# Two-dimensional Dirac operators with singular interactions supported on closed curves

Jussi Behrndt, Markus Holzmann, Thomas Ourmières-Bonafos, Konstantin Pankrashkin

# Two-dimensional Dirac operators with singular interactions supported on closed curves

Jussi Behrndt
Institut für Angewandte Mathematik, Technische Universität Graz
Steyrergasse 30, 8010 Graz, Austria
E-mail: behrndt@tugraz.at
Webpage: http://www.math.tugraz.at/~behrndt/

Markus Holzmann
Institut für Angewandte Mathematik, Technische Universität Graz
Steyrergasse 30, 8010 Graz, Austria
E-mail: holzmann@math.tugraz.at
Webpage: http://www.math.tugraz.at/~holzmann/

Thomas Ourmières-Bonafos
CNRS & Université Paris-Dauphine, PSL University, CEREMADE,
Place de Lattre de Tassigny, 75016 Paris, France
E-mail: ourmieres-bonafos@ceremade.dauphine.fr
Webpage: http://www.ceremade.dauphine.fr/~ourmieres/

Konstantin Pankrashkin
Laboratoire de Mathématiques d'Orsay, Univ. Paris-Sud, CNRS,
Université Paris-Saclay, 91405 Orsay, France
E-mail: konstantin.pankrashkin@math.u-psud.fr
Webpage: http://www.math.u-psud.fr/~pankrashkin/

July 12, 2019

**Abstract**

This paper is devoted to the study of the two-dimensional Dirac operator with an arbitrary combination of an electrostatic and a Lorentz scalar $\delta$-interaction of constant strengths supported on a closed curve. For any combination of the coupling constants a rigorous description of the self-adjoint realization of the operators is given and the spectral properties are described. For a non-zero mass and a critical combination of coupling constants the operator appears to have an additional point in the essential spectrum, which is

related to a loss of regularity in the operator domain, and the position of this point is expressed in terms of the coupling constants.

# Contents



# 1 Introduction

## 1.1 Motivations and state of the art

Initially introduced to model the effects of special relativity on the behavior of quantum particles of spin $\frac{1}{2}$ (such as electrons), the Dirac operator also comes into play as an effective operator when studying low-energy electrons in a single layered material like graphene. In order to model the interaction of the particles with external forces, the Dirac operator is coupled to a potential and the understanding of the spectral features of the resulting Hamiltonian translates into dynamical properties of the quantum system.

In the last few years a class of singular potentials has been extensively studied in this relativistic setting. These potentials, which are called $\delta$-interactions, are supported on sets of Lebesgue measure zero and used as idealized replacements for regular potentials localized in thin neighborhoods of the interaction supports in the

ambient Euclidean space. In nonrelativistic quantum mechanics these interactions were successfully studied in the case of Schrödinger operators with point interactions in [1] or with $\delta$-interactions supported on hypersurfaces in $\mathbb{R}^d$, e.g., in [10, 13, 21]. In the relativistic setting also the one dimensional case, i.e. Dirac operators with $\delta$-potentials supported on points in $\mathbb{R}$, were investigated first, see [1, 16, 24, 31]. Then, the case of potentials supported on surfaces in $\mathbb{R}^3$ was discussed in [3–7, 9, 20, 25, 29, 30], a recent contribution in the two-dimensional case is [32]. In the works mentioned above, it was observed that there are critical interaction strengths for which the self-adjoint realization of the operator shows a loss of regularity in the operator domain and, as a result, may have different spectral properties as in the non-critical case. This critical case is still not fully understood in the three dimensional case.

In this paper we want to study Dirac operators in $\mathbb{R}^2$ with electrostatic and Lorentz scalar $\delta$-potentials supported on loops. We provide a systematic approach combining the general theory of boundary triples and pseudodifferential calculus for matrix-valued singular integral operators supported on loops, which is inspired by the analysis in [9] and the paper [15], where similar questions for sign-changing Laplacians are studied. We show the self-adjointness of the Dirac operators with these singular potentials and discuss spectral properties for all possible combinations of interaction strengths. Unlike the previous work in $\mathbb{R}^3$, we are able to deal with more general $\delta$-interactions and there is no restriction on the geometry of the loops. This answers fully [29, Open Problem 11] in space dimension two.

In the following we describe the problem setting in more details. To set the stage, let $\Sigma$ be a connected and closed $C^\infty$-smooth curve which splits $\mathbb{R}^2$ into a bounded domain $\Omega_+$ and an unbounded domain $\Omega_-$, and let $\nu = (\nu_1, \nu_2)$ be the unit normal vector field at $\Sigma$ pointing outwards of $\Omega_+$. For a $\mathbb{C}^2$-valued function $f$ defined on $\mathbb{R}^2$ we will often use the notation $f_\pm := f \restriction \Omega_\pm$. Then, the distribution $\delta_\Sigma f$ with a function $f$ having a discontinuity along $\Sigma$ is defined in the symmetric form by

$$\langle \delta_\Sigma f, \varphi \rangle := \int_\Sigma \frac{1}{2} \big( \mathcal{T}_+^D f_+ + \mathcal{T}_-^D f_- \big) \cdot \varphi \, \mathrm{d}s,$$

where $\mathcal{T}_\pm^D f_\pm$ denotes the Dirichlet trace of $f_\pm$ at $\Sigma$ and $\mathrm{d}s$ means the integration with respect to the arc-length. We study Dirac operators $A_{\eta,\tau}$ in $L^2(\mathbb{R}^2; \mathbb{C}^2)$ which correspond to the formal differential expression

$$D_{\eta,\tau} := -\mathrm{i}\big(\sigma_1 \partial_1 + \sigma_2 \partial_2\big) + m\sigma_3 + (\eta\sigma_0 + \tau\sigma_3)\delta_\Sigma,$$

where $\sigma_0$ is the identity matrix in $\mathbb{C}^{2\times 2}$, $\sigma_1, \sigma_2, \sigma_3$ are the $\mathbb{C}^{2\times 2}$-valued Pauli spin matrices defined in (1.4), and $m, \eta, \tau \in \mathbb{R}$. Following the standard language [38] one may interpret $\eta$ and $\tau$ as the strengths of the electrostatic and Lorentz scalar interactions on $\Sigma$, respectively, while the parameter $m$ is usually interpreted as the mass. Integration by parts shows that if the distribution $D_{\eta,\tau} f$ is generated by an $L^2$-function, then $f$ has to fulfil the transmission condition

$$-\mathrm{i}\,(\sigma_1\nu_1 + \sigma_2\nu_2)\,(\mathcal{T}_+^D f_+ - \mathcal{T}_-^D f_-) = \frac{1}{2}(\eta\sigma_0 + \tau\sigma_3)(\mathcal{T}_+^D f_+ + \mathcal{T}_-^D f_-). \tag{1.1}$$

Our approach to study the self-adjointness and the spectral properties of $A_{\eta,\tau}$ is to define this operator as an extension of a certain symmetric operator and to use a suitable boundary triple to investigate the mentioned properties. Boundary triples are an abstract approach in the extension and spectral theory of symmetric and self-adjoint operators in Hilbert spaces [8, 14, 17, 18]. In the one and three dimensional setting they were applied successfully to study similar operators as $A_{\eta,\tau}$ in [6,9,16,31]. In the present paper we follow ideas from [9] to construct a boundary triple which allows us to show the self-adjointness and study the spectral properties of $A_{\eta,\tau}$ for all possible combinations of interaction strengths $\eta$ and $\tau$.

The second main ingredient in the study of $A_{\eta,\tau}$ are explicit properties of integral operators associated to the Green function corresponding to the unperturbed Dirac operator. Similar objects played a key role in the investigation of Dirac operators with singular potentials supported on surfaces in $\mathbb{R}^3$ in [3–7,9,30]. Since to the best of our knowledge these integral operators are not studied in the two dimensional case in detail, we provide the necessary results. In this analysis the properties of several well-known periodic pseudodifferential operators, such as the Cauchy and Hilbert transforms on $\Sigma$, play a crucial role and linking them to the Dirac operator is an important finding in this paper.

## 1.2 Main results

Let us pass to the formulation and discussion of the main results of this paper. To define the operator $A_{\eta,\tau}$ rigorously, we denote for an open set $\Omega \subset \mathbb{R}^2$

$$H(\sigma,\Omega) = \big\{ f \in L^2(\Omega;\mathbb{C}^2) : (\sigma_1\partial_1 + \sigma_2\partial_2) f \in L^2(\Omega;\mathbb{C}^2) \big\},$$

where the derivatives are understood in the distributional sense. One can show that functions $f_\pm$ in $H(\sigma,\Omega_\pm)$ admit Dirichlet traces $\mathfrak{T}_\pm^D f_\pm$ in $H^{-1/2}(\Sigma;\mathbb{C}^2)$. With these notations in hand we define now, following (1.1), for $\eta,\tau \in \mathbb{R}$ the operator $A_{\eta,\tau}$ in $L^2(\mathbb{R}^2;\mathbb{C}^2)$ by

$$\begin{aligned} A_{\eta,\tau} f &:= \big( -\mathrm{i}(\sigma_1\partial_1 + \sigma_2\partial_2) + m\sigma_3 \big) f_+ \oplus \big( -\mathrm{i}(\sigma_1\partial_1 + \sigma_2\partial_2) + m\sigma_3 \big) f_-, \\ \operatorname{dom} A_{\eta,\tau} &:= \Big\{ f = f_+ \oplus f_- \in H(\sigma,\Omega_+) \oplus H(\sigma,\Omega_-) : \\ &\qquad -\mathrm{i}\,(\sigma_1\nu_1 + \sigma_2\nu_2)\big(\mathfrak{T}_+^D f_+ - \mathfrak{T}_-^D f_-\big) = \frac{1}{2}\,(\eta\sigma_0 + \tau\sigma_3)\big(\mathfrak{T}_+^D f_+ + \mathfrak{T}_-^D f_-\big) \Big\}. \end{aligned} \tag{1.2}$$

In the analysis of $A_{\eta,\tau}$ it turns out that the special combination $\eta^2 - \tau^2 = 4$ of interaction strengths is *critical* in the sense that the $A_{\eta,\tau}$ has different properties than in the so called *non-critical* case $\eta^2 - \tau^2 \neq 4$. This phenomenon was also observed in the three dimensional case in [7], see also [3, 6, 9, 30].

In the non-critical case $\eta^2 - \tau^2 \neq 4$ the basic properties of $A_{\eta,\tau}$ are the following:

**Theorem 1.1.** *Let $\eta,\tau \in \mathbb{R}$ be such that $\eta^2 - \tau^2 \neq 4$. Then $A_{\eta,\tau}$ is self-adjoint in $L^2(\mathbb{R}^2;\mathbb{C}^2)$ with $\operatorname{dom} A_{\eta,\tau} \subset H^1(\mathbb{R}^2 \setminus \Sigma;\mathbb{C}^2)$, the essential spectrum of $A_{\eta,\tau}$ is*

$$\operatorname{spec}_{\mathrm{ess}}(A_{\eta,\tau}) = \big(-\infty, -|m|\big] \cup \big[|m|, \infty\big),$$

*and the discrete spectrum of $A_{\eta,\tau}$ in $(-|m|,|m|)$ is finite.*

The proof of Theorem 1.1 is given in Section 4.2. There, also some additional properties of $A_{\eta,\tau}$ like a Krein-type resolvent formula, an abstract version of the Birman-Schwinger principle, and some symmetry relations in the point spectrum of $A_{\eta,\tau}$ are shown. Similar results are known in the three dimensional case, see [7].

Our main results in the critical case $\eta^2 - \tau^2 = 4$ are stated in the following theorem. In particular, this shows that there is a loss of regularity in the domain of $A_{\eta,\tau}$ and that there is an additional point in the essential spectrum. Hence, $A_{\eta,\tau}$ has indeed different properties in the critical as in the non-critical case.

**Theorem 1.2.** *Let $\eta, \tau \in \mathbb{R}$ be such that $\eta^2 - \tau^2 = 4$. Then $A_{\eta,\tau}$ is self-adjoint in $L^2(\mathbb{R}^2;\mathbb{C}^2)$ with $\operatorname{dom} A_{\eta,\tau} \not\subset H^s(\mathbb{R}^2\setminus\Sigma;\mathbb{C}^2)$ for any $s>0$, and the restriction of $A_{\eta,\tau}$ onto the set $\operatorname{dom} A_{\eta,\tau} \cap H^1(\mathbb{R}^2\setminus\Sigma;\mathbb{C}^2)$ is essentially self-adjoint in $L^2(\mathbb{R}^2;\mathbb{C}^2)$. The essential spectrum of $A_{\eta,\tau}$ is*

$$\operatorname{spec}_{\rm ess}(A_{\eta,\tau}) = \big(-\infty, -|m|\big] \cup \left\{-\frac{\tau}{\eta}m\right\} \cup \big[|m|,\infty\big).$$

Theorem 1.2 is the main result of this paper and shown in Section 4.3. There, also a Krein type resolvent formula, a Birman Schwinger principle, and several symmetry relations in the point spectrum of $A_{\eta,\tau}$ are shown. We would like to point out that the corresponding properties in dimension three are only known in the case of purely electrostatic interactions, i.e. when $\eta = \pm 2$ and $\tau = 0$, see [9,30]. In particular, the fact that the new point $-\frac{\tau}{\eta}m$ of the essential spectrum of $A_{\eta,\tau}$ can be any value in $(-|m|,|m|)$ was not observed previously. We remark that several papers addressed the question of presence of a non-empty essential spectrum for Dirac operators in bounded domains with various boundary conditions, see e.g. [12,23,36]. Our results can also be regarded as a contribution in this direction.

By a minor modification of the argument, one can also deal with an interaction supported on several loops. Let $N \geq 1$ and consider a family of non-intersecting $C^\infty$-smooth loops $\Sigma_1, \dots, \Sigma_N$ with normals $\nu_j$, $j \in \{1,\dots,N\}$. We set $\Sigma := \bigcup_{j=1}^N \Sigma_j$, and for $f \in H(\sigma, \mathbb{R}^2\setminus\Sigma)$ we denote its Dirichlet traces on the two sides of $\Sigma_j$ as $\mathcal{T}^D_{\pm,j} f$, where $-$ corresponds to the side to which $\nu_j$ is directed. In addition, consider a family of pairs of real parameters

$$\mathcal{P} := \big((\eta_j,\tau_j)\big)_{j\in\{1,\dots,N\}}, \quad \eta_j,\, \tau_j \in \mathbb{R},$$

and define the associated operator $A_{\Sigma,\mathcal{P}}$ by

$$\begin{aligned} A_{\Sigma,\mathcal{P}} f &:= \big(-\mathrm{i}(\sigma_1\partial_1 + \sigma_2\partial_2) + m\sigma_3\big) f \qquad \text{in } \mathbb{R}^2\setminus\Sigma, \\ \operatorname{dom} A_{\Sigma,\mathcal{P}} &:= \Big\{ f \in H(\sigma,\mathbb{R}^2\setminus\Sigma) : \\ &\qquad -\mathrm{i}\,(\sigma_1\nu_{j,1} + \sigma_2\nu_{j,2})\big(\mathcal{T}^D_{+,j} f - \mathcal{T}^D_{-,j} f\big) = \frac{1}{2}\,(\eta\sigma_0 + \tau\sigma_3)\big(\mathcal{T}^D_{+,j} f + \mathcal{T}^D_{-,j} f\big) \\ &\qquad\qquad \text{for each } j \in \{1,\dots,N\}\Big\}. \end{aligned} \tag{1.3}$$

Then the preceding results can be extended as follows:

**Theorem 1.3.** *Denote*

$$\mathcal{I}_{\mathrm{crit}} := \big\{ j \in \{1, \dots, N\} : \eta_j^2 - \tau_j^2 = 4 \big\}.$$

*Then the following is true:*

(i) *If* $\mathcal{I}_{\mathrm{crit}} = \emptyset$, *then* $A_{\Sigma,\mathcal{P}}$ *is self-adjoint with* $\operatorname{dom} A_{\Sigma,\mathcal{P}} \subset H^1(\mathbb{R}^2 \setminus \Sigma; \mathbb{C}^2)$, *the essential spectrum of* $A_{\Sigma,\mathcal{P}}$ *is*

$$\operatorname{spec}_{\mathrm{ess}}(A_{\Sigma,\mathcal{P}}) = \big( -\infty, -|m| \big] \cup \big[ |m|, \infty \big),$$

*and the discrete spectrum of* $A_{\Sigma,\mathcal{P}}$ *in* $(-|m|, |m|)$ *is finite.*

(ii) *If* $\mathcal{I}_{\mathrm{crit}} \neq \emptyset$, *then* $A_{\Sigma,\mathcal{P}}$ *is self-adjoint with* $\operatorname{dom} A_{\Sigma,\mathcal{P}} \not\subset H^s(\mathbb{R}^2 \setminus \Sigma; \mathbb{C}^2)$ *for any* $s > 0$, *and the restriction of* $A_{\Sigma,\mathcal{P}}$ *onto the set* $\operatorname{dom} A_{\Sigma,\mathcal{P}} \cap H^1(\mathbb{R}^2 \setminus \Sigma; \mathbb{C}^2)$ *is essentially self-adjoint in* $L^2(\mathbb{R}^2; \mathbb{C}^2)$. *The essential spectrum of* $A_{\Sigma,\mathcal{P}}$ *is*

$$\sigma_{\mathrm{ess}}(A_{\Sigma,\mathcal{P}}) = \big( -\infty, -|m| \big] \bigcup_{j \in \mathcal{I}_{\mathrm{crit}}} \Big\{ -\frac{\tau_j}{\eta_j} m \Big\} \cup \big[ |m|, +\infty \big).$$

Necessary modifications for the proof of Theorem 1.3 are given in Subsection 4.4.

## 1.3 Structure of the paper

Let us shortly describe the structure of the paper. First, in Section 2 we recall some well-known facts on periodic pseudodifferential operators on curves, boundary triples, and Schur complements of block operator matrices. With that we study then in Section 3 integral operators, which are associated to the Green function corresponding to the free Dirac operator in $\mathbb{R}^2$, and construct a boundary triple which is suitable to study the properties of $A_{\eta,\tau}$. Eventually, Section 4 is devoted to the proofs of the main results of this paper, Theorems 1.1–1.3.

## 1.4 Notations

In this paper we denote the identity matrix in $\mathbb{C}^{2\times 2}$ by $\sigma_0$ and the $\mathbb{C}^{2\times 2}$-Pauli spin matrices by

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.4}$$

It is not difficult to see that the Pauli matrices fulfil

$$\sigma_j \sigma_k + \sigma_k \sigma_j = 2\delta_{jk} \sigma_0, \quad j, k \in \{1, 2, 3\}. \tag{1.5}$$

For $x = (x_1, x_2) \in \mathbb{C}^2$ we write $\sigma \cdot x = \sigma_1 x_1 + \sigma_2 x_2$ and in this sense $\sigma \cdot \nabla = \sigma_1 \partial_1 + \sigma_2 \partial_2$.

Next, $\Sigma \subset \mathbb{R}^2$ is always a $C^\infty$-loop of length $\ell > 0$, which splits $\mathbb{R}^2$ into a bounded domain $\Omega_+$ and an unbounded domain $\Omega_-$ with common boundary $\Sigma$. By $\nu$ we denote the unit normal vector field at $\Sigma$ which points outwards of $\Omega_+$, and $\mathbf{t}$ denotes

the unit tangent vector at $\Sigma$. If $\gamma : [0, \ell] \to \mathbb{R}^2$ is an arc length parametrization of $\Sigma$ with positive orientation, then we have $\mathbf{t} = \gamma'$ and $\nu = (\gamma_2', -\gamma_1')$. We sometimes identify the vector $\mathbf{t} \in \mathbb{R}^2$ with the complex number $T = \mathbf{t}_1 + i\mathbf{t}_2$.

If $\Omega$ is a measurable set, we write, as usual, $L^2(\Omega)$ for the classical $L^2$-spaces and $L^2(\Omega; \mathbb{C}^2) := L^2(\Omega) \otimes \mathbb{C}^2$. If $\Omega = \Sigma$, then $L^2(\Sigma)$ is based on the inner product, where the integrals are taken with respect to the arc-length. By $H^s(\Omega)$ we denote Sobolev spaces of order $s \in \mathbb{R}$ on $\Omega$, and the Sobolev spaces on the curve $\Sigma$ are reviewed in Section 2.1.

Next, we set

$$\mathbb{T} := \mathbb{R}/\mathbb{Z}.$$

Then $C^\infty(\mathbb{T})$ is the space of all $C^\infty(\mathbb{R})$-functions which are 1-periodic. For $\alpha \in \mathbb{R}$ we denote the set of periodic pseudodifferential operators on $\mathbb{T}$ by $\Psi^\alpha$ and the set of periodic pseudodifferential operators on $\Sigma$ by $\Psi^\alpha_\Sigma$, see Definitions 2.1 and 2.3.

For a linear operator $A$ in a Hilbert space $\mathcal{H}$ we write $\operatorname{dom} A$, $\operatorname{ran} A$, and $\ker A$ for its domain, range, and kernel, respectively. The identity operator is often denoted by $\mathbb{1}$. If $A$ is self-adjoint, then we denote by $\operatorname{res}(A), \operatorname{spec}(A), \operatorname{spec}_{\mathrm{p}}(A)$, and $\operatorname{spec}_{\mathrm{ess}}(A)$ the resolvent set, spectrum, point, and essential spectrum, respectively. If $A$ is self-adjoint and bounded from below, then $\mathcal{N}(A, z)$ is the number of eigenvalues smaller than $z$ taking multiplicities into account. For $z > \inf \operatorname{spec}_{\mathrm{ess}}(A)$ this is understood as $\mathcal{N}(A, z) = \infty$.

Finally, $K_j$ stands for the modified Bessel function of the second kind and order $j$.


## Acknowledgments

Thomas Ourmières-Bonafos and Konstantin Pankrashkin were supported in part by the PHC Amadeus 37853TB funded by the French Ministry of Foreign Affairs and the French Ministry of Higher Education, Research and Innovation. Jussi Behrndt and Markus Holzmann were supported by the Austrian Agency for International Cooperation in Education and Research (OeAD) within the project FR 01/2017. Thomas Ourmières-Bonafos was also supported by the ANR "Défi des autres savoirs (DS10) 2017" programm, reference ANR-17-CE29-0004, project molQED.


# 2 Preliminaries

In this section we provide some preliminary material from functional analysis and operator theory. First, in Section 2.1 we recall the definition and some properties of periodic pseudodifferential operators on smooth curves and some special integral operators of this form. Afterwards, in Section 2.2 a theorem on the Schur complement of block operator matrices is recalled and finally, in Section 2.3 boundary triples and their $\gamma$-fields and Weyl functions are briefly discussed.

## 2.1 Sobolev spaces and periodic pseudodifferential operators on closed curves

In this section some properties of periodic pseudodifferential operators on closed curves are discussed. Special realizations of such operators will play an important role in the analysis of Dirac operators with singular interactions later. The presentation in this section follows closely the one in [35, Chapters 5 and 7].

Throughout this section $\Sigma \subset \mathbb{R}^2$ is always a $C^\infty$-smooth loop of length $\ell$ and $\gamma : [0, \ell] \to \Sigma$ is an arc length parametrization of this curve, i.e. one has $|\gamma'(s)| = 1$ for all $s \in [0, \ell]$. First, we will introduce Sobolev spaces on $\Sigma$. For that we recall some constructions for Sobolev spaces of periodic functions on the unit interval. Denote

$$\mathbb{T} := \mathbb{R}/\mathbb{Z}.$$

For a distribution $f \in \mathcal{D}'(\mathbb{T}) := C^\infty(\mathbb{T})'$ (in [35] this space is denoted by $\mathcal{D}'_1(\mathbb{R})$) we write, as usual,

$$\widehat{f}(n) := \langle f, e_{-n} \rangle_{\mathcal{D}'(\mathbb{T}), \mathcal{D}(\mathbb{T})}, \quad e_n(t) = e^{2\pi n i t}, \qquad n \in \mathbb{Z},$$

for its Fourier coefficients. Recall that a distribution $f \in \mathcal{D}'(\mathbb{T})$ can be reconstructed from its Fourier coefficients by

$$f = \sum_{n \in \mathbb{Z}} \widehat{f}(n) e_n, \tag{2.1}$$

where the series converges in $\mathcal{D}'(\mathbb{T})$, see [35, Theorem 5.2.1]. For two distributions $f, g \in \mathcal{D}'(\mathbb{T})$ we denote by $f \star g$ their convolution which is defined (via its Fourier coefficients) by

$$\widehat{f \star g}(n) = \widehat{f}(n)\, \widehat{g}(n), \quad n \in \mathbb{N}.$$

In particular, for $f, g \in L^1(\mathbb{T})$ one simply has

$$f \star g = \int_{\mathbb{T}} f(s) g(\cdot - s) \, \mathrm{d}s.$$

For convenience we set

$$\underline{n} := \begin{cases} 1, & n = 0, \\ |n|, & n \neq 0, \end{cases} \qquad n \in \mathbb{Z}.$$

Then for $s \in \mathbb{R}$, the Sobolev space $H^s(\mathbb{T})$ consists of the distributions $f \in \mathcal{D}'(\mathbb{T})$ with

$$\|f\|^2_{H^s(\mathbb{T})} := \sum_{n \in \mathbb{Z}} \underline{n}^{2s} \big|\widehat{f}(n)\big|^2 < \infty.$$

The set $H^s(\mathbb{T})$ endowed with the above norm becomes a Hilbert space. If $s < t$, then $H^t(\mathbb{T})$ is compactly embedded into $H^s(\mathbb{T})$.

Having the definition of Sobolev spaces on $\mathbb{T}$, we can translate this to Sobolev spaces of order $s \in \mathbb{R}$ on $\Sigma$. For that we define on $\mathcal{D}'(\Sigma) := C^\infty(\Sigma)'$ the linear map

$$U : \mathcal{D}'(\Sigma) \to \mathcal{D}'(\mathbb{T}), \quad (Uf)(\varphi) = f\left(\ell^{-1} \varphi(\ell^{-1} \gamma^{-1}(\cdot))\right), \quad \varphi \in C^\infty(\mathbb{T}). \tag{2.2}$$

It is not difficult to verify that

$$Uf(t) = f(\gamma(\ell t)), \quad f \in L^1(\Sigma),\ t \in \mathbb{T}; \tag{2.3}$$

this property will often be used. For $s \in \mathbb{R}$ we define the space

$$H^s(\Sigma) := \big\{ f \in \mathcal{D}'(\Sigma) : Uf \in H^s(\mathbb{T}) \big\},$$

which, endowed with the norm

$$\|f\|_{H^s(\Sigma)} := \|Uf\|_{H^s(\mathbb{T})}, \quad f \in H^s(\Sigma),$$

is a Hilbert space. By construction the induced map

$$U : H^s(\Sigma) \to H^s(\mathbb{T}), \quad s \in \mathbb{R}, \tag{2.4}$$

is unitary. For $f \in H^0(\Sigma)$ it is useful to observe that

$$\|f\|^2_{H^0(\Sigma)} = \|Uf\|^2_{H^0(\mathbb{T})} = \sum_{n\in\mathbb{Z}} \big|(Uf, e_n)_{L^2(\mathbb{T})}\big|^2 = \|Uf\|^2_{L^2(\mathbb{T})} = \ell^{-1}\|f\|^2_{L^2(\Sigma)}.$$

Note also that the definition of $H^s(\Sigma)$ implies that $C^\infty(\Sigma)$ is dense in $H^s(\Sigma)$ for all $s \in \mathbb{R}$.

