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Coloring tournaments: from local to global *

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Abstract

The *chromatic number* of a directed graph D is the minimum number of colors needed to color the vertices of D such that each color class of D induces an acyclic subdigraph. Thus, the chromatic number of a tournament T is the minimum number of transitive subtournaments which cover the vertex set of T . We show in this paper that tournaments are significantly simpler than graphs with respect to coloring. Indeed, while undirected graphs can be altogether “locally simple” (every neighborhood is a stable set) and have large chromatic number, we show that locally simple tournaments are indeed simple. In particular, there is a function f such that if the out-neighborhood of every vertex in a tournament T has chromatic number at most c , then T has chromatic number at most $f(c)$. This answers a question of Berger et al.

Keywords: chromatic number of tournaments, Erdős-Hajnal conjecture, digraph coloring

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1 Introduction

A directed graph is said to be *acyclic* if it does not contain any directed cycles. Given a loopless digraph D , a k -coloring of D is a coloring of each of the vertices of D with one of the colors from the set $\{1, \dots, k\}$ such that each color class induces an acyclic subdigraph. The *chromatic number* $\vec{\chi}(D)$ of D is the smallest number k for which D admits a k -coloring. This digraph invariant was introduced by Neumann-Lara [13], and naturally generalizes many results on the graph chromatic number (see, for example, [4], [9] [10], [11], [12]). In this paper, we study the chromatic number of a class of tournaments where the out-neighborhood of every vertex has bounded chromatic number.

A *tournament* is a loopless digraph such that for every pair of distinct vertices u, v , exactly one of uv, vu is an arc. Given a tournament T , a subset X of $V(T)$ is *transitive* if the subtournament of T induced by X contains no directed cycle. Thus, $\vec{\chi}(T)$ is the minimum k such that $V(T)$ can be colored with k colors where each color class is a transitive set. The coloring of tournaments has close relationship with the celebrated Erdős–Hajnal conjecture (cf. [1, 8]) and has been studied in [3, 5, 6, 2, 7].

Given $t \geq 1$, a tournament T is *t-local* if for every vertex v , the subtournament of T induced by the set of out-neighbors of v has chromatic number at most t . The following conjecture was raised in [3] (Conjecture 2.6) and settled for $t = 2$ in [7].

Conjecture 1. *There is a function f such that every t -local tournament T satisfies $\vec{\chi}(T) \leq f(t)$.*

The goal of this note is to provide a proof of Conjecture 1 for all t .

Given a set $S \subset V(T)$, we say that S is a *dominating set* of T if every vertex in $V \setminus S$ has an in-neighbor in S . The *dominating number* $\gamma(T)$ of a tournament T is the smallest number k such that T has a dominating set of size k . The main tool to prove Conjecture 1 is the following theorem, which seems more interesting than our original goal.

Theorem 2. *For every integer $k \geq 1$, there exist integers K and ℓ such that every tournament T with dominating number at least K contains a subtournament on ℓ vertices and chromatic number at least k .*

Roughly speaking, Theorem 2 asserts that if the dominating number of a tournament is sufficiently large, then it contains a bounded-size subtournament with large chromatic number. One may ask whether high dominating number is enough to force an induced copy of a specific (high chromatic number) subtournament. The following tournaments may be potential candidates. Let S_1 be the tournament with a single vertex. For every $i > 1$,

let S_i be the tournament (with $2^i - 1$ vertices) obtained by blowing up two vertices of an oriented triangle into two copies of S_{i-1} . It is easy to check that $\vec{\chi}(S_i) \geq i$. The following problem is trivial for $i \leq 2$ and verified for $i = 3$ in [7], while still open for all $i \geq 4$.

Problem 3. *For every integer $i \geq 1$, there exist $f(i)$ such that every tournament T with dominating number at least $f(i)$ contains an isomorphic copy of S_i .*

On another note, it is natural to ask whether Theorem 2 still holds with a weaker hypothesis. In particular, is it true that for every k , if the chromatic number of a tournament is huge, then it contains a bounded-size subtournament with chromatic number at least k ? Unfortunately, the answer is negative for any $k \geq 3$. It is well-known that for any ℓ , there is an undirected simple graph G with arbitrarily high chromatic number and girth at least $\ell + 1$. We fix an arbitrary enumeration of vertices of G and create a tournament T as follows: If ij with $i < j$ is an edge of G then ij is an arc of T ; otherwise, ji is an arc of T . Then T has arbitrarily high chromatic number while every subtournament of T of size ℓ has chromatic number at most 2. However, a similar question for dominating number is still open.

