*. One can use the arguments in part 4.3 of [24] to see that it is well-defined. We denote by $\mathbb{P}_\nu$ the law of $(\eta_t)_{t\in[0,+\infty[}$ when the initial configuration has law ν.

For any $q' \in [0,1]$, we denote $\nu_{q'}$ the product Bernoulli$(1-q')$ measure. Since the constraint at a site does not depend on the state of the site, it can be easily checked that ν_q satisfies the detailed balance with respect to the dynamics, hence is reversible and invariant. ν_q is called equilibrium measure of the dynamics.

We will say that a function $f : \{0,1\}^{\mathbb{Z}^d} \mapsto \mathbb{R}$ is *local* if its output depends only on the states of a finite set of sites, and we then denote $\|f\|_\infty = \sup_{\eta\in\{0,1\}^{\mathbb{Z}^d}} |f(\eta)|$ its norm.

Theorem 3. *If $d = 1$ or 2, for any supercritical update family $\mathcal{U}$, for any $q' \in]0,1]$, there exists $q_0 = q_0(\mathcal{U}, q') \in [0,1[$ such that for any $q \in [q_0, 1]$, for any local function $f : \{0,1\}^{\mathbb{Z}^d} \mapsto \mathbb{R}$, there exist two constants $c = c(\mathcal{U}, q') > 0$ and $C = C(\mathcal{U}, q', f) > 0$ such that for any $t \in [0,+\infty[$,*

$$\left|\mathbb{E}_{\nu_{q'}}(f(\eta_t)) - \nu_q(f)\right| \leq Ce^{-ct}.$$

Remark 4. We expect theorem 3 to hold also for $d \geq 3$. However, our proof relies on proposition 13, which is easy for $d = 1$ and was proven in [6] for $d = 2$, but for which there is no equivalent for $d \geq 3$. Such an equivalent would extend our result to $d \geq 3$.

The remainder of this article is devoted to the proof of theorem 3. The argument is based on the proof given in [21] for the particular case of the Fredrickson-Andersen one-spin facilitated model, but brings in novel ideas in order to accommodate the much greater complexity of general supercritical models. From now on, we fix $d = 1$ or 2 and $\mathcal{U}$ a supercritical update family in dimension d. We begin in section 3 by using the notion of dual paths to reduce the proof of theorem 3 to the simpler proof of proposition 7. Then in section 4 we use the concept of codings to simplify it further, reducing it to the proof of proposition 11. In section 5 we introduce an auxiliary oriented percolation process, that we use in section 6 to prove proposition 11 hence theorem 3.

3. Dual paths

In this section, we use the concept of *dual paths* to reduce the proof of theorem 3 to the easier proof of proposition 7. Let $q, q' \in [0,1]$. We notice that the Harris graphical construction allows us to couple a process $(\eta_t)_{t\in[0,+\infty[}$ with initial law $\nu_{q'}$ and a process $(\tilde{\eta}_t)_{t\in[0,+\infty[}$ with initial law ν_q by using the same clock rings but different initial configurations (independent of the clock rings). We denote the joint law by $\mathbb{P}_{q',q}$. We notice that since ν_q is an invariant measure for the dynamics, $\tilde{\eta}_t$ has law ν_q for all $t \in [0,+\infty[$. To prove theorem 3, it is actually enough to show

Proposition 5. *For any $q' \in]0,1]$, there exists $q_0 = q_0(\mathcal{U}, q') \in [0,1[$ such that for any $q \in [q_0, 1]$, there exist two constants $c_1 = c_1(\mathcal{U}, q') > 0$ and $C_1 = C_1(\mathcal{U}, q') > 0$ such that for any $x \in \mathbb{Z}^d$ and $t \in [0,+\infty[$, $\mathbb{P}_{q',q}(\eta_t(x) \neq \tilde{\eta}_t(x)) \leq C_1 e^{-c_1 t}$.*

Indeed, if $f : \{0,1\}^{\mathbb{Z}^d} \mapsto \mathbb{R}$ is a local function depending of a finite set of sites S,

$$\left|\mathbb{E}_{\nu_{q'}}(f(\eta_t)) - \nu_q(f)\right| = \left|\mathbb{E}_{q',q}(f(\eta_t)) - \mathbb{E}_{q',q}(f(\tilde{\eta}_t))\right| \leq \mathbb{E}_{q',q}(|f(\eta_t) - f(\tilde{\eta}_t)|)$$
$$\leq 2\|f\|_\infty \mathbb{P}_{q',q}((\eta_t)_S \neq (\tilde{\eta}_t)_S) \leq 2\|f\|_\infty \sum_{x\in S} \mathbb{P}_{q',q}(\eta_t(x) \neq \tilde{\eta}_t(x)).$$

Therefore we will work on proving proposition 5.

In order to do that, we need to introduce dual paths. We define the *range* ρ of $\mathcal{U}$ by

$$\rho = \max\{\|x\|_\infty \,|\, x \in X, X \in \mathcal{U}\}.$$

For any $x \in \mathbb{Z}^d$, $t > 0$ and $0 \leq t' \leq t$, a dual path of length t' starting at (x,t) (see figure 1) is a right-continuous path $(\Gamma(s))_{0\leq s\leq t'}$ that starts at site x at time t, goes backwards, is allowed to jump only when there is a clock ring, and only to a site within ℓ^∞-distance ρ. To write it rigorously, the path satisfies $\Gamma(0) = x$ and there exists a sequence of times $0 = s_0 < s_1 < \cdots < s_n = t'$ satisfying the following properties: for all $0 \leq k \leq n-1$ and all $s \in [s_k, s_{k+1}[$, $\Gamma(s) = \Gamma(s_k)$, $\Gamma(s_n) = \Gamma(s_{n-1})$ and for all $0 \leq k < n-1$, $t - s_{k+1} \in \mathcal{P}^0_{\Gamma(s_k)} \cup \mathcal{P}^1_{\Gamma(s_k)}$ and $\|\Gamma(s_{k+1}) - \Gamma(s_k)\|_\infty \leq \rho$.

We denote $\mathcal{D}(x,t,t')$ the (random) set of all dual paths of length t' starting from (x,t). A dual path $\Gamma \in \mathcal{D}(x,t,t')$ is called an *activated path* if it "encounters a point at which both processes are at 0", i.e. if there exists $s \in [0,t']$ such that $\eta_{t-s}(\Gamma(s)) = \tilde{\eta}_{t-s}(\Gamma(s)) = 0$. The set of all activated paths in $\mathcal{D}(x,t,t')$ is called $\mathcal{A}(x,t,t')$. We have the

Lemma 6. *For any $x \in \mathbb{Z}^d$ and $t > 0$, if $\eta_t(x) \neq \tilde{\eta}_t(x)$, then for all $0 \leq t' \leq t$, $\mathcal{A}(x,t,t') \neq \mathcal{D}(x,t,t')$.*

Sketch of proof. The proof is the same as for lemma 1 of [21], apart from the fact that if the path is at y, it does not necessarily jump to a neighbor of y, but to an element of $y + X$, $X \in \mathcal{U}$. The idea of the proof is to start a dual path at (x,t), where the two processes disagree, and, staying at x, to go backwards in time until the processes agree at x. At this time, there was an update at x in one

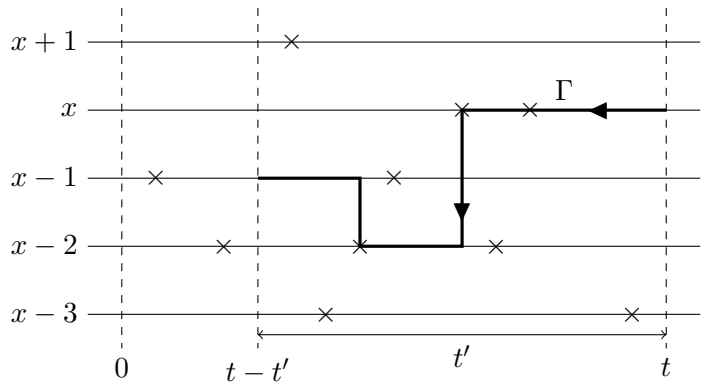

FIGURE 1. Illustration of a dual path Γ of length t' starting at (x,t) for $d=1$ and $\rho=2$. Each horizontal line represents the timeline of a site of $\mathbb{Z}$, the $\times$ representing the clock rings. Γ is the thick polygonal line; it starts at t and ends at $t-t'$. It can jump only when there is a clock ring, and never at a distance greater than $\rho=2$.

process but not in the other, hence an update rule $x+X$ that was full of zeroes in one process but not in the other, thus a site at distance at most ρ of x at which the two processes disagree. We jump to this site and continue to go backwards. This construction yields a dual path along which the two processes disagree, hence they can not be both at zero, so the path is not activated. $\square$

Lemma 6 implies that to prove proposition 5 hence theorem 3, it is enough to prove

Proposition 7. *For any $q' \in]0,1]$, there exists $q_0 = q_0(\mathcal{U}, q') \in [0,1[$ such that for any $q \in [q_0,1]$, there exist two constants $c_2 = c_2(\mathcal{U},q') > 0$ and $C_2 = C_2(\mathcal{U},q') > 0$ such that for any $x \in \mathbb{Z}^d$, $t \in [0,+\infty[$, there exists $0 \le t' \le t$ such that $\mathbb{P}_{q',q}(\mathcal{A}(x,t,t') \neq \mathcal{D}(x,t,t')) \le C_2 e^{-c_2 t}$.*

The remainder of the article will be devoted to the proof of proposition 7.