Next, we recall the definition of periodic pseudodifferential operators on $\mathbb{T}$ and translate this concept to periodic pseudodifferential operators on $\Sigma$. For that we define for a function $F : \mathbb{Z} \to \mathbb{C}$

$$(\omega F)(n) = (\omega_n F)(n) := F(n+1) - F(n), \quad n \in \mathbb{Z}. \tag{2.5}$$

The subscript $n$ is used, if the function $F$ depends on more than one variable to clarify on which variable $\omega$ is acting.

**Definition 2.1.** *A linear operator $H$ acting on $C^\infty(\mathbb{T})$ is called a* periodic pseudodifferential operator of order $\alpha \in \mathbb{R}$, *if there exists a function $h : \mathbb{T}\times\mathbb{Z} \to \mathbb{C}$ with $h(\cdot, n) \in C^\infty(\mathbb{T})$ for each $n \in \mathbb{Z}$ and*

$$Hu(t) = \sum_{n\in\mathbb{Z}} h(t,n)\,\widehat{u}(n)\ e_n(t) \text{ for all } u \in C^\infty(\mathbb{T}), \tag{2.6}$$

*and for all $k, l \in \mathbb{N}_0$ there exist constants $c_{k,l} > 0$ such that*

$$\left|\frac{\partial^k}{\partial t^k}\,\omega_n^l h(t,n)\right| \le c_{k,l}\,\underline{n}^{\alpha-l} \quad \text{for all } n \in \mathbb{Z}.$$

*The class of all periodic pseudodifferential operators of order $\alpha$ is denoted by $\Psi^\alpha$. Furthermore, we set*

$$\Psi^{-\infty} := \bigcap_{\alpha\in\mathbb{R}} \Psi^\alpha.$$

We note that one has the obvious inclusions $\Psi^\alpha \subset \Psi^\beta$ for $\alpha < \beta$. Moreover, in the spirit of (2.1) the periodic pseudodifferential operator $H$ is determined by its Fourier coefficients

$$\widehat{Hu}(m) = \sum_{n\in\mathbb{Z}} \widehat{u}(n)\langle h(\cdot,n)e_n, e_{-m}\rangle_{\mathcal{D}'(\mathbb{T}),\mathcal{D}(\mathbb{T})}.$$

In particular, if $h$ is independent of $t$, then we simply have $\widehat{Hu}(n) = h(n)\widehat{u}(n)$. The following properties of periodic pseudodifferential operators can be found in [35, Theorem 7.3.1 and Theorem 7.8.1].

**Proposition 2.2.** (i) *Let $H \in \Psi^\alpha$. Then for any $s \in \mathbb{R}$ the operator $H$ uniquely extends by continuity to a bounded operator $H^s(\mathbb{T}) \to H^{s-\alpha}(\mathbb{T})$; this extension will be denoted by the same symbol $H$.*

(ii) *Let $H \in \Psi^\alpha$ and $G \in \Psi^\beta$. Then $H + G \in \Psi^{\max\{\alpha,\beta\}}$, $HG \in \Psi^{\alpha+\beta}$, and $HG - GH \in \Psi^{\alpha+\beta-1}$.*

Having the definition of periodic pseudodifferential operators on $\mathbb{T}$ and the bijective map $U$ in (2.2) it is now straightforward to define periodic pseudodifferential operators on the loop $\Sigma$.

**Definition 2.3.** *A linear map $H : C^\infty(\Sigma) \to \mathcal{D}'(\Sigma)$ is called a periodic pseudodifferential operator of order $\alpha \in \mathbb{R}$ on $\Sigma$, if there exists a periodic pseudodifferential operator $H_0$ of order $\alpha$ on $\mathbb{T}$ such that*

$$H = U^{-1}H_0U.$$

*We denote by $\Psi^\alpha_\Sigma$ the linear space of all periodic pseudodifferential operators of order $\alpha \in \mathbb{R}$ on $\Sigma$ and set*

$$\Psi^{-\infty}_\Sigma := \bigcap_{\alpha\in\mathbb{R}} \Psi^\alpha_\Sigma.$$

In view of Proposition 2.2 and the fact that $U$ in (2.4) is unitary it is clear that each $H \in \Psi^\alpha_\Sigma$ induces a unique bounded operator $H : H^s(\Sigma) \to H^{s-\alpha}(\Sigma)$.

In what follows we discuss several special periodic pseudodifferential operators and their mapping properties which will play an important role in the analysis in the main part of this paper. First, let $c_0 > 0$ be a constant and consider the operator

$$L^\alpha u(t) = \sum_{n\in\mathbb{Z}} \left(c_0^2 + |n|\right)^{\alpha/2} \widehat{u}(n)\ e_n(t), \quad u \in C^\infty(\mathbb{T}), \quad \alpha \in \mathbb{R}, \tag{2.7}$$

on $C^\infty(\mathbb{T})$. Note that the Fourier coefficients of $L^\alpha u$ are $\widehat{L^\alpha u}(n) = (c_0^2 + |n|)^{\alpha/2}\,\widehat{u}(n)$ for $n \in \mathbb{Z}$. One can show that $L^\alpha \in \Psi^{\alpha/2}$ and hence $L^\alpha$ induces an isomorphism from $H^s(\mathbb{T})$ to $H^{s-\alpha/2}(\mathbb{T})$ for any $s \in \mathbb{R}$. The operator $L = L^1$ will be of particular importance in the following.

Using the operator $U$ from (2.2) we introduce

$$\Lambda^\alpha := U^{-1}L^\alpha U \in \Psi^{\alpha/2}_\Sigma, \quad \alpha \in \mathbb{R}, \tag{2.8}$$

and conclude that $\Lambda^\alpha : H^s(\Sigma) \to H^{s-\alpha/2}(\Sigma)$ is an isomorphism for any $\alpha, s \in \mathbb{R}$. Moreover, the above definition of $\Lambda$ implies that $\Lambda^\alpha \Lambda^\beta = \Lambda^{\alpha+\beta}$ for all $\alpha, \beta \in \mathbb{R}$. We note that the realization of $\Lambda = \Lambda^1$ for $s = \frac{1}{2}$ is viewed as an unbounded self-adjoint operator in $L^2(\Sigma)$ satisfying $\Lambda \geq c_0$. In particular, by varying $c_0$ we get that $\Lambda$ is a uniformly positive operator and that its lower bound can be arbitrarily large.

With the aid of $\Lambda$ we can prove now the following lemma.

**Lemma 2.4.** *Let $H \in \Psi^\alpha_\Sigma$, consider the associated linear operator in $L^2(\Sigma)$ defined by*

$$H_\infty u = Hu, \quad \operatorname{dom} H_\infty = C^\infty(\Sigma),$$

*and assume that $H_\infty$ is symmetric. Then the adjoint $H_\infty^*$ is given by*

$$H_\infty^* f = Hf, \quad \operatorname{dom} H_\infty^* = \big\{ f \in L^2(\Sigma) : Hf \in L^2(\Sigma) \big\}.$$

**Proof**. The result is trivial for $\alpha \leq 0$ due to the boundedness of $H_\infty$; cf. Proposition 2.2. Hence, we may assume that $\alpha > 0$. Recall that $f \in \operatorname{dom} H_\infty^*$ if and only if the mapping

$$C^\infty(\Sigma) \ni u \mapsto (H_\infty u, f)_{L^2(\Sigma)} \tag{2.9}$$

can be extended to a bounded functional on $L^2(\Sigma)$.

Let $f \in L^2(\Sigma)$ and $f_n \in C^\infty(\Sigma)$ such that $f_n \to f$ in $L^2(\Sigma)$. For $u \in C^\infty(\Sigma)$ and the map $U$ from (2.2)-(2.4) one has

$$\begin{aligned}(H_\infty u, f)_{L^2(\Sigma)} &= \lim_{n\to\infty} (H_\infty u, f_n)_{L^2(\Sigma)} = \lim_{n\to\infty} (u, Hf_n)_{L^2(\Sigma)} = \lim_{n\to\infty} \ell(Uu, UHf_n)_{L^2(\mathbb{T})} \\ &= \lim_{n\to\infty} \ell(L^{2\alpha}Uu, L^{-2\alpha}UHf_n)_{L^2(\mathbb{T})} = \ell(L^{2\alpha}Uu, L^{-2\alpha}UHf)_{L^2(\mathbb{T})},\end{aligned}$$

where we have used in the last step that $L^{-2\alpha}UH = L^{-2\alpha}UHU^{-1}U$ gives rise to a bounded operator from $L^2(\Sigma) \to L^2(\mathbb{T})$ due to $L^{-2\alpha} \in \Psi^{-\alpha}$, $UHU^{-1} \in \Psi^\alpha$, and Proposition 2.2. Therefore, if $f \in L^2(\Sigma)$ is such that $Hf \in L^2(\Sigma)$, then

$$\ell(L^{2\alpha}Uu, L^{-2\alpha}UHf)_{L^2(\mathbb{T})} = \ell(Uu, UHf)_{L^2(\mathbb{T})} = (u, Hf)_{L^2(\Sigma)}$$

and the functional in (2.9) is bounded,

$$|(H_\infty u, f)_{L^2(\Sigma)}| = |(u, Hf)_{L^2(\Sigma)}| \leq \|u\|_{L^2(\Sigma)} \|Hf\|_{L^2(\Sigma)},$$

and hence, $f \in \operatorname{dom} H_\infty^*$ and $H_\infty^* f = Hf$.

On the other hand, for $f \in \operatorname{dom} H_\infty^*$ and every $u \in C^\infty(\Sigma)$ the functional in (2.9) is bounded. For the special choice

$$u_k = \sum_{|n|\leq k} \widehat{UHf}(n) U^{-1} e_n \in C^\infty(\Sigma), \quad k \in \mathbb{N},$$

one has $\widehat{Uu_k}(n) = \widehat{UHf}(n)$ for $|n| \leq k$ and $\widehat{Uu_k}(n) = 0$ for $|n| > k$, and hence

$$\begin{aligned}(H_\infty u_k, f)_{L^2(\Sigma)} &= \ell(L^{2\alpha}Uu_k, L^{-2\alpha}UHf)_{L^2(\mathbb{T})} \\ &= \ell \sum_{n\in\mathbb{Z}} \widehat{L^{2\alpha}Uu_k}(n)\overline{\widehat{L^{-2\alpha}UHf}(n)} \\ &= \ell \sum_{n\in\mathbb{Z}} (c_0^2+|n|)^{\alpha}\widehat{Uu_k}(n)\overline{(c_0^2+|n|)^{-\alpha}\widehat{UHf}(n)} \\ &= \ell \sum_{|n|\leq k} \left|\widehat{UHf}(n)\right|^2.\end{aligned}$$

Sending $k \to \infty$ we see that a necessary condition for the functional in (2.9) to be bounded on $L^2(\Sigma)$ is given by

$$\sum_{n\in\mathbb{Z}} \left|\widehat{UHf}(n)\right|^2 < \infty,$$

i.e. $UHf \in L^2(\mathbb{T})$, and hence $Hf \in L^2(\Sigma)$. We have shown that $f \in \operatorname{dom} H_\infty^*$ if and only if $Hf \in L^2(\Sigma)$, which finishes the proof of this lemma. □

Next, we discuss that several types of integral operators on $\mathbb{T}$ are in fact periodic pseudodifferential operators, which allows us to deduce their mapping properties from the general theory. Note that via the isomorphism $U$ from (2.2) the results can be translated to integral operators on $\Sigma$. To formulate the following first result, recall the definition of the map $\omega$ from (2.5); the proof of this proposition can be found in [35, Theorem 7.6.1].

**Proposition 2.5.** *Let $\alpha \in \mathbb{R}$ and $\kappa \in \mathcal{D}'(\mathbb{T})$ such that for any $j \in \mathbb{N}_0$ there exists $c_j > 0$ with $\left|\omega^j \widehat{\kappa}(n)\right| \leq c_j \underline{n}^{\alpha-j}$ for all $n \in \mathbb{Z}$. Let $h \in C^\infty(\mathbb{T}^2)$ and let the operator $H$ be defined on $C^\infty(\mathbb{T})$ by*

$$(Hu)(t) := \kappa \star \big(h(t,\cdot)u\big), \quad u \in C^\infty(\mathbb{T}). \tag{2.10}$$

*Then $H \in \Psi^\alpha$.*

We remark that for $\kappa \in L^1(\mathbb{T})$ the operator $H$ in (2.10) is an integral operator acting as

$$(Hu)(t) := \int_{\mathbb{T}} \kappa(t-s)h(t,s)u(s)\,\mathrm{d}s, \quad u \in C^\infty(\mathbb{T}).$$

As a corollary we obtain:

**Corollary 2.6.** *Let $h \in C^\infty(\mathbb{T}^2)$. Then the integral operator acting as*

$$Hu(t) := \int_{\mathbb{T}} h(t,s)u(s)\,\mathrm{d}s, \quad u \in C^\infty(\mathbb{T}),$$

*belongs to $\Psi^{-\infty}$.*

In the following proposition we discuss a class of integral operators that appear quite frequently in our applications.

**Proposition 2.7.** *Let $m \in \mathbb{N}_0$, let*

$$a : \mathbb{T}^2 \to \mathbb{C} \quad \textit{and} \quad \rho : \mathbb{T} \to \mathbb{C}$$

*be $C^\infty$-functions, assume that $\rho$ is injective with $\rho'(t) \neq 0$ for all $t \in \mathbb{T}$, set $\kappa_m(z) := z^m \log|z|$ for $z \in \mathbb{C} \setminus \{0\}$, and define the integral operator*

$$H_m u(t) := \int_{\mathbb{T}} \kappa_m\big(\rho(t) - \rho(s)\big)\, a(t,s)\, u(s)\, \mathrm{d}s, \quad u \in C^\infty(\mathbb{T}).$$

*Then $H_m \in \Psi^{-m-1}$. Furthermore, in the special case $a \equiv 1$ and $m = 0$ one has*

$$\mathbb{1} + 2LH_0L \in \Psi^{-1}, \tag{2.11}$$

*where the operator $L$ is defined by* (2.7).

**Proof**. First, we treat the case $m = 0$. For that we introduce the auxiliary function $\chi_0 : \mathbb{T} \to \mathbb{R}$ by $\chi_0(t) := \log\big|\sin(\pi t)\big|$. Then the Fourier coefficients of $\chi_0$ are

$$\widehat{\chi_0}(n) = \begin{cases} -\log 2, & n = 0, \\ -\dfrac{1}{2|n|}, & n \neq 0, \end{cases} \tag{2.12}$$

see [35, Example 5.6.1]. Next, one has

$$\log\big(|\rho(t) - \rho(s)|\big) = \log(|\sin(\pi(t-s))|) + a_0(t,s) \tag{2.13}$$

with

$$a_0(t,s) = \log\left(\left|\frac{\rho(t)-\rho(s)}{\sin(\pi(t-s))}\right|\right), \quad t \neq s, \quad \text{and} \quad a_0(t,t) = \log\left(\frac{|\rho'(t)|}{\pi}\right).$$

Using Taylor series expansions one sees that there exist smooth functions $f_1$ and $f_2$ such that

$$\frac{1}{\sin(\pi(t-s))} = \frac{1}{\pi(t-s)} f_1(t,s) \quad \text{and} \quad \rho(t) - \rho(s) = (t-s) f_2(t,s),$$

and since $\rho$ is injective, we have $\frac{\rho(t)-\rho(s)}{\sin(\pi(t-s))} \neq 0$. From this one concludes that $a_0 : \mathbb{T}^2 \to \mathbb{C}$ is a $C^\infty$-function. Now we decompose $H_0 = C_0 + D_0$, where

$$\begin{aligned} C_0 u(t) &= \int_{\mathbb{T}} \chi_0(t-s)\, a(t,s)\, u(s)\, \mathrm{d}s = (\chi_0 \star (a(t,\cdot)u))(t), \\ D_0 u(t) &= \int_{\mathbb{T}} a_0(t,s)\, a(t,s)\, u(s)\, \mathrm{d}s. \end{aligned}$$

It follows from (2.12) and Proposition 2.5 that $C_0 \in \Psi^{-1}$ and by Corollary 2.6 we have $D_0 \in \Psi^{-\infty}$. Hence $H_0 \in \Psi^{-1}$ by Proposition 2.2.

To show (2.11) consider $LH_0L = LC_0L + LD_0L$ and note that the second term in the sum belongs to $\Psi^{-\infty}$. Furthermore, for $a \equiv 1$ the Fourier coefficients of $C_0Lu$ are given by

$$\widehat{C_0Lu}(n) = \widehat{\chi_0}(n)\widehat{Lu}(n) = \widehat{\chi_0}(n)(c_0^2 + |n|)^{1/2}\widehat{u}(n),$$

and hence one finds with the aid of (2.12)

$$\widehat{LC_0Lu}(n) = (c_0^2 + |n|)^{1/2}\widehat{\chi_0}(n)(c_0^2 + |n|)^{1/2}\widehat{u}(n) = b(n)\widehat{u}(n)$$

with

$$b(n) = (c_0^2 + |n|)\widehat{\chi_0}(n) = \begin{cases} -c_0^2 \log 2, & n = 0, \\ -\dfrac{1}{2} - \dfrac{c_0^2}{2|n|}, & n \neq 0, \end{cases}$$

which shows that the action of the operator $K := \mathbb{1} + 2LC_0L$ is determined by

$$\widehat{Ku}(n) = k(n)\widehat{u}(n) \quad \text{with} \quad k(n) = \begin{cases} 1 - 2c_0^2 \log 2, & n = 0, \\ -\dfrac{c_0^2}{|n|}, & n \neq 0. \end{cases}$$

Therefore, one can show with the help of Proposition 2.5 that $K \in \Psi^{-1}$.

To study the case $m \geq 1$ we consider

$$\rho(t) - \rho(s) = \left(e^{-2\pi \mathrm{i}(t-s)} - 1\right) a_1(t,s)$$

with the $C^\infty$-function

$$a_1(t,s) = \frac{\rho(t) - \rho(s)}{e^{-2\pi \mathrm{i}(t-s)} - 1}, \ \ t \neq s, \quad \text{and} \quad a_1(t,t) = \frac{\rho'(t)}{-2\pi \mathrm{i}}$$

and note, as for $a_0$, that $a_1 \in C^\infty(\mathbb{T}^2)$. Then using the decomposition (2.13) we write

$$\begin{aligned} &(\rho(t) - \rho(s))^m \log(|\rho(t) - \rho(s)|) \\ &\qquad = \left(e^{-2\pi \mathrm{i}(t-s)} - 1\right)^m \log(|\sin(\pi(t-s))|) a_1(t,s)^m \\ &\qquad\qquad + \left(e^{-2\pi \mathrm{i}(t-s)} - 1\right)^m a_0(t,s) a_1(t,s)^m. \end{aligned}$$

This shows that $H_m = C_m + D_m$, where $C_m$ and $D_m$ are integral operators

$$\begin{aligned} C_m u(t) &= \int_{\mathbb{T}} \left(e^{-2\pi \mathrm{i}(t-s)} - 1\right)^m \log(|\sin(\pi(t-s))|) a_1(t,s)^m \, a(t,s) u(s) \, \mathrm{d}s, \\ D_m u(t) &= \int_{\mathbb{T}} \left(e^{-2\pi \mathrm{i}(t-s)} - 1\right)^m a_0(t,s) a_1(t,s)^m a(t,s) \, u(s) \, \mathrm{d}s. \end{aligned}$$

The integral kernel of $D_m$ is smooth, which implies by Corollary 2.6 that $D_m \in \Psi^{-\infty}$. It is remains to show that $C_m \in \Psi^{-(m+1)}$. For that consider the function

$$\chi_m : \mathbb{T} \to \mathbb{C}, \quad \chi_m(t) := \left(e^{-2\pi \mathrm{i}t} - 1\right)^m \log(|\sin(\pi t)|).$$

Using the map $\omega$ from (2.5) and $\chi_0$ one obtains that $\widehat{\chi_m}(n) = \big(\omega^m \widehat{\chi_0}\big)(n)$. Now can show with the help of (2.12) that

$$|\omega^j \widehat{\chi_m}(n)| = |\omega^{m+j} \widehat{\chi_0}(n)| \leq c_j \underline{n}^{-m-1-j}.$$

By Proposition 2.5 this yields $C_m \in \Psi^{-(m+1)}$, which completes the proof of this proposition. □

Next, recall that the *Hilbert transform* $T_0$ on $\mathbb{T}$ is defined by

$$T_0 u(t) := \mathrm{i}\ \text{p.v.} \int_{\mathbb{T}} \cot\big(\pi(t-s)\big) u(s) \mathrm{d}s = (\kappa \star u)(t), \quad \kappa = \mathrm{i}\,\text{p.v.}\cot(\pi\cdot), \tag{2.14}$$

where p.v. means the principal value of the integral. By [35, Section 5.7] the distribution $\kappa$ satisfies

$$\widehat{\kappa}(n) = \operatorname{sgn} n = \begin{cases} -1, & n < 0, \\ 0, & n = 0, \\ 1, & n > 0. \end{cases}$$

It follows that $\widehat{T_0^2 u}(n) = (1 - \delta_{0,n})\widehat{u}(n)$, and

$$T_0 \in \Psi^0, \quad T_0^2 - \mathbb{1} \in \Psi^{-\infty}. \tag{2.15}$$

In the following assume that $a \in C^\infty(\mathbb{T}^2)$. Then the operator

$$(T_1 u)(t) = \mathrm{i}\ \text{p.v.} \int_{\mathbb{T}} \cot\big(\pi(t-s)\big)\, a(s,t)\, u(s)\, \mathrm{d}s$$

satisfies for $a_0(t) := a(t,t)$ the relation

$$T_1 - a_0 T_0 \in \Psi^{-\infty}, \tag{2.16}$$

see Section 7.6.2 in [35]. Since the commutator $T_2 := a_0 T_0 - T_0 a_0$, which acts as

$$T_2 u(t) = \mathrm{i}\ \text{p.v.} \int_{\mathbb{T}} \cot\big(\pi(t-s)\big)\, \big(a(t,t) - a(s,s)\big)\, u(s)\, \mathrm{d}s,$$

has a $C^\infty$ integral kernel, the principal value can be dropped, as the integral is convergent, and Corollary 2.6 implies that $T_2 \in \Psi^{-\infty}$. Hence, we also have

$$T - T_0 a_0 \in \Psi^{-\infty}. \tag{2.17}$$

**Corollary 2.8.** *Let $\rho : \mathbb{T} \to \mathbb{C}$ be $C^\infty$-smooth and injective with $\rho'(t) \neq 0$ for all $t \in \mathbb{T}$. Then the operator $C$ given by*

$$Cu(t) = \frac{\mathrm{i}}{\pi}\ \text{p.v.} \int_{\mathbb{T}} \frac{u(s)}{\rho(t) - \rho(s)}\, \mathrm{d}s, \quad u \in C^\infty(\mathbb{T}),$$

*satisfies*

$$C - \frac{1}{\rho'} T_0 \in \Psi^{-\infty} \qquad \textit{and} \qquad C - T_0 \frac{1}{\rho'} \in \Psi^{-\infty}. \tag{2.18}$$

**Proof**. We write

$$\frac{1}{\pi}\frac{1}{\rho(t)-\rho(s)} = \cot\big(\pi(t-s)\big)a(t,s) \quad \text{with} \quad a(t,s) = \frac{1}{\pi}\frac{\tan\big(\pi(t-s)\big)}{\rho(t)-\rho(s)}, \quad t \neq s,$$

and $a(t,t) = 1/\rho'(t)$. Then $a \in C^\infty(\mathbb{T}^2)$ and $a_0(t) = a(t,t) = 1/\rho'(t)$. Thus (2.18) follows from (2.16) and (2.17). □

Finally we introduce the *Cauchy transform* $C_\Sigma$ *on* $\Sigma$. For that we identify $\mathbb{R}^2$ with $\mathbb{C}$ and use the notation

$$\mathbb{R}^2 \ni x = (x_1, x_2)^\top \sim x_1 + ix_2 =: \xi \in \mathbb{C},$$
$$\mathbb{R}^2 \ni y = (y_1, y_2)^\top \sim y_1 + iy_2 =: \zeta \in \mathbb{C}.$$

Then

$$C_\Sigma u(\xi) := \frac{i}{\pi}\,\text{p.v.}\int_\Sigma \frac{u(\zeta)}{\xi-\zeta}\mathrm{d}\zeta, \quad u \in C^\infty(\Sigma),\ \xi \in \Sigma, \tag{2.19}$$

where the complex line integral is understood as its principal value. With an arc length parametrization $\gamma$ of $\Sigma$ and $x = \gamma(t), y = \gamma(s)$ it follows that $C_\Sigma$ acts as

$$C_\Sigma u(\gamma(t)) = \frac{i}{\pi}\,\text{p.v.}\int_0^\ell \frac{(\gamma_1'(s)+i\gamma_2'(s))u(\gamma(s))}{(\gamma_1(t)+i\gamma_2(t))-(\gamma_1(s)+i\gamma_2(s))}\mathrm{d}s.$$

Recall that for the tangent vector field $\mathbf{t}$ at $\Sigma$ and $y = \gamma(s) \in \Sigma$ we use the notation $T(y) := \mathbf{t}_1(y) + i\mathbf{t}_2(y) = \gamma_1'(s) + i\gamma_2'(s)$. We shall also view $y \mapsto T(y)$ as a function on $\Sigma$ or $s \mapsto T(\gamma(s))$ as a function on $[0,\ell]$. The same holds for the function $\overline{T}(y) := \mathbf{t}_1(y) - i\mathbf{t}_2(y) = \gamma_1'(s) - i\gamma_2'(s)$, and we will also denote the corresponding multiplication operators by $T$ and $\overline{T}$. With this we see for $u \in C^\infty(\Sigma)$ and $x = \gamma(t) \in \Sigma$ that

$$\begin{aligned}(C_\Sigma \overline{T} u)(x) &= \frac{i}{\pi}\,\text{p.v.}\int_0^\ell \frac{(\gamma_1'(s)+i\gamma_2'(s))(\gamma_1'(s)-i\gamma_2'(s))u(\gamma(s))}{(\gamma_1(t)+i\gamma_2(t))-(\gamma_1(s)+i\gamma_2(s))}\mathrm{d}s \\ &= \frac{i}{\pi}\,\text{p.v.}\int_\Sigma \frac{u(y)}{(x_1+ix_2)-(y_1+iy_2)}\mathrm{d}s(y).\end{aligned} \tag{2.20}$$

In our considerations also the formal dual $C_\Sigma'$ of $C_\Sigma$ in $L^2(\Sigma)$, which acts as

$$C_\Sigma' u(\gamma(t)) = \frac{i}{\pi}\,\text{p.v.}\int_0^\ell \frac{(\gamma_1'(t)-i\gamma_2'(t))u(\gamma(s))}{(\gamma_1(t)-i\gamma_2(t))-(\gamma_1(s)-i\gamma_2(s))}\mathrm{d}s \tag{2.21}$$

for $u \in C^\infty(\Sigma)$ and $x = \gamma(t) \in \Sigma$ will play an important role. Note that $C_\Sigma'$ is the operator which satisfies $(C_\Sigma u, v)_{L^2(\Sigma)} = (u, C_\Sigma' v)_{L^2(\Sigma)}$ for all $u, v \in C^\infty(\Sigma)$. Similarly as in (2.20) we have

$$\begin{aligned}(TC_\Sigma' u)(x) &= \frac{i}{\pi}\,\text{p.v.}\int_0^\ell \frac{(\gamma_1'(t)+i\gamma_2'(t))(\gamma_1'(t)-i\gamma_2'(t))u(\gamma(s))}{(\gamma_1(t)-i\gamma_2(t))-(\gamma_1(s)-i\gamma_2(s))}\mathrm{d}s \\ &= \frac{i}{\pi}\,\text{p.v.}\int_\Sigma \frac{u(y)}{(x_1-ix_2)-(y_1-iy_2)}\mathrm{d}s(y).\end{aligned} \tag{2.22}$$

In the following proposition we summarize the basic properties of $C_\Sigma$ and $C_\Sigma'$ which are needed for our further considerations. They basically follow directly from (2.20), (2.22), Corollary 2.8, and (2.15).

