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Coloring tournaments: from local to global∗

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Abstract

The chromatic number of a directed graph $D$ is the minimum number of colors needed to color the vertices of $D$ such that each color class of $D$ induces an acyclic subdigraph. Thus, the chromatic number of a tournament $T$ is the minimum number of transitive subtournaments which cover the vertex set of $T$. We show in this paper that tournaments are significantly simpler than graphs with respect to coloring. Indeed, while undirected graphs can be altogether “locally simple” (every neighborhood is a stable set) and have large chromatic number, we show that locally simple tournaments are indeed simple. In particular, there is a function $f$ such that if the out-neighborhood of every vertex in a tournament $T$ has chromatic number at most $c$, then $T$ has chromatic number at most $f(c)$. This answers a question of Berger et al.

Keywords: chromatic number of tournaments, Erdős-Hajnal conjecture, digraph coloring

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1 Introduction

A directed graph is said to be acyclic if it does not contain any directed cycles. Given a loopless digraph $D$, a $k$-coloring of $D$ is a coloring of each of the vertices of $D$ with one of the colors from the set $\{1, ..., k\}$ such that each color class induces an acyclic subdigraph. The chromatic number $\overrightarrow{\chi}(D)$ of $D$ is the smallest number $k$ for which $D$ admits a $k$-coloring. This digraph invariant was introduced by Neumann-Lara [13], and naturally generalizes many results on the graph chromatic number (see, for example, [4], [9] [10], [11], [12]). In this paper, we study the chromatic number of a class of tournaments where the out-neighborhood of every vertex has bounded chromatic number.

A tournament is a loopless digraph such that for every pair of distinct vertices $u, v$, exactly one of $uv, vu$ is an arc. Given a tournament $T$, a subset $X$ of $V(T)$ is transitive if the subtournament of $T$ induced by $X$ contains no directed cycle. Thus, $\chi(T)$ is the minimum $k$ such that $V(T)$ can be colored with $k$ colors where each color class is a transitive set. The coloring of tournaments has close relationship with the celebrated Erdős–Hajnal conjecture (cf. [1, 8]) and has been studied in [3, 5, 6, 2, 7].

Given $t \geq 1$, a tournament $T$ is $t$-local if for every vertex $v$, the subtournament of $T$ induced by the set of out-neighbors of $v$ has chromatic number at most $t$. The following conjecture was raised in [3] (Conjecture 2.6) and settled for $t = 2$ in [7].

Conjecture 1. There is a function $f$ such that every $t$-local tournament $T$ satisfies $\overrightarrow{\chi}(T) \leq f(t)$.

The goal of this note is to provide a proof of Conjecture 1 for all $t$.

Given a set $S \subset V(T)$, we say that $S$ is a dominating set of $T$ if every vertex in $V \setminus S$ has an in-neighbor in $S$. The dominating number $\gamma(T)$ of a tournament $T$ is the smallest number $k$ such that $T$ has a dominating set of size $k$. The main tool to prove Conjecture 1 is the following theorem, which seems more interesting than our original goal.

Theorem 2. For every integer $k \geq 1$, there exist integers $K$ and $\ell$ such that every tournament $T$ with dominating number at least $K$ contains a subtournament on $\ell$ vertices and chromatic number at least $k$.

Roughly speaking, Theorem 2 asserts that if the dominating number of a tournament is sufficiently large, then it contains a bounded-size subtournament with large chromatic number. One may ask whether high dominating number is enough to force an induced copy of a specific (high chromatic number) subtournament. The following tournaments may be potential candidates. Let $S_1$ be the tournament with a single vertex. For every $i > 1$,
let \( S_i \) be the tournament (with \( 2^i - 1 \) vertices) obtained by blowing up two vertices of an oriented triangle into two copies of \( S_{i-1} \). It is easy to check that \( \overline{\chi}(S_i) \geq i \). The following problem is trivial for \( i \leq 2 \) and verified for \( i = 3 \) in [7], while still open for all \( i \geq 4 \).

**Problem 3.** For every integer \( i \geq 1 \), there exist \( f(i) \) such that every tournament \( T \) with dominating number at least \( f(i) \) contains an isomorphic copy of \( S_i \).

On another note, it is natural to ask whether Theorem 2 still holds with a weaker hypothesis. In particular, is it true that for every \( k \), if the chromatic number of a tournament is huge, then it contains a bounded-size subtournament with chromatic number at least \( k \)? Unfortunately, the answer is negative for any \( k \geq 3 \). It is well-known that for any \( \ell \), there is an undirected simple graph \( G \) with arbitrarily high chromatic number and girth at least \( \ell + 1 \). We fix an arbitrary enumeration of vertices of \( G \) and create a tournament \( T \) as follows: If \( ij \) with \( i < j \) is an edge of \( G \) then \( ij \) is an arc of \( T \); otherwise, \( ji \) is an arc of \( T \). Then \( T \) has arbitrarily high chromatic number while every subtournament of \( T \) of size \( \ell \) has chromatic number at most 2. However, a similar question for dominating number is still open.