4. Codings

This section is devoted to the reduction of the proof of proposition 7 (hence of theorem 3) to the simpler proof of proposition 11, via the use of *codings*. The idea is the following: in order to prove proposition 7, it is enough to show that along each dual path, the two processes are at zero at one of the discrete times 0, K, $2K$, etc. hence we only need to consider the positions of the path at these times, which will make up the coding of the path. Let $K \ge 2$ and $t \ge K$. A coding is a sequence $(y_k)_{k\in\{0,\dots,\lfloor \frac{t}{K^2}\rfloor\}}$ of sites in $\mathbb{Z}^d$. Moreover, for $x \in \mathbb{Z}^d$ and $\Gamma \in \mathcal{D}(x,t,\frac{t}{K})$, the coding $\bar{\Gamma}$ of Γ is the sequence $\{\Gamma(kK)\}_{k\in\{0,\dots,\lfloor \frac{t}{K^2}\rfloor\}}$. If $\gamma = (y_k)_{k\in\{0,\dots,\lfloor \frac{t}{K^2}\rfloor\}}$ is a coding, we define the event $G(\gamma) = \left\{\exists k \in \left\{0,\dots,\left\lfloor \frac{t}{K^2}\right\rfloor\right\}, \eta_{t-kK}(y_k) = \tilde{\eta}_{t-kK}(y_k) = 0\right\}$. If $G(\bar{\Gamma})$ is satisfied, Γ is an activated path.

Therefore, to prove proposition 7 hence theorem 3, it is enough to prove

Proposition 8. *For any $q' \in]0,1]$, there exists $q_0 = q_0(\mathcal{U},q') \in [0,1[$ such that for any $q \in [q_0,1]$, there exist two constants $c_3 = c_3(\mathcal{U},q') > 0$ and $C_3 = C_3(\mathcal{U},q') > 0$ and a constant $K = K(\mathcal{U},q') \ge 2$ such that for any $x \in \mathbb{Z}^d$ and $t \ge 2K^2$, $\mathbb{P}_{q',q}(\exists \Gamma \in \mathcal{D}(x,t,\frac{t}{K}), G(\bar{\Gamma})^c) \le C_3 e^{-c_3 t}$.*

Proposition 8 holds only for t greater than a constant, but this is enough, since we only have to enlarge C_3 to obtain a bound valid for all t.

In order to prove proposition 8, we will define a set $C_K^N(x,t)$ of "reasonable codings" and prove that the probability that there exists a dual path whose coding is not in $C_K^N(x,t)$ decays exponentially in t (lemma 9). Then we will count the number of codings in $C_K^N(x,t)$ (lemma 10). Therefore it will be enough to give a bound on $\mathbb{P}_{q',q}(G(\gamma)^c)$ for any $\gamma \in C_K^N(x,t)$ to prove proposition 8 hence theorem 3. Such a bound is stated in proposition 11 and will be proven in section 6.

For any constant $N > 0$, for any $K \geq 2$, $x \in \mathbb{Z}^d$ and $t \geq K$, the set $C_K^N(x,t)$ of "reasonable codings" is defined as the set of $(y_{j_1+\cdots+j_k})_{k\in\{0,\ldots,\lfloor \frac{t}{K^2}\rfloor\}}$ where $(y_i)_{i\in\{0,\ldots,I\}}$ is a sequence of sites satisfiying $y_0 = x$, $I \leq \frac{Nt}{K}$ and $\|y_{i+1} - y_i\|_\infty \leq \rho$ for all $i \in \{0,\ldots,I-1\}$ and where $j_1,\ldots,j_{\lfloor \frac{t}{K^2}\rfloor} \in \mathbb{N}$ satisfy $j_1 + \cdots + j_{\lfloor \frac{t}{K^2}\rfloor} \leq I$. We can now state lemmas 9 and 10, as well as proposition 11. These statements together prove proposition 8.

Lemma 9. *For any $q' \in [0,1]$, there exists $N = N(\mathcal{U}) > 0$ such that for any $K \geq 2$, $q \in [0,1]$, there exists a constant $\check{c} = \check{c}(\mathcal{U},K) > 0$ such that for all $x \in \mathbb{Z}^d$ and $t \geq K$, $\mathbb{P}_{q',q}(\exists \Gamma \in \mathcal{D}(x,t,\frac{t}{K}), \bar{\Gamma} \notin C_K^N(x,t)) \leq e^{-\check{c}t}$.*

In the following, N will always be the N given by lemma 9.

Lemma 10. *There exist constants $\lambda > 0$ and $\beta = \beta(\mathcal{U}) > 0$ such that for any $K \geq 2$, $x \in \mathbb{Z}^d$ and $t \geq 2K^2$, $|C_K^N(x,t)| \leq \lambda(\beta K)^{(d+1)\frac{t}{K^2}}$.*

Proposition 11. *For any $q' \in [0,1]$, there exists a constant $K_0 = K_0(\mathcal{U}) \geq 2$ such that for any $K \geq K_0$, there exists $q_K \in [0,1[$ such that for any $q \in [q_K,1]$, there exist two constants $c_4 = c_4(\mathcal{U},q') > 0$ and $C_4 = C_4(\mathcal{U},K) > 0$ such that for any $x \in \mathbb{Z}^d$, $t \geq K$ and $\gamma \in C_K^N(x,t)$, $\mathbb{P}_{q',q}(G(\gamma)^c) \leq C_4 e^{-c_4\frac{t}{K}}$.*

We are now going to prove lemmas 9 and 10. After that, it will suffice to prove proposition 11 to prove theorem 3.

Sketch of proof of lemma 9. This can be proven with the argument of the lemma 5 of [21]; the idea is that if there exists $\Gamma \in \mathcal{D}(x,t,\frac{t}{K})$ with $\bar{\Gamma} \notin C_K^N(x,t)$, there are so many clock rings that the probability becomes very small. Indeed, let us say Γ visits the sites $y_0 = x, y_1, \ldots, y_{j_1}$ in the time interval $[0,K]$, then the sites $y_{j_1},\ldots,y_{j_1+j_2}$ in the time interval $[K,2K]$, etc. until the sites $y_{j_1+\cdots+j_{\lfloor \frac{t}{K^2}\rfloor}},\ldots,y_{j_1+\cdots+j_{\lfloor \frac{t}{K^2}\rfloor+1}}$ in the time interval $[\lfloor \frac{t}{K^2}\rfloor K, (\lfloor \frac{t}{K^2}\rfloor + 1)K]$. Then the coding of Γ is $\bar{\Gamma} = (y_{j_1+\cdots+j_k})_{k\in\{0,\ldots,\lfloor \frac{t}{K^2}\rfloor\}}$, hence $\bar{\Gamma} \notin C_K^N(x,t)$ implies $j_1 + \cdots + j_{\lfloor \frac{t}{K^2}\rfloor+1} > \frac{Nt}{K}$. It yields that Γ visits more than $\frac{Nt}{K}$ sites in a time $\frac{t}{K}$, and there must be successive clock rings at these sites. The proof of lemma 5 of [21] yields that we can choose N large enough depending on ρ, hence on $\mathcal{U}$, such that the probability of this event is at most $e^{-\check{c}t}$ with $\check{c} = \check{c}(\mathcal{U},N,K) = \check{c}(\mathcal{U},K) > 0$. □

To prove lemma 10, we need the following classical combinatorial result, which will also be used in the proof of lemma 19.

Lemma 12. *For any $I,J \in \mathbb{N}$, $\binom{I}{I} + \binom{I+1}{I} + \cdots + \binom{I+J}{I} = \binom{I+J+1}{I+1}$. Moreover, for any $I,J \in \mathbb{N}$, $|\{(j_1,\ldots,j_I) \in \mathbb{N}^I \,|\, j_1 + \cdots + j_I = J\}| = \binom{I+J-1}{I-1}$.*

The proof of the first part of lemma 12 can be found just before the section 2 of [16] and the proof of the second part in section 1.2 of [23] (weak compositions).

Proof of lemma 10. Let $K \geq 2$, $x \in \mathbb{Z}^d$ and $t \geq 2K^2$. By definition, elements of $C_K^N(x,t)$ have the form $(y_{j_1+\cdots+j_k})_{k\in\{0,\ldots,\lfloor \frac{t}{K^2}\rfloor\}}$ with $(y_i)_{i\in\{0,\ldots,I\}}$ satisfiying $y_0 = x$, $I \leq \frac{Nt}{K}$ and $\|y_{i+1} - y_i\|_\infty \leq \rho$ for all

$i \in \{0, \dots, I-1\}$, and with $j_1, \dots, j_{\lfloor \frac{t}{K^2} \rfloor} \in \mathbb{N}$ satisfying $j_1 + \dots + j_{\lfloor \frac{t}{K^2} \rfloor} \leq I$. Therefore, to count the number of elements of $C_K^N(x,t)$, it is enough to count the number of possible $(j_k)_{k \in \{1,\dots,\lfloor \frac{t}{K^2} \rfloor\}}$ and the number of possible $(y_{j_1+\dots+j_k})_{k \in \{0,\dots,\lfloor \frac{t}{K^2} \rfloor\}}$ given $(j_k)_{k \in \{1,\dots,\lfloor \frac{t}{K^2} \rfloor\}}$.