**Proposition 2.9.** *Let $C_\Sigma$ and $C'_\Sigma$ be defined by* (2.19) *and* (2.21)*, let $U$ be given by* (2.2)*, and let the Hilbert transform $T_0$ be defined by* (2.14)*. Then the following is true:*

(i) $C_\Sigma - U^{-1}T_0U \in \Psi_\Sigma^{-\infty}$*. In particular, $C_\Sigma \in \Psi_\Sigma^0$ and for all $s \in \mathbb{R}$ the operator $C_\Sigma$ gives rise to a bounded operator in $H^s(\Sigma)$.*

(ii) $C'_\Sigma - U^{-1}T_0U \in \Psi_\Sigma^{-\infty}$*. In particular, $C'_\Sigma \in \Psi_\Sigma^0$ and for all $s \in \mathbb{R}$ the operator $C'_\Sigma$ gives rise to a bounded operator in $H^s(\Sigma)$.*

*Furthermore, one has $C'_\Sigma C_\Sigma - \mathbb{1} \in \Psi_\Sigma^{-\infty}$ and $C_\Sigma C'_\Sigma - \mathbb{1} \in \Psi_\Sigma^{-\infty}$.*

**Proof**. Let us prove (i). Note first that the multiplication operators $T$ and $\overline{T}$ that multiply with the functions $s \mapsto T(\gamma(s)) = \gamma_1'(s) + i\gamma_2'(s)$ and $s \mapsto \overline{T}(\gamma(s)) = \gamma_1'(s) - i\gamma_2'(s)$ belong to $\Psi_\Sigma^0$, see [35, Section 7.2]. Hence (i) is equivalent to

$$C_\Sigma\overline{T} - U^{-1}T_0U\overline{T} = C_\Sigma\overline{T} - U^{-1}T_0\overline{T}(\gamma(\ell\cdot))U \in \Psi_\Sigma^{-\infty}$$

which in turn is equivalent, by definition, to

$$UC_\Sigma\overline{T}U^{-1} - T_0\overline{T}(\gamma(\ell\cdot)) \in \Psi^{-\infty}.$$

For $v \in C^\infty(\mathbb{T})$ and $t \in \mathbb{T}$, we compute $\big(UC_\Sigma\overline{T}U^{-1}v\big)(t)$. Remark that for $x = (x_1, x_2)^\top \in \Sigma$ and $w(x) := (U^{-1}v)(x)$, (2.3) and (2.20) gives

$$\begin{aligned}(C_\Sigma\overline{T}w)(x) &= \frac{\mathrm{i}}{\pi}\,\mathrm{p.v.}\int_0^\ell \frac{w(\gamma(s))}{(x_1+ix_2)-(\gamma_1(s)+i\gamma_2(s))}ds \\ &= \frac{\mathrm{i}}{\pi}\,\mathrm{p.v.}\int_0^\ell \frac{v(\ell^{-1}s)}{(x_1+ix_2)-(\gamma_1(s)+i\gamma_2(s))}\,\mathrm{d}s.\end{aligned}$$

Hence, a change of variable yields

$$(UC_\Sigma\overline{T}U^{-1}v)(t) = \ell\frac{\mathrm{i}}{\pi}\,\mathrm{p.v.}\int_{\mathbb{T}}\frac{v(s)}{\rho(t)-\rho(s)}\,\mathrm{d}s$$

with $\rho(t) := \gamma_1(\ell t)+i\gamma_2(\ell t)$. Remark that for all $t \in \mathbb{T}$ we have $\rho'(t) = \ell T(\gamma(\ell t)) \neq 0$ and $\frac{1}{\rho'(t)} = \ell^{-1}\overline{T}(\gamma(\ell t))$. Corollary 2.8 gives

$$\ell^{-1}UC_\Sigma\overline{T}U^{-1} - \ell^{-1}T_0\overline{T}(\ell\cdot) \in \Psi^{-\infty}$$

which completes the proof of (i). Item (ii) is proved in a similar fashion and the last statement is a consequence of (i), (ii), and (2.15). This can be seen by the equivalences

$$T_0^2 - \mathbb{1} \in \Psi^{-\infty} \Longleftrightarrow UC'_\Sigma U^{-1}UC_\Sigma U^{-1} - \mathbb{1} \in \Psi^{-\infty} \Longleftrightarrow C'_\Sigma C_\Sigma - \mathbb{1} \in \Psi_\Sigma^{-\infty},$$

and a similar argument shows $C_\Sigma C'_\Sigma - \mathbb{1} \in \Psi_\Sigma^{-\infty}$. This completes the proof. □

## 2.2 Schur complement of block operators

Let $W_{jk}$, $j,k \in \{1,2\}$, be closable densely defined operators in a Hilbert space $\mathcal{H}$. Define a linear operator $W$ in $\mathcal{H} \oplus \mathcal{H}$ by

$$W := \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}, \quad \operatorname{dom} W = (\operatorname{dom} W_{11} \cap \operatorname{dom} W_{21}) \oplus (\operatorname{dom} W_{12} \cap \operatorname{dom} W_{22}).$$

Assume that $\operatorname{dom} W_{11} \subset \operatorname{dom} W_{21}$ and that $W_{11}$ is invertible. Then one can define the Schur complement $\mathcal{S}(W)$ of $W$ as an operator in $\mathcal{H}$ by

$$\mathcal{S}(W) := W_{22} - W_{21} W_{11}^{-1} W_{12}, \tag{2.23}$$

and one has the factorization

$$W = \begin{pmatrix} \mathbb{1} & 0 \\ W_{21} W_{11}^{-1} & \mathbb{1} \end{pmatrix} \begin{pmatrix} W_{11} & 0 \\ 0 & \mathcal{S}(W) \end{pmatrix} \begin{pmatrix} \mathbb{1} & W_{11}^{-1} W_{12} \\ 0 & \mathbb{1} \end{pmatrix}. \tag{2.24}$$

We will use the following facts, which follow from Theorem 2.2.14 and Theorem 2.4.6 in the monograph [39].

**Proposition 2.10.** *Assume that* $0 \in \operatorname{res}(W_{11})$*, that* $\operatorname{dom} W_{11} \subset \operatorname{dom} W_{21}$ *and that* $W_{11}^{-1} W_{12}$ *is bounded on* $\operatorname{dom} W_{12}$*. Then* $W$ *is closable/closed if and only if its Schur complement* $\mathcal{S}(W)$ *is closable/closed, with*

$$\overline{W} = \begin{pmatrix} \mathbb{1} & 0 \\ W_{21} W_{11}^{-1} & \mathbb{1} \end{pmatrix} \begin{pmatrix} W_{11} & 0 \\ 0 & \overline{\mathcal{S}(W)} \end{pmatrix} \begin{pmatrix} \mathbb{1} & \overline{W_{11}^{-1} W_{12}} \\ 0 & \mathbb{1} \end{pmatrix},$$

*and*

$$\operatorname{dom} \overline{W} = \left\{ (x_1, x_2) \in \mathcal{H} \times \mathcal{H} : x_1 + \overline{W_{11}^{-1} W_{12}}\, x_2 \in \operatorname{dom} W_{11},\ x_2 \in \operatorname{dom} \overline{\mathcal{S}(W)} \right\}.$$

*Moreover, if* $\overline{W}$ *is self-adjoint, then* $0 \in \operatorname{spec}_{\mathrm{ess}}(\overline{W})$ *if and only if* $0 \in \operatorname{spec}_{\mathrm{ess}} \big(\overline{\mathcal{S}(W)}\big)$.

## 2.3 Boundary triples and their Weyl functions

We recall some basic facts about boundary triples following the first chapter of the paper [14], in which the proofs for all statements can be found. We also refer the reader to [17, 18] and the monographs [8, 19] for more details and applications. Throughout this abstract section $\mathcal{H}$ is always a separable Hilbert space.

**Definition 2.11.** *Let* $S$ *be a densely defined closed symmetric operator in* $\mathcal{H}$*. A* boundary triple *for* $S^*$ *is a triple* $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ *consisting of a Hilbert space* $\mathcal{G}$ *and two linear maps* $\Gamma_0, \Gamma_1 : \operatorname{dom} S^* \to \mathcal{G}$ *satisfying the following two conditions:*

(i) *For all* $f, g \in \operatorname{dom} S^*$

$$(S^* f, g)_{\mathcal{H}} - (f, S^* g)_{\mathcal{H}} = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{G}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{G}}$$

*holds.*

(ii) *The map* $\operatorname{dom} S^* \ni f \mapsto (\Gamma_0 f, \Gamma_1 f)^\top \in \mathcal{G} \times \mathcal{G}$ *is surjective.*

A boundary triple for $S^*$ exists if and only if $S$ admits self-adjoint extensions in $\mathcal{H}$. From now on we assume that this is satisfied and pick a boundary triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$. This induces a number of additional objects. First, the operator

$$B_0 := S^* \restriction \ker \Gamma_0$$

is self-adjoint, and for any $z \in \operatorname{res}(B_0)$ one has the direct sum decomposition

$$\operatorname{dom} S^* = \operatorname{dom} B_0 \dotplus \ker(S^* - z) = \ker \Gamma_0 \dotplus \ker(S^* - z), \tag{2.25}$$

showing that $\Gamma_0 \restriction \ker(S^* - z)$ is bijective. This allows to define the $\gamma$-field $G$ and the Weyl function $M$ associated to $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ by

$$\operatorname{res}(B_0) \ni z \mapsto G_z := \big(\Gamma_0 \restriction \ker(S^* - z)\big)^{-1} : \mathcal{G} \to \mathcal{H}$$

and

$$\operatorname{res}(B_0) \ni z \mapsto M_z := \Gamma_1 G_z : \mathcal{G} \to \mathcal{G}.$$

It is not difficult to show that the operators $G_z$ and $M_z$, $z \in \operatorname{res}(B_0)$, are bounded, that the adjoints of these operators are given by

$$G_z^* = \Gamma_1 (B_0 - \overline{z})^{-1} \quad \text{and} \quad M_z^* = M_{\bar{z}},$$

and that $z \mapsto G_z$ and $z \mapsto M_z$ are holomorphic in $z \in \operatorname{res}(B_0)$.

Boundary triples are designed as a tool to handle operators with boundary conditions in an abstract framework via the boundary mappings $\Gamma_0$ and $\Gamma_1$. To make this more precise, assume that $\mathcal{G} = \mathcal{G}_\Pi \oplus \mathcal{G}_\Pi^\perp$ with some closed subspace $\mathcal{G}_\Pi$, let $\Pi : \mathcal{G} \to \mathcal{G}_\Pi$ be the orthogonal projection onto $\mathcal{G}_\Pi$, and let $\Pi^* : \mathcal{G}_\Pi \to \mathcal{G}$ be the canonical embedding of $\mathcal{G}_\Pi$ in $\mathcal{G}$. Assume that $\Theta$ is a linear operator in the Hilbert space $\mathcal{G}_\Pi$ viewed with the induced inner product. In the following we are interested in extensions of $S$ (formally) given by

$$B_{\Pi,\Theta} := S^* \restriction \ker(\Pi\Gamma_1 - \Theta\Gamma_0). \tag{2.26}$$

More precisely, the operator $B_{\Pi,\Theta}$ is the restriction of $S^*$ onto the set

$$\operatorname{dom} B_{\Pi,\Theta} = \big\{ f \in \operatorname{dom} S^* : \Pi\Gamma_1 f = \Theta\Pi\Gamma_0 f,\ (1 - \Pi^*\Pi)\Gamma_0 f = 0 \big\},$$

where the boundary condition $\Pi\Gamma_1 f = \Theta\Pi\Gamma_0 f$ in $\operatorname{dom} B_{\Pi,\Theta}$ also contains the condition $\Pi\Gamma_0 f \in \operatorname{dom}\Theta$. A number of properties of $B_{\Pi,\Theta}$ are encoded in $\Theta$. The most important of them for our purposes are summarized in the following theorem:

**Theorem 2.12.** *Let $S$ be a densely defined closed symmetric operator in $\mathcal{H}$, let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triple for $S^*$ with $\gamma$-field $G_z$ and Weyl function $M_z$, and let $B_0 = S^* \restriction \ker \Gamma_0$. Moreover, let $\Pi : \mathcal{G} \to \mathcal{G}_\Pi$ be an orthogonal projection, let $\Theta$ be a linear operator in $\mathcal{G}_\Pi$, and let $B_{\Pi,\Theta}$ be defined by* (2.26)*. Then $B_{\Pi,\Theta}$ is (essentially) self-adjoint in $\mathcal{H}$ if and only if $\Theta$ is (essentially) self-adjoint in $\mathcal{G}_\Pi$. Furthermore, if $\Theta$ is self-adjoint and $z \in \operatorname{res}(B_0)$, then the following assertions hold:*

(i) $z \in \operatorname{spec}(B_{\Pi,\Theta})$ *if and only if* $0 \in \operatorname{spec}(\Theta - \Pi M_z \Pi^*)$.

(ii) $z \in \operatorname{spec}_{\mathrm{p}}(B_{\Pi,\Theta})$ *if and only if* $0 \in \operatorname{spec}_{\mathrm{p}}(\Theta - \Pi M_z \Pi^*)$ *and the eigenspace is* $\ker(B_{\Pi,\Theta} - z) = G_z \Pi^* \ker(\Theta - \Pi M_z \Pi^*)$.

(iii) $z \in \operatorname{spec}_{\mathrm{ess}}(B_{\Pi,\Theta})$ *if and only if* $0 \in \operatorname{spec}_{\mathrm{ess}}(\Theta - \Pi M_z \Pi^*)$.

(iii) *For all* $z \in \operatorname{res}(B_{\Pi,\Theta}) \cap \operatorname{res}(B_0)$ *one has*

$$(B_{\Pi,\Theta} - z)^{-1} = (B_0 - z)^{-1} + G_z \Pi^* (\Theta - \Pi M_z \Pi^*)^{-1} \Pi G_{\bar{z}}^*.$$

Finally we recall a special approach for the construction of boundary triples using abstract trace maps developed in [33] and [34], see also [14, Section 1.4.2]. Let $B$ be a self-adjoint operator in the Hilbert space $\mathcal{H}$, let $\mathcal{G}$ be another Hilbert space, and assume that

$$\mathcal{T} : \operatorname{dom} B \to \mathcal{G}$$

is a *surjective* linear operator which is bounded with respect to the graph norm of $B$ and such that $\ker \mathcal{T}$ is a dense subspace of the initial Hilbert space $\mathcal{H}$. Then

$$S := B \restriction \ker \mathcal{T}$$

is a densely defined closed symmetric operator. Next, define for any $z \in \operatorname{res}(B)$ the injective operator

$$G_z := \big(\mathcal{T}(B - \bar{z})^{-1}\big)^*, \tag{2.27}$$

which is bounded from $\mathcal{G}$ to $\mathcal{H}$. Then one has $\operatorname{ran} G_z = \ker(S^* - z)$ for $z \in \operatorname{res}(B)$ and (2.25) leads to the direct sum decomposition

$$\operatorname{dom} S^* = \operatorname{dom} B \dot{+} \operatorname{ran} G_z, \quad z \in \operatorname{res}(B), \tag{2.28}$$

which shows that for all $f \in \operatorname{dom} S^*$ there exist unique $f_z \in \operatorname{dom} B$ and $\xi \in \mathcal{G}$ such that $f = f_z + G_z \xi$; one can show that the component $\xi$ does not depend on the choice of $z$. Having these notations in hand we can formulate now the following proposition:

**Proposition 2.13.** *Let* $\zeta \in \operatorname{res}(B)$ *be fixed and define the mappings* $\Gamma_0, \Gamma_1 : \operatorname{dom} S^* \to \mathcal{G}$ *for* $f = f_\zeta + G_\zeta \xi = f_{\bar{\zeta}} + G_{\bar{\zeta}} \xi \in \operatorname{dom} S^*$ *by*

$$\Gamma_0 f := \xi \quad \textit{and} \quad \Gamma_1 f := \frac{1}{2} \mathcal{T}(f_\zeta + f_{\bar{\zeta}}).$$

*Then* $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ *is a boundary triple for* $S^*$ *with* $S^* \restriction \ker \Gamma_0 = B$. *Moreover, the* $\gamma$*-field and the Weyl function are given by* (2.27) *and*

$$M_z = \mathcal{T}\big(G_z - \frac{1}{2}(G_\zeta + G_{\bar{\zeta}})\big), \quad z \in \operatorname{res}(B).$$

# 3 The free Dirac operator and a boundary triple for Dirac operators in $\mathbb{R}^2$

In this section we first recall the definition of the free Dirac operator in $\mathbb{R}^2$, a minimal and a maximal realization of the Dirac operator in $\mathbb{R}^2 \setminus \Sigma$, and we introduce and study some families of integral operators which will play an important role in our analysis in Section 4. Afterwards, we define a boundary triple which is useful in the treatment of Dirac operators with singular $\delta$-interactions.

## 3.1 The free, the minimal, and the maximal Dirac operator and some associated integral operators

For $m \in \mathbb{R}$ the free Dirac operator in $\mathbb{R}^2$ is defined by

$$A_0 f = -\mathrm{i} \sum_{j=1}^{2} \sigma_j \partial_j f + m\sigma_3 f = -\mathrm{i}\, \sigma \cdot \nabla f + m\sigma_3 f, \quad \operatorname{dom} A_0 = H^1(\mathbb{R}^2; \mathbb{C}^2), \tag{3.1}$$

where $\sigma := (\sigma_1, \sigma_2)$ and $\sigma_3$ are the $\mathbb{C}^{2\times 2}$-valued Pauli spin matrices in (1.4). First, we provide some basic properties of $A_0$. We refer to the monograph [38] for a detailed discussion of these facts in the three dimensional case; the modifications to the present two dimensional situation are left to the reader. Using the Fourier transform and (1.5) one verifies that $A_0$ is self-adjoint in $L^2(\mathbb{R}^2; \mathbb{C}^2)$ and that its spectrum is purely essential,

$$\operatorname{spec}(A_0) = \operatorname{spec}_{\mathrm{ess}}(A_0) = \big(-\infty, -|m|\big] \cup \big[|m|, +\infty\big).$$

In particular, $\operatorname{spec}(A_0) = \mathbb{R}$ for $m = 0$. Due to the identity

$$(A_0 - z)(A_0 + z) = (-\Delta + m^2 - z^2)\sigma_0$$

one can express the resolvent of $A_0$ through the resolvent of the free Laplacian. Recall that for $z \notin \operatorname{spec}(-\Delta) = [0, \infty)$ the resolvent $(-\Delta - z)^{-1}$ is the integral operator

$$(-\Delta - z)^{-1} f(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} K_0\big(\sqrt{-z}|x-y|\big)\, f(y)\, dy,$$

where $K_j$ stands for the modified Bessel function of second kind of order $j$, and we take the principal square root function, i.e. for $z \in \mathbb{C} \setminus [0, \infty)$ the number $\sqrt{z}$ is determined by $\operatorname{Re}\sqrt{z} > 0$. For $z \in \operatorname{res}(A_0)$ one gets

$$(A_0 - z)^{-1} = (A_0 + z)\big(-\Delta - (z^2 - m^2)\big)^{-1}\sigma_0,$$

which leads to

$$(A_0 - z)^{-1} f(x) = \int_{\mathbb{R}^2} \phi_z(x-y) f(y)\, \mathrm{d}y, \quad f \in L^2(\mathbb{R}^2; \mathbb{C}^2),$$

where

$$\begin{aligned}\phi_z(x) = \mathrm{i}\frac{\sqrt{m^2-z^2}}{2\pi}K_1\big(\sqrt{m^2-z^2}|x|\big)\left(\sigma\cdot\frac{x}{|x|}\right)\\ +\frac{1}{2\pi}K_0\big(\sqrt{m^2-z^2}|x|\big)\big(m\sigma_3+z\sigma_0\big).\end{aligned} \tag{3.2}$$

Next we introduce a minimal symmetric operator $S$ which is suitable for our purposes. More precisely, let $S$ be the restriction of $A_0$ to the functions vanishing at $\Sigma$, i.e.

$$Sf = (-i\sigma\cdot\nabla + m\sigma_3)f, \quad \operatorname{dom} S = H_0^1(\mathbb{R}^2\setminus\Sigma;\mathbb{C}^2). \tag{3.3}$$

We remark that the $A_{\eta,\tau}$ defined in (1.2) is an extension of the symmetric operator $S$. One verifies in the same way as in the three dimensional case (see, e.g., [9, Proposition 3.1]) that the adjoint $S^*$ is the maximal realization of the Dirac operator

$$\begin{aligned}S^*f &= (-i\sigma\cdot\nabla + m\sigma_3)f_+ \oplus (-i\sigma\cdot\nabla + m\sigma_3)f_-,\\ \operatorname{dom} S^* &= \big\{f = f_+\oplus f_-\in L^2(\Omega_+;\mathbb{C}^2)\oplus L^2(\Omega_-;\mathbb{C}^2):\ f_\pm\in H(\sigma,\Omega_\pm)\big\},\end{aligned} \tag{3.4}$$

where

$$H(\sigma,\Omega_\pm) = \big\{f_\pm\in L^2(\Omega_\pm;\mathbb{C}^2) : (-i\sigma\cdot\nabla + m\sigma_3)f_\pm\in L^2(\Omega_\pm;\mathbb{C}^2)\big\}, \tag{3.5}$$

and the derivatives in the above formula are understood in the distributional sense. It is not difficult to see that $H(\sigma,\Omega_\pm)$ endowed with the norm

$$\|f_\pm\|^2_{H(\sigma,\Omega_\pm)} := \|f\|^2_{L^2(\Omega_\pm;\mathbb{C}^4)} + \big\|(-i\sigma\cdot\nabla + m\sigma_3)f_\pm\big\|^2_{L^2(\Omega_\pm;\mathbb{C}^2)}$$

is a Hilbert space which is actually independent of $m$; cf. [11, Lemma 2.1]. For our further considerations, it is useful to extend the Dirichlet trace operator onto $H(\sigma;\Omega_\pm)$. In the following lemma we summarize several known results from [11, Lemma 2.3 and Lemma 2.4]:

**Lemma 3.1.** *The trace map $\mathfrak{T}^D_{\pm,0}: H^1(\Omega_\pm;\mathbb{C}^2)\to H^{1/2}(\Sigma;\mathbb{C}^2), \mathfrak{T}^D_{\pm,0}f = f|_\Sigma$, can be extended to a bounded linear operator*

$$\mathfrak{T}^D_\pm : H(\sigma,\Omega_\pm)\to H^{-1/2}(\Sigma;\mathbb{C}^2).$$

*Moreover, if $\mathfrak{T}^D_\pm f\in H^{1/2}(\Sigma;\mathbb{C}^2)$ for $f\in H(\sigma,\Omega_\pm)$, then $f\in H^1(\Omega_\pm;\mathbb{C}^2)$.*

In the next result we show that any self-adjoint extension $A$ of $S$ with $\operatorname{dom} A\subset H^s(\mathbb{R}^2\setminus\Sigma;\mathbb{C}^2)$ for some $s>0$ has only finitely many discrete eigenvalues in $(-|m|,|m|)$.

**Proposition 3.2.** *Let $A$ be a self-adjoint extension of the symmetric operator $S$ in $L^2(\mathbb{R}^2;\mathbb{C}^2)$ and assume that $\operatorname{dom} A\subset H^s(\mathbb{R}^2\setminus\Sigma;\mathbb{C}^2)$ holds for some $s>0$. Then $\operatorname{spec}(A)\cap(-|m|,|m|)$ is purely discrete and the number of the discrete eigenvalues of $A$ in $(-|m|,|m|)$ is finite.*

**Proof**. It is sufficient to show that $A^2$ has at most finitely many eigenvalues in $(-\infty, m^2)$. For that, consider the quadratic form

$$a[f,f] = \int_{\mathbb{R}^2} |Af|^2 \,\mathrm{d}x, \quad \operatorname{dom} a = \operatorname{dom} A.$$

Since $A$ is self-adjoint and hence closed, also the densely defined nonnegative form $a$ is closed. The self-adjoint operator associated to $a$ via the first representation theorem is $A^2$. Next, take $0 < r < R$ with $r$ chosen sufficiently large, such that the open ball $B_r = \{x \in \mathbb{R}^2 : |x| < r\}$ contains $\overline{\Omega_+}$ in its interior, and choose $\varphi_1, \varphi_2 \in C^\infty(\mathbb{R}^2)$ which satisfy

$$0 \le \varphi_1, \varphi_2 \le 1, \quad \varphi_1^2 + \varphi_2^2 = 1, \quad \varphi_1 = 1 \text{ in } B_r, \quad \varphi_2 = 1 \text{ in } \mathbb{R}^2 \setminus B_R.$$

Let $f \in \operatorname{dom} A$ be fixed. Then by construction one has $\varphi_j f \in \operatorname{dom} A$ and

$$A(\varphi_j f) = \varphi_j A f - \mathrm{i}\sigma \cdot (\nabla \varphi_j) f.$$

In particular, we note that $\varphi_2 f \in H(\sigma, \Omega_-)$ with $\mathfrak{T}_-^D f = 0 \in H^{1/2}(\Sigma; \mathbb{C}^2)$. Thus, it follows from Lemma 3.1 that $\varphi_2 f \in H^1(\Omega_-; \mathbb{C}^2)$.