**Problem 4.** For every integer \( k \geq 1 \), there exist integers \( K \) and \( \ell \) such that every tournament \( T \) with dominating number at least \( K \) contains a subtournament with \( \ell \) vertices and dominating number at least \( k \).

## 2 Proof of Conjecture 1

For every vertex \( v \) in a tournament \( T \), we denote by \( N^+_T(v) \) the set of out-neighbors of \( v \) in \( T \). Given a subset \( X \) of \( V(T) \), let \( N^+_T(X) \) denote the union of all \( N^+_T(v) \), for \( v \in X \), and denote by \( N^+_T[X] := X \cup N^+_T(X) \). For every subset \( X \) of \( V(T) \), let \( \overline{\chi}_T(X) \) denote the chromatic number of the subtournament of \( T \) induced by \( X \).

Given a tournament \( T \) and a subset \( X \) of \( V(T) \), we say a set \( R \subseteq V(T) \) (not necessary disjoint from \( X \)) is a dominating set of \( X \) in \( T \) if every vertex in \( X \setminus R \) has an in-neighbor in \( R \). The dominating number \( \gamma_T(X) \) of \( X \) in \( T \) is the smallest number \( k \) such that \( X \) has a dominating set of size \( k \). When it is clear in the context, we omit the subscript \( T \) in the notation.

Let \( T \) be a tournament and \( X, Y \subseteq V(T) \). The following inequalities are straightforward:

\[
\gamma_T(N^+[X]) \leq |X|, \tag{1}
\]

and

\[
\gamma_T(Y) \leq \gamma_T(X) + \gamma_T(Y \setminus X). \tag{2}
\]
Let us restate Theorem 2.

**Theorem 5.** For every integer \( k \geq 1 \), there exist integers \( K \) and \( \ell \) such that every tournament \( T \) with \( \gamma(T) \geq K \) contains a subtournament \( A \) on \( \ell \) vertices and \( \overline{\chi}(A) \geq k \).

**Proof.** We proceed by induction on \( k \). The claim is trivial for \( k = 1 \). For \( k = 2 \), we can choose \( K = 2 \) and \( \ell = 3 \). Indeed, if a tournament \( T \) satisfies \( \gamma(T) \geq K = 2 \), then \( T \) is not transitive and thus it contains an oriented triangle \( A \) of size \( \ell = 3 \) and \( \overline{\chi}(A) \geq k = 2 \).

Assuming now that \((K, \ell)\) exists for \( k \), we want to find \((K', \ell')\) for \( k + 1 \). For this, we set \( K' := k(K + \ell + 1) + K \), and fix \( \ell' \) later. Let \( T \) be a tournament such that \( \gamma(T) \geq K' \). Let \( D \) be a dominating set of \( T \) of minimum size. Consider a subset \( W \) of \( D \) of size \( k(K + \ell + 1) \). From (1) and (2) we have

\[
\gamma(V \setminus N^+[W]) \geq \gamma(T) - \gamma(N^+[W]) \geq K' - |W| \geq K,
\]

where \( V \) is the vertex set of \( T \). Thus by induction hypothesis applied to \( k \), one can find a set \( A \subseteq V \setminus N^+[W] \) such that \( A \) has \( \ell \) vertices and \( \overline{\chi}(A) \geq k \). Note that by construction, \( A \cap W = \emptyset \) and all arcs between \( A \) and \( W \) are directed from \( A \) to \( W \).

Consider now a subset \( S \) of \( W \) of size \( K + \ell + 1 \). We claim that \( \gamma(N^+(S)) \geq K + \ell \). If not, we can choose a dominating set \( S' \) of \( N^+(S) \) of size at most \( K + \ell - 1 \). Note that \( x \) dominates \( S \) for any \( x \in A \), and so \( S' \cup \{x\} \) dominates \( N^+[S] \). Hence \( (D \setminus S) \cup S' \cup \{x\} \) would be a dominating set of \( T \) of size less than \( |D| \), which contradicts the minimality of \( |D| \). Therefore \( \gamma(N^+(S)) \geq K + \ell \).

Let \( N' \) be the set of vertices \( N^+(S) \setminus N^+(A) \). From (1) and (2) we have

\[
\gamma(N') \geq \gamma(N^+(S)) - \gamma(N^+(A)) \geq K + \ell - |A| = K.
\]

Thus by induction hypothesis applied to \( k \), there is a subset \( A_S \) of \( N' \) such that \( |A_S| = \ell \) and \( \overline{\chi}(A_S) \geq k \). Note that by construction, \( A_S \cap A = \emptyset \) and all arcs between \( A_S \) and \( A \) are directed from \( A_S \) to \( A \).