We begin by counting the number of possible $(j_k)_{k \in \{1,\dots,\lfloor \frac{t}{K^2} \rfloor\}}$. We have $j_1 + \dots + j_{\lfloor \frac{t}{K^2} \rfloor} \leq \frac{Nt}{K}$. Moreover, by the second part of lemma 12, for any integer $0 \leq J \leq \frac{Nt}{K}$, the number of possible sequences of integers $(j_k)_{k \in \{1,\dots,\lfloor \frac{t}{K^2} \rfloor\}}$ such that $j_1 + \dots + j_{\lfloor \frac{t}{K^2} \rfloor} = J$ is at most $\binom{\lfloor \frac{t}{K^2} \rfloor + J - 1}{\lfloor \frac{t}{K^2} \rfloor - 1}$, hence the number of possible $(j_k)_{k \in \{1,\dots,\lfloor \frac{t}{K^2} \rfloor\}}$ is at most $\sum_{J=0}^{\lfloor \frac{Nt}{K} \rfloor} \binom{\lfloor \frac{t}{K^2} \rfloor + J - 1}{\lfloor \frac{t}{K^2} \rfloor - 1} = \binom{\lfloor \frac{t}{K^2} \rfloor + \lfloor \frac{Nt}{K} \rfloor}{\lfloor \frac{t}{K^2} \rfloor}$ by the first part of lemma 12. Furthermore $\binom{\lfloor \frac{t}{K^2} \rfloor + \lfloor \frac{Nt}{K} \rfloor}{\lfloor \frac{t}{K^2} \rfloor} \leq \frac{(\lfloor \frac{t}{K^2} \rfloor + \lfloor \frac{Nt}{K} \rfloor)^{\lfloor \frac{t}{K^2} \rfloor}}{(\lfloor \frac{t}{K^2} \rfloor)!} \leq \lambda \left(\frac{e(\lfloor \frac{t}{K^2} \rfloor + \lfloor \frac{Nt}{K} \rfloor)}{\lfloor \frac{t}{K^2} \rfloor} \right)^{\lfloor \frac{t}{K^2} \rfloor} \leq \lambda \left(e + e \frac{\lfloor \frac{Nt}{K} \rfloor}{\lfloor \frac{t}{K^2} \rfloor} \right)^{\frac{t}{K^2}}$ by the Stirling formula, where $\lambda > 0$ is a constant. In addition, since $t \geq 2K^2$, $\lfloor \frac{t}{K^2} \rfloor \geq \frac{t}{2K^2}$, hence the number of possible $(j_k)_{k \in \{1,\dots,\lfloor \frac{t}{K^2} \rfloor\}}$ is at most $\lambda \left(e + e \frac{Nt}{K} \frac{2K^2}{t} \right)^{\frac{t}{K^2}} = \lambda \left(e + 2eKN \right)^{\frac{t}{K^2}} \leq \lambda (3eKN)^{\frac{t}{K^2}}$ as $K \geq 2$ and N is large.

We now fix a sequence $(j_k)_{k \in \{1,\dots,\lfloor \frac{t}{K^2} \rfloor\}}$ and count the possible $(y_{j_1+\dots+j_k})_{k \in \{0,\dots,\lfloor \frac{t}{K^2} \rfloor\}}$. We know that $y_0 = x$. Moreover, for all $i \in \{0, \dots, j_1 + \dots + j_{\lfloor \frac{t}{K^2} \rfloor} - 1\}$, $\|y_{i+1} - y_i\|_\infty \leq \rho$, hence for each $k \in \{0, \dots, \lfloor \frac{t}{K^2} \rfloor - 1\}$, we have $\|y_{j_1+\dots+j_{k+1}} - y_{j_1+\dots+j_k}\|_\infty \leq \rho j_{k+1}$, so there are at most $(2\rho j_{k+1} + 1)^d$ choices for $y_{j_1+\dots+j_{k+1}}$ given $y_{j_1+\dots+j_k}$. Therefore the number of choices for $(y_{j_1+\dots+j_k})_{k \in \{0,\dots,\lfloor \frac{t}{K^2} \rfloor\}}$ is at most $\prod_{k=1}^{\lfloor \frac{t}{K^2} \rfloor} (2\rho j_k + 1)^d$. Moreover, for any $n \in \mathbb{N}^*$ and any positive $x_1, \dots, x_n$, we have $x_1 \dots x_n \leq (\frac{x_1 + \dots + x_n}{n})^n$, therefore the number of choices is bounded by

$$\left(\frac{\sum_{k=1}^{\lfloor \frac{t}{K^2} \rfloor} (2\rho j_k + 1)}{\lfloor \frac{t}{K^2} \rfloor} \right)^{d \lfloor \frac{t}{K^2} \rfloor} = \left(\frac{2\rho \sum_{k=1}^{\lfloor \frac{t}{K^2} \rfloor} j_k + \lfloor \frac{t}{K^2} \rfloor}{\lfloor \frac{t}{K^2} \rfloor} \right)^{d \lfloor \frac{t}{K^2} \rfloor} \leq \left(\frac{2\rho \frac{Nt}{K} + \lfloor \frac{t}{K^2} \rfloor}{\lfloor \frac{t}{K^2} \rfloor} \right)^{d \frac{t}{K^2}}$$

since $\sum_{k=1}^{\lfloor \frac{t}{K^2} \rfloor} j_k \leq \frac{Nt}{K}$. As $t \geq 2K^2$, $\lfloor \frac{t}{K^2} \rfloor \geq \frac{t}{2K^2}$, thus the number of choices for $(y_{j_1+\dots+j_k})_{k \in \{0,\dots,\lfloor \frac{t}{K^2} \rfloor\}}$ given $(j_k)_{k \in \{1,\dots,\lfloor \frac{t}{K^2} \rfloor\}}$ is bounded by $\left(2\rho \frac{Nt}{K} \frac{2K^2}{t} + 1 \right)^{d \frac{t}{K^2}} = (4\rho NK + 1)^{d \frac{t}{K^2}} \leq (5\rho NK)^{d \frac{t}{K^2}}$. □

5. An auxiliary process

In order to prove proposition 11, we need to find a mechanism for the zeroes to spread in the KCM process; this mechanism uses novel ideas to deal with the complexity of general supercritical models. We begin in section 5.1 by using the bootstrap percolation results of [6] to find a mechanism allowing the zeroes to spread locally (proposition 13). Then we use it in section 5.2 to define an auxiliary oriented percolation process which guarantees that if certain conditions are met, the KCM process is at zero at a given time (proposition 15). Finally, in section 5.3 we prove some properties of this auxiliary process that we will use in section 6.

5.1. Local spread of zeroes.

This is the place where we need the supercriticality of $\mathcal{U}$. Indeed, since $\mathcal{U}$ is supercritical, the results of [6] yield the following proposition (see figure 2):

Proposition 13 ([6]). *For $d = 1$ or 2, there exists $u \in S^{d-1}$, a rectangle R of the following form:*

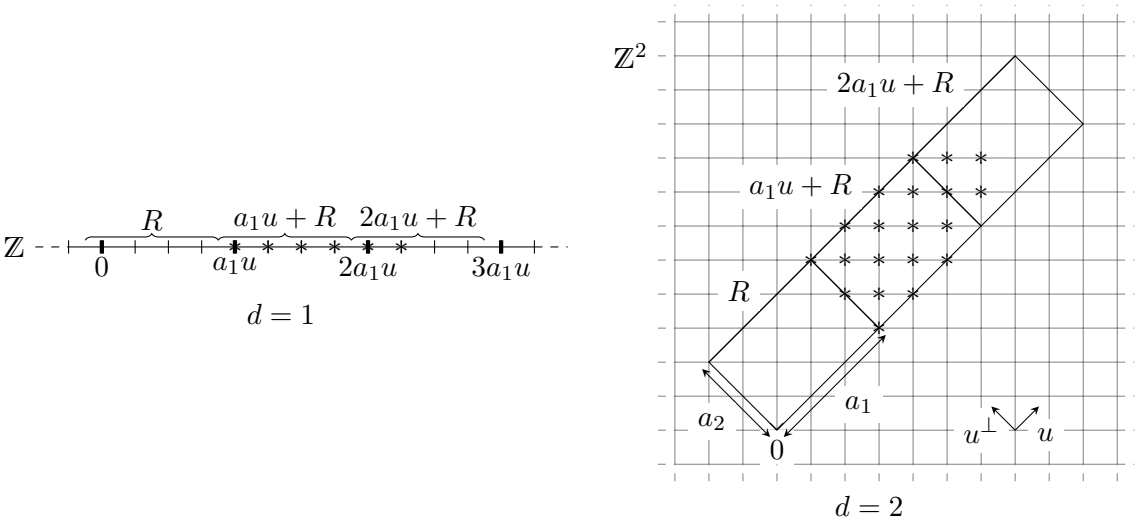

FIGURE 2. Illustration of proposition 13. The $*$ represent the sites $x_1, \dots, x_m$. If we start with the sites of R at zero and there are successive 0-clock rings at $x_1, \dots, x_m$ while there is no 1-clock ring in $R \cup \{x_1, \dots, x_m\}$, these clock rings will put $x_1, \dots, x_m$ at zero, hence the sites of $a_1u + R$ will be put at zero.

- *if* $d = 1$, $R = [0, a_1u[\cap \mathbb{Z}$ *with* $a_1u \in \mathbb{Z}$*;*
- *if* $d = 2$, $R = ([0, a_1[u + [0, a_2]u^\perp) \cap \mathbb{Z}^2$ *with* $a_1u \in \mathbb{Z}^2$*, where* $u^\perp$ *is a vector orthogonal to* u*,*

and a sequence of sites $(x_i)_{1 \le i \le m}$ *in* $(a_1u + R) \cup (2a_1u + R)$ *such that if the sites of* R *are at zero and there are successive 0-clock rings at* $x_1, x_2, \dots, x_m$ *while there is no 1-clock ring in* $R \cup \{x_1, \dots, x_m\}$*, the sites of* $a_1u + R$ *are at zero afterwards.*

Remark 14. For $d \ge 3$, we expect a similar proposition to hold, maybe with $R = [0, a_1[u + \bar{R}$, $\bar{R}$ contained in the hyperplane orthogonal to u, but we can not prove it because an equivalent of the construction of [6] is not available yet. Proving such a construction would be enough to extend our result to any dimension.