Next, we remark that $\nabla \varphi_j$ is supported in $\overline{B_R} \setminus B_r$. Hence, we have for $j \in \{1,2\}$

$$a[\varphi_j f, \varphi_j f] = \int_{\mathbb{R}^2} \left( \varphi_j^2 |Af|^2 + |\mathrm{i}\sigma \cdot (\nabla \varphi_j) f|^2 \right) \mathrm{d}x + \mathfrak{I}_j,$$

where

$$\begin{aligned}
\mathfrak{I}_j &= \int_{B_R \setminus B_r} 2 \operatorname{Re} \left( \varphi_j (-i\sigma \cdot \nabla + m\sigma_3) f, -\mathrm{i}\sigma \cdot (\nabla \varphi_j) f \right)_{\mathbb{C}^2} \mathrm{d}x \\
&= \int_{B_R \setminus B_r} 2 \operatorname{Re} \left( (-i\sigma \cdot \nabla + m\sigma_3) f, -\mathrm{i}\sigma \cdot (\varphi_j \nabla \varphi_j) f \right)_{\mathbb{C}^2} \mathrm{d}x \\
&= \int_{B_R \setminus B_r} \operatorname{Re} \left( (-i\sigma \cdot \nabla + m\sigma_3) f, -\mathrm{i}\sigma \cdot \nabla (\varphi_j^2) f \right)_{\mathbb{C}^2} \mathrm{d}x.
\end{aligned}$$

From $\varphi_1^2 + \varphi_2^2 = 1$ we obtain $\nabla(\varphi_1^2) = -\nabla(\varphi_2^2)$ and hence $\mathfrak{I}_1 = -\mathfrak{I}_2$. Moreover, using (1.5) one verifies $|\mathrm{i}\sigma \cdot (\nabla \varphi_j) f|^2 = |\nabla \varphi_j|^2 |f|^2$ for $j \in \{1,2\}$. Therefore, it follows that

$$\begin{aligned}
&a[\varphi_1 f, \varphi_1 f] + a[\varphi_2 f, \varphi_2 f] \\
&\quad = \int_{\mathbb{R}^2} (\varphi_1^2 + \varphi_2^2) |Af|^2 \,\mathrm{d}x + \int_{\mathbb{R}^2} \left( |\nabla \varphi_1|^2 + |\nabla \varphi_2|^2 \right) |f|^2 \right) \mathrm{d}x \\
&\quad = \int_{\mathbb{R}^2} |Af|^2 \,\mathrm{d}x + \int_{\mathbb{R}^2} V |f|^2 \,\mathrm{d}x,
\end{aligned}$$

where we have used the abbreviation $V := |\nabla \varphi_1|^2 + |\nabla \varphi_2|^2$ in the last step; note that $V$ is supported in $\overline{B_R} \setminus B_r$. This leads to

$$a[f,f] = a[\varphi_1 f, \varphi_1 f] - \int_{\mathbb{R}^2} V |\varphi_1 f|^2 \,\mathrm{d}x + a[\varphi_2 f, \varphi_2 f] - \int_{\mathbb{R}^2} V |\varphi_2 f|^2 \,\mathrm{d}x. \tag{3.6}$$

In the following we will often restrict functions in $\operatorname{dom} a$ to $B_R$ or $\mathbb{R}^2\setminus\overline{B_r}$ and view them as elements in $L^2(B_R;\mathbb{C}^2)$ or $L^2(\mathbb{R}^2\setminus\overline{B_r};\mathbb{C}^2)$, or we will extend $L^2$-functions on $B_R$ or $\mathbb{R}^2\setminus\overline{B_r}$ by zero onto $\mathbb{R}^2$ and view them as elements in $L^2(\mathbb{R}^2;\mathbb{C}^2)$. We find it convenient to use the same letter for the original and the restricted or extended function.

Let $a_1$ be the quadratic form in $L^2(B_R;\mathbb{C}^2)$ defined by

$$\operatorname{dom} a_1 = \big\{g\in\operatorname{dom} a : \operatorname{supp} g\subset\overline{B_R}\big\},\quad a_1[g,g] = a[g,g] - \int_{B_R} V|g|^2\,\mathrm{d}x.$$

As $V$ is bounded and $a$ is nonnegative it follows that $a_1$ is semibounded from below. It is also clear that $a_1$ is densely defined in $L^2(B_R;\mathbb{C}^2)$. To see that $a_1$ is closed consider $g_n\in\operatorname{dom} a_1$ such that $g_n\to g$ in $L^2(B_R;\mathbb{C}^2)$ for $n\to\infty$ and $a_1(g_n-g_m,g_n-g_m)\to 0$ for $n,m\to\infty$. Since $V$ is bounded it follows that the zero extensions $g_n$ and $g$ satisfy $g_n\to g$ in $L^2(\mathbb{R}^2;\mathbb{C}^2)$ for $n\to\infty$ and $a(g_n-g_m,g_n-g_m)\to 0$ for $n,m\to\infty$. As $a$ is closed we conclude $g\in\operatorname{dom} a$ and $a(g_n-g,g_n-g)\to 0$ for $n\to\infty$. Furthermore, as $\operatorname{supp} g\subset\overline{B_R}$ we have $g\in\operatorname{dom} a_1$ and $a_1(g_n-g,g_n-g)\to 0$ for $n\to\infty$, thus $a_1$ is closed. Let $A_1$ be the self-adjoint operator in $L^2(B_R;\mathbb{C}^2)$ corresponding to $a_1$. Then $A_1$ has a compact resolvent since the form domain $\operatorname{dom} a_1\subset H^s(B_R;\mathbb{C}^2)$ is compactly embedded in $L^2(B_R;\mathbb{C}^2)$ for $s>0$. Hence, the number of eigenvalues $\mathcal{N}(A_1,m^2)$ of $A_1$ below $m^2$ is finite, that is, $\mathcal{N}(A_1,m^2)<\infty$.

Next, let $a_2$ be the quadratic form in $L^2(\mathbb{R}^2\setminus\overline{B_r};\mathbb{C}^2)$ defined by

$$\operatorname{dom} a_2 = H^1_0\big(\mathbb{R}^2\setminus\overline{B_r};\mathbb{C}^2\big),\quad a_2[g,g] = a[g,g] - \int_{\mathbb{R}^2\setminus B_r} V|g|^2\,\mathrm{d}x.$$

As above it is clear that $a_2$ is densely defined and semibounded from below. Using integration by parts and (1.5) one sees for $g\in C_0^\infty(\mathbb{R}^2\setminus\overline{B_r};\mathbb{C}^2)$ that

$$\begin{aligned} a[g,g] &= \int_{\mathbb{R}^2\setminus B_r} |(-i\sigma\cdot\nabla + m\sigma_3)g|^2\,\mathrm{d}x \\ &= \int_{\mathbb{R}^2\setminus B_r} \big(g,(-i\sigma\cdot\nabla+m\sigma_3)^2 g\big)_{\mathbb{C}^2}\,\mathrm{d}x \\ &= \int_{\mathbb{R}^2\setminus B_r} \big(g,(-\Delta+m^2)g\big)_{\mathbb{C}^2}\,\mathrm{d}x \\ &= \int_{\mathbb{R}^2\setminus B_r} \big(|\nabla g|^2+m^2|g|^2\big)\,\mathrm{d}x, \end{aligned}$$

which then extends by density to all $g\in H^1_0\big(\mathbb{R}^2\setminus\overline{B_r};\mathbb{C}^2\big)$. Therefore, the form $a_2$ is closed and the self-adjoint operator associated to $a_2$ is $A_2 = -\Delta^D + m^2 - V$, where $-\Delta^D$ denotes the Dirichlet Laplacian in $\mathbb{R}^2\setminus\overline{B_r}$. Hence, it follows that $\mathcal{N}(m^2,A_2)<\infty$, as $V$ has compact support.[1]

[1] In fact, to see that a compactly supported potential $V$ leads only to finitely many eigenvalues of $A_2$ below $m^2$ one may argue as follows: Decompose $A_2$ in a similar way as in the proof of [25, Proposition 3.6 (a)] in an operator $A_3$ acting in $L^2(B_{2R}\setminus\overline{B_r};\mathbb{C}^2)$ and an operator $A_4$ acting in

Now, we can conclude that $A^2$ has only finitely many eigenvalues below $m^2$. For this consider

$$J : L^2(\mathbb{R}^2;\mathbb{C}^2) \to L^2(B_R;\mathbb{C}^2) \oplus L^2(\mathbb{R}^2 \setminus \overline{B_r};\mathbb{C}^2), \qquad Jf = \varphi_1 f \oplus \varphi_2 f.$$

Due to the properties of $\varphi_1$ and $\varphi_2$ we get that $J$ is an isometry. Moreover, with the above considerations we see $J(\operatorname{dom} a) \subset \operatorname{dom} a_1 \oplus \operatorname{dom} a_2$, and with the equality (3.6) we obtain

$$\frac{a[f,f]}{\|f\|^2_{L^2(\mathbb{R}^2;\mathbb{C}^2)}} = \frac{(a_1 \oplus a_2)[Jf,Jf]}{\|Jf\|^2_{L^2(B_R;\mathbb{C}^2)\oplus L^2(\mathbb{R}^2\setminus\overline{B_r};\mathbb{C}^2)}}.$$

It follows from the min-max principle that

$$\mathcal{N}(m^2, A^2) \leq \mathcal{N}(m^2, A_1 \oplus A_2) = \mathcal{N}(m^2, A_1) + \mathcal{N}(m^2, A_2).$$

As we have seen above, the quantity on the right hand side is finite and hence $\mathcal{N}(m^2, A^2) < \infty$. This completes the proof. □

In the following we introduce some families of integral operators associated to the Green function $\phi_z$ associated to $A_0$ given by (3.2). Let us denote the Dirichlet trace operator on $H^1(\mathbb{R}^2;\mathbb{C}^2)$ by $\mathcal{T}^D : H^1(\mathbb{R}^2;\mathbb{C}^2) \to H^{1/2}(\Sigma;\mathbb{C}^2)$. It is well-known that $\mathcal{T}^D$ is bounded, surjective, and $\ker \mathcal{T}^D = H^1_0(\mathbb{R}^2 \setminus \Sigma;\mathbb{C}^2)$; cf. [26, Theorems 3.37 and 3.40]. For $z \in \operatorname{res}(A_0)$ we first define the bounded operator

$$\Phi'_z := \mathcal{T}^D(A_0 - \bar{z})^{-1} : L^2(\mathbb{R}^2;\mathbb{C}^2) \to H^{1/2}(\Sigma;\mathbb{C}^2) \tag{3.7}$$

and its anti-dual

$$\Phi_z := \big(\mathcal{T}^D(A_0 - \bar{z})^{-1}\big)' : H^{-1/2}(\Sigma;\mathbb{C}^2) \to L^2(\mathbb{R}^2;\mathbb{C}^2). \tag{3.8}$$

The basic properties of $\Phi_z$ are stated in the following proposition:

**Proposition 3.3.** *Let $z \in \operatorname{res}(A_0)$ and consider the operator $\Phi_z$ in (3.8). Then for $\varphi \in L^2(\Sigma;\mathbb{C}^2)$ one has*

$$\Phi_z\varphi(x) = \int_\Sigma \phi_z(x-y)\varphi(y)\mathrm{d}s(y) \quad \text{for a.e. } x \in \mathbb{R}^2 \setminus \Sigma.$$

*Moreover, $\Phi_z$ is a bounded bijective operator from $H^{-1/2}(\Sigma;\mathbb{C}^2)$ onto $\ker(S^* - z)$.*

**Proof**. First, due to the properties of the trace map it is clear that $\Phi'_z$ defined by (3.7) is surjective and

$$\ker \Phi'_z = \big\{ f \in L^2(\mathbb{R}^2;\mathbb{C}^2) : (A_0 - \bar{z})^{-1} f \in H^1_0(\mathbb{R}^2 \setminus \Sigma;\mathbb{C}^2) \big\} = \operatorname{ran}(S - \bar{z}),$$

$L^2(\mathbb{R}^2 \setminus \overline{B_{2R}};\mathbb{C}^2)$ by introducing a Neumann boundary condition on $\partial B_{2R}$. The quadratic forms associated to $A_3$ and $A_4$ are denoted by $a_3$ and $a_4$, respectively. Then, as for $A_1$, the operator $A_3$ has a compact resolvent, as $\operatorname{dom} A_3 \subset \operatorname{dom} a_3 \subset H^1(B_{2R} \setminus \overline{B_r};\mathbb{C}^2)$, and hence only finitely many discrete eigenvalues below $m^2$. Since $V$ is supported in $\overline{B_R} \setminus B_r$, the form $a_4$ corresponds to the Neumann Laplacian in $L^2(\mathbb{R}^2 \setminus \overline{B_{2R}};\mathbb{C}^2)$ shifted by $m^2$ and hence it has no eigenvalues below $m^2$. Thus, by the min-max principle also $A_2$ has only finitely many eigenvalues below $m^2$.

as $S = A_0 \upharpoonright H^1_0(\mathbb{R}^2 \setminus \Sigma; \mathbb{C}^2)$. Using the closed range theorem, $(\operatorname{ran} \Phi_z)^\perp = \ker \Phi_z'$, and the fact that $\ker(S^* - z) = (\operatorname{ran}(S - \bar{z}))^\perp$ is closed we conclude that

$$\Phi_z : H^{-1/2}(\Sigma; \mathbb{C}^2) \to \ker(S^* - z)$$

is a bounded bijective operator. To prove the integral representation consider $\varphi \in L^2(\Sigma; \mathbb{C}^2)$ and $f \in L^2(\mathbb{R}^2; \mathbb{C}^2)$. A direct computation using Fubini's theorem shows

$$\begin{aligned}
(f, \Phi_z \varphi)_{L^2(\mathbb{R}^2;\mathbb{C}^2)} &= (\Phi_z' f, \varphi)_{L^2(\Sigma;\mathbb{C}^2)} \\
&= \big(\mathcal{T}^D (A_0 - \bar{z})^{-1} f, \varphi\big)_{L^2(\Sigma;\mathbb{C}^2)} \\
&= \int_\Sigma \bigg( \int_{\mathbb{R}^2} \phi_{\bar{z}}(x - y) f(y) \, \mathrm{d}y, \varphi(x) \bigg)_{\mathbb{C}^2} \mathrm{d}s(x) \\
&= \int_{\mathbb{R}^2} \bigg( f(y), \int_\Sigma \phi_{\bar{z}}(x - y)^* \varphi(x) \, \mathrm{d}s(x) \bigg)_{\mathbb{C}^2} \mathrm{d}y \\
&= \int_{\mathbb{R}^2} \bigg( f(y), \int_\Sigma \phi_z(y - x) \varphi(x) \, \mathrm{d}s(x) \bigg)_{\mathbb{C}^2} \mathrm{d}y,
\end{aligned}$$

where the symmetry property $\phi_{\bar{z}}(x - y)^* = \phi_z(y - x)$ was used in the last equality. This implies the representation for $\Phi_z \varphi$, $\varphi \in L^2(\Sigma; \mathbb{C}^2)$, and completes the proof of this proposition. □

We will also need a family of boundary integral operators with integral kernel $\phi_z$. To introduce these operators, we study first the structure of the Green function $\phi_z$ in more detail:

**Lemma 3.4.** *Let $z \in \operatorname{res}(A_0)$ and consider the function $\phi_z$ in (3.2). Then there exist scalar analytic functions $g_1, g_2, g_3$, and $g_4$ and a constant $c_1 < 0$ such that*

$$\begin{aligned}
\phi_z(x) = {} & \frac{\mathrm{i}}{2\pi} \sigma \cdot \frac{x}{|x|^2} - \frac{1}{2\pi} \Big( \log|x| + \log\sqrt{m^2 - z^2} + c_1 \Big) (m\sigma_3 + z\sigma_0) \\
& + \frac{\mathrm{i}}{2\pi} (m^2 - z^2) \Big[ g_1\big((m^2 - z^2)|x|^2\big) \big( \log\sqrt{m^2 - z^2} + \log|x| \big) \\
& \qquad + g_2\big((m^2 - z^2)|x|^2\big) \Big] (\sigma \cdot x) \qquad (3.9) \\
& + \frac{1}{2\pi} (m^2 - z^2)|x|^2 \Big[ g_3\big((m^2 - z^2)|x|^2\big) \big( \log\sqrt{m^2 - z^2} + \log|x| \big) \\
& \qquad + g_4\big((m^2 - z^2)|x|^2\big) \Big] (m\sigma_3 + z\sigma_0).
\end{aligned}$$

*In particular, there exist $C^\infty$-smooth matrix valued functions $f_1$ and $f_2$ such that*

$$\phi_z(x) = \frac{\mathrm{i}}{2\pi} \begin{pmatrix} 0 & \dfrac{1}{x_1 + \mathrm{i}x_2} \\ \dfrac{1}{x_1 - \mathrm{i}x_2} & 0 \end{pmatrix} + f_1(x) \log|x| + f_2(x). \qquad (3.10)$$

**Proof**. In order to prove the claimed results, let us recall the series representations of $K_j$ from, e.g., §10.25.2, 10.31.1 and 10.31.2 in [28], which read

$$I_\mu(t) = \frac{t^\mu}{2^\mu} \sum_{k=0}^{\infty} \frac{t^{2k}}{4^k k! \Gamma(\mu + k + 1)}, \quad \mu \in \{0, 1\},$$

$$K_1(t) = \frac{1}{t} + (\log t - \log 2) I_1(t) - \frac{t}{4} \sum_{k=0}^{\infty} \big(\psi(k+1) + \psi(k+2)\big) \frac{t^{2k}}{4^k k!(k+1)!},$$

$$K_0(t) = -(\log t - \log 2 + \gamma) I_0(t) + \sum_{k=1}^{\infty} \sum_{j=1}^{k} \frac{1}{j} \frac{t^{2k}}{4^k (k!)^2},$$

with $\psi(t) = \dfrac{\Gamma'(t)}{\Gamma(t)}$ and $\gamma = -\psi(1) < \log 2$. This implies first that

$$I_0(t) = 1 + t^2 h_0(t^2) \quad \text{and} \quad I_1(t) = t h_1(t^2)$$

with some analytic functions $h_0$ and $h_1$. Furthermore, with some analytic functions $k_0$ and $k_1$ we have

$$\begin{aligned} K_1(t) &= \frac{1}{t} + (\log t - \log 2) I_1(t) + t k_1(t^2) \\ &= \frac{1}{t} + t h_1(t^2) \log t + t\big(k_1(t^2) - h_1(t^2) \log 2\big) \end{aligned}$$

and

$$\begin{aligned} K_0(t) &= -(\log t - \log 2 + \gamma) I_0(t) + t^2 k_0(t^2) \\ &= -\log t - c_1 - t^2 h_0(t^2) \log t - c_1 t^2 h_0(t^2) + t^2 k_0(t^2) \end{aligned}$$

with $c_1 := \gamma - \log 2 < 0$. This can be rewritten in a simplified form as

$$\begin{aligned} K_1(t) &= \frac{1}{t} + t g_1(t^2) \log t + t g_2(t^2), \\ K_0(t) &= -\log t - c_1 + t^2 g_3(t^2) \log t + t^2 g_4(t^2), \end{aligned}$$

where $g_1, g_2, g_3$, and $g_4$ are analytic functions and $c_1 < 0$. Using now the explicit expression for $\phi_z$ we decompose

$$\begin{aligned} \phi_z(x) &= \mathrm{i} \frac{\sqrt{m^2 - z^2}}{2\pi} K_1\big(\sqrt{m^2 - z^2}|x|\big) \left(\sigma \cdot \frac{x}{|x|}\right) + \frac{1}{2\pi} K_0\big(\sqrt{m^2 - z^2}|x|\big)\big(m\sigma_3 + z\sigma_0\big) \\ &= \mathrm{i} \frac{\sqrt{m^2 - z^2}}{2\pi} \Big\{ \frac{1}{\sqrt{m^2 - z^2}|x|} \\ &\qquad + \sqrt{m^2 - z^2}|x| g_1\big((m^2 - z^2)|x|^2\big) \log\big(\sqrt{m^2 - z^2}|x|\big) \\ &\qquad\qquad + \sqrt{m^2 - z^2}|x| g_2\big((m^2 - z^2)|x|^2\big) \Big\} \left(\sigma \cdot \frac{x}{|x|}\right) \\ &\quad + \frac{1}{2\pi} \Big\{ -\log\big(\sqrt{m^2 - z^2}|x|\big) - c_1 \\ &\qquad + (m^2 - z^2)|x|^2 g_3\big((m^2 - z^2)|x|^2\big) \log\big(\sqrt{m^2 - z^2}|x|\big) \\ &\qquad\qquad + (m^2 - z^2)|x|^2 g_4\big((m^2 - z^2)|x|^2\big) \Big\} \big(m\sigma_3 + z\sigma_0\big), \end{aligned}$$

which leads to the decomposition (3.9). The representation (3.10) follows from (3.9) after noting that

$$\frac{\mathrm{i}}{2\pi}\,\sigma\cdot\frac{x}{|x|^2}=\frac{\mathrm{i}}{2\pi}\begin{pmatrix}0 & \dfrac{1}{x_1+\mathrm{i}x_2}\\ \dfrac{1}{x_1-\mathrm{i}x_2} & 0\end{pmatrix}.$$

□

For $z\in\mathrm{res}(A_0)$ we introduce the operator

$$\mathcal{C}_z\varphi(x):=\text{p.v.}\int_\Sigma \phi_z(x-y)\varphi(y)\mathrm{d}s(y),\quad \varphi\in C^\infty(\Sigma;\mathbb{C}^2),\ x\in\Sigma. \tag{3.11}$$

The basic properties of $\mathcal{C}_z$ are stated in the following proposition. For the formulation of the result, recall the definition of the operator $\Lambda$ from (2.8) and of the Cauchy transform $C_\Sigma$ and its dual $C'_\Sigma$ from (2.19) and (2.21), respectively.

**Proposition 3.5.** *Let $z\in\mathrm{res}(A_0)$ and consider the operator $\mathcal{C}_z$ in (3.11). Then $\mathcal{C}_z\in\Psi^0_\Sigma$ and, in particular, $\mathcal{C}_z$ gives rise to a bounded operator in $H^s(\Sigma;\mathbb{C}^2)$ for any $s\in\mathbb{R}$. The realization in $L^2(\Sigma;\mathbb{C}^2)$ satisfies $\mathcal{C}_z^*=\mathcal{C}_{\bar z}$. Moreover, if $\mathbf{t}=(\mathbf{t_1},\mathbf{t_2})$ is the tangent vector field at $\Sigma$ and $T=\mathbf{t}_1+i\mathbf{t}_2$, $\overline{T}=\mathbf{t}_1-i\mathbf{t}_2$, then one has*

$$\Lambda\mathcal{C}_z\Lambda=\frac{1}{2}\begin{pmatrix}0 & \Lambda C_\Sigma\overline{T}\Lambda\\ \Lambda T C'_\Sigma\Lambda & 0\end{pmatrix}+\frac{\ell}{4\pi}\begin{pmatrix}(z+m)\mathbb{1} & 0\\ 0 & (z-m)\mathbb{1}\end{pmatrix}+\Psi \tag{3.12}$$

*with $\Psi\in\Psi^{-1}_\Sigma$.*

**Proof**. We make use of (3.9) to decompose $\phi_z$ in the form

$$\phi_z(x)=\chi_1(x)+\chi_2(x)+\chi_3(x),$$

where

$$\begin{aligned}
\chi_1(x)&=\frac{\mathrm{i}}{2\pi}\begin{pmatrix}0 & \dfrac{1}{x_1+\mathrm{i}x_2}\\ \dfrac{1}{x_1-\mathrm{i}x_2} & 0\end{pmatrix},\\
\chi_2(x)&=-\frac{1}{2\pi}\begin{pmatrix}z+m & 0\\ 0 & z-m\end{pmatrix}\log|x|\\
\chi_3(x)&=\big[h_1\big(|x|^2\big)\log|x|+h_2\big(|x|^2\big)\big]\,(\sigma\cdot x)\\
&\qquad+\big[|x|^2h_3\big(|x|^2\big)\log|x|+h_4\big(|x|^2\big)\big]\,(m\sigma_3+z\sigma_0),
\end{aligned}$$

and $h_1,h_2,h_3$, and $h_4$ are analytic functions. In the following we will use the corresponding decomposition $\mathcal{C}_z=P_1+P_2+P_3$, where

$$\begin{aligned}
(P_1\varphi)(x)&=\text{p.v.}\int_\Sigma\chi_1(x-y)\varphi(y)\,\mathrm{d}s(y),\\
(P_2\varphi)(x)&=\int_\Sigma\chi_2(x-y)\varphi(y)\,\mathrm{d}s(y),\\
(P_3\varphi)(x)&=\int_\Sigma\chi_3(x-y)\varphi(y)\,\mathrm{d}s(y).
\end{aligned}$$

Here we have removed the principal value from the integral operators $P_2$ and $P_3$, since these integrals converge almost everywhere by [22, Proposition 3.10].

Let us discuss the operator $P_1$ first. With the help of (2.20) and (2.22) we obtain

$$P_1 = \frac{1}{2}\begin{pmatrix} 0 & C_\Sigma \overline{T} \\ T C_\Sigma' & 0 \end{pmatrix} \tag{3.13}$$

and since $T, \overline{T} \in \Psi^0_\Sigma$ we conclude $P_1 \in \Psi^0_\Sigma$ from Proposition 2.9.

Next, we claim that the integral operator $P_2$ admits the representation

$$P_2 = \frac{\ell}{4\pi}\begin{pmatrix} (z+m)\Lambda^{-2} & 0 \\ 0 & (z-m)\Lambda^{-2} \end{pmatrix} + \Psi_1 \tag{3.14}$$

with some $\Psi_1 \in \Psi^{-2}_\Sigma$ and $\Lambda^{-2} = U^{-1}L^{-2}U \in \Psi^{-1}_\Sigma$, so that $P_2 \in \Psi^{-1}_\Sigma$. In fact, using a parametrization $\gamma : [0,\ell] \to \mathbb{R}^2$ of $\Sigma$ we find

$$(UP_2 f)(t) = -\frac{\ell}{2\pi}\begin{pmatrix} z+m & 0 \\ 0 & z-m \end{pmatrix} \int_{\mathbb{T}} \log\big|\gamma(\ell t) - \gamma(\ell s)\big|\, f\big(\gamma(\ell s)\big)\, \mathrm{d}s$$

for $f \in C^\infty(\Sigma)$. Therefore, with $f = U^{-1}u$ and $\rho(\cdot) = \gamma_1(\ell\cdot) + i\gamma_2(\ell\cdot) \equiv \gamma(\ell\cdot)$ we conclude

$$\begin{aligned}(UP_2U^{-1}u)(t) &= -\frac{\ell}{2\pi}\begin{pmatrix} z+m & 0 \\ 0 & z-m \end{pmatrix} \int_{\mathbb{T}} \log\big|\rho(t) - \rho(s)\big|\, u(s)\, \mathrm{d}s \\ &= -\frac{\ell}{2\pi}\begin{pmatrix} z+m & 0 \\ 0 & z-m \end{pmatrix} H_0 u(t)\end{aligned}$$

with $H_0$ as in Proposition 2.7. Now it follows from Proposition 2.7 (with $m = 0$, $a \equiv 1$, and $\rho$ as above) that $H_0 \in \Psi^{-1}$ and $\mathbb{1} + 2LH_0L \in \Psi^{-1}$. Furthermore, Proposition 2.2 (ii) and $L^{-1} \in \Psi^{-1/2}$ yield $\frac{1}{2}L^{-2} + H_0 \in \Psi^{-2}$ and hence

$$-\frac{\ell}{4\pi}\begin{pmatrix} (z+m)L^{-2} & 0 \\ 0 & (z-m)L^{-2} \end{pmatrix} + UP_2U^{-1} \in \Psi^{-2}.$$

We then conclude

$$-\frac{\ell}{4\pi}\begin{pmatrix} (z+m)\Lambda^{-2} & 0 \\ 0 & (z-m)\Lambda^{-2} \end{pmatrix} + P_2 \in \Psi^{-2}_\Sigma,$$

which leads to (3.14).