Problem 4. *For every integer $k \geq 1$, there exist integers K and ℓ such that every tournament T with dominating number at least K contains a subtournament with ℓ vertices and dominating number at least k .*

2 Proof of Conjecture 1

For every vertex v in a tournament T , we denote by $N_T^+(v)$ the set of out-neighbors of v in T . Given a subset X of $V(T)$, let $N_T^+(X)$ denote the union of all $N_T^+(v)$, for $v \in X$, and denote by $N_T^+[X] := X \cup N_T^+(X)$. For every subset X of $V(T)$, let $\vec{\chi}_T(X)$ denote the chromatic number of the subtournament of T induced by X .

Given a tournament T and a subset X of $V(T)$, we say a set $R \subseteq V(T)$ (not necessary disjoint from X) is a dominating set of X in T if every vertex in $X \setminus R$ has an in-neighbor in R . The *dominating number* $\gamma_T(X)$ of X in T is the smallest number k such that X has a dominating set of size k . When it is clear in the context, we omit the subscript T in the notation.

Let T be a tournament and $X, Y \subseteq V(T)$. The following inequalities are straightforward:

$$\gamma_T(N^+[X]) \leq |X|, \tag{1}$$

and

$$\gamma_T(Y) \leq \gamma_T(X) + \gamma_T(Y \setminus X). \tag{2}$$

Let us restate Theorem 2.

Theorem 5. *For every integer $k \geq 1$, there exist integers K and ℓ such that every tournament T with $\gamma(T) \geq K$ contains a subtournament A on ℓ vertices and $\vec{\chi}(A) \geq k$.*

Proof. We proceed by induction on k . The claim is trivial for $k = 1$. For $k = 2$, we can choose $K = 2$ and $\ell = 3$. Indeed, if a tournament T satisfies $\gamma(T) \geq K = 2$, then T is not transitive and thus it contains an oriented triangle A of size $\ell = 3$ and $\vec{\chi}(A) \geq k = 2$.

Assuming now that (K, ℓ) exists for k , we want to find (K', ℓ') for $k + 1$. For this, we set $K' := k(K + \ell + 1) + K$, and fix ℓ' later. Let T be a tournament such that $\gamma(T) \geq K'$. Let D be a dominating set of T of minimum size. Consider a subset W of D of size $k(K + \ell + 1)$. From (1) and (2) we have

$$\gamma(V \setminus N^+[W]) \geq \gamma(T) - \gamma(N^+[W]) \geq K' - |W| \geq K,$$

where V is the vertex set of T . Thus by induction hypothesis applied to k , one can find a set $A \subseteq V \setminus N^+[W]$ such that A has ℓ vertices and $\vec{\chi}(A) \geq k$. Note that by construction, $A \cap W = \emptyset$ and all arcs between A and W are directed from A to W .

Consider now a subset S of W of size $K + \ell + 1$. We claim that $\gamma(N^+(S)) \geq K + \ell$. If not, we can choose a dominating set S' of $N^+(S)$ of size at most $K + \ell - 1$. Note that x dominates S for any $x \in A$, and so $S' \cup \{x\}$ dominates $N^+[S]$. Hence $(D \setminus S) \cup S' \cup \{x\}$ would be a dominating set of T of size less than $|D|$, which contradicts the minimality of $|D|$. Therefore $\gamma(N^+(S)) \geq K + \ell$.

Let N' be the set of vertices $N^+(S) \setminus N^+(A)$. From (1) and (2) we have

$$\gamma(N') \geq \gamma(N^+(S)) - \gamma(N^+(A)) \geq K + \ell - |A| = K.$$

Thus by induction hypothesis applied to k , there is a subset A_S of N' such that $|A_S| = \ell$ and $\vec{\chi}(A_S) \geq k$. Note that by construction, $A_S \cap A = \emptyset$ and all arcs between A_S and A are directed from A_S to A .