Proof of proposition 13. We begin with the case $d = 1$. Since $\mathcal{U}$ is supercritical there exists u an unstable direction. Without loss of generality we can say that $u = 1$, therefore there exists an update rule X contained in $-\mathbb{N}^*$. This yields the mechanism illustrated by figure 3(a): if $R = \{0, \dots, \ell\}$ is sufficiently large and full of zeroes, $(\ell + 1) + X$ is full of zeroes, hence if the site $\ell + 1$ receives a 0-clock ring, this clock ring puts it at zero. Then $(\ell + 2) + X$ is full of zeroes, thus if $\ell + 2$ receives a 0-clock ring, this clock ring puts it at zero. In the same way, if the sites $\ell + 3, \dots, 2\ell + 1$ receive successive 0-clock rings, these clock rings will put them successively at zero, therefore $\{\ell + 1, \dots, 2\ell + 1\} = (\ell + 1) + R$ will be at zero. This yields the result with $a_1 = \ell + 1$ and $(x_i)_{1 \le i \le m} = \ell + 1, \ell + 2, \dots, 2\ell + 1$.

We now consider the case $d = 2$. Since $\mathcal{U}$ is supercritical, there exists a semicircle in S^1 that contains no stable direction; we call u its middle. The results of section 5 of [6] (see in particular figure 5 and lemma 5.5 therein) prove that there exists a set of sites, called a *droplet*, such that in the bootstrap percolation dynamics, if we start with all the sites of the droplet infected, other sites in the direction u can be infected, creating a bigger infected droplet with the same shape (see figure 3(b)). We can enlarge this droplet into a rectangle $R = [0, a_1[u + [0, a_2]u^\perp$ as in figure 3(c); furthermore u can be

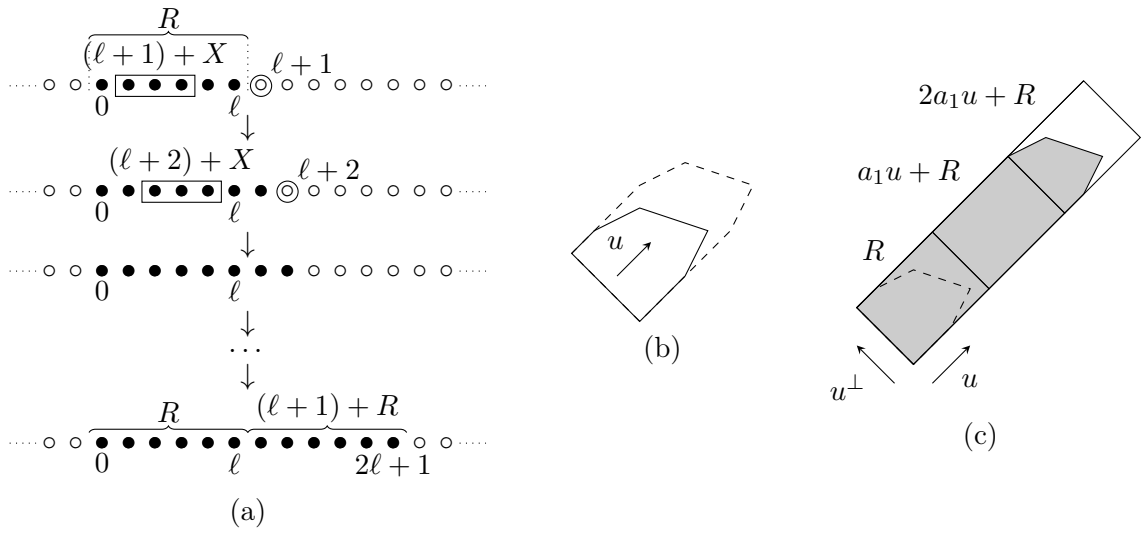

FIGURE 3. The proof of proposition 13. (a) The mechanism for $d = 1$; the $\bullet$ represent zeroes and the $\circ$ represent ones. (b) The shape delimited by the solid line is the droplet of [6]; if it is infected in the bootstrap percolation dynamics, the infection can grow to the shape delimited by the dashed line. (c) R contains the original droplet (dashed line), hence if R is infected, the infection can propagate to a bigger droplet (in gray) that contains $a_1u + R$ and is contained in $R \cup (a_1u + R) \cup (2a_1u + R)$.

chosen rational[1], hence we may enlarge R enough so that $a_1u \in \mathbb{Z}^2$. Now, since R contains the original droplet, if R is infected the infection can grow from the droplet into a droplet big enough to contain a_1u+R while staying in $R\cup(a_1u+R)\cup(2a_1u+R)$ (see figure 3(c)). We call $x_1, \dots, x_m$ the sites that are successively infected during this growth (sites infected at the same time are ordered arbitrarily). Since x_1 is the first site infected by the bootstrap percolation dynamics starting with the sites of R infected, there exists an update rule X such that $x_1 + X \subset R$, therefore if the KCM dynamics starts with all the sites of R at zero and there is a 0-clock ring at x_1, this clock ring sets x_1 to zero. Then, if there is a 0-clock ring at x_2, it will set x_2 to zero for the same reason, and successive 0-clock rings at $x_3, \dots, x_m$ will set them successively to 0, which puts $a_1u + R$ at zero. □

5.2. **Definition of the auxiliary process.** Let $K > 0$, $q \in [0, 1]$ and $t \geq K$. For any $y \in \mathbb{Z}^d$ and $k \in \{0, \dots, \lfloor \frac{t}{K} \rfloor\}$, we will define an oriented percolation process $\zeta^{y,k}$ on $\mathbb{Z}$, from time zero to time $n^{y,k} = \lfloor \frac{t}{K} \rfloor - k$ (see [9] for an introduction to oriented percolation). For $n \in \{1, \dots, n^{y,k}\}$ and $r \in \mathbb{Z}$ with $r + n$ even, the *bonds* $(r-1, n-1) \to (r, n)$ and $(r+1, n-1) \to (r, n)$ can be *open* or *closed*; we set $\zeta_0^{y,k}(r) = \mathbb{1}_{\{r=0\}}$, and for any $n \in \{1, \dots, n^{y,k}\}$, $r \in \mathbb{Z}$ with $r + n$ even, $\zeta_n^{y,k}(r) = 1$ if and only if $\zeta_{n-1}^{y,k}(r-1) = 1$ and the bond $(r-1, n-1) \to (r, n)$ is open or $\zeta_{n-1}^{y,k}(r+1) = 1$ and the bond $(r+1, n-1) \to (r, n)$ is open.

The state of the bonds is defined as follows. For any $n \in \{1, \dots, n^{y,k}\}$, $r \in \mathbb{Z}$ with $r + n$ even:

[1]Indeed, theorem 1.10 of [6] states that the set of stable directions is a finite union of closed intervals with rational endpoints, hence the semicircle containing no stable direction can be chosen with rational endpoints.

- $(r-1,n-1)\to(r,n)$ is open if and only if
$$\left\{\forall x\in y+\frac{r-n}{2}a_1u+R,]t-(k+n)K,t-(k+n-1)K]\cap\mathcal{P}^1_x=\emptyset\right\},$$
i.e. there is no 1-clock ring in $y+\frac{r-n}{2}a_1u+R$ during the time interval $]t-(k+n)K,t-(k+n-1)K]$;
- $(r+1,n-1)\to(r,n)$ is open if and only if
$$\left\{\exists t-(k+n)K<t_1<\cdots<t_m\le t-(k+n-1)K,\forall i\in\{1,\dots,m\},t_i\in\mathcal{P}^0_{y+\frac{r-n}{2}a_1u+x_i}\right\}$$
$$\cap\left\{\forall x\in y+\frac{r-n}{2}a_1u+R\cup\{x_1,\dots,x_m\},]t-(k+n)K,t-(k+n-1)K]\cap\mathcal{P}^1_x=\emptyset\right\},$$
i.e. there are successive 0-clock rings in the equivalent of $x_1,\dots,x_m$ for $y+\frac{r-n}{2}a_1u+R$ during the time interval $]t-(k+n)K,t-(k+n-1)K]$, and no 1-clock ring at these sites or in $y+\frac{r-n}{2}a_1u+R$ in this time interval.

We notice that if all the sites of $y+\frac{r-n}{2}a_1u+R$ are at zero at time $t-(k+n)K$ and $(r-1,n-1)\to(r,n)$ is open, the sites of $y+\frac{r-n}{2}a_1u+R$ are still at zero at time $t-(k+n-1)K$. Moreover, by proposition 13, if the sites of $y+\frac{r-n}{2}a_1u+R$ are at zero at time $t-(k+n)K$ and $(r+1,n-1)\to(r,n)$ is open, the sites of $a_1u+(y+\frac{r-n}{2}a_1u+R)=y+\frac{(r+1)-(n-1)}{2}a_1u+R$ are at zero at time $t-(k+n-1)K$. This allows us to deduce (see figure 4 for an illustration of the mechanism):

Proposition 15. *If there exists $r_0\in\mathbb{Z}$ such that $\zeta^{y,k}_{n^{y,k}}(r_0)=1$ and the sites of $y+\frac{r_0-n^{y,k}}{2}a_1u+R$ are at zero at time $t-\lfloor\frac{t}{K}\rfloor K$, then the sites of $y+R$ are at zero at time $t-kK$.*

5.3. **Properties of the auxiliary process.** In this subsection we state the two oriented percolation properties of $\zeta^{y,k}$, propositions 17 and 18, that we will use in section 6 to prove proposition 11. In order to do that, we need a bound on the probability that a bond is closed; this will be lemma 16. It is there that we need q bigger than a $q_0>0$; this is necessary so that the probability that there is no 1-clock ring at the sites we consider is large. For any $K>0$, we set $q_K=1+\frac{1}{3K|R|}\ln(1-e^{-K})$. We can then state