It will be shown now that $P_3 \in \Psi^{-2}_\Sigma$. Indeed, setting again $\rho(\cdot) = \gamma_1(\ell\cdot) + i\gamma_2(\ell\cdot) \equiv \gamma(\ell\cdot)$ we see that $\chi_3$ can be written in the form

$$\chi_3(\rho(t) - \rho(s)) = \log|\rho(t) - \rho(s)|\, a_1(t,s)\begin{pmatrix} 0 & \overline{\rho(t) - \rho(s)} \\ \rho(t) - \rho(s) & 0 \end{pmatrix} + a_2(t,s)$$

with the $C^\infty$-smooth matrix valued functions

$$\begin{aligned}a_1(t,s) &:= h_1\big(|\rho(t) - \rho(s)|^2\big)\sigma_0 \\ &\quad + h_3\big(|\rho(t) - \rho(s)|^2\big)(m\sigma_3 + z\sigma_0)\begin{pmatrix} 0 & \overline{\rho(t) - \rho(s)} \\ \rho(t) - \rho(s) & 0 \end{pmatrix}\end{aligned}$$

and

$$a_2(t,s) := h_2\big(|\rho(t)-\rho(s)|^2\big)\big] \begin{pmatrix} 0 & \overline{\rho(t)-\rho(s)} \\ \rho(t)-\rho(s) & 0 \end{pmatrix} + h_4\big(|\rho(t)-\rho(s)|^2\big)\,(m\sigma_3 + z\sigma_0).$$

Hence, it follows as above in the proof of (3.14) with Proposition 2.7 applied in the case $m = 1$ that $UP_3U^{-1} = H_1 \in \Psi^{-2}$, so that $P_3 \in \Psi_\Sigma^{-2}$. Together with (3.13) and (3.14) this implies first $\mathcal{C}_z \in \Psi_\Sigma^0$ and in a second step, together with Proposition 2.2 (i) and $\Lambda \in \Psi_\Sigma^{1/2}$, that also (3.12) is true.

Finally, since $\phi_z(y-x)^* = \phi_{\bar z}(x-y)$, we find that the realization of $\mathcal{C}_z$ in $L^2(\Sigma;\mathbb{C}^2)$ satisfies $\mathcal{C}_z^* = \mathcal{C}_{\bar z}$. Hence, all claims have been shown.

□

Finally, we prove a result on how $\Phi_z$ and $\mathcal{C}_z$ are related to each other by taking traces. Recall that $\mathcal{T}_\pm^D$ is the Dirichlet trace operator on $H(\sigma,\Omega_\pm)$, see Lemma 3.1.

**Proposition 3.6.** *For $\varphi \in H^{-1/2}(\Sigma;\mathbb{C}^2)$ one has*

$$\mathcal{T}_\pm^D \Phi_z \varphi = \mp \frac{\mathrm{i}}{2}\,(\sigma\cdot\nu)\,\varphi + \mathcal{C}_z\varphi. \tag{3.15}$$

**Proof**. First we note that it suffices to prove (3.15) for $\varphi \in C^\infty(\Sigma;\mathbb{C}^2)$; by continuity this implies the claim for any $\varphi \in H^{-1/2}(\Sigma;\mathbb{C}^2)$. The assertion essentially follows from the classical Plemelj-Sokhotskii formula, see, e.g., [35, Theorem 4.1.1], which states that the holomorphic function

$$\mathbb{C}\setminus\Sigma \ni \xi \mapsto \Phi(\xi) = \frac{1}{2\pi\mathrm{i}} \int_\Sigma \frac{\varphi(\zeta)}{\zeta-\xi}\,\mathrm{d}\zeta$$

satisfies

$$\mathcal{T}_\pm^D \Phi(\xi) = \frac{1}{2\pi i}\,\mathrm{p.v.}\int_\Sigma \frac{\varphi(\zeta)}{\zeta-\xi}\,\mathrm{d}\zeta \pm \frac{1}{2}\,\varphi(\xi), \quad \xi\in\Sigma. \tag{3.16}$$

In order to use it, recall that by (3.10) we can write $\phi_z(x) = \chi_1(x) + \widetilde{\chi}_2(x)$ with

$$\chi_1(x) = -\frac{1}{2\pi\mathrm{i}} \begin{pmatrix} 0 & \dfrac{1}{x_1+\mathrm{i}x_2} \\ \dfrac{1}{x_1-\mathrm{i}x_2} & 0 \end{pmatrix} \quad \text{and} \quad \widetilde{\chi}_2(x) = f_1(x)\log|x| + f_2(x),$$

where $f_1$ and $f_2$ are $C^\infty$-smooth matrix functions. In a corresponding way we decompose $\Phi_z = \Psi_1 + \Psi_2$ with

$$\Psi_1\varphi(x) = \int_\Sigma \chi_1(x-y)\,\varphi(y)\,\mathrm{d}s(y) \quad \text{and} \quad \Psi_2\varphi(x) = \int_\Sigma \widetilde{\chi}_2(x-y)\,\varphi(y)\,\mathrm{d}s(y),$$

and $\mathcal{C}_z = P_1 + P_2$ with

$$P_1\varphi(x) = \mathrm{p.v.}\int_\Sigma \chi_1(x-y)\,\varphi(y)\,\mathrm{d}s(y) \quad \text{and} \quad P_2\varphi(x) = \int_\Sigma \widetilde{\chi}_2(x-y)\,\varphi(y)\,\mathrm{d}s(y).$$

As in the proof of Proposition 3.5 we have removed the principal value from the integral operator $P_2$, since the integral exists almost everywhere. One sees easily that $\Psi_2\varphi$ is continuous on $\mathbb{R}^2$, and its value on $\Sigma$ coincides with $P_2\varphi$, i.e.

$$\mathcal{T}_\pm^D \Psi_2 \varphi = P_2 \varphi. \tag{3.17}$$

In order to find the relation between $\Psi_1\varphi$ and $P_1\varphi$, we write the normal vector field as a complex number $N = \nu_1 + \mathrm{i}\nu_2 = \gamma_2' - \mathrm{i}\gamma_1'$ and use the relation $\mathrm{d}(y_1 + \mathrm{i}y_2) = \mathrm{i}N(y)\,\mathrm{d}s(y)$ of the complex and the classical line element on $\Sigma$. With $\varphi = (\varphi_1, \varphi_2)$ we get then

$$\begin{aligned}
\Psi_1\varphi(x) &= \frac{1}{2\pi\mathrm{i}} \int_\Sigma \begin{pmatrix} 0 & \dfrac{1}{(y_1 + \mathrm{i}y_2) - (x_1 + \mathrm{i}x_2)} \\ \dfrac{1}{(y_1 - \mathrm{i}y_2) - (x_1 - \mathrm{i}x_2)} & 0 \end{pmatrix} \begin{pmatrix} \varphi_1(y) \\ \varphi_2(y) \end{pmatrix} \mathrm{d}s(y) \\
&= \begin{pmatrix} \dfrac{1}{2\pi\mathrm{i}} \displaystyle\int_\Sigma \frac{\varphi_2(y)}{(y_1 + \mathrm{i}y_2) - (x_1 + \mathrm{i}x_2)}\,\mathrm{d}s(y) \\ \overline{\dfrac{1}{2\pi\mathrm{i}} \displaystyle\int_\Sigma \frac{\overline{\varphi_1(y)}}{(y_1 + \mathrm{i}y_2) - (x_1 + \mathrm{i}x_2)}\,\mathrm{d}s(y)} \end{pmatrix} \\
&= \begin{pmatrix} \dfrac{1}{2\pi\mathrm{i}} \displaystyle\int_\Sigma \frac{-\mathrm{i}\overline{N(y)}\varphi_2(y)}{(y_1 + \mathrm{i}y_2) - (x_1 + \mathrm{i}x_2)}\,\mathrm{d}(y_1 + \mathrm{i}y_2) \\ -\overline{\dfrac{1}{2\pi\mathrm{i}} \displaystyle\int_\Sigma \frac{-\mathrm{i}\overline{N(y)\varphi_1(y)}}{(y_1 + \mathrm{i}y_2) - (x_1 + \mathrm{i}x_2)}\,\mathrm{d}(y_1 + \mathrm{i}y_2)} \end{pmatrix}.
\end{aligned}$$

Applying now (3.16) to each component of this vector we find that

$$\begin{aligned}
\mathcal{T}_\pm^D \Psi_1\varphi(x) &= \begin{pmatrix} \dfrac{1}{2\pi\mathrm{i}}\,\mathrm{p.v.} \displaystyle\int_\Sigma \frac{-\mathrm{i}\overline{N(y)}\varphi_2(y)}{(y_1 + \mathrm{i}y_2) - (x_1 + \mathrm{i}x_2)}\,\mathrm{d}(y_1 + \mathrm{i}y_2) \\ -\overline{\dfrac{1}{2\pi\mathrm{i}}\,\mathrm{p.v.} \displaystyle\int_\Sigma \frac{-\mathrm{i}\overline{N(y)\varphi_1(y)}}{(y_1 + \mathrm{i}y_2) - (x_1 + \mathrm{i}x_2)}\,d(y_1 + \mathrm{i}y_2)} \end{pmatrix} \pm \frac{1}{2} \begin{pmatrix} -\mathrm{i}\overline{N(x)}\varphi_2(x) \\ -\mathrm{i}N(x)\varphi_1(x) \end{pmatrix} \\
&= \begin{pmatrix} -\dfrac{1}{2\pi\mathrm{i}}\,\mathrm{p.v.} \displaystyle\int_\Sigma \frac{\varphi_2(y)}{(x_1 + \mathrm{i}x_2) - (y_1 + \mathrm{i}y_2)}\,\mathrm{d}s(y) \\ -\dfrac{1}{2\pi\mathrm{i}}\,\mathrm{p.v.} \displaystyle\int_\Sigma \frac{\varphi_1(y)}{(x_1 - \mathrm{i}x_2) - (y_1 - \mathrm{i}y_2)}\,\mathrm{d}s(y) \end{pmatrix} \mp \frac{\mathrm{i}}{2} \begin{pmatrix} \overline{N(x)}\varphi_2(x) \\ N(x)\varphi_1(x) \end{pmatrix} \\
&= P_1\varphi(x) \mp \frac{\mathrm{i}}{2}\,(\sigma \cdot \nu(x))\,\varphi(x).
\end{aligned}$$

A combination of this and (3.17) leads to the claim of this proposition. □

## 3.2 A boundary triple for Dirac operators with singular interactions supported on loops

In this section we follow the strategy from Section 2.3 to introduce a boundary triple which is suitable to study Dirac operators in $L^2(\mathbb{R}^2; \mathbb{C}^2)$ with singular interactions

supported on the loop $\Sigma$. To get an explicit representation of the boundary mappings the results from Section 3.1 play an important role. We remark that the obtained boundary triple is closely related to the one used in [9] to study Dirac operators in the three dimensional case.

Recall the definitions of the free Dirac operator $A_0$, the symmetric operator $S$, and its adjoint $S^*$ from (3.1), (3.3), and (3.4), respectively. Moreover, $\mathfrak{T}_\pm^D$ is the Dirichlet trace operator defined on $\operatorname{dom} S^*$ from Lemma 3.1, the integral operators $\Phi_z$ and $\mathcal{C}_z$ are introduced for $z \in \operatorname{res}(A_0)$ in (3.8) and (3.11), respectively. The operator $\Lambda \in \Psi_\Sigma^{1/2}$ is given by (2.8) and will sometimes be viewed as an isomorphism from $L^2(\Sigma;\mathbb{C}^2) \to H^{-1/2}(\Sigma;\mathbb{C}^2)$ or from $H^{1/2}(\Sigma;\mathbb{C}^2) \to L^2(\Sigma;\mathbb{C}^2)$, and is also regarded as an unbounded uniformly positive self-adjoint operator $L^2(\Sigma;\mathbb{C}^2)$.

**Proposition 3.7.** *Let $\zeta \in \operatorname{res}(A_0)$ be fixed and define $\Gamma_0, \Gamma_1 : \operatorname{dom} S^* \to L^2(\Sigma;\mathbb{C}^2)$ by*

$$\begin{aligned} \Gamma_0 f &= \mathrm{i}\Lambda^{-1}(\sigma\cdot\nu)\big(\mathfrak{T}_+^D f_+ - \mathfrak{T}_-^D f_-\big), \\ \Gamma_1 f &= \frac{1}{2}\,\Lambda\Big((\mathfrak{T}_+^D f_+ + \mathfrak{T}_-^D f_-) - (\mathcal{C}_\zeta + \mathcal{C}_{\bar\zeta})\Lambda\Gamma_0 f\Big), \qquad f = f_+ \oplus f_- \in \operatorname{dom} S^*. \end{aligned} \tag{3.18}$$

*Then $\{L^2(\Sigma;\mathbb{C}^2), \Gamma_0, \Gamma_1\}$ is a boundary triple for $S^*$ such that $A_0 = S^* \restriction \ker\Gamma_0$. Moreover, the corresponding $\gamma$-field is*

$$\operatorname{res}(A_0) \ni z \mapsto G_z = \Phi_z\Lambda$$

*and the Weyl function is*

$$\operatorname{res}(A_0) \ni z \mapsto M_z = \Lambda\Big(\mathcal{C}_z - \frac{1}{2}\big(\mathcal{C}_\zeta + \mathcal{C}_{\bar\zeta}\big)\Big)\Lambda.$$

**Proof**. Recall that the Dirichlet trace operator $\mathfrak{T}^D : H^1(\mathbb{R}^2;\mathbb{C}^2) \to H^{1/2}(\Sigma;\mathbb{C}^2)$ is bounded, surjective, and one has $\ker\mathfrak{T}^D = H_0^1(\mathbb{R}^2\setminus\Sigma;\mathbb{C}^2)$. Hence,

$$\mathfrak{T} := \Lambda\mathfrak{T}^D : H^1(\mathbb{R}^2;\mathbb{C}^2) = \operatorname{dom} A_0 \to L^2(\Sigma;\mathbb{C}^2)$$

is bounded and surjective with $\ker\mathfrak{T} = \operatorname{dom} S$. Following the constructions in Section 2.3 for $B = A_0$ we consider for $z \in \operatorname{res}(A_0)$

$$\mathfrak{T}(A_0 - \bar z)^{-1} = \Lambda\mathfrak{T}^D(A_0 - \bar z)^{-1} = \Lambda\Phi_z'$$

with $\Phi_z'$ given by (3.7), so that the operator $G_z$ from (2.27) in the present context is given by

$$G_z = \Phi_z\Lambda. \tag{3.19}$$

Let $\zeta \in \operatorname{res}(A_0)$ be fixed. Then, by (2.28) any $f \in \operatorname{dom} S^*$ can be written as

$$f = f_\zeta + G_\zeta\xi = f_{\bar\zeta} + G_{\bar\zeta}\xi$$

for some $\xi \in L^2(\Sigma;\mathbb{C}^2)$ and $f_\zeta, f_{\bar\zeta} \in H^1(\mathbb{R}^2;\mathbb{C}^2)$, and according to Proposition 2.13

$$\Gamma_0 f = \xi \quad\text{and}\quad \Gamma_1 f = \frac{1}{2}\big(\mathfrak{T} f_\zeta + \mathfrak{T} f_{\bar\zeta}\big)$$

defines a boundary triple for $S^*$ such that $A_0 = S^* \restriction \ker \Gamma_0$.

Next we show that the above boundary maps coincide with the more explicit representations of $\Gamma_0$ and $\Gamma_1$ stated in the proposition. Let $f = f_\zeta + G_\zeta \xi = f_\zeta + \Phi_\zeta \Lambda \xi$ with $\xi \in L^2(\Sigma; \mathbb{C}^2)$ and $f_\zeta \in H^1(\mathbb{R}^2; \mathbb{C}^2)$ be fixed. Using that the jump of the trace of $f_\zeta \in H^1(\mathbb{R}^2; \mathbb{C}^2)$ at $\Sigma$ is zero and the trace formula from Proposition 3.6 we find

$$\begin{aligned}
\mathcal{T}_+^D f_+ - \mathcal{T}_-^D f_- &= \mathcal{T}_+^D \big(f_\zeta + \Phi_\zeta \Lambda \xi\big)_+ - \mathcal{T}_-^D \big(f_\zeta + \Phi_\zeta \Lambda \xi\big)_- \\
&= \mathcal{T}_+^D \big(\Phi_\zeta \Lambda \xi\big)_+ - \mathcal{T}_-^D \big(\Phi_\zeta \Lambda \xi\big)_- \\
&= -\frac{\mathrm{i}}{2}(\sigma \cdot \nu)\Lambda \xi + \mathcal{C}_\zeta \Lambda \xi - \frac{\mathrm{i}}{2}(\sigma \cdot \nu)\Lambda \xi - \mathcal{C}_\zeta \Lambda \xi \\
&= -\mathrm{i}(\sigma \cdot \nu)\Lambda \xi
\end{aligned}$$

and hence we conclude

$$\Gamma_0 f = \xi = \mathrm{i}\Lambda^{-1}(\sigma \cdot \nu)\big(\mathcal{T}_+^D f_+ - \mathcal{T}_-^D f_-\big),$$

which is the claimed formula for $\Gamma_0 f$. Employing again Proposition 3.6 we find

$$\begin{aligned}
\mathcal{T}^D f_\zeta &= \frac{1}{2}\big(\mathcal{T}_+^D f_{\zeta,+} + \mathcal{T}_-^D f_{\zeta,-}\big) \\
&= \frac{1}{2}\big(\mathcal{T}_+^D (f - \Phi_\zeta \Lambda \xi)_+ + \mathcal{T}_-^D (f - \Phi_\zeta \Lambda \xi)_-\big) \\
&= \frac{1}{2}\left(\mathcal{T}_+^D f_+ - \mathcal{C}_\zeta \Lambda \xi + \frac{\mathrm{i}}{2}(\sigma \cdot \nu)\Lambda \xi + \mathcal{T}_-^D f_- - \mathcal{C}_\zeta \Lambda \xi - \frac{\mathrm{i}}{2}(\sigma \cdot \nu)\Lambda \xi\right) \qquad (3.20) \\
&= \frac{1}{2}\left(\mathcal{T}_+^D f_+ + \mathcal{T}_-^D f_-\right) - \mathcal{C}_\zeta \Lambda \xi \\
&= \frac{1}{2}\left(\mathcal{T}_+^D f_+ + \mathcal{T}_-^D f_-\right) - \mathcal{C}_\zeta \Lambda \Gamma_0 f
\end{aligned}$$

and analogously

$$\mathcal{T}^D f_{\bar\zeta} = \frac{1}{2}\left(\mathcal{T}_+^D f_+ + \mathcal{T}_-^D f_-\right) - \mathcal{C}_{\bar\zeta} \Lambda \Gamma_0 f. \qquad (3.21)$$

By summing up the last two formulae (3.20) and (3.21) we find

$$\Gamma_1 f = \frac{1}{2}\big(\mathcal{T} f_\zeta + \mathcal{T} f_{\bar\zeta}\big) = \frac{1}{2}\Lambda\big(\mathcal{T}^D f_\zeta + \mathcal{T}^D f_{\bar\zeta}\big) = \frac{1}{2}\Lambda\Big((\mathcal{T}_+^D f_+ + \mathcal{T}_-^D f_-) - (\mathcal{C}_\zeta + \mathcal{C}_{\bar\zeta})\Lambda \Gamma_0 f\Big)$$

which is the claimed formula for $\Gamma_1$ in (3.18).

Finally, the claimed representation of the $\gamma$-field follows from Proposition 2.13 and (3.19). Using again Proposition 3.6, we can simplify the formula for the Weyl function $M_z$ from Proposition 2.13 and get for $\varphi \in L^2(\Sigma; \mathbb{C}^2)$

$$\begin{aligned}
M_z \varphi &= \mathcal{T}\left(G_z - \frac{1}{2}(G_\zeta + G_{\bar\zeta})\right)\varphi \\
&= \Lambda \mathcal{T}_+^D \left(\Phi_z - \frac{1}{2}(\Phi_\zeta + \Phi_{\bar\zeta})\right)\Lambda \varphi \\
&= \Lambda\left(\mathcal{C}_z - \frac{\mathrm{i}}{2}(\sigma \cdot \nu) - \frac{1}{2}\left(\mathcal{C}_\zeta - \frac{\mathrm{i}}{2}(\sigma \cdot \nu) + \mathcal{C}_{\bar\zeta} - \frac{\mathrm{i}}{2}(\sigma \cdot \nu)\right)\right)\Lambda \varphi \\
&= \Lambda\left(\mathcal{C}_z - \frac{1}{2}\left(\mathcal{C}_\zeta + \mathcal{C}_{\bar\zeta}\right)\right)\Lambda \varphi.
\end{aligned}$$

In the above calculation we used the regularization property $(G_z - \frac{1}{2}(G_\zeta + G_{\bar\zeta}))\varphi \in \operatorname{dom} A_0 = H^1(\mathbb{R}^2;\mathbb{C}^2)$, which holds automatically by the abstract theory (see the formula for the Weyl function in Proposition 2.13), and hence $\mathfrak{T}^D$ and $\mathfrak{T}^D_+$ lead to the same trace in the second equality above. Therefore, all claimed statements have been shown. □

Finally, we state an auxiliary regularity result that will be used later.

**Lemma 3.8.** *Let $f \in \operatorname{dom} S^*$. Then $f \in H^1(\mathbb{R}^2 \setminus \Sigma;\mathbb{C}^2)$ if and only if $\Gamma_0 f \in H^1(\Sigma;\mathbb{C}^2)$.*

**Proof**. First, if $f = f_+ \oplus f_- \in H^1(\mathbb{R}^2 \setminus \Sigma;\mathbb{C}^2)$, then one has $\mathfrak{T}^D_\pm f_\pm \in H^{1/2}(\Sigma;\mathbb{C}^2)$ implying $\mathfrak{T}^D_+ f_+ - \mathfrak{T}^D_- f_- \in H^{1/2}(\Sigma;\mathbb{C}^2)$. As $\sigma\cdot\nu$ is a $C^\infty$-matrix function it follows that $\mathrm{i}(\sigma\cdot\nu)\big(\mathfrak{T}^D_+ f_+ - \mathfrak{T}^D_- f_-\big) \in H^{1/2}(\Sigma;\mathbb{C}^2)$. Using that $\Lambda$ is a bijection from $H^s(\Sigma)$ to $H^{s-1/2}(\Sigma)$ for all $s\in\mathbb{R}$, this yields

$$\Gamma_0 f = \mathrm{i}\Lambda^{-1}(\sigma\cdot\nu)\big(\mathfrak{T}^D_+ f_+ - \mathfrak{T}^D_- f_-\big) \in H^1(\Sigma;\mathbb{C}^2).$$

Conversely, let $f = f_+ \oplus f_- \in \operatorname{dom} S^*$ with $\Gamma_0 f \in H^1(\Sigma;\mathbb{C}^2)$. Since $\Lambda : H^1(\Sigma) \to H^{1/2}(\Sigma)$ is bijective and the $C^\infty$-matrix function $\sigma\cdot\nu$ is invertible we conclude from the definition of $\Gamma_0$ that

$$\mathfrak{T}^D_+ f_+ - \mathfrak{T}^D_- f_- \in H^{1/2}(\Sigma;\mathbb{C}^2). \tag{3.22}$$

By Proposition 3.5 the operators $\mathcal{C}_\zeta$ and $\mathcal{C}_{\bar\zeta}$ are bounded in $H^{1/2}(\Sigma;\mathbb{C}^2)$, which gives $(\mathcal{C}_\zeta + \mathcal{C}_{\bar\zeta})\Lambda\Gamma_0 f \in H^{1/2}(\Sigma;\mathbb{C}^2)$. In addition, $\Gamma_1 f \in L^2(\Sigma;\mathbb{C}^2)$ implies $\Lambda^{-1}\Gamma_1 \in H^{1/2}(\Sigma;\mathbb{C}^2)$. With the definition of $\Gamma_1$ this yields

$$\frac{1}{2}\big(\mathfrak{T}^D_+ f_+ + \mathfrak{T}^D_- f_-\big) = \Lambda^{-1}\Gamma_1 f + \frac{1}{2}(\mathcal{C}_\zeta + \mathcal{C}_{\bar\zeta})\Lambda\Gamma_0 f \in H^{1/2}(\Sigma;\mathbb{C}^2).$$

Hence, together with (3.22) this implies $\mathfrak{T}^D_\pm f_\pm \in H^{1/2}(\Sigma;\mathbb{C}^2)$. Finally, Lemma 3.1 shows $f_\pm \in H^1(\Omega_\pm;\mathbb{C}^2)$. □

# 4 Dirac operators with singular interactions

In this section we study the Dirac operator $A_{\eta,\tau}$ introduced in (1.2) and we prove the main results of this paper. First, in Section 4.1 we show how $A_{\eta,\tau}$ is related to the boundary triple $\{L^2(\Sigma;\mathbb{C}^2),\Gamma_0,\Gamma_1\}$ from Proposition 3.7. Then, in Section 4.2, we show the self-adjointness of $A_{\eta,\tau}$ for non-critical interaction strengths, i.e. when $\eta^2 - \tau^2 \neq 4$, and investigate the spectral properties of $A_{\eta,\tau}$ in this setting. In Section 4.3 we the study the self-adjointness and the spectral properties of $A_{\eta,\tau}$ in the case of critical interaction strengths. Finally, in Section 4.4 we provide a sketch of the proof of Theorem 1.3.