We now construct our subtournament of T with chromatic number at least $k + 1$. For this we consider the set of vertices $A \cup W$ to which we add the collection of A_S , for all subsets $S \subseteq W$ of size $K + \ell + 1$. Call A' this new tournament and observe that its number of vertices is at most

$$\ell' := \ell + k(K + \ell + 1) + \ell \binom{k(K + \ell + 1)}{K + \ell + 1}.$$

To conclude, it is sufficient to show that $\vec{\chi}(A') \geq k + 1$. Suppose not, and for contradiction, take a k -coloring of A' . Since $|W| = k(K + \ell + 1)$ there

is a monochromatic set S in W of size $K + \ell + 1$ (say, colored 1). Recall that we have all arcs from A_S to A and all arcs from A to S , and note that since $\vec{\chi}(A) \geq k$ and $\vec{\chi}(A_S) \geq k$, both A and A_S have a vertex of each of the k colors. Hence there are $u \in A$ and $w \in A_S$ colored 1. Since $A_S \subseteq N^+(S)$, there is $v \in S$ such that vw is an arc. We then obtain the monochromatic cycle uvw of color 1, a contradiction. Thus, $\vec{\chi}(A') \geq k + 1$, completing the proof. \square

We now show that Conjecture 1 is true.

Theorem 6. *There is a function f such that every t -local tournament T satisfies $\vec{\chi}(T) \leq f(t)$.*

Proof. Let (K, ℓ) satisfy Theorem 5 for $k := t + 1$. Let T be a t -local tournament. Thus, if $\gamma(T) \geq K$ then T contains a set A of ℓ vertices and $\vec{\chi}(A) \geq t + 1$. If a vertex $v \in V(T) \setminus A$ does not have an in-neighbor in A , then $A \subseteq N^+(v)$, and so $t + 1 \leq \vec{\chi}(A) \leq \vec{\chi}(N^+(v)) \leq t$, a contradiction. Hence, A is a dominating set of T . Note that

$$\vec{\chi}(N^+[v]) \leq \vec{\chi}(N^+(v)) + \vec{\chi}(\{v\}) \leq t + 1$$

for every $v \in V(T)$. Thus

$$\vec{\chi}(T) = \vec{\chi}(N^+[A]) \leq \sum_{v \in A} \vec{\chi}(N^+[v]) \leq (t + 1)|A| = (t + 1)\ell.$$

Otherwise, $\gamma(T) < K$. Let D be a dominating set of T with minimum size. Then

$$\vec{\chi}(T) = \vec{\chi}(N^+[D]) \leq \sum_{v \in D} \vec{\chi}(N^+[v]) \leq (t + 1)|D| < (t + 1)K.$$

Consequently, t -local tournaments have chromatic number at most $f(t) := \max((t + 1)K, (t + 1)\ell)$. \square

The implication of our result is that we are possibly missing a key-definition of what is a “large” (or “dense”) hypergraph (i.e., a set of subsets). It could be that for a suitable definition of “large” (for which “large” intersecting “large” would be “large”), we would obtain that for any tournament T on vertex set V , the set of out-neighborhoods of vertices of T is “large”, and in addition the set of subsets of vertices of a K -chromatic tournament inducing at least chromatic number k is also “large”. Hence, if two large sets are intersecting in a non-empty way, one could find an out-neighborhood with chromatic number k .

If such a notion would exist, it should decorrelate the two large sets (out-neighborhoods and k -chromatic), and thus imply the following: If T_1, T_2 are tournaments on the same set of vertices and $\vec{\chi}(T_1)$ is huge, then there is a vertex v such that T_1 induces on $N_{T_2}^+(v)$ a subtournament of large chromatic number. A very similar conjecture was proposed by Alex Scott and Paul Seymour.

Conjecture 7. [14] *For every k , there exists K such that if T and G are respectively a tournament and a graph on the same set of vertices with G of chromatic number at least K , then there is a vertex v such that G induces on $N_T^+(v)$ a subgraph of G of chromatic number at least k .*

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