Lemma 16. *There exists a constant $K_p=K_p(\mathcal{U})>0$ such that for $K\ge K_p$, $q\in[q_K,1]$, $t\ge K$, $y\in\mathbb{Z}^d$ and $k\in\{0,\dots,\lfloor\frac{t}{K}\rfloor\}$, the probability that any given bond is closed for the process $\zeta^{y,k}$ is smaller than $e^{-\frac{K}{4}}$.*

Proof. Let $K>0$, $q\in[q_K,1]$, $t\ge K$, $y\in\mathbb{Z}^d$ and $k\in\{0,\dots,\lfloor\frac{t}{K}\rfloor\}$. Let $n\in\{1,\dots,n^{y,k}\}$, $r\in\mathbb{Z}$ with $r+n$ even. We notice that if the bond $(r-1,n-1)\to(r,n)$ is closed, the bond $(r+1,n-1)\to(r,n)$ is also closed, hence it is enough to bound the probability that $(r+1,n-1)\to(r,n)$ is closed. Denoting $E_1=\{\forall x\in y+\frac{r-n}{2}a_1u+R\cup\{x_1,\dots,x_m\},]t-(k+n)K,t-(k+n-1)K]\cap\mathcal{P}^1_x=\emptyset\}$ and $E_2=\{\exists t-(k+n)K<t_1<\cdots<t_m\le t-(k+n-1)K,\forall i\in\{1,\dots,m\},t_i\in\mathcal{P}^0_{y+\frac{r-n}{2}a_1u+x_i}\}$, we need to bound the probabilities of E_1^c and E_2^c. We begin with E_1^c. The events $]t-(k+n)K,t-(k+n-1)K]\cap\mathcal{P}^1_x=\emptyset$ are independent and have probability $e^{-(1-q)K}$ each; moreover, $x_1,\dots,x_m$ belong to $(a_1u+R)\cup(2a_1u+R)$, so $|R\cup\{x_1,\dots,x_m\}|\le 3|R|$; we deduce the probability of E_1 is
$$e^{-|R\cup\{x_1,\dots,x_m\}|(1-q)K}\ge e^{-3|R|(1-q)K}\ge e^{-3|R|(1-q_K)K}$$
$$\ge e^{-3|R|\left(1-\left(1+\frac{1}{3K|R|}\ln(1-e^{-K})\right)\right)K}=e^{\ln(1-e^{-K})}=1-e^{-K},$$

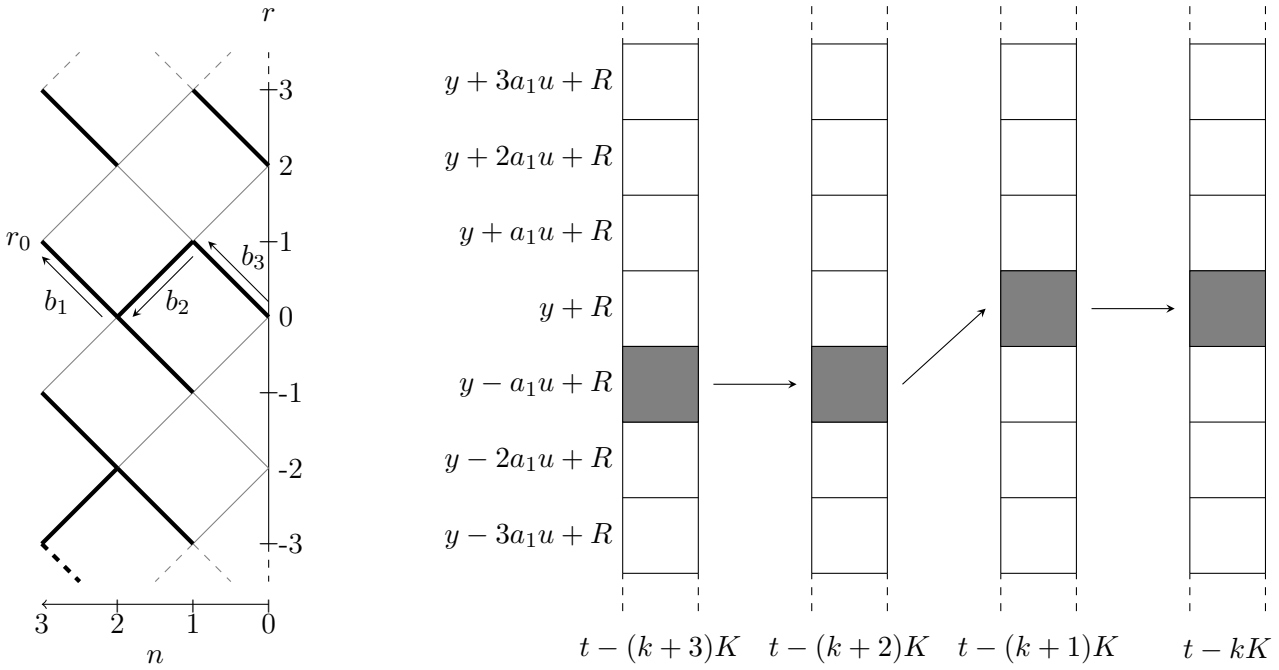

FIGURE 4. An illustration of proposition 15 with $n^{y,k} = 3$ and $r_0 = 1$. The figure at the left represents the bonds of the oriented percolation process $\zeta^{y,k}$; the open bonds are the thick ones, and the path of open bonds allowing $\zeta^{y,k}_{n^{y,k}}(r) = 1$ is outlined by arrows. The figure at the right represents the consequences on the KCM process; each vertical strip represents the state of $\bigcup_{i\in\mathbb{Z}}(y + ia_1u + R)$ at a certain time. If at time $t-(k+3)K$ the rectangle $y+\frac{1-n^{y,k}}{2}a_1u+R = y-a_1u+R$ is at zero (in gray), since the bond $(0,2) \to (1,3)$ (bond b_1) is open, $y - a_1u + R$ is still at zero at time $t-(k+2)K$. Moreover, since $(1,1) \to (0,2)$ (bond b_2) is open and $y - a_1u + R$ is at zero at time $t-(k+2)K$, $a_1u + (y - a_1u + R) = y + R$ is at zero at time $t-(k+1)K$. Finally, since $(0,0) \to (1,1)$ (bond b_3) is open and $y + R$ is at zero at time $t-(k+1)K$, $y + R$ is still at zero at time $t - kK$.

thus the probability of E_1^c is at most e^{-K}. Moreover, the probability of E_2^c is the probability that a Poisson point process of parameter q has strictly less than m elements in an interval of length K, hence it is $\sum_{i=0}^{m-1} e^{-qK}\frac{(qK)^i}{i!}$. When K is large enough, $q \in [1/2, 1]$, hence this probability is smaller than $e^{-\frac{1}{2}K}\sum_{i=0}^{m-1}\frac{K^i}{i!}$, which is smaller than $e^{-\frac{K}{3}}$ when K is large enough depending on m, hence on $\mathcal{U}$. Consequently, when K is large enough depending on $\mathcal{U}$, the probability that $(r+1, n-1) \to (r, n)$ is closed is smaller than $e^{-K} + e^{-\frac{K}{3}}$, which is smaller than $e^{-\frac{K}{4}}$ when K is large enough. □

Thanks to lemma 16, it is possible to prove two oriented percolation properties of $\zeta^{y,k}$. Firstly, for any $K > 0$, $q \in [q_K, 1]$, $t \geq K$, $y \in \mathbb{Z}^d$ and $k \in \{0, \dots, \lfloor \frac{t}{K} \rfloor\}$, we define $\tau^{y,k} = \inf\{n \in \{0, \dots, n^{y,k}\} \,|\, \forall r \in \mathbb{Z}, \zeta^{y,k}_n(r) = 0\}$ the time of death of the process $\zeta^{y,k}$ (if the set is empty, $\tau^{y,k}$ is infinite). Since $\zeta^{y,k}_0(r) = \mathbb{1}_{\{r=0\}}$, which is not identically zero, $\tau^{y,k} \geq 1$. Then we have

Proposition 17. *For any $q' \in [0,1]$, there exists a constant $K_c = K_c(\mathcal{U}) > 0$ such that for any $K \geq K_c$, $q \in [q_K, 1]$, $t \geq K$, $y \in \mathbb{Z}^d$, $k \in \{0, \dots, \lfloor \frac{t}{K} \rfloor\}$, $n \in \{0, \dots, n^{y,k}\}$, $\mathbb{P}_{q',q}(n \leq \tau^{y,k} < +\infty) \leq 23^{2n} e^{-\frac{Kn}{24}}$.*

Sketch of proof. The proposition can be proven by a classical contour method like the one presented in section 10 of [9]. The idea is that if $n \leq \tau^{y,k} < +\infty$ we can draw a "contour of closed bonds" around the connected component of ones in $\zeta^{y,k}$, and this contour will have length $\Omega(n)$. Furthermore, it can be seen that bonds separated by at least 5 bonds from each other are independent, because they depend on clock rings in disjoint space-time intervals. Therefore if we keep one bond out of 6, we extract $\Omega(n)$ independent closed bonds from the contour, each of them having probability $e^{-\frac{K}{4}}$ from lemma 16 when $K \geq K_p$, hence the bound. □

$\zeta^{y,k}$ also satisfies a second property. For any $K > 0$, $q \in [q_K, 1]$, $t \geq K$, $y \in \mathbb{Z}^d$ and $k \in \{0, \dots, \lfloor \frac{t}{K} \rfloor\}$, we denote $\mathcal{X}^{y,k} = \{r \in \{-\lfloor \frac{n^{y,k}}{2} \rfloor, \dots, \lfloor \frac{n^{y,k}}{2} \rfloor\} \,|\, \zeta^{y,k}_{n^{y,k}}(r) = 1\}$. Then we have