## 4.1 Definition of $A_{\eta,\tau}$ via the boundary triple

Recall the definition of the space $H(\sigma,\Omega_\pm)$ from (3.5), the trace maps $\mathcal{T}_\pm^D$ on $H(\sigma,\Omega_\pm)$ in Lemma 3.1, and that the operator $A_{\eta,\tau}$ in (1.2) is defined by

$$\begin{aligned} A_{\eta,\tau} f &= (-i\sigma\cdot\nabla + m\sigma_3) f_+ \oplus (-i\sigma\cdot\nabla + m\sigma_3) f_-, \\ \operatorname{dom} A_{\eta,\tau} &= \Big\{ f = f_+ \oplus f_- \in H(\sigma,\Omega_+)\oplus H(\sigma,\Omega_-) : \\ &\qquad -\mathrm{i}(\sigma\cdot\nu)\big(\mathcal{T}_+^D f_+ - \mathcal{T}_-^D f_-\big) = \frac{1}{2}(\eta\sigma_0+\tau\sigma_3)\big(\mathcal{T}_+^D f_+ + \mathcal{T}_-^D f_-\big)\Big\}. \end{aligned} \tag{4.1}$$

Before analyzing the properties of $A_{\eta,\tau}$ we would like to mention that for special values of the interaction strengths $A_{\eta,\tau}$ decouples in Dirac operators in $L^2(\Omega_+;\mathbb{C}^2)$ and $L^2(\Omega_-;\mathbb{C}^2)$ subject to certain boundary conditions. Similar effects are known from dimension three, see [20, Section V], [4, Section 5], and [7, Lemma 3.1]. The result reads as follows:

**Lemma 4.1.** *Let $\eta,\tau\in\mathbb{R}$. Then the following holds:*

(i) *If $\eta^2-\tau^2\neq -4$, then there is an invertible matrix $M$ (explicitly given below in (4.4)) such that $f=f_+\oplus f_-\in\operatorname{dom} A_{\eta,\tau}$ if and only if*

$$\mathcal{T}_+^D f_+ = M\mathcal{T}_-^D f_-.$$

(ii) *If $\eta^2-\tau^2=-4$, then $A_{\eta,\tau}=A_+\oplus A_-$, where $A_\pm$ is a Dirac operator in $L^2(\Omega_\pm;\mathbb{C}^2)$ and $f_\pm\in\operatorname{dom} A_\pm$ if and only if*

$$\mathcal{T}_\pm^D f_\pm = \pm\frac{i}{2}(\sigma\cdot\nu)\,(\eta\sigma_0+\tau\sigma_3)\,\mathcal{T}_\pm^D f_\pm. \tag{4.2}$$

*Remark* 4.2. Assume that $\eta^2-\tau^2=-4$, which is equivalent to $\frac{\eta^2}{\tau^2}+\frac{4}{\tau^2}=1$. Thus, there exists $\vartheta\in[0,2\pi]\setminus\{\frac{\pi}{2},\frac{3\pi}{2}\}$ such that

$$\frac{\eta}{\tau} = -\sin\vartheta \quad\text{and}\quad \frac{2}{\tau}=\cos\vartheta.$$

Using (1.5) we see that (4.2) for $f_+$ is equivalent to

$$\begin{aligned} 0 &= \frac{2i}{\tau}\sigma_3(\sigma\cdot\nu)\left(\sigma_0 - \frac{i}{2}(\sigma\cdot\nu)\,(\eta\sigma_0+\tau\sigma_3)\right)\mathcal{T}_+^D f_+ \\ &= \big(\sigma_0 + i\sigma_3(\sigma\cdot\nu)\cos\vartheta - \sin\vartheta\sigma_3\big)\mathcal{T}_+^D f_+, \end{aligned}$$

i.e. the operators $A_+$ in the bounded domain $\Omega_+$ are exactly those investigated in [11]. The case $\vartheta=0$ corresponds to the well-known infinite mass boundary condition (also called MIT bag) studied in [2, 27, 37]. We would like to point out that our results on $A_{\eta,\tau}$ obtained later in Section 4.2 can be used for a deeper understanding for $A_\pm$.

*Proof of Lemma 4.1.* The transmission condition in the definition of $A_{\eta,\tau}$ can be written in the form

$$\Big(\mathrm{i}(\sigma\cdot\nu)+\frac{1}{2}(\eta\sigma_0+\tau\sigma_3)\Big)\,\mathfrak{T}_+^D f_+ = \Big(\mathrm{i}(\sigma\cdot\nu)-\frac{1}{2}(\eta\sigma_0+\tau\sigma_3)\Big)\,\mathfrak{T}_-^D f_-.$$

Multiplying this equation with $-\mathrm{i}\,(\sigma\cdot\nu)$ we obtain the equivalent form

$$(\sigma_0 - R)\,\mathfrak{T}_+^D f_+ = (\sigma_0 + R)\,\mathfrak{T}_-^D f_- \tag{4.3}$$

with

$$R := \frac{\mathrm{i}}{2}\,(\sigma\cdot\nu)(\eta\sigma_0+\tau\sigma_3) = \frac{\mathrm{i}}{2}\,(\eta\sigma_0-\tau\sigma_3)(\sigma\cdot\nu),$$

where (1.5) was used. One computes

$$R^2 = \frac{\mathrm{i}}{2}\,(\eta\sigma_0-\tau\sigma_3)(\sigma\cdot\nu)\,\frac{\mathrm{i}}{2}\,(\sigma\cdot\nu)(\eta\sigma_0+\tau\sigma_3) = -\frac{\eta^2-\tau^2}{4}\,\sigma_0,$$

which implies

$$(\sigma_0 - R)(\sigma_0 + R) = \sigma_0 - R^2 = \sigma_0 + \frac{\eta^2-\tau^2}{4}\,\sigma_0.$$

Assume now $\eta^2-\tau^2 \neq -4$. Then both $\sigma_0 \pm R$ are invertible with inverses $(\sigma_0\pm R)^{-1} = \frac{1}{4+\eta^2-\tau^2}(\sigma_0\mp R)$. Therefore, the transmission condition can be equivalently rewritten as

$$\mathfrak{T}_+^D f_+ = (\sigma_0 - R)^{-1}(\sigma_0 + R)\,\mathfrak{T}_-^D f_- \quad \text{or} \quad \mathfrak{T}_-^D f_- = (\sigma_0 + R)^{-1}(\sigma_0 - R)\,\mathfrak{T}_+^D f_+, \tag{4.4}$$

which shows assertion (i). On the other hand, for $\eta^2-\tau^2 = -4$ one has $R^2 = \sigma_0$ and multiplying (4.3) by $\sigma_0 - R$ or $\sigma_0 + R$ leads to the two conditions

$$\mathfrak{T}_\pm^D f_\pm = \pm R\mathfrak{T}_\pm^D f_\pm.$$

It follows that the operator $A_{\eta,\tau}$ decouples in a orthogonal sum of operators $A_\pm$ acting in $\Omega_\pm$ and hence, also statement (ii) has been shown. □

We are going to represent $A_{\eta,\tau}$ using the boundary triple $\{L^2(\Sigma;\mathbb{C}^2),\Gamma_0,\Gamma_1\}$ constructed in Proposition 3.7. Note that the definition of $\Gamma_0$ and $\Gamma_1$ can be rewritten as

$$\mathrm{i}(\sigma\cdot\nu)\left(\mathfrak{T}_+^D f_+ - \mathfrak{T}_-^D f_-\right) = \Lambda\Gamma_0 f, \tag{4.5}$$

$$\frac{1}{2}\left(\mathfrak{T}_+^D f_+ + \mathfrak{T}_-^D f_-\right) = \Lambda^{-1}\Gamma_1 f + \frac{1}{2}\,(\mathcal{C}_\zeta + \mathcal{C}_{\bar\zeta})\Lambda\Gamma_0 f. \tag{4.6}$$

With this we can identify the parameter in $L^2(\Sigma;\mathbb{C}^2)$ that corresponds to the operator $A_{\eta,\tau}$.

**Proposition 4.3.** *Let $\eta,\tau\in\mathbb{R}$. Then the following holds:*

(i) *Assume* $|\eta| \neq |\tau|$ *and let* $\Theta$ *be the linear operator in* $L^2(\Sigma;\mathbb{C}^2)$ *obtained as the maximal realization of the periodic pseudodifferential operator* $\theta \in \Psi^1_\Sigma$ *given by*

$$\theta = -\Lambda\left[\frac{1}{\eta^2-\tau^2}(\eta\sigma_0 - \tau\sigma_3) + \frac{1}{2}\left(\mathcal{C}_\zeta + \mathcal{C}_{\bar{\zeta}}\right)\right]\Lambda, \tag{4.7}$$

*i.e.* $\operatorname{dom}\Theta = \big\{\varphi \in L^2(\Sigma;\mathbb{C}^2) : \theta\varphi \in L^2(\Sigma;\mathbb{C}^2)\big\}$ *and* $\Theta\varphi = \theta\varphi$. *Then*

$$\operatorname{dom} A_{\eta,\tau} = \big\{f \in \operatorname{dom} S^* : \Gamma_0 f \in \operatorname{dom}\Theta,\ \Gamma_1 f = \Theta\Gamma_0 f\big\}. \tag{4.8}$$

(ii) *Assume* $\eta = \tau \neq 0$, *let*

$$\Pi_+ : L^2(\Sigma;\mathbb{C}^2) \to L^2(\Sigma), \quad \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} \mapsto \varphi_1,$$

*and let* $\Theta_+$ *be the linear operator in* $L^2(\Sigma)$ *obtained as the maximal realization of the periodic pseudodifferential operator* $\theta_+ \in \Psi^1_\Sigma$ *given by*

$$\theta_+ = -\Lambda\Big(\frac{1}{2\eta} + \Pi_+\frac{1}{2}(\mathcal{C}_\zeta + \mathcal{C}_{\bar{\zeta}})\Pi_+^*\Big)\Lambda, \tag{4.9}$$

*i.e.* $\operatorname{dom}\Theta_+ = \big\{\varphi \in L^2(\Sigma) : \theta_+\varphi \in L^2(\Sigma)\big\}$ *and* $\Theta_+\varphi = \theta_+\varphi$. *Then*

$$\operatorname{dom} A_{\eta,\tau} = \Big\{f \in \operatorname{dom} S^* : \Pi_+\Gamma_1 f = \Theta_+\Pi_+\Gamma_0 f,\ (\sigma_0 - \Pi_+^*\Pi_+)\Gamma_0 f = 0\Big\}. \tag{4.10}$$

(iii) *Assume* $\eta = -\tau \neq 0$, *let*

$$\Pi_- : L^2(\Sigma;\mathbb{C}^2) \to L^2(\Sigma), \quad \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} \mapsto \varphi_2,$$

*and let* $\Theta_-$ *be the linear operator in* $L^2(\Sigma)$ *obtained as the maximal realization of the periodic pseudodifferential operator* $\theta_- \in \Psi^1_\Sigma$ *given by*

$$\theta_- = -\Lambda\Big(\frac{1}{2\eta} + \Pi_-\frac{1}{2}(\mathcal{C}_\zeta + \mathcal{C}_{\bar{\zeta}})\Pi_-^*\Big)\Lambda, \tag{4.11}$$

*i.e.* $\operatorname{dom}\Theta_- = \big\{\varphi \in L^2(\Sigma) : \theta_-\varphi \in L^2(\Sigma)\big\}$ *and* $\Theta_-\varphi = \theta_-\varphi$. *Then*

$$\operatorname{dom} A_{\eta,\tau} = \Big\{f \in \operatorname{dom} S^* : \Pi_-\Gamma_1 f = \Theta_-\Pi_-\Gamma_0 f,\ (\sigma_0 - \Pi_-^*\Pi_-)\Gamma_0 f = 0\Big\}. \tag{4.12}$$

Note that the case $\eta = \tau = 0$ is not discussed in the previous statement because $A_{\eta,\tau}$ simply becomes the free Dirac operator $A_0$ introduced in (3.1).

*Remark* 4.4. (i) The operators $\Theta$ and $\Theta_\pm$ in Proposition 4.3 are well-defined due to the fact that $\theta$ and $\theta_\pm$ are periodic pseudodifferential operators of order 1. For example $\theta\varphi$ makes sense as an element of $H^{-1}(\Sigma;\mathbb{C}^2)$ for any $\varphi \in L^2(\Sigma;\mathbb{C}^2)$, and $H^1(\Sigma;\mathbb{C}^2) \subset \operatorname{dom}\Theta$.

(ii) In assertions (ii) and (iii) of the above proposition we decomposed $\mathcal{G} = L^2(\Sigma;\mathbb{C}^2) = \mathcal{G}_{\Pi_+} \oplus \mathcal{G}_{\Pi_-}$, where

$$\mathcal{G}_{\Pi_+} := \left\{ \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \in L^2(\Sigma;\mathbb{C}^2) : \varphi_2 = 0 \right\} \simeq L^2(\Sigma)$$

and

$$\mathcal{G}_{\Pi_-} := \left\{ \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \in L^2(\Sigma;\mathbb{C}^2) : \varphi_1 = 0 \right\} \simeq L^2(\Sigma).$$

**Proof**. With the help of (4.5) and (4.6) the transmission condition in (4.1) can be rewritten as

$$-\Lambda\Gamma_0 f = (\eta\sigma_0 + \tau\sigma_3)\Big(\Lambda^{-1}\Gamma_1 f + \frac{1}{2}\,(\mathcal{C}_\zeta + \mathcal{C}_{\bar\zeta})\Lambda\Gamma_0 f\Big). \tag{4.13}$$

Now let us distinguish between several cases.

(i) For $|\eta| \neq |\tau|$ the matrix $\eta\sigma_0 + \tau\sigma_3$ is invertible with

$$(\eta\sigma_0 + \tau\sigma_3)^{-1} = \frac{1}{\eta^2 - \tau^2}(\eta\sigma_0 - \tau\sigma_3).$$

Hence, we can rewrite the equality (4.13) as

$$\Gamma_1 f = -\Lambda\left[\frac{1}{\eta^2 - \tau^2}(\eta\sigma_0 - \tau\sigma_3) + \frac{1}{2}\,(\mathcal{C}_\zeta + \mathcal{C}_{\bar\zeta})\right]\Lambda\Gamma_0 f = \Theta\Gamma_0 f,$$

which gives the claimed representation in (4.8)

The cases (ii) are and (iii) are almost identical, so we only give a proof for (ii). By (4.13) we have that $f \in \operatorname{dom} A_{\eta,\tau}$ if and only if

$$\begin{aligned} -\Lambda\Gamma_0 f &= (\eta\sigma_0 + \tau\sigma_3)\Big(\Lambda^{-1}\Gamma_1 f + \frac{1}{2}\,(\mathcal{C}_\zeta + \mathcal{C}_{\bar\zeta})\Lambda\Gamma_0 f\Big) \\ &= \begin{pmatrix} 2\eta & 0 \\ 0 & 0 \end{pmatrix}\Big(\Lambda^{-1}\Gamma_1 f + \frac{1}{2}\,(\mathcal{C}_\zeta + \mathcal{C}_{\bar\zeta})\Lambda\Gamma_0 f\Big) \\ &= 2\eta\Pi_+^*\Pi_+\Big(\Lambda^{-1}\Gamma_1 f + \frac{1}{2}\,(\mathcal{C}_\zeta + \mathcal{C}_{\bar\zeta})\Lambda\Gamma_0 f\Big). \end{aligned}$$

Writing this equation in components it follows that this boundary condition is equivalent to the conditions

$$(\sigma_0 - \Pi_+^*\Pi_+)\Gamma_0 f = 0$$

and

$$\begin{aligned} \Pi_+\Gamma_1 f &= -\Lambda\Big(\frac{1}{2\eta} + \Pi_+\,\frac{1}{2}(\mathcal{C}_\zeta + \mathcal{C}_{\bar\zeta})\Big)\Lambda\Gamma_0 f \\ &= -\Lambda\Big(\frac{1}{2\eta} + \Pi_+\,\frac{1}{2}(\mathcal{C}_\zeta + \mathcal{C}_{\bar\zeta})\Pi_+^*\Big)\Lambda\Pi_+\Gamma_0 f \\ &= \Theta_+\Pi_+\Gamma_0 f. \end{aligned}$$

Hence, we find that (4.10) is true. □

In view of the general theory of boundary triples, see Subsection 2.3, many properties of $A_{\eta,\tau}$ can be deduced from the respective properties of the operators $\Theta$ and $\Theta_\pm$ from Proposition 4.3. We prefer to consider separately the non-critical case $\eta^2 - \tau^2 \neq 4$ and the critical case $\eta^2 - \tau^2 = 4$, where the latter one is more involved.

## 4.2 Non-critical case

Throughout this subsection we assume that

$$\eta^2 - \tau^2 \neq 4.$$

In order to show the self-adjointness of $A_{\eta,\tau}$ we use Theorem 2.12. For that it is necessary to investigate the operators $\Theta$ and $\Theta_\pm$ in Proposition 4.3.

**Lemma 4.5.** *Let $\eta, \tau \in \mathbb{R}$ with $\eta^2 - \tau^2 \neq 4$. Then the following holds:*

(i) *If $\eta^2 - \tau^2 \neq 0$, then $\operatorname{dom}\Theta = H^1(\Sigma;\mathbb{C}^2)$ and $\Theta$ is self-adjoint in $L^2(\Sigma;\mathbb{C}^2)$.*

(ii) *If $\eta = \pm\tau$, then $\operatorname{dom}\Theta_\pm = H^1(\Sigma)$ and $\Theta_\pm$ is self-adjoint in $L^2(\Sigma)$.*

**Proof**. (i) Let us consider the restriction $\Theta_1 := \Theta \restriction H^1(\Sigma;\mathbb{C}^2)$. Since $\theta \in \Psi^1_\Sigma$, the operator $\Theta_1$ is well-defined as an operator in $L^2(\Sigma;\mathbb{C}^2)$. We show $\Theta = \Theta_1$ and that $\Theta_1$ is self-adjoint in $L^2(\Sigma;\mathbb{C}^2)$.

First, it follows from Proposition 3.5 that $(\mathcal{C}_\zeta + \mathcal{C}_{\bar\zeta})^* = \mathcal{C}_{\bar\zeta} + \mathcal{C}_\zeta$ and hence $\Theta_1$ is a symmetric operator in $L^2(\Sigma;\mathbb{C}^2)$. Moreover, since $\Theta_1$ is a symmetric extension of the symmetric operator $\Theta_\infty := \Theta \restriction C^\infty(\Sigma;\mathbb{C}^2)$ Lemma 2.4 implies $\Theta_1^* \subset \Theta_\infty^* = \Theta$. Hence, $\Theta = \Theta_1$ and $\Theta_1 = \Theta_1^*$ follows if we show $\Theta \subset \Theta_1$, for which it suffices to check the inclusion

$$\operatorname{dom}\Theta \subset \operatorname{dom}\Theta_1 = H^1(\Sigma;\mathbb{C}^2). \tag{4.14}$$

To see (4.14) fix some $\varphi \in \operatorname{dom}\Theta$. Then $\theta\varphi \in L^2(\Sigma;\mathbb{C}^2)$. Using Proposition 3.5 we find that

$$\theta\varphi = -\frac{1}{2}\Lambda P \Lambda\varphi + \widehat{\Psi}\varphi, \quad \text{where } P = \begin{pmatrix} \dfrac{2}{\eta+\tau} & C_\Sigma\overline{T} \\ TC'_\Sigma & \dfrac{2}{\eta-\tau} \end{pmatrix} \text{ and } \widehat{\Psi} \in \Psi^0_\Sigma.$$

Hence, $\Lambda P\Lambda\varphi \in L^2(\Sigma;\mathbb{C}^2)$ and as $\Lambda : H^{1/2}(\Sigma;\mathbb{C}^2) \to L^2(\Sigma;\mathbb{C}^2)$ is bijective, this amounts to $P\Lambda\varphi \in H^{1/2}(\Sigma;\mathbb{C}^2)$. Since $C_\Sigma, C'_\Sigma \in \Psi^0_\Sigma$ by Proposition 2.9, these operators give rise to bounded operators in $H^{1/2}(\Sigma;\mathbb{C}^2)$, which implies that

$$\begin{aligned} &\begin{pmatrix} \dfrac{2}{\eta-\tau} & -C_\Sigma\overline{T} \\ -TC'_\Sigma & \dfrac{2}{\eta+\tau} \end{pmatrix} \begin{pmatrix} \dfrac{2}{\eta+\tau} & C_\Sigma\overline{T} \\ TC'_\Sigma & \dfrac{2}{\eta-\tau} \end{pmatrix} \Lambda\varphi \\ &\qquad = \begin{pmatrix} \dfrac{4}{\eta^2-\tau^2} - C_\Sigma\overline{T}TC'_\Sigma & 0 \\ 0 & \dfrac{4}{\eta^2-\tau^2} - TC'_\Sigma C_\Sigma\overline{T} \end{pmatrix} \Lambda\varphi \in H^{1/2}(\Sigma;\mathbb{C}^2). \end{aligned}$$

Now we use that $\overline{T}T = T\overline{T}$ is the multiplication operator with the constant function $\mathbb{1}$ and that $C_\Sigma C'_\Sigma - \mathbb{1}, C'_\Sigma C_\Sigma - \mathbb{1} \in \Psi^{-\infty}_\Sigma$ by Proposition 2.9. We then obtain from the last line that

$$\frac{4-\eta^2+\tau^2}{\eta^2-\tau^2}\Lambda\varphi + \widetilde{\Psi}\varphi \in H^{1/2}(\Sigma;\mathbb{C}^2)$$

with some $\widetilde{\Psi} \in \Psi_\Sigma^{-\infty}$ and hence $\frac{4-\eta^2+\tau^2}{\eta^2-\tau^2}\Lambda\varphi \in H^{1/2}(\Sigma;\mathbb{C}^2)$. Since $\eta^2-\tau^2 \neq 4$ by assumption, this implies $\Lambda\varphi \in H^{1/2}(\Sigma;\mathbb{C}^2)$ and thus, $\varphi \in H^1(\Sigma;\mathbb{C}^2)$. We have shown (4.14). This completes the proof of (i)

(ii) We consider the case $\eta = \tau$, the other one being similar. Recall that $\Theta_+$ is the maximal operator in $L^2(\Sigma)$ associated to the periodic pseudodifferential operator

$$\theta_+ = -\frac{1}{2}\,\Lambda\Big(\frac{1}{\eta} + \Pi_+(\mathcal{C}_\zeta + \mathcal{C}_{\bar\zeta})\Pi_+^*\Big)\Lambda.$$

Using Proposition 3.5 we find for $\varphi \in \operatorname{dom}\Theta_+$ that

$$\Theta_+\varphi = -\frac{1}{2\eta}\Lambda^2\varphi - \frac{1}{2}\Pi_+\begin{pmatrix} 0 & \Lambda C_\Sigma\overline{T}\Lambda \\ \Lambda T C_\Sigma'\Lambda & 0\end{pmatrix}\Pi_+^*\varphi + \widehat{\Psi}\varphi = -\frac{1}{2\eta}\Lambda^2\varphi + \widehat{\Psi}\varphi$$

with some symmetric operator $\widehat{\Psi} \in \Psi_\Sigma^0$. This implies $\operatorname{dom}\Theta_+ = \operatorname{dom}\Lambda^2 = H^1(\Sigma;\mathbb{C})$ and since $\Lambda^2$ is self-adjoint we conclude that also $\Theta_+$ is self-adjoint in $L^2(\Sigma)$. □

After the preparatory considerations in Lemma 4.5 we are now ready to show the self-adjointness of $A_{\eta,\tau}$ for non-critical interaction strengths. To formulate the result we recall the definitions of the free Dirac operator $A_0$ from (3.1), of $\Phi_z$ and $\Phi_z'$ from (3.8) and (3.7), and of $\mathcal{C}_z$ in (3.11), respectively.

**Theorem 4.6.** *Assume that $\eta,\tau \in \mathbb{R}$ with $\eta^2-\tau^2 \neq 4$ and $(\eta,\tau) \neq (0,0)$. Then the operator $A_{\eta,\tau}$ is self-adjoint in $L^2(\mathbb{R}^2;\mathbb{C}^2)$ with $\operatorname{dom}A_{\eta,\tau} \subset H^1(\mathbb{R}^2\setminus\Sigma;\mathbb{C}^2)$. Moreover, for all $z \in \operatorname{res}(A_{\eta,\tau}) \cap \operatorname{res}(A_0)$ the operator $\sigma_0 + (\eta\sigma_0+\tau\sigma_3)\mathcal{C}_z$ is bounded and boundedly invertible in $H^{1/2}(\Sigma;\mathbb{C}^2)$ and*

$$(A_{\eta,\tau}-z)^{-1} = (A_0-z)^{-1} - \Phi_z\big(\sigma_0 + (\eta\sigma_0+\tau\sigma_3)\mathcal{C}_z\big)^{-1}(\eta\sigma_0+\tau\sigma_3)\Phi_{\bar z}' \tag{4.15}$$

*holds.*

**Proof**. First, according to Theorem 2.12 the self-adjointness of $\Theta$ and $\Theta_\pm$ in $L^2(\Sigma;\mathbb{C}^2)$ and $L^2(\Sigma)$, respectively, implies the self-adjointness of $A_{\eta,\tau}$ in $L^2(\mathbb{R}^2;\mathbb{C}^2)$. In addition, since $\operatorname{dom}\Theta = H^1(\Sigma;\mathbb{C}^2)$ and $\operatorname{dom}\Theta_\pm = H^1(\Sigma)$, Lemma 3.8 yields $\operatorname{dom}A_{\eta,\tau} \subset H^1(\mathbb{R}^2\setminus\Sigma;\mathbb{C}^2)$.

It remains to show the Krein type resolvent formula in (4.15). First, for $|\eta| \neq |\tau|$ we have by Theorem 2.12 that $\Theta - M_z$, $z \in \operatorname{res}(A_{\eta,\tau}) \cap \operatorname{res}(A_0)$, is boundedly invertible in $L^2(\Sigma;\mathbb{C}^2)$ and

$$(A_{\eta,\tau}-z)^{-1} = (A_0-z)^{-1} + G_z\big(\Theta - M_z\big)^{-1}G_{\bar z}^*.$$

Taking the special form of $\Theta$ and $M_z = \Lambda\big(\mathcal{C}_z - \frac{1}{2}\big(\mathcal{C}_\zeta + \mathcal{C}_{\bar\zeta}\big)\big)\Lambda$ into account and using $\frac{1}{\eta^2-\tau^2}(\eta\sigma_0-\tau\sigma_3) = (\eta\sigma_0+\tau\sigma_3)^{-1}$, we find

$$\begin{aligned}\Theta - M_z &= -\Lambda\bigg[\frac{1}{\eta^2-\tau^2}(\eta\sigma_0-\tau\sigma_3) + \frac{1}{2}\,(\mathcal{C}_\zeta+\mathcal{C}_{\bar\zeta})\bigg]\Lambda - \Lambda\Big(\mathcal{C}_z - \frac{1}{2}\big(\mathcal{C}_\zeta+\mathcal{C}_{\bar\zeta}\big)\Big)\Lambda \\ &= -\Lambda\bigg[\frac{1}{\eta^2-\tau^2}(\eta\sigma_0-\tau\sigma_3) + \mathcal{C}_z\bigg]\Lambda \\ &= -\Lambda(\eta\sigma_0+\tau\sigma_3)^{-1}\big(\sigma_0 + (\eta\sigma_0+\tau\sigma_3)\mathcal{C}_z\big)\Lambda.\end{aligned} \tag{4.16}$$

As $\Theta - M_z$ is a bijective operator in $L^2(\Sigma;\mathbb{C}^2)$ defined on $\operatorname{dom}\Theta = H^1(\Sigma;\mathbb{C}^2)$ this implies that $\sigma_0 + (\eta\sigma_0 + \tau\sigma_3)\mathcal{C}_z$ is bijective in $H^{1/2}(\Sigma;\mathbb{C}^2)$. In particular, the inverse $(\sigma_0 + (\eta\sigma_0 + \tau\sigma_3)\mathcal{C}_z)^{-1}$ is well-defined and bounded in $H^{1/2}(\Sigma;\mathbb{C}^2)$. Using $G_z = \Phi_z\Lambda$ and $G_{\bar z}^* = \Lambda\Phi_{\bar z}'$ we get

$$\begin{aligned} G_z\big(\Theta - M_z\big)^{-1}G_{\bar z}^* &= -\Phi_z\Lambda\Lambda^{-1}\big(\sigma_0 + (\eta\sigma_0 + \tau\sigma_3)\mathcal{C}_z\big)^{-1}(\eta\sigma_0 + \tau\sigma_3)\Lambda^{-1}\Lambda\Phi_{\bar z}' \\ &= -\Phi_z\big(\sigma_0 + (\eta\sigma_0 + \tau\sigma_3)\mathcal{C}_z\big)^{-1}(\eta\sigma_0 + \tau\sigma_3)\Phi_{\bar z}', \end{aligned} \tag{4.17}$$

which leads to (4.15).