We now construct our subtournament of \( T \) with chromatic number at least \( k + 1 \). For this we consider the set of vertices \( A \cup W \) to which we add the collection of \( A_S \), for all subsets \( S \subseteq W \) of size \( K + \ell + 1 \). Call \( A' \) this new tournament and observe that its number of vertices is at most

\[
\ell' := \ell + k(K + \ell + 1) + \ell \left( \frac{k(K + \ell + 1)}{K + \ell + 1} \right).
\]

To conclude, it is sufficient to show that \( \overline{\chi}(A') \geq k + 1 \). Suppose not, and for contradiction, take a \( k \)-coloring of \( A' \). Since \( |W| = k(K + \ell + 1) \) there
is a monochromatic set \( S \) in \( W \) of size \( K + \ell + 1 \) (say, colored 1). Recall that we have all arcs from \( A_S \) to \( A \) and all arcs from \( A \) to \( S \), and note that since \( \chi(A) \geq k \) and \( \chi(A_S) \geq k \), both \( A \) and \( A_S \) have a vertex of each of the \( k \) colors. Hence there are \( u \in A \) and \( v \in A_S \) colored 1. Since \( A_S \subseteq N^+(S) \), there is \( v \in S \) such that \( uv \) is an arc. We then obtain the monochromatic cycle \( uvw \) of color 1, a contradiction. Thus, \( \chi(A') \geq k + 1 \), completing the proof.

We now show that Conjecture 1 is true.

**Theorem 6.** There is a function \( f \) such that every \( t \)-local tournament \( T \) satisfies \( \chi(T) \leq f(t) \).

**Proof.** Let \((K, \ell)\) satisfy Theorem 5 for \( k := t + 1 \). Let \( T \) be a \( t \)-local tournament. Thus, if \( \gamma(T) \geq K \) then \( T \) contains a set \( A \) of \( \ell \) vertices and \( \chi(A) \geq t + 1 \). If a vertex \( v \in V(T) \setminus A \) does not have an in-neighbor in \( A \), then \( A \subseteq N^+(v) \), and so \( t + 1 \leq \chi(A) \leq \chi(N^+(v)) \leq t \), a contradiction. Hence, \( A \) is a dominating set of \( T \). Note that

\[
\chi(N^+[v]) \leq \chi(N^+(v)) + \chi(\{v\}) \leq t + 1
\]

for every \( v \in V(T) \). Thus

\[
\chi(T) = \chi(N^+[A]) \leq \sum_{v \in A} \chi(N^+[v]) \leq (t + 1)|A| = (t + 1)\ell.
\]

Otherwise, \( \gamma(T) < K \). Let \( D \) be a dominating set of \( T \) with minimum size. Then

\[
\chi(T) = \chi(N^+[D]) \leq \sum_{v \in D} \chi(N^+[v]) \leq (t + 1)|D| < (t + 1)K.
\]

Consequently, \( t \)-local tournaments have chromatic number at most \( f(t) := \max \left( (t + 1)K, (t + 1)\ell \right) \).

The implication of our result is that we are possibly missing a key-definition of what is a “large” (or “dense”) hypergraph (i.e., a set of subsets). It could be that for a suitable definition of “large” (for which “large” intersecting “large” would be “large”), we would obtain that for any tournament \( T \) on vertex set \( V \), the set of out-neighborhoods of vertices of \( T \) is “large”, and in addition the set of subsets of vertices of a \( K \)-chromatic tournament inducing at least chromatic number \( k \) is also “large”. Hence, if two large sets are intersecting in a non-empty way, one could find an out-neighborhood with chromatic number \( k \).
If such a notion would exist, it should decorrelate the two large sets (out-neighborhoods and \( k \)-chromatic), and thus imply the following: If \( T_1, T_2 \) are tournaments on the same set of vertices and \( \chi(T_1) \) is huge, then there is a vertex \( v \) such that \( T_1 \) induces on \( N_{T_2}^+(v) \) a subtournament of large chromatic number. A very similar conjecture was proposed by Alex Scott and Paul Seymour.

**Conjecture 7.** \([14]\) For every \( k \), there exists \( K \) such that if \( T \) and \( G \) are respectively a tournament and a graph on the same set of vertices with \( G \) of chromatic number at least \( K \), then there is a vertex \( v \) such that \( G \) induces on \( N_T^+(v) \) a subgraph of \( G \) of chromatic number at least \( k \).

**References**