Proposition 18. *For any $q' \in [0,1]$, $\alpha \in]0,1[$, there exists a constant $K_g(\alpha) = K_g(\mathcal{U}, \alpha) > 0$ such that for any $K \geq K_g(\alpha)$, there exist constants $c_g > 0$ and $C_g = C_g(\mathcal{U}, K, \alpha) > 0$ such that for any $q \in [q_K, 1]$, $t \geq K$, $y \in \mathbb{Z}^d$ and $k \in \{0, \dots, \lfloor \frac{t}{K} \rfloor\}$, $\mathbb{P}_{q',q}\left(\tau^{y,k} = +\infty, |\mathcal{X}^{y,k}| \leq \frac{\alpha}{2} n^{y,k}\right) \leq C_g e^{-c_g n^{y,k}}$.*

Sketch of proof. This proposition comes from classical results in oriented percolation. Firstly, if the process survives until time $n^{y,k}$, it has a big "range", which means that if we define $r^{y,k} = \sup\{r \in \mathbb{Z} \,|\, \zeta^{y,k}_{n^{y,k}}(r) = 1\}$ and $\ell^{y,k} = \inf\{r \in \mathbb{Z} \,|\, \zeta^{y,k}_{n^{y,k}}(r) = 1\}$, $r^{y,k}$ and $|\ell^{y,k}|$ are so large $\{-\lfloor \frac{n^{y,k}}{2} \rfloor, \dots, \lfloor \frac{n^{y,k}}{2} \rfloor\} \subset \{\ell^{y,k}, \dots, r^{y,k}\}$; this can be proven with the contour argument in section 11 of [9]. Moreover, the argument that proves (1) in [9] also proves that in $\{\ell^{y,k}, \dots, r^{y,k}\}$, $\zeta^{y,k}_{n^{y,k}}$ coincides with the oriented percolation process that has the same bonds, but which starts with all sites at 1 instead of just the origin. Finally, the end of section 5 of [10] contains a contour argument for the latter process which allows to prove that it has a lot of ones; we can use this argument with the same adaptations we used for the contours of proposition 17. □

6. Proof of proposition 11

In this section we use the auxiliary process defined in section 5 to give a proof of proposition 11. In order to do that, we need some definitions. For any $q' \in]0,1]$, $K \geq 2$, $q \in [q_K, 1]$, $x \in \mathbb{Z}^d$, $t \geq K$ and $\gamma = (y_k)_{k \in \{0, \dots, \lfloor \frac{t}{K^2} \rfloor\}} \in C^N_K(x,t)$, we define $k(\gamma) = \inf\{k \in \{0, \dots, \lfloor \frac{t}{K^2} \rfloor\} \,|\, \tau^{y_k,k} = +\infty\}$ if such a k exists; in this case we also denote $y(\gamma) = y_{k(\gamma)}$ (in the following, when we write $k(\gamma)$ or $y(\gamma)$ without more precision, we always assume that they exist). For any $r \in \mathcal{X}^{y(\gamma),k(\gamma)}$ we define the events

$$W^{\gamma,\eta}(r) = \left\{ \left(\eta_{t - \lfloor \frac{t}{K} \rfloor K}\right)_{y(\gamma) + \frac{r - n^{y(\gamma),k(\gamma)}}{2} a_1 u + R} = 0 \right\}, W^{\gamma,\tilde{\eta}}(r) = \left\{ \left(\tilde{\eta}_{t - \lfloor \frac{t}{K} \rfloor K}\right)_{y(\gamma) + \frac{r - n^{y(\gamma),k(\gamma)}}{2} a_1 u + R} = 0 \right\}.$$

By proposition 15, if $\{\exists r \in \mathcal{X}^{y(\gamma),k(\gamma)}, W^{\gamma,\eta}(r)\} \cap \{\exists r \in \mathcal{X}^{y(\gamma),k(\gamma)}, W^{\gamma,\tilde{\eta}}(r)\}$, then the sites of $y(\gamma) + R$ are at zero at time $t - k(\gamma)K$ in both processes $(\eta_t)_{t \in [0,+\infty[}$ and $(\tilde{\eta}_t)_{t \in [0,+\infty[}$, in particular $y(\gamma)$ is at

zero at time $t-k(\gamma)K$ in both processes, therefore $G(\gamma)$ is satisfied. Consequently,

$$\begin{aligned}\mathbb{P}_{q',q}(G(\gamma)^c) \leq & \mathbb{P}_{q',q}\left(k(\gamma) \text{ does not exist}\right) + \mathbb{P}_{q',q}\left(k(\gamma) \text{ exists}, |\mathcal{X}^{y(\gamma),k(\gamma)}| \leq \frac{t}{6K}\right) \\ & + \mathbb{P}_{q',q}\left(\left\{|\mathcal{X}^{y(\gamma),k(\gamma)}| > \frac{t}{6K}\right\} \cap \left\{\forall r \in \mathcal{X}^{y(\gamma),k(\gamma)}, W^{\gamma,\eta}(r)^c\right\}\right) \\ & + \mathbb{P}_{q',q}\left(\left\{|\mathcal{X}^{y(\gamma),k(\gamma)}| > \frac{t}{6K}\right\} \cap \left\{\forall r \in \mathcal{X}^{y(\gamma),k(\gamma)}, W^{\gamma,\tilde{\eta}}(r)^c\right\}\right).\end{aligned}$$

Therefore we only have to prove the following lemmas 19, 20 and 21 to prove proposition 11, thus ending the proof of theorem 3:

Lemma 19. *For any $q' \in]0,1]$, there exists a constant $K_1 = K_1(\mathcal{U}) \geq 2$ such that for any $K \geq K_1$, $q \in [q_K, 1]$, there exist constants $\breve{c}_1 > 0$ and $\breve{C}_1 = \breve{C}_1(K) > 0$ such that for any $x \in \mathbb{Z}^d$, $t \geq K$, $\gamma \in C_K^N(x,t)$, we have $\mathbb{P}_{q',q}(k(\gamma)$ does not exist$) \leq \breve{C}_1 e^{-\breve{c}_1 \frac{t}{K}}$.*

Lemma 20. *For any $q' \in]0,1]$, there exists a constant $K_2 = K_2(\mathcal{U}) \geq 2$ such that for any $K \geq K_2$, $q \in [q_K, 1]$, there exist constants $\breve{c}_2 > 0$ and $\breve{C}_2 = \breve{C}_2(\mathcal{U}, K) > 0$ such that for any $x \in \mathbb{Z}^d$, $t \geq K$, $\gamma \in C_K^N(x,t)$, $\mathbb{P}_{q',q}(k(\gamma)$ exists, $|\mathcal{X}^{y(\gamma),k(\gamma)}| \leq \frac{t}{6K}) \leq \breve{C}_2 e^{-\breve{c}_2 \frac{t}{K}}$.*

Lemma 21. *For any $q' \in]0,1]$, $K \geq 2$, $q \in [q_K, 1]$, there exists a constant $\breve{c}_3 = \breve{c}_3(\mathcal{U}, q') > 0$ such that for any $x \in \mathbb{Z}^d$, $t \geq K$, $\gamma \in C_K^N(x,t)$, we get $\mathbb{P}_{q',q}(\{|\mathcal{X}^{y(\gamma),k(\gamma)}| > \frac{t}{6K}\} \cap \{\forall r \in \mathcal{X}^{y(\gamma),k(\gamma)}, W^{\gamma,\eta}(r)^c\}) \leq e^{-\breve{c}_3 \frac{t}{K}}$ and $\mathbb{P}_{q',q}(\{|\mathcal{X}^{y(\gamma),k(\gamma)}| > \frac{t}{6K}\} \cap \{\forall r \in \mathcal{X}^{y(\gamma),k(\gamma)}, W^{\gamma,\tilde{\eta}}(r)^c\}) \leq e^{-\breve{c}_3 \frac{t}{K}}$.*

Proof of lemma 19. We set $K_1 = \max(K_c, 48(\ln 36 + 1))$, which depends only on $\mathcal{U}$. Let $q' \in]0,1]$, $K \geq K_1$, $q \in [q_K, 1]$, $x \in \mathbb{Z}^d$, $t \geq K$ and $\gamma = (y_k)_{k \in \{0,\dots,\lfloor \frac{t}{K^2} \rfloor\}} \in C_K^N(x,t)$. If $k(\gamma)$ does not exist, $\tau^{y_k,k}$ is finite for $k \in \{0,\dots,\lfloor \frac{t}{K^2} \rfloor\}$, therefore if we call $k_1 = 0$ and $k_i = \sum_{j=1}^{i-1} \tau^{y_{k_j},k_j}$ for $i \geq 2$, $\tau^{y_{k_i},k_i}$ is finite as long as $k_i \leq \lfloor \frac{t}{K^2} \rfloor$. We will use proposition 17 to bound the probability that this happens. We call $L = \max\{i \geq 1 \,|\, k_i \leq \lfloor \frac{t}{K^2} \rfloor\}$; we then have $\sum_{i=1}^{L} \tau^{y_{k_i},k_i} > \lfloor \frac{t}{K^2} \rfloor$, hence if n_L is the integer satisfying $n_L = \lfloor \frac{t}{K^2} \rfloor - \sum_{i=1}^{L-1} \tau^{y_{k_i},k_i}$, we have $n_L \leq \tau^{y_{k_L},k_L} < +\infty$. Furthermore, if $n_1,\dots,n_{L-1}$ are integers satisfiying $n_i = \tau^{y_{k_i},k_i}$ for $i \in \{1,\dots,L-1\}$, we get $n_1 + \dots + n_L = \lfloor \frac{t}{K^2} \rfloor$, $k_i = \sum_{j=1}^{i-1} n_j$ for all $i \in \{1,\dots,L\}$ (we denote $\sum_{j=1}^{i-1} n_j = N_i$). In addition, since $\tau^{y_k,k} \geq 1$ for any $k \in \{0,\dots,\lfloor \frac{t}{K^2} \rfloor\}$, $L \leq \lfloor \frac{t}{K^2} \rfloor + 1$. We deduce

$$\begin{aligned}& \mathbb{P}_{q',q}(k(\gamma) \text{ does not exist}) \\ \leq & \sum_{M \leq \left\lfloor \frac{t}{K^2} \right\rfloor + 1, n_1 + \dots + n_M = \left\lfloor \frac{t}{K^2} \right\rfloor} \mathbb{P}_{q',q}(L = M, \forall 1 \leq i \leq M-1, \tau^{y_{N_i},N_i} = n_i, n_M \leq \tau^{y_{N_M},N_M} < +\infty).\end{aligned}$$