The proof of (4.15) for $|\eta| = |\tau| \neq 0$ is similar as above. First, one notes in the same way as in (4.16) that

$$\Theta_\pm - \Pi_\pm M_z\Pi_\pm^* = -\Lambda\bigg(\frac{1}{2\eta} + \Pi_\pm\mathcal{C}_z\Pi_\pm^*\bigg)\Lambda = -\frac{1}{2\eta}\Pi_\pm\Lambda\big(\sigma_0 + 2\eta\Pi_\pm^*\Pi_\pm\mathcal{C}_z\big)\Lambda\Pi_\pm^*, \tag{4.18}$$

which implies with $2\eta\Pi_\pm^*\Pi_\pm = \eta\sigma_0 + \tau\sigma_3$

$$\begin{aligned} \Pi_\pm^*(\Theta_\pm - \Pi_\pm M_z\Pi_\pm^*)^{-1}\Pi_\pm &= \Lambda^{-1}\Pi_\pm^*\big(\Pi_\pm(\sigma_0 + 2\eta\Pi_\pm^*\Pi_+\mathcal{C}_z)\Pi_\pm^*\big)^{-1}2\eta\Pi_\pm\Lambda^{-1} \\ &= \Lambda^{-1}\big(\Pi_\pm^*\Pi_\pm(\sigma_0 + 2\eta\Pi_\pm^*\Pi_+\mathcal{C}_z)\big)^{-1}2\eta\Pi_\pm^*\Pi_\pm\Lambda^{-1} \\ &= \Lambda^{-1}\big(\sigma_0 + (\eta\sigma_0 + \tau\sigma_3)\mathcal{C}_z\big)^{-1}(\eta\sigma_0 + \tau\sigma_3)\Lambda^{-1}. \end{aligned}$$

With this observation and the same ideas as above one shows (4.15) also in the case $|\eta| = |\tau|$. This finishes the proof of this theorem. □

In the following proposition we discuss the basic spectral properties of $A_{\eta,\tau}$:

**Proposition 4.7.** *Let $\eta, \tau \in \mathbb{R}$ be such that $\eta^2 - \tau^2 \neq 4$. Then the following holds:*

(i) *For the essential spectrum of $A_{\eta,\tau}$ we have*

$$\operatorname{spec}_{\mathrm{ess}}(A_{\eta,\tau}) = \big(-\infty, -|m|\big] \cup \big[|m|, \infty\big).$$

*In particular, for $m = 0$ we have $\operatorname{spec}(A_{\eta,\tau}) = \operatorname{spec}_{\mathrm{ess}}(A_{\eta,\tau}) = \mathbb{R}$.*

(ii) *Assume $m \neq 0$. Then $z \in (-|m|, |m|)$ is a discrete eigenvalue of $A_{\eta,\tau}$ if and only if there exists $\varphi \in H^{1/2}(\Sigma;\mathbb{C}^2)$ such that $\big(\sigma_0 + (\eta\sigma_0 + \tau\sigma_3)\mathcal{C}_z\big)\varphi = 0$.*

(iii) *If $m \neq 0$, then $A_{\eta,\tau}$ has at most finitely many eigenvalues in $\big(-|m|, |m|\big)$.*

**Proof**. Item (i) is a direct consequence of (4.15). In fact, by (3.7) and Theorem 4.6 the operator

$$\big(\sigma_0 + (\eta\sigma_0 + \tau\sigma_3)\mathcal{C}_z\big)^{-1}(\eta\sigma_0 + \tau\sigma_3)\Phi_{\bar z}' : L^2(\mathbb{R}^2;\mathbb{C}^2) \to H^{1/2}(\Sigma;\mathbb{C}^2)$$

is bounded. Since the embedding $H^{1/2}(\Sigma;\mathbb{C}^2) \hookrightarrow H^{-1/2}(\Sigma;\mathbb{C}^2)$ is compact and $\Phi_z : H^{-1/2}(\Sigma;\mathbb{C}^2) \to L^2(\mathbb{R}^2;\mathbb{C}^2)$ is bounded by definition, we conclude from (4.15) that the resolvent difference

$$(A_{\eta,\tau} - z)^{-1} - (A_0 - z)^{-1} = -\Phi_z\big(\sigma_0 + (\eta\sigma_0 + \tau\sigma_3)\mathcal{C}_z\big)^{-1}(\eta\sigma_0 + \tau\sigma_3)\Phi_{\bar z}'$$

is a compact operator. Hence, $\operatorname{spec}_{\rm ess}(A_{\eta,\tau}) = \operatorname{spec}_{\rm ess}(A_0) = (-\infty, -|m|] \cup [|m|, \infty)$.

Next, we prove assertion (ii) for $|\eta| \neq |\tau|$. First we note that by Theorem 2.12 a number $z \in \operatorname{res}(A_0)$ is an eigenvalue of $A_{\eta,\tau}$ if and only if zero is an eigenvalue of $\Theta - M_z$. Using (4.16) this means that $z \in \operatorname{res}(A_0)$ is an eigenvalue of $A_{\eta,\tau}$ if and only if there exists $\psi \in \operatorname{dom}\Theta = H^1(\Sigma;\mathbb{C}^2)$ such that

$$-\Lambda(\eta\sigma_0 + \tau\sigma_3)^{-1}\big(\sigma_0 + (\eta\sigma_0 + \tau\sigma_3)\mathcal{C}_z\big)\Lambda\psi = 0,$$

i.e. if and only if $\varphi := \Lambda\psi \in H^{1/2}(\Sigma;\mathbb{C}^2)$ satisfies

$$\big(\sigma_0 + (\eta\sigma_0 + \tau\sigma_3)\mathcal{C}_z\big)\varphi = 0.$$

The proof of item (ii) for $|\eta| = |\tau|$ is similar, one just has to use (4.18) instead of (4.16).

Finally, assertion (iii) is an immediate consequence of Proposition 3.2, as $\operatorname{dom} A_{\eta,\tau} \subset H^1(\mathbb{R}^2 \setminus \Sigma;\mathbb{C}^2)$ by Theorem 4.6. □

Finally, we provide some symmetry relations for the point spectrum of $A_{\eta,\tau}$, which can be seen as consequences of commutator relations of $A_{\eta,\tau}$. The following results are the two dimensional analogues of [7, Proposition 4.2].

**Proposition 4.8.** *Let $\eta, \tau \in \mathbb{R}$ and assume that $\eta^2 - \tau^2 \neq 4$. Then the following holds:*

(i) *If $|\eta| \neq |\tau|$, then $z \in \operatorname{spec}_{\rm p}(A_{-4\eta/(\eta^2-\tau^2),-4\tau/(\eta^2-\tau^2)})$ if and only if $z \in \operatorname{spec}_{\rm p}(A_{\eta,\tau})$.*

(ii) *$z \in \operatorname{spec}_{\rm p}(A_{\eta,\tau})$ if and only if $-z \in \operatorname{spec}_{\rm p}(A_{-\eta,\tau})$.*

**Proof**. (i) Consider the unitary and self-adjoint operator

$$U : L^2(\Omega_+;\mathbb{C}^2)\oplus L^2(\Omega_-;\mathbb{C}^2) \to L^2(\Omega_+;\mathbb{C}^2)\oplus L^2(\Omega_-;\mathbb{C}^2), \quad U(f_+\oplus f_-) = f_+\oplus(-f_-).$$

We claim that

$$A_{\eta,\tau} = U A_{-4\eta/(\eta^2-\tau^2),-4\tau/(\eta^2-\tau^2)} U. \tag{4.19}$$

For this purpose we note first that $f = f_+ \oplus f_- \in H^1(\Omega_+;\mathbb{C}^2) \oplus H^1(\Omega_-;\mathbb{C}^2)$ belongs to $\operatorname{dom} A_{\eta,\tau}$, if and only if

$$-\mathrm{i}(\sigma\cdot\nu)\big(\mathfrak{T}_+^D f_+ - \mathfrak{T}_-^D f_-\big) = \frac{1}{2}(\eta\sigma_0 + \tau\sigma_3)\big(\mathfrak{T}_+^D f_+ + \mathfrak{T}_-^D f_-\big), \tag{4.20}$$

which is equivalent to

$$-\mathrm{i}(\sigma\cdot\nu)\big(\mathfrak{T}_+^D (Uf)_+ + \mathfrak{T}_-^D (Uf)_-\big) = \frac{1}{2}(\eta\sigma_0 + \tau\sigma_3)\big(\mathfrak{T}_+^D (Uf)_+ - \mathfrak{T}_-^D (Uf)_-\big).$$

By multiplying the last equation with $(\eta\sigma_0 + \tau\sigma_3)^{-1} = \frac{1}{\eta^2-\tau^2}(\eta\sigma_0 - \tau\sigma_3)$ and using (1.5) we find that $f \in \operatorname{dom} A_{\eta,\tau}$ if and only if

$$-\mathrm{i}(\sigma\cdot\nu)\frac{1}{\eta^2-\tau^2}(\eta\sigma_0 + \tau\sigma_3)\big(\mathfrak{T}_+^D (Uf)_+ + \mathfrak{T}_-^D (Uf)_-\big) = \frac{1}{2}\big(\mathfrak{T}_+^D (Uf)_+ - \mathfrak{T}_-^D (Uf)_-\big),$$

which is equivalent to

$$-\frac{4}{\eta^2-\tau^2}(\eta\sigma_0+\tau\sigma_3)\frac{1}{2}\big(\mathfrak{T}_+^D(Uf)_+ + \mathfrak{T}_-^D(Uf)_-\big) = -\mathrm{i}(\sigma\cdot\nu)\big(\mathfrak{T}_+^D(Uf)_+ - \mathfrak{T}_-^D(Uf)_-\big)$$

i.e. $Uf \in \operatorname{dom} A_{-4\eta/(\eta^2-\tau^2),-4\tau/(\eta^2-\tau^2)}$. Hence, we have shown the equality $\operatorname{dom} A_{\eta,\tau} = \operatorname{dom} A_{-4\eta/(\eta^2-\tau^2),-4\tau/(\eta^2-\tau^2)}U$. Moreover, a straightforward calculation shows $UA_{\eta,\tau}f = A_{-4\eta/(\eta^2-\tau^2),-4\tau/(\eta^2-\tau^2)}Uf$ for any $f \in \operatorname{dom} A_{\eta,\tau}$. This gives (4.19), which yields (i).

(ii) Define the nonlinear charge conjugation operator

$$Cf = \sigma_1\overline{f}, \qquad f \in L^2(\mathbb{R}^2;\mathbb{C}^2).$$

Then we see immediately $C^2f = f$ for all $f \in L^2(\mathbb{R}^2;\mathbb{C}^2)$. We claim that

$$CA_{\eta,\tau} = -A_{-\eta,\tau}C, \tag{4.21}$$

which yields then the claim of statement (ii). To prove (4.21), we note first by taking the complex conjugate of equation (4.20) that $f \in \operatorname{dom} A_{\eta,\tau}$ if and only if

$$\mathrm{i}(\overline{\sigma}\cdot\nu)\big(\mathfrak{T}_+^D\overline{f_+} - \mathfrak{T}_-^D\overline{f_-}\big) = \frac{1}{2}(\eta\sigma_0+\tau\sigma_3)\big(\mathfrak{T}_+^D\overline{f_+} + \mathfrak{T}_-^D\overline{f_-}\big), \tag{4.22}$$

where $\overline{\sigma} = (\overline{\sigma_1},\overline{\sigma_2})$ and $\overline{\sigma_j}$ is the matrix with the complex conjugate entries of $\sigma_j$. By multiplying this equation by $\sigma_1$ and using (1.5), $\overline{\sigma_1} = \sigma_1$, and $\overline{\sigma_2} = -\sigma_2$ we find that (4.22) is equivalent to

$$\mathrm{i}(\sigma\cdot\nu)\big(\mathfrak{T}_+^D(\sigma_1\overline{f_+}) - \mathfrak{T}_-^D(\sigma_1\overline{f_-})\big) = \frac{1}{2}(\eta\sigma_0-\tau\sigma_3)\big(\mathfrak{T}_+^D(\sigma_1\overline{f_+}) + \mathfrak{T}_-^D(\sigma_1\overline{f_-})\big),$$

i.e. $Cf \in \operatorname{dom} A_{-\eta,\tau}$. Moreover, using again (1.5) and $\overline{\sigma_2} = -\sigma_2$ we get

$$\begin{aligned}(-i\sigma\cdot\nabla + m\sigma_3)Cf &= (-i\sigma\cdot\nabla + m\sigma_3)\sigma_1\overline{f}\\ &= \sigma_1(-i\overline{\sigma}\cdot\nabla - m\sigma_3)\overline{f}\\ &= -\sigma_1\overline{(-i\sigma\cdot\nabla + m\sigma_3)f}\\ &= -C\big(-i\sigma\cdot\nabla + m\sigma_3)f\big),\end{aligned}$$

which implies (4.21). □

## 4.3 Critical case

In this subsection we study the self-adjointness and the spectral properties of $A_{\eta,\tau}$ for the critical interaction strengths, i.e. when $\eta^2-\tau^2 = 4$. To show the self-adjointness of $A_{\eta,\tau}$ we prove that the corresponding operator $\Theta$ in Proposition 4.3 is self-adjoint in $L^2(\Sigma;\mathbb{C}^2)$.

**Lemma 4.9.** *Let $\eta,\tau \in \mathbb{R}$ be such that $\eta^2-\tau^2 = 4$. Then the operator $\Theta$ is self-adjoint in $L^2(\Sigma;\mathbb{C}^2)$ and the restriction of $\Theta$ onto $H^1(\Sigma;\mathbb{C}^2)$ is essentially self-adjoint in $L^2(\Sigma;\mathbb{C}^2)$.*

*Remark* 4.10. According to Lemma 4.9 the operator $\Theta$ is essentially self-adjoint on $H^1(\Sigma;\mathbb{C}^2)$. It will turn out later in the proof of Proposition 4.12 that $\operatorname{spec}_{\mathrm{ess}}(\Theta)$ is non-empty. Hence, one has $\operatorname{dom}\Theta \not\subset H^s(\Sigma;\mathbb{C}^2)$ for all $s>0$.

*Proof of Lemma 4.9.* As in the proof of Lemma 4.5 we consider the restriction $\Theta_1 := \Theta \restriction H^1(\Sigma;\mathbb{C}^2)$. It follows in the same way as in the proof of Lemma 4.5 that $\Theta_1$ is a symmetric operator in $L^2(\Sigma;\mathbb{C}^2)$ and together with Lemma 2.4 we see $\overline{\Theta}_1 \subset \Theta_1^* \subset \Theta$. To see $\Theta \subset \overline{\Theta_1}$, which then implies the claims, we will show (the slightly stronger fact) that

$$\operatorname{dom}\Theta = \operatorname{dom}\overline{\Theta_1}. \tag{4.23}$$

For this we consider the associated periodic pseudodifferential operator $\theta$ defined in (4.7) and recall that with the aid of Proposition 3.5 we have

$$\theta = -\frac{1}{2}\,\upsilon + \Psi, \quad \text{where } \upsilon = \begin{pmatrix} \dfrac{2}{\eta+\tau}\Lambda^2 & \Lambda C_\Sigma \overline{T}\Lambda \\ \Lambda T C_\Sigma' \Lambda & \dfrac{2}{\eta-\tau}\,\Lambda^2 \end{pmatrix}, \tag{4.24}$$

with some operator $\Psi \in \Psi^0_\Sigma$, which is symmetric and hence self-adjoint in $L^2(\Sigma;\mathbb{C}^2)$. In the following we denote by $\Upsilon$ the maximal realization of $\upsilon$ in $L^2(\Sigma;\mathbb{C}^2)$, that is

$$\Upsilon\varphi = \upsilon\varphi, \quad \operatorname{dom}\Upsilon = \big\{\varphi \in L^2(\Sigma;\mathbb{C}^2) : \upsilon\varphi \in L^2(\Sigma;\mathbb{C}^2)\big\} = \operatorname{dom}\Theta,$$

and $\Upsilon_1 = \Upsilon \restriction H^1(\Sigma;\mathbb{C}^2)$. Note that $\operatorname{dom}\overline{\Upsilon_1} = \operatorname{dom}\overline{\Theta_1}$. In the same way as in Subsection 2.2 we use the Schur complement to decompose $\upsilon$ (on a formal level in the sense of periodic pseudodifferential operators without specification of the operator domains) as

$$\upsilon = \begin{pmatrix} \mathbb{1} & 0 \\ \dfrac{\eta+\tau}{2}\,\Lambda T C_\Sigma' \Lambda^{-1} & \mathbb{1} \end{pmatrix} \begin{pmatrix} \dfrac{2}{\eta+\tau}\,\Lambda^2 & 0 \\ 0 & \mathcal{S}(\upsilon) \end{pmatrix} \begin{pmatrix} \mathbb{1} & \dfrac{\eta+\tau}{2}\Lambda^{-1} C_\Sigma \overline{T}\Lambda \\ 0 & \mathbb{1} \end{pmatrix}, \tag{4.25}$$

where the Schur complement has the form

$$\mathcal{S}(\upsilon) = \frac{2}{\eta-\tau}\,\Lambda^2 - \frac{\eta+\tau}{2}\Lambda T C_\Sigma' \Lambda (\Lambda^2)^{-1} \Lambda C_\Sigma \overline{T}\Lambda = \frac{2}{\eta-\tau}\,\Lambda^2 - \frac{\eta+\tau}{2}\Lambda T C_\Sigma' C_\Sigma \overline{T}\Lambda.$$

Using that $C_\Sigma' C_\Sigma = \mathbb{1} + R$ with $R \in \Psi^{-\infty}_\Sigma$, see Proposition 2.9, we can rewrite this expression as

$$\mathcal{S}(\upsilon) = \frac{2}{\eta-\tau}\Lambda^2 - \frac{\eta+\tau}{2}\Lambda T\overline{T}\Lambda - \frac{\eta+\tau}{2}\Lambda T R \overline{T}\Lambda = -\frac{\eta+\tau}{2}\Lambda T R \overline{T}\Lambda \in \Psi^{-\infty}_\Sigma,$$

where we used in the last step that $T\overline{T}$ is the multiplication operator with the constant function $\mathbb{1}$ and $\eta^2 - \tau^2 = 4$. From this, (4.25), and $\operatorname{dom}\Lambda^2 = H^1(\Sigma)$ we obtain now

$$\operatorname{dom}\Theta = \operatorname{dom}\Upsilon = \left\{(\varphi_1,\varphi_2)^\top \in L^2(\Sigma;\mathbb{C}^2) : \varphi_1 + \frac{\eta+\tau}{2}\Lambda^{-1} C_\Sigma \overline{T}\Lambda\varphi_2 \in H^1(\Sigma)\right\}.$$

Let us now consider the operator realizations $\Theta_1, \Upsilon_1$ of $\theta, \upsilon$ and their closures $\overline{\Theta_1}, \overline{\Upsilon_1}$ in $L^2(\Sigma;\mathbb{C}^2)$. We leave it to the reader to check that the assumptions in Proposition 2.10 are satisfied when each entry of the pseudodifferential operators in the matrix representation of $\upsilon$ in (4.24) is defined on $H^1(\Sigma)$; in particular, note that the upper left corner is a boundedly invertible self-adjoint operator in $L^2(\Sigma)$ with domain $H^1(\Sigma)$. Then it follows from Proposition 2.10 that $\operatorname{dom}\overline{\mathfrak{S}(\Xi_1)} = L^2(\Sigma)$ and

$$\begin{aligned}\operatorname{dom}\overline{\Theta_1} &= \operatorname{dom}\overline{\Xi_1}\\ &= \left\{(\varphi_1,\varphi_2)^\top \in L^2(\Sigma;\mathbb{C}^2) : \varphi_1 + \frac{\eta+\tau}{2}\Lambda^{-1}C_\Sigma\overline{T}\Lambda\varphi_2 \in H^1(\Sigma)\right\} = \operatorname{dom}\Theta\end{aligned}$$

hold. Hence, we have shown (4.23), which finishes the proof of this proposition. □

With Lemma 4.9 we are now ready to show the self-adjointness of $A_{\eta,\tau}$ for critical interaction strengths. To formulate the result we recall the definitions of the free Dirac operator $A_0$ from (3.1), of $\Phi_z$ and $\Phi'_z$ from (3.8) and (3.7), and of $\mathcal{C}_z$ in (3.11), respectively.

**Theorem 4.11.** *Assume that $\eta,\tau\in\mathbb{R}$ with $\eta^2-\tau^2=4$. Then the operator $A_{\eta,\tau}$ is self-adjoint in $L^2(\mathbb{R}^2;\mathbb{C}^2)$ and the restriction to $\operatorname{dom}A_{\eta,\tau}\cap H^1(\mathbb{R}^2\setminus\Sigma;\mathbb{C}^2)$ is essentially self-adjoint in $L^2(\mathbb{R}^2;\mathbb{C}^2)$. Moreover, for all $z\in\operatorname{res}(A_{\eta,\tau})\cap\operatorname{res}(A_0)$ the operator $\sigma_0+(\eta\sigma_0+\tau\sigma_3)\mathcal{C}_z$ admits a bounded inverse from $H^{1/2}(\Sigma;\mathbb{C}^2)$ to $H^{-1/2}(\Sigma;\mathbb{C}^2)$ and*

$$(A_{\eta,\tau}-z)^{-1} = (A_0-z)^{-1} - \Phi_z\big(\sigma_0+(\eta\sigma_0+\tau\sigma_3)\mathcal{C}_z\big)^{-1}(\eta\sigma_0+\tau\sigma_3)\Phi'_{\bar z} \tag{4.26}$$

*holds.*

**Proof**. First, according to Theorem 2.12 the self-adjointness of $\Theta$ in $L^2(\Sigma;\mathbb{C}^2)$ implies the self-adjointness of $A_{\eta,\tau}$ in $L^2(\mathbb{R}^2;\mathbb{C}^2)$, and the essential self-adjointness of $\Theta_1=\Theta\restriction H^1(\Sigma;\mathbb{C}^2)$ in $L^2(\Sigma;\mathbb{C}^2)$ implies the essential self-adjointness of the restriction of $A_{\eta,\tau}$ to $\operatorname{dom}A_{\eta,\tau}\cap H^1(\mathbb{R}^2\setminus\Sigma;\mathbb{C}^2)$ in $L^2(\mathbb{R}^2;\mathbb{C}^2)$. For the latter observation we have also used that by Lemma 3.8

$$S^*\restriction\ker(\Gamma_1-\Theta_1\Gamma_0) = A_{\eta,\tau}\restriction\big(\operatorname{dom}A_{\eta,\tau}\cap H^1(\mathbb{R}^2\setminus\Sigma;\mathbb{C}^2)\big).$$

It remains to verify the Krein type resolvent formula in (4.26). By Theorem 2.12 we have that $\Theta-M_z$ is boundedly invertible in $L^2(\Sigma;\mathbb{C}^2)$ and

$$(A_{\eta,\tau}-z)^{-1} = (A_0-z)^{-1} + G_z\big(\Theta-M_z\big)^{-1}G_{\bar z}^*.$$

Taking the special form of $\Theta$ and $M_z=\Lambda\big(\mathcal{C}_z-\frac{1}{2}\big(\mathcal{C}_\zeta+\mathcal{C}_{\bar\zeta}\big)\big)\Lambda$ into account we find with a similar calculation as in (4.16)-(4.17) that

$$(\Theta-M_z)^{-1} = -\Lambda^{-1}\big(\sigma_0+(\eta\sigma_0+\tau\sigma_3)\mathcal{C}_z\big)^{-1}(\eta\sigma_0+\tau\sigma_3)\Lambda^{-1}.$$

As $(\Theta-M_z)^{-1}$ is bounded in $L^2(\Sigma;\mathbb{C}^2)$ we deduce that $(\sigma_0+(\eta\sigma_0+\tau\sigma_3)\mathcal{C}_z)^{-1}$ is bounded from $H^{1/2}(\Sigma;\mathbb{C}^2)$ to $H^{-1/2}(\Sigma;\mathbb{C}^2)$. Using $G_z=\Phi_z\Lambda$ and $G_{\bar z}^*=\Lambda\Phi'_{\bar z}$ we get

$$\begin{aligned}G_z\big(\Theta-M_z\big)^{-1}G_{\bar z}^* &= -\Phi_z\Lambda\Lambda^{-1}\big(\sigma_0+(\eta\sigma_0+\tau\sigma_3)\mathcal{C}_z\big)^{-1}(\eta\sigma_0+\tau\sigma_3)\Lambda^{-1}\Lambda\Phi'_{\bar z}\\ &= -\Phi_z\big(\sigma_0+(\eta\sigma_0+\tau\sigma_3)\mathcal{C}_z\big)^{-1}(\eta\sigma_0+\tau\sigma_3)\Phi'_{\bar z},\end{aligned}$$

and thus (4.26). □

In the next proposition we analyze the essential spectrum of the self-adjoint operator $\Theta$. Note that our assumption $\eta^2 - \tau^2 = 4$ implies $|\tau| < |\eta|$, and hence $-\frac{\tau}{\eta}\, m \in (-|m|, |m|)$.