Moreover, the events $\{\tau^{y_{k_{N_i}},N_i} = n_i\}$, $i \in \{1,\dots,M-1\}$ and $\{n_M \leq \tau^{y_{k_{N_M}},N_M} < +\infty\}$ depend only on clock rings in the time intervals $]t-(N_i+n_i)K, t-N_iK] =]t-N_{i+1}K, t-N_iK]$, $i \in \{1,\dots,M-1\}$ and $]t-(N_M+n_M)K, t-N_MK]$, which are disjoint, thus the events are independent, hence

$$\begin{aligned}& \mathbb{P}_{q',q}(L = M, \forall 1 \leq i \leq M-1, \tau^{y_{N_i},N_i} = n_i, n_M \leq \tau^{y_{N_M},N_M} < +\infty) \\ \leq & \left(\prod_{i=1}^{M-1} \mathbb{P}_{q',q}\left(\tau^{y_{N_i},N_i} = n_i\right)\right) \mathbb{P}_{q',q}\left(n_M \leq \tau^{y_{N_M},N_M} < +\infty\right) \leq \prod_{i=1}^{M} \mathbb{P}_{q',q}\left(n_i \leq \tau^{y_{N_i},N_i} < +\infty\right)\end{aligned}$$

$$\leq \prod_{i=1}^{M} 23^{2n_i} e^{-\frac{Kn_i}{24}} = 2^M 3^{2\sum_{i=1}^M n_i} e^{-\frac{K}{24}\sum_{i=1}^M n_i} = 2^M 3^{2\left\lfloor \frac{t}{K^2}\right\rfloor} e^{-\frac{K}{24}\left\lfloor \frac{t}{K^2}\right\rfloor}$$

by proposition 17 and since $n_1 + \cdots + n_M = \left\lfloor \frac{t}{K^2}\right\rfloor$. Consequently,

$$\mathbb{P}_{q',q}(k(\gamma) \text{ does not exist}) \leq \sum_{M \leq \left\lfloor \frac{t}{K^2}\right\rfloor + 1, n_1 + \cdots + n_M = \left\lfloor \frac{t}{K^2}\right\rfloor} 2^M 3^{2\left\lfloor \frac{t}{K^2}\right\rfloor} e^{-\frac{K}{24}\left\lfloor \frac{t}{K^2}\right\rfloor}.$$

In addition, lemma 12 yields that for any $M \in \{1, \ldots, \lfloor \frac{t}{K^2}\rfloor + 1\}$, we have $|\{(n_1, \ldots, n_M) \in \mathbb{N}^M \mid n_1 + \cdots + n_M = \lfloor \frac{t}{K^2}\rfloor\}| = \binom{M + \lfloor \frac{t}{K^2}\rfloor - 1}{M-1} = \binom{M + \lfloor \frac{t}{K^2}\rfloor - 1}{\lfloor \frac{t}{K^2}\rfloor}$, and by the Stirling formula there exists a constant $\lambda > 0$ such that

$$\binom{M + \lfloor \frac{t}{K^2}\rfloor - 1}{\lfloor \frac{t}{K^2}\rfloor} \leq \frac{\left(M + \lfloor \frac{t}{K^2}\rfloor - 1\right)^{\lfloor \frac{t}{K^2}\rfloor}}{\lfloor \frac{t}{K^2}\rfloor!} \leq \lambda \left(\frac{e\left(M + \lfloor \frac{t}{K^2}\rfloor - 1\right)}{\lfloor \frac{t}{K^2}\rfloor} \right)^{\lfloor \frac{t}{K^2}\rfloor} \leq \lambda \left(\frac{e\left(\lfloor \frac{t}{K^2}\rfloor + \lfloor \frac{t}{K^2}\rfloor\right)}{\lfloor \frac{t}{K^2}\rfloor} \right)^{\lfloor \frac{t}{K^2}\rfloor}$$

since $M \leq \lfloor \frac{t}{K^2}\rfloor + 1$. We deduce $|\{(n_1, \ldots, n_M) \in \mathbb{N}^M \mid n_1 + \cdots + n_M = \lfloor \frac{t}{K^2}\rfloor\}| \leq \lambda (2e)^{\lfloor \frac{t}{K^2}\rfloor}$. Therefore

$$\mathbb{P}_{q',q}(k(\gamma) \text{ does not exist}) \leq \sum_{M=1}^{\left\lfloor \frac{t}{K^2}\right\rfloor + 1} \lambda (2e)^{\left\lfloor \frac{t}{K^2}\right\rfloor} 2^M 3^{2\left\lfloor \frac{t}{K^2}\right\rfloor} e^{-\frac{K}{24}\left\lfloor \frac{t}{K^2}\right\rfloor}$$

$$\leq \lambda (2e)^{\left\lfloor \frac{t}{K^2}\right\rfloor} 2^{\left\lfloor \frac{t}{K^2}\right\rfloor + 2} 3^{2\left\lfloor \frac{t}{K^2}\right\rfloor} e^{-\frac{K}{24}\left\lfloor \frac{t}{K^2}\right\rfloor} = 4\lambda \left(36 e e^{-\frac{K}{24}}\right)^{\left\lfloor \frac{t}{K^2}\right\rfloor}.$$

In addition, since $K \geq 48(\ln 36 + 1)$, $36ee^{-\frac{K}{48}} \leq 36ee^{-\ln 36 - 1} = 1$, so $36ee^{-\frac{K}{24}} \leq e^{-\frac{K}{48}}$, hence

$$\mathbb{P}_{q',q}(k(\gamma) \text{ does not exist}) \leq 4\lambda e^{-\frac{K}{48}\left\lfloor \frac{t}{K^2}\right\rfloor} \leq 4\lambda e^{-\frac{K}{48}\left(\frac{t}{K^2} - 1\right)} = 4\lambda e^{\frac{K}{48}} e^{-\frac{t}{48K}},$$

which is the lemma. □

Proof of lemma 20. This proof is an application of proposition 18. We set $K_2 = \max(4, K_g(1/2))$, which depends only on $\mathcal{U}$. Let $q' \in]0,1]$, $K \geq K_2$, $q \in [q_K, 1]$ and $x \in \mathbb{Z}^d$. It is enough to prove the lemma for $t \geq \max(K, \frac{3K^2}{K-3})$; indeed, if the lemma holds for $t \geq \max(K, \frac{3K^2}{K-3})$, one has only to enlarge $\breve{C}_2$ to prove it for $t \geq K$. Therefore we set $t \geq \max(K, \frac{3K^2}{K-3})$ and $\gamma = (y_k)_{k \in \{0, \ldots, \lfloor \frac{t}{K^2}\rfloor\}} \in C_K^N(x,t)$. If $k(\gamma)$ exists but $|\mathcal{X}^{y(\gamma),k(\gamma)}| \leq \frac{t}{6K}$, we have $\tau^{y(\gamma),k(\gamma)} = +\infty$ and $|\mathcal{X}^{y(\gamma),k(\gamma)}| \leq \frac{t}{6K}$, hence

$$\mathbb{P}_{q',q}\left(k(\gamma) \text{ exists}, |\mathcal{X}^{y(\gamma),k(\gamma)}| \leq \frac{t}{6K}\right) \leq \sum_{k=0}^{\left\lfloor \frac{t}{K^2}\right\rfloor} \mathbb{P}_{q',q}\left(\tau^{y_k,k} = +\infty, |\mathcal{X}^{y_k,k}| \leq \frac{t}{6K}\right).$$

We are going to bound the term on the right. For any $k \in \{0, \ldots, \lfloor \frac{t}{K^2}\rfloor\}$, we have $n^{y_k,k} = \lfloor \frac{t}{K}\rfloor - k \geq \lfloor \frac{t}{K}\rfloor - \lfloor \frac{t}{K^2}\rfloor \geq \frac{t}{K} - 1 - \frac{t}{K^2}$, and since $t \geq \frac{3K^2}{K-3}$, $(K-3)t \geq 3K^2$ thus $\frac{1}{3}\frac{t}{K} - \frac{t}{K^2} \geq 1$, so $n^{y_k,k} \geq \frac{2}{3}\frac{t}{K}$, hence if we choose $\alpha = \frac{1}{2}$ we have $\frac{\alpha}{2} n^{y_k,k} \geq \frac{t}{6K}$. Therefore by proposition 18,

$$\mathbb{P}_{q',q}\left(\tau^{y_k,k} = +\infty, |\mathcal{X}^{y_k,k}| \leq \frac{t}{6K}\right) \leq C_g e^{-c_g n^{y_k,k}} \leq C_g e^{-c_g \frac{2}{3}\frac{t}{K}}$$

since $n^{y_k,k} \geq \frac{2}{3}\frac{t}{K}$. Consequently

$$\mathbb{P}_{q',q}\left(k(\gamma) \text{ exists}, |\mathcal{X}^{y(\gamma),k(\gamma)}| \leq \frac{t}{6K}\right) \leq \left(\left\lfloor \frac{t}{K^2}\right\rfloor + 1\right) C_g e^{-\frac{2c_g}{3}\frac{t}{K}} \leq \left(\frac{t}{K}+1\right) C_g e^{-\frac{2c_g}{3}\frac{t}{K}},$$

which yields lemma 20. □

Proof of lemma 21. Let $q' \in]0,1]$, $K \geq 2$, $q \in [q_K, 1]$, $x \in \mathbb{Z}^d$, $t \geq K$ and $\gamma \in C_K^N(x,t)$. The argument is elementary: we notice that there is a positive probability that a rectangle is full of zeroes in the initial configurations of the two processes since they have laws $\nu_{q'}$ and ν_q, as well as a positive probability that there is no 1-clock ring in the rectangle in the time interval $[0, t - K\lfloor \frac{t}{K}\rfloor]$. Therefore there is a positive probability that a rectangle is full of zeroes in both processes at time $t - K\lfloor \frac{t}{K}\rfloor$, so if there are $\frac{t}{6K}$ elements in $\mathcal{X}^{y(\gamma),k(\gamma)}$, the probability that none of the corresponding rectangles is full of zeroes in both processes at time $t - K\lfloor \frac{t}{K}\rfloor$ is of order $e^{-\check{c}_3 \frac{t}{K}}$.

We notice that $\mathcal{X}^{y(\gamma),k(\gamma)}$ depends only on clock rings in the time interval $]t - K\lfloor \frac{t}{K}\rfloor, t]$, hence if $\mathcal{F}$ is the σ-algebra generated by the clock rings in $]t - K\lfloor \frac{t}{K}\rfloor, t]$, for $\hat{\eta} = \eta$ or $\tilde{\eta}$, we have

$$\begin{aligned} &\mathbb{P}_{q',q}\left(\left\{|\mathcal{X}^{y(\gamma),k(\gamma)}| > \frac{t}{6K}\right\} \cap \{\forall r \in \mathcal{X}^{y(\gamma),k(\gamma)}, W^{\gamma,\hat{\eta}}(r)^c\}\right) \\ &= \mathbb{E}_{q',q}\left(\mathbb{1}_{\{|\mathcal{X}^{y(\gamma),k(\gamma)}| > \frac{t}{6K}\}} \mathbb{P}_{q',q}(\forall r \in \mathcal{X}^{y(\gamma),k(\gamma)}, W^{\gamma,\hat{\eta}}(r)^c | \mathcal{F})\right). \end{aligned} \tag{1}$$

Moreover,

$$\begin{aligned} &\mathbb{P}_{q',q}(\forall r \in \mathcal{X}^{y(\gamma),k(\gamma)}, W^{\gamma,\hat{\eta}}(r)^c | \mathcal{F}) \\ &= \mathbb{P}_{q',q}\left(\forall r \in \mathcal{X}^{y(\gamma),k(\gamma)}, \exists x' \in y(\gamma) + \frac{r - n^{y(\gamma),k(\gamma)}}{2} a_1 u + R, \hat{\eta}_{t - \lfloor \frac{t}{K}\rfloor K}(x') \neq 0 \,\middle|\, \mathcal{F}\right) \leq \\ &\mathbb{P}_{q',q}\left(\forall r \in \mathcal{X}^{y(\gamma),k(\gamma)}, \exists x' \in y(\gamma) + \frac{r - n^{y(\gamma),k(\gamma)}}{2} a_1 u + R, \hat{\eta}_0(x') \neq 0 \text{ or } \mathcal{P}^1_{x'} \cap \left[0, t - \left\lfloor \frac{t}{K}\right\rfloor K\right] \neq \emptyset \,\middle|\, \mathcal{F}\right) \\ &= \prod_{r \in \mathcal{X}^{y(\gamma),k(\gamma)}} \mathbb{P}_{q',q}\left(\exists x' \in y(\gamma) + \frac{r - n^{y(\gamma),k(\gamma)}}{2} a_1 u + R, \hat{\eta}_0(x') \neq 0 \text{ or } \mathcal{P}^1_{x'} \cap \left[0, t - \left\lfloor \frac{t}{K}\right\rfloor K\right] \neq \emptyset\right) \end{aligned}$$

since the events $\{\exists x' \in y(\gamma) + \frac{r - n^{y(\gamma),k(\gamma)}}{2} a_1 u + R, \hat{\eta}_0(x') \neq 0 \text{ or } \mathcal{P}^1_{x'} \cap [0, t - \lfloor \frac{t}{K}\rfloor K] \neq \emptyset\}$ depend only on the state of $\hat{\eta}_0$ and on the clock rings of the time interval $[0, t - K\lfloor \frac{t}{K}\rfloor]$ at the sites of $y(\gamma) + \frac{r - n^{y(\gamma),k(\gamma)}}{2} a_1 u + R$, so they are mutually independent and independent of $\mathcal{F}$. Therefore the invariance by translation yields

$$\begin{aligned} \mathbb{P}_{q',q}(\forall r \in \mathcal{X}^{y(\gamma),k(\gamma)}, W^{\gamma,\hat{\eta}}(r)^c | \mathcal{F}) &\leq \mathbb{P}_{q',q}\left(\exists x' \in R, \hat{\eta}_0(x') \neq 0 \text{ or } \mathcal{P}^1_{x'} \cap \left[0, t - \left\lfloor \frac{t}{K}\right\rfloor K\right] \neq \emptyset\right)^{|\mathcal{X}^{y(\gamma),k(\gamma)}|} \\ &= \left(1 - \mathbb{P}_{q',q}\left(\forall x' \in R, \hat{\eta}_0(x') = 0, \mathcal{P}^1_{x'} \cap \left[0, t - \left\lfloor \frac{t}{K}\right\rfloor K\right] = \emptyset\right)\right)^{|\mathcal{X}^{y(\gamma),k(\gamma)}|} \\ &= \left(1 - \left(\mathbb{P}_{q',q}\left(\hat{\eta}_0(0) = 0\right) \mathbb{P}_{q',q}\left(\mathcal{P}^1_0 \cap \left[0, t - \left\lfloor \frac{t}{K}\right\rfloor K\right] = \emptyset\right)\right)^{|R|}\right)^{|\mathcal{X}^{y(\gamma),k(\gamma)}|}. \end{aligned}$$

Furthermore, since $t - \lfloor \frac{t}{K} \rfloor K \leq K$ and $q \geq q_K = 1 + \frac{1}{3K|R|} \ln(1 - e^{-K})$,

$$\mathbb{P}_{q',q}\left(\mathcal{P}_0^1 \cap \left[0, t - \left\lfloor \frac{t}{K} \right\rfloor K\right] = \emptyset\right) = e^{-(1-q)\left(t - \lfloor \frac{t}{K} \rfloor K\right)} \geq e^{\frac{1}{3K|R|}\ln(1-e^{-K})K} = (1-e^{-K})^{\frac{1}{3|R|}} \geq \left(\frac{1}{2}\right)^{\frac{1}{3|R|}}$$

since $K \geq 2$. This implies

$$\mathbb{P}_{q',q}(\forall r \in \mathcal{X}^{y(\gamma),k(\gamma)}, W^{\gamma,\hat{\eta}}(r)^c | \mathcal{F}) \leq \left(1 - \mathbb{P}_{q',q}\left(\hat{\eta}_0(0) = 0\right)^{|R|} \left(\frac{1}{2}\right)^{\frac{1}{3}}\right)^{|\mathcal{X}^{y(\gamma),k(\gamma)}|}.$$

In addition, if $\hat{\eta} = \eta$, $\mathbb{P}_{q',q}(\hat{\eta}_0(0) = 0) = q'$, so $1 - \mathbb{P}_{q',q}(\eta_0(0) = 0)^{|R|}(\frac{1}{2})^{\frac{1}{3}} = 1 - (q')^{|R|}2^{-\frac{1}{3}}$, and if $\hat{\eta} = \tilde{\eta}$, $1 - \mathbb{P}_{q',q}(\hat{\eta}_0(0) = 0)^{|R|}(\frac{1}{2})^{\frac{1}{3}} = 1 - q^{|R|}2^{-\frac{1}{3}}$. Moreover, since $K \geq 2$, $q \geq q_K = 1 + \frac{1}{3K|R|}\ln(1 - e^{-K}) \geq 1 + \frac{1}{6|R|}\ln(1 - e^{-2}) \geq \frac{1}{2}$, hence $1 - \mathbb{P}_{q',q}(\tilde{\eta}_0(0) = 0)^{|R|}(\frac{1}{2})^{\frac{1}{3}} \leq 1 - 2^{-|R|-\frac{1}{3}}$. This implies that if $\breve{c}_3'$ is the minimum of $-\ln(1 - (q')^{|R|}2^{-\frac{1}{3}})$ and $-\ln(1 - 2^{-|R|-\frac{1}{3}})$ (which depends only on $\mathcal{U}$ and q'), for $\hat{\eta} = \eta$ or $\tilde{\eta}$ we have $\mathbb{P}_{q',q}(\forall r \in \mathcal{X}^{y(\gamma),k(\gamma)}, W^{\gamma,\hat{\eta}}(r)^c | \mathcal{F}) \leq e^{-\breve{c}_3'|\mathcal{X}^{y(\gamma),k(\gamma)}|}$. Consequently, (1) yields

$$\mathbb{P}_{q',q}\left(\left\{|\mathcal{X}^{y(\gamma),k(\gamma)}| > \frac{t}{6K}\right\} \cap \{\forall r \in \mathcal{X}^{y(\gamma),k(\gamma)}, W^{\gamma,\hat{\eta}}(r)^c\}\right)$$
$$\leq \mathbb{E}_{q',q}\left(\mathbb{1}_{\{|\mathcal{X}^{y(\gamma),k(\gamma)}| > \frac{t}{6K}\}} e^{-\breve{c}_3'|\mathcal{X}^{y(\gamma),k(\gamma)}|}\right) \leq e^{-\breve{c}_3'\frac{t}{6K}},$$

which is the lemma. □

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