**Proposition 4.12.** *Let $\eta, \tau \in \mathbb{R}$ be such that $\eta^2 - \tau^2 = 4$ and let $m \neq 0$. Then for $z \in (-|m|, |m|)$ one has $0 \in \operatorname{spec}_{\mathrm{ess}}(M_z - \Theta)$ if and only if $z = -\frac{\tau}{\eta}\, m$.*

**Proof**. Throughout the proof we assume that $z \in (-|m|, |m|)$. In particular, $M_z$ is a bounded self-adjoint operator in $L^2(\Sigma; \mathbb{C}^2)$. Recall that

$$M_z - \Theta = \Lambda \frac{1}{\eta^2 - \tau^2}(\eta\sigma_0 - \tau\sigma_3)\Lambda + \Lambda \mathcal{C}_z \Lambda,$$

and using Proposition 3.5 we decompose this self-adjoint operator in $M_z - \Theta = \Xi_1 + \Xi_2$, where

$$\Xi_1 := \begin{pmatrix} \dfrac{1}{\eta + \tau}\, \Lambda^2 + \dfrac{\ell}{4\pi}(z + m)\mathbb{1} & \dfrac{1}{2}\Lambda C_\Sigma \overline{T} \Lambda \\ \dfrac{1}{2}\Lambda T C_\Sigma' \Lambda & \dfrac{1}{\eta - \tau}\, \Lambda^2 + \dfrac{\ell}{4\pi}(z - m)\mathbb{1} \end{pmatrix}$$

and $\Xi_2 \in \Psi_\Sigma^{-1}$ is a compact self-adjoint operator in $L^2(\Sigma; \mathbb{C}^2)$. We note that $\Xi_1$ defined on $\operatorname{dom}(M_z - \Theta) = \operatorname{dom}\Theta$ is a self-adjoint operator in $L^2(\Sigma; \mathbb{C}^2)$. It follows that $\operatorname{spec}_{\mathrm{ess}}(M_z - \Theta) = \operatorname{spec}_{\mathrm{ess}}(\Xi_1)$ and, in particular,

$$0 \in \operatorname{spec}_{\mathrm{ess}}(M_z - \Theta) \text{ if and only if } 0 \in \operatorname{spec}_{\mathrm{ess}}(\Xi_1).$$

In the following we will show that $0 \in \operatorname{spec}_{\mathrm{ess}}(\Xi_1)$ if and only if $z = -\frac{\tau}{\eta}\, m$. For this, the Schur complement of $\Xi_1$ and Proposition 2.10 will be used. To proceed, we shall use the operator $\Lambda \in \Psi_\Sigma^{1/2}$ from (2.8) (see also (2.7)). Recall also that $\Lambda^2 \geq c_0^2$ for $c_0 > 0$. Now we choose $c_0$ such that $c_0^2 > \frac{|m|\ell}{2\pi}|\eta + \tau|$. Then the upper left corner of $\Xi_1$

$$\frac{1}{\eta + \tau}\, \Lambda^2 + \frac{\ell}{4\pi}(z + m)\mathbb{1}$$

is boundedly invertible in $L^2(\Sigma)$. We leave it to the reader to check that the other assumptions in Proposition 2.10 are also satisfied for the block operator matrix $\Xi_1$. Therefore, we have $0 \in \operatorname{spec}_{\mathrm{ess}}(\Xi_1)$ if and only if $0 \in \operatorname{spec}_{\mathrm{ess}}(\mathcal{S})$, where $\mathcal{S} := \mathcal{S}(\Xi_1)$ is the Schur complement

$$\mathcal{S} = \frac{1}{\eta - \tau}\, \Lambda^2 + \frac{\ell(z - m)}{4\pi}\mathbb{1} - \frac{\eta + \tau}{4}\, \Lambda T C_\Sigma' \Lambda \left( \Lambda^2 + \frac{\ell(z + m)(\eta + \tau)}{4\pi}\mathbb{1} \right)^{-1} \Lambda C_\Sigma \overline{T} \Lambda.$$

To simplify the last summand in the above expression of $\mathcal{S}$ we use the identity

$$(\Lambda^2 + a\mathbb{1})^{-1} = \Lambda^{-2} - a\Lambda^{-1}(\Lambda^2 + a\mathbb{1})^{-1}\Lambda^{-1} = \Lambda^{-2} - a\Lambda^{-2}(\Lambda^2 + a\mathbb{1})^{-1} \tag{4.27}$$

and rewrite $\mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2$ with

$$\mathcal{S}_1 = \frac{1}{\eta - \tau}\, \Lambda^2 + \frac{\ell(z - m)}{4\pi}\mathbb{1} - \frac{\eta + \tau}{4}\, \Lambda T C_\Sigma' C_\Sigma \overline{T} \Lambda$$

and

$$\mathcal{S}_2 = \frac{(\eta+\tau)^2}{4} \cdot \frac{\ell(z+m)}{4\pi} \Lambda T C_\Sigma' \left( \Lambda^2 + \frac{\ell(z+m)(\eta+\tau)}{4\pi} \mathbb{1} \right)^{-1} C_\Sigma \overline{T} \Lambda.$$

By Proposition 2.9 one has $C_\Sigma' C_\Sigma = \mathbb{1} + K_1$ with $K_1 \in \Psi_\Sigma^{-\infty}$, so

$$\frac{\eta+\tau}{4} \Lambda T C_\Sigma' C_\Sigma \overline{T} \Lambda = \frac{\eta+\tau}{4} \Lambda^2 + K_2$$

with $K_2 \in \Psi_\Sigma^{-\infty}$. This gives because of $\eta^2 - \tau^2 = 4$

$$\mathcal{S}_1 = \frac{1}{\eta-\tau} \Lambda^2 + \frac{\ell(z-m)}{4\pi} \mathbb{1} - \frac{\eta+\tau}{4} \Lambda^2 - K_2 = \frac{\ell(z-m)}{4\pi} \mathbb{1} - K_2.$$

In order to deal with $\mathcal{S}_2$ we use again the identity (4.27), which gives

$$\frac{4}{(\eta+\tau)^2} \cdot \frac{4\pi}{\ell(z+m)} \mathcal{S}_2 = \Lambda T C_\Sigma' \left( \Lambda^2 + \frac{\ell(z+m)(\eta+\tau)}{4\pi} \mathbb{1} \right)^{-1} C_\Sigma \overline{T} \Lambda = K_3 + K_4,$$

where

$$K_3 = \Lambda T C_\Sigma' \Lambda^{-2} C_\Sigma \overline{T} \Lambda$$

and

$$K_4 = -\frac{\ell(z+m)(\eta+\tau)}{4\pi} \Lambda T C_\Sigma' \Lambda^{-2} \left( \Lambda^2 + \frac{\ell(z+m)(\eta+\tau)}{4\pi} \mathbb{1} \right)^{-1} C_\Sigma \overline{T} \Lambda.$$

Using Proposition 2.2 one finds that $K_4 \in \Psi_\Sigma^{-1}$ and hence this operator is compact in $L^2(\Sigma; \mathbb{C}^2)$. In order to simplify $K_3$ we note first that

$$K_5 := T C_\Sigma' \Lambda^{-2} - \Lambda^{-2} T C_\Sigma' \in \Psi_\Sigma^{-2}$$

by Proposition 2.2 (ii). Hence,

$$K_3 = \Lambda \Lambda^{-2} T C_\Sigma' C_\Sigma \overline{T} \Lambda + \Lambda K_5 C_\Sigma \overline{T} \Lambda =: \Lambda \Lambda^{-2} T C_\Sigma' C_\Sigma \overline{T} \Lambda + K_6$$

with $K_6 \in \Psi_\Sigma^{-1}$. Using again $C_\Sigma' C_\Sigma - \mathbb{1} \in \Psi^{-\infty}$, see Proposition 2.9, we arrive at $K_3 = \mathbb{1} + K_7$ with $K_7 \in \Psi_\Sigma^{-1}$. With this we find

$$\mathcal{S}_2 = \frac{(\eta+\tau)^2}{4} \cdot \frac{\ell(z+m)}{4\pi} (K_3 + K_4) = \frac{(\eta+\tau)^2}{4} \cdot \frac{\ell(z+m)}{4\pi} \mathbb{1} + K_8$$

with $K_8 \in \Psi_\Sigma^{-1}$. Using this in the expression of the Schur complement $\mathcal{S}$ we conclude, with some $K_9 \in \Psi_\Sigma^{-1}$, that

$$\begin{aligned} \mathcal{S} &= \mathcal{S}_1 + \mathcal{S}_2 \\ &= \Big( \frac{\ell(z-m)}{4\pi} + \frac{(\eta+\tau)^2}{4} \cdot \frac{\ell(z+m)}{4\pi} \Big) \mathbb{1} + K_9 \\ &= \frac{\ell}{4\pi} \Big[ \Big( \frac{(\eta+\tau)^2}{4} + 1 \Big) z + \Big( \frac{(\eta+\tau)^2}{4} - 1 \Big) m \Big] \mathbb{1} + K_9. \end{aligned}$$

As $K_9$ is compact and symmetric, it does not influence the essential spectrum, and we have

$$0 \in \operatorname{spec}_{\mathrm{ess}}(\mathcal{S}) \text{ if and only if } z = -\frac{(\eta+\tau)^2-4}{(\eta+\tau)^2+4}\, m.$$

With $\eta^2 - \tau^2 = 4$ we can simplify the last expression to

$$\frac{(\eta+\tau)^2-4}{(\eta+\tau)^2+4} = \frac{\eta^2+\tau^2+2\eta\tau-\eta^2+\tau^2}{\eta^2+\tau^2+2\eta\tau+\eta^2-\tau^2} = \frac{2\tau^2+2\eta\tau}{2\eta^2+2\eta\tau} = \frac{2\tau(\eta+\tau)}{2\eta(\eta+\tau)} = \frac{\tau}{\eta}.$$

Hence, $0 \in \operatorname{spec}_{\mathrm{ess}}(\mathcal{S})$ if and only if $z = -\dfrac{\tau}{\eta}\, m$. This finishes the proof of this proposition. □

We are now ready to describe the spectral properties of $A_{\eta,\tau}$ for critical interaction strengths. Compared to Proposition 4.7, the following theorem shows that the spectral properties of $A_{\eta,\tau}$ differ significantly from the non-critical case.

**Theorem 4.13.** *Let $\eta, \tau \in \mathbb{R}$ be such that $\eta^2 - \tau^2 = 4$. Then the following holds:*

(i) *The essential spectrum of $A_{\eta,\tau}$ is*

$$\operatorname{spec}_{\mathrm{ess}}(A_{\eta,\tau}) = \big(-\infty, -|m|\big] \cup \big\{-\tfrac{\tau}{\eta}\, m\big\} \cup \big[|m|, +\infty\big).$$

*In particular, for $m = 0$ we have $\operatorname{spec}(A_{\eta,\tau}) = \operatorname{spec}_{\mathrm{ess}}(A_{\eta,\tau}) = \mathbb{R}$.*

(ii) *Assume $m \neq 0$. Then $z \notin \operatorname{spec}_{\mathrm{ess}}(A_{\eta,\tau})$ is a discrete eigenvalue of $A_{\eta,\tau}$ if and only if there exists $\varphi \in H^{-1/2}(\Sigma; \mathbb{C}^2)$ such that $\big(\sigma_0 + (\eta\sigma_0 + \tau\sigma_3)\mathcal{C}_z\big)\varphi = 0$.*

(iii) *For all $s > 0$ we have $\operatorname{dom} A_{\eta,\tau} \not\subset H^s(\mathbb{R}^2 \setminus \Sigma; \mathbb{C}^2)$.*

*Remark* 4.14. Item (ii) in the above theorem is slightly weaker as Proposition 4.7 (ii), since one has to search for eigenfunctions $\varphi$ of the Birman-Schwinger operator $\sigma_0 + (\eta\sigma_0 + \tau\sigma_3)\mathcal{C}_z$ in the larger space $H^{-1/2}(\Sigma; \mathbb{C}^2)$. However, as there is no Sobolev regularity in $\operatorname{dom} A_{\eta,\tau}$ the smoothness of the eigenfunctions of $\sigma_0 + (\eta\sigma_0 + \tau\sigma_3)\mathcal{C}_z$ can not be improved.

*Proof of Theorem 4.13.* In order to verify assertion (i) we note that the inclusion

$$\big(-\infty, -|m|\big] \cup \big[|m|, \infty\big) \subset \operatorname{spec}_{\mathrm{ess}}(A_{\eta,\tau}) \tag{4.28}$$

can be shown in the same way as in [9, Theorem 5.7 (i)], where the three dimensional situation is discussed. In fact, for a fixed $z \in (-\infty, -|m|) \cup (|m|, \infty)$ one verifies that

$$f_n(x_1, x_2) := \frac{1}{n}\chi\left(\frac{1}{n}|x - y_n|\right) e^{\mathrm{i}\sqrt{z^2-m^2}x_1}\big(\sqrt{z^2-m^2}\sigma_1 + m\sigma_3 + z\sigma_0\big)\zeta$$

is a singular sequence for $A_{\eta,\tau}$ and $z$. Here $\chi : \mathbb{R} \to [0,1]$ is a $C^\infty$-function such that $\chi(t) = 1$ for $|t| \leq \frac{1}{2}$ and $\chi(t) = 0$ for $|t| \geq 1$, $\zeta \in \mathbb{C}^2$ is chosen such that $(\sqrt{z^2-m^2}\sigma_1 + m\sigma_3 + z\sigma_0)\zeta \neq 0$, $R > 0$ is such that $\mathbb{R}^3 \setminus \overline{B(0,R)} \subset \Omega_-$, and

$y_n := (R + n^2, 0)$, $n \in \mathbb{N}$. The reader is referred to the proof of [9, Theorem 5.7 (i)] for more details. Moreover, according to Theorem 2.12 we have $z \in \operatorname{spec}_{\mathrm{ess}}(A_{\eta,\tau}) \cap (-|m|, |m|)$ if and only if $0 \in \operatorname{spec}_{\mathrm{ess}}(\Theta - M_z)$, that is, by Proposition 4.12 we have $z \in \operatorname{spec}_{\mathrm{ess}}(A_{\eta,\tau}) \cap (-|m|, |m|)$ if and only if $z = -\frac{\tau}{\eta}\, m$. Together with (4.28) this implies (i).

To prove item (ii) we note first that by Theorem 2.12 a point $z \in \operatorname{res}(A_0)$ is an eigenvalue of $A_{\eta,\tau}$ if and only if zero is an eigenvalue of $\Theta - M_z$. Using a similar calculation as in (4.16) this shows that $z \in \operatorname{res}(A_0)$ is an eigenvalue of $A_{\eta,\tau}$ if and only if there exists $\psi \in \operatorname{dom} \Theta \subset L^2(\Sigma; \mathbb{C}^2)$ such that

$$-\Lambda(\eta\sigma_0 + \tau\sigma_3)^{-1}\big(\sigma_0 + (\eta\sigma_0 + \tau\sigma_3)\mathcal{C}_z\big)\Lambda\psi = 0,$$

i.e. if and only if $\varphi := \Lambda\psi \in H^{-1/2}(\Sigma; \mathbb{C}^2)$ satisfies $\big(\sigma_0 + (\eta\sigma_0 + \tau\sigma_3)\mathcal{C}_z\big)\varphi = 0$.

Eventually, since $\operatorname{dom} A_{\eta,\tau}$ is independent of $m$, it suffices to prove statement (iii) for $m \neq 0$. In this case the claim is a consequence of Proposition 3.2, as $\operatorname{spec}_{\mathrm{ess}}(A_{\eta,\tau}) \cap (-|m|, |m|) \neq \emptyset$. □

Finally, we state several symmetry relations in the spectrum of $A_{\eta,\tau}$. The following proposition is the counterpart of Proposition 4.8 for critical interaction strengths.

**Proposition 4.15.** *Let $\eta, \tau \in \mathbb{R}$ and assume that $\eta^2 - \tau^2 = 4$. Then the following holds:*

(i) $z \in \operatorname{spec}_{\mathrm{p}}(A_{\eta,\tau})$ *if and only if* $z \in \operatorname{spec}_{\mathrm{p}}(A_{-\eta,-\tau})$.

(ii) $z \in \operatorname{spec}_{\mathrm{p}}(A_{\eta,\tau})$ *if and only if* $-z \in \operatorname{spec}_{\mathrm{p}}(A_{-\eta,\tau})$.

**Proof**. In the following set $A^1_{\eta,\tau} := A_{\eta,\tau} \restriction (\operatorname{dom} A_{\eta,\tau} \cap H^1(\mathbb{R}^2 \setminus \Sigma; \mathbb{C}^2))$. Then by Theorem 4.11 the operator $A^1_{\eta,\tau}$ is essentially self-adjoint in $L^2(\mathbb{R}^2; \mathbb{C}^2)$ and, in particular, $\overline{A^1_{\eta,\tau}} = A_{\eta,\tau}$.

(i) Consider the unitary and self-adjoint mapping

$$U : L^2(\Omega_+; \mathbb{C}^2) \oplus L^2(\Omega_-; \mathbb{C}^2) \to L^2(\Omega_+; \mathbb{C}^2) \oplus L^2(\Omega_-; \mathbb{C}^2), \quad U(f_+ \oplus f_-) = f_+ \oplus (-f_-).$$

As in the proof of Proposition 4.8 (i) one verifies $A^1_{\eta,\tau} = U A^1_{-\eta,-\tau} U$. By taking closures we find $A_{\eta,\tau} = U A_{-\eta,-\tau} U$ and hence the claim follows.

(ii) Consider the nonlinear charge conjugation operator

$$Cf = \sigma_1 \overline{f}, \qquad f \in L^2(\mathbb{R}^2; \mathbb{C}^2).$$

Then $C^2 f = f$ for $f \in L^2(\mathbb{R}^2; \mathbb{C}^2)$ and in the same way as in the proof of Proposition 4.8 (ii) one obtains $C A^1_{\eta,\tau} = -A^1_{-\eta,\tau} C$. Taking closures leads to $C A_{\eta,\tau} = -A_{-\eta,\tau} C$, which implies (ii). □

## 4.4 Sketch of the proof of Theorem 1.3

To prove Theorem 1.3 we use similar constructions as in the case of one loop. We give some comments on necessary modifications in this subsection. Let $N \geq 1$ and let $\Sigma_j, j \in \{1, \dots, N\}$, be non-intersecting $C^\infty$-smooth loops with normals $\nu_j$. We set $\Sigma := \bigcup_{j=1}^N \Sigma_j$, and for $f \in H(\sigma, \mathbb{R}^2 \setminus \Sigma)$ we denote its Dirichlet traces from Lemma 3.1 on the two sides of $\Sigma_j$ by $\mathcal{T}^D_{\pm,j} f$, where $-$ corresponds to the side to which $\nu_j$ is directed. The Sobolev spaces on $\Sigma$ are defined by $H^s(\Sigma) := \bigoplus_{j=1}^N H^s(\Sigma_j)$, and for $\varphi \in H^s(\Sigma)$ we denote by $\varphi_j$ its restriction on $\Sigma_j$. Furthermore, if $\Lambda_j$ denotes the isomorphism defined in (2.8) on $\Sigma_j$, then we set $\Lambda := \bigoplus_{j=1}^N \Lambda_j$. As in the case of one loop one starts with the symmetric operator $S := A_0 \restriction H^1_0(\mathbb{R}^2 \setminus \Sigma; \mathbb{C}^2)$. For $z \in \mathrm{res}(A_0)$ and $\varphi \in L^2(\Sigma; \mathbb{C}^2)$ we introduce

$$\Phi_z \varphi(x) = \int_\Sigma \phi_z(x-y)\varphi(y)\, \mathrm{d}s(y), \quad x \in \mathbb{R}^2 \setminus \Sigma.$$

As for the single loop in Proposition 3.3 one shows that $\Phi_z$ extends to a bounded map $\Phi_z : H^{-1/2}(\Sigma; \mathbb{C}^2) \to L^2(\mathbb{R}^2; \mathbb{C}^2)$ with $\operatorname{ran} \Phi_z = \ker(S^* - z)$. The associated principal value operator $\mathcal{C}_z$,

$$\big(\mathcal{C}_z \varphi\big)(x) := \text{p.v.} \int_\Sigma \phi_z(x-y)\varphi(y)\, \mathrm{d}s(y), \quad \varphi \in C^\infty(\Sigma; \mathbb{C}^2),\ x \in \Sigma,$$

has a block structure of the form

$$(\mathcal{C}_z \varphi)_j(x) := \mathcal{C}^j_z \varphi_j(x) + \sum_{k \neq j} (\mathcal{K}^{j,k}_z \varphi_k)(x), \quad \varphi \in C^\infty(\Sigma; \mathbb{C}^2),\ x \in \Sigma_j, \tag{4.29}$$

$$(\mathcal{C}^j_z \varphi_j)(x) = \text{p.v.} \int_{\Sigma_j} \phi_z(x-y)\varphi_j(y)\, \mathrm{d}s(y), \quad x \in \Sigma_j, \tag{4.30}$$

$$(\mathcal{K}^{j,k}_z \varphi_k)(x) = \int_{\Sigma_k} \phi_z(x-y)\varphi_k(y)\, \mathrm{d}s(y), \quad x \in \Sigma_j. \tag{4.31}$$

The operators $\mathcal{C}^j_z$ are the same as in the one loop case, while the operators $\mathcal{K}^{j,k}_z$ have smooth integral kernels; hence, they define bounded operators from $H^s(\Sigma_k, \mathbb{C}^2)$ to $H^t(\Sigma_j, \mathbb{C}^2)$ for any $s, t \in \mathbb{R}$. With the help of Proposition 3.6 one can show now the trace equality

$$\mathcal{T}^D_{\pm,j} \Phi_z \varphi = \mp \frac{\mathrm{i}}{2} (\sigma \cdot \nu_j)\, \varphi_j + \big(\mathcal{C}_z \varphi\big)_j.$$

The construction of the boundary triple takes then literally the same form as for a single loop. Let $\zeta \in \mathrm{res}(A_0)$ be fixed and set $(\mathcal{T}^D_\pm f) := (\mathcal{T}^D_{\pm,j} f)_{j=1}^N$. Then $\{L^2(\Sigma; \mathbb{C}^2), \Gamma_0, \Gamma_1\}$ with

$$\begin{aligned} \Gamma_0 f &= \mathrm{i}\Lambda^{-1}(\sigma \cdot \nu)\big(\mathcal{T}^D_+ f - \mathcal{T}^D_- f\big), \\ \Gamma_1 f &= \frac{1}{2}\Lambda\Big((\mathcal{T}^D_+ f_+ + \mathcal{T}^D_- f_-) - (\mathcal{C}_\zeta + \mathcal{C}_{\bar\zeta})\Lambda\Gamma_0 f\Big), \end{aligned}$$

is a boundary triple for $S^*$. The corresponding $\gamma$-field $G$ and Weyl function $M$ are $z \mapsto G_z = \Phi_z \Lambda$ and

$$z \mapsto M_z = \Lambda\Big(\mathcal{C}_z - \frac{1}{2}\big(\mathcal{C}_\zeta + \mathcal{C}_{\bar\zeta}\big)\Big)\Lambda.$$

Assume first that $|\eta_j| \neq |\tau_j|$ for any $j$. We define the linear operator $\Theta$ in $L^2(\Sigma;\mathbb{C}^2)$ by

$$\Theta = -\Lambda\left[\Xi + \frac{1}{2}\left(\mathcal{C}_\zeta + \mathcal{C}_{\bar\zeta}\right)\right]\Lambda, \quad (\Xi\varphi)_j := \frac{1}{\eta_j^2 - \tau_j^2}(\eta_j\sigma_0 - \tau_j\sigma_3)\,\varphi_j,$$

on its maximal domain in $L^2(\Sigma;\mathbb{C}^2)$. Then the operator $A_{\Sigma,\mathcal{P}}$ defined in (1.3) corresponds to the boundary condition $\Gamma_1 f = \Theta\Gamma_0 f$. Using (4.29) one sees that $\Theta$ can be written as $\Theta = \bigoplus_{j=1}^N \Theta_j + \widetilde{\Theta}$, where $\Theta_j$ is the operator in $L^2(\Sigma_j;\mathbb{C}^2)$ acting as

$$\Theta_j = -\Lambda_j\left[\frac{1}{\eta_j^2 - \tau_j^2}(\eta_j\sigma_0 - \tau_j\sigma_3) + \frac{1}{2}\left(\mathcal{C}_\zeta^j + \mathcal{C}_{\bar\zeta}^j\right)\right]\Lambda_j,$$

with maximal domain, while $\widetilde{\Theta}$ is a bounded operator from $H^s(\Sigma,\mathbb{C}^2)$ to $H^t(\Sigma,\mathbb{C}^2)$ for any $s,t \in \mathbb{R}$ which is self-adjoint in $L^2(\Sigma;\mathbb{C}^2)$. Hence, the self-adjointness of $\Theta$ is determined by the self-adjointness of $\bigoplus_{j=1}^N \Theta_j$, and each $\Theta_j$ is exactly of the form as in the single-loop case. Hence, $\Theta_j$ is self-adjoint by Lemma 4.5 and Lemma 4.9 and thus, also $\Theta$ is self-adjoint in $L^2(\Sigma;\mathbb{C}^2)$. This implies also the statements concerning the domain regularity.

In order to study the essential spectrum we decompose $M_z$ to blocks as in (4.29) and remark that the terms $\mathcal{K}_z^{j,k}$ produce compact remainders, which do not influence the essential spectrum. Hence, the condition $0 \in \mathrm{spec}_{\mathrm{ess}}(M_z - \Theta)$ is equivalent to

$$0 \in \mathrm{spec}_{\mathrm{ess}}\left(\bigoplus_{j=1}^N \left(\Lambda_j \frac{1}{\eta_j^2 - \tau_j^2}(\eta_j\sigma_0 - \tau_j\sigma_3)\Lambda_j + \Lambda_j\mathcal{C}_z^j\Lambda_j\right)\right).$$

As each of the terms on the right-hand side is covered by the analysis of the single-loop case, the statement on the essential spectrum of $M_z - \Theta$ and thus, with the help of Theorem 2.12, also of $A_{\Sigma,\mathcal{P}}$, follows.

If for some $j$ one has $|\eta_j| = |\tau_j|$, then the analysis can be done in a similar way following the strategy from Section 4.2. The details are left to the reader.

# References


[1] S. Albeverio, F. Gesztesy, R. Høegh-Krohn, H. Holden: *Solvable Models in Quantum Mechanics.* With an Appendix by Pavel Exner, 2nd ed., Amer. Math. Soc. Chelsea Publishing, Providence, RI, 2005.

[2] N. Arrizabalaga, L. Le Treust, A. Mas, N. Raymond: *The MIT Bag Model as an infinite mass limit.* J. Éc. polytech. Math. **6** (2019) 329–365.

[3] N. Arrizabalaga, A. Mas, L. Vega: *Shell interactions for Dirac operators.* J. Math. Pures Appl. **102** (2014) 617–639.

[4] N. Arrizabalaga, A. Mas, L. Vega: *Shell interactions for Dirac operators: on the point spectrum and the confinement.* SIAM J. Math. Anal. **47** (2015) 1044–1069.

[5] N. Arrizabalaga, A. Mas, L. Vega: *An isoperimetric-type inequality for electrostatic shell interactions for Dirac operators.* Comm. Math. Phys. **344** (2016) 483–505.

[6] J. Behrndt, P. Exner, M. Holzmann, V. Lotoreichik: *On the spectral properties of Dirac operators with electrostatic $\delta$-shell interactions.* J. Math. Pures Appl. **111** (2018) 47–78.

[7] J. Behrndt, P. Exner, M. Holzmann, V. Lotoreichik: *On Dirac operators in $\mathbb{R}^3$ with electrostatic and Lorentz scalar $\delta$-shell interactions.* To appear in Quantum Stud.: Math. Found., DOI: `https://doi.org/10.1007/s40509-019-00186-6`.

[8] J. Behrndt, S. Hassi, H.S.V. de Snoo: *Boundary Value Problems, Weyl Functions, and Differential Operators.* Monographs in Mathematics, Springer, *to appear.*

[9] J. Behrndt, M. Holzmann: *On Dirac operators with electrostatic $\delta$-shell interactions of critical strength.* To appear in J. Spect. Theory. Preprint arXiv:1612.02290v2.

[10] J. Behrndt, M. Langer, V. Lotoreichik: *Schrödinger operators with $\delta$ and $\delta'$-potentials supported on hypersurfaces.* Ann. Henri Poincaré **14** (2013) 385–423.

[11] R. D. Benguria, S. Fournais, E. Stockmeyer, H. Van Den Bosch: *Self-adjointness of two-dimensional Dirac operators on domains.* Ann. Henri Poincaré **18** (2017) 1371–1383.

[12] R. D. Benguria, S. Fournais, E. Stockmeyer, H. Van Den Bosch: *Spectral gaps of Dirac operators describing graphene quantum dots.* Math. Phys. Anal. Geom. **20**:2 (2017) Art. 11.

[13] J. Brasche, P. Exner, Y. Kuperin, P. Šeba: *Schrödinger operators with singular interactions.* J. Math. Anal. Appl. **184** (1994) 112–139.

[14] J. Brüning, V. Geyler, K. Pankrashkin: *Spectra of self-adjoint extensions and applications to solvable Schrödinger operators.* Rev. Math. Phys. **20** (2008) 1–70.

[15] C. Cacciapuoti, K. Pankrashkin, A. Posilicano: *Self-adjoint indefinite Laplacians.* To appear in J. Anal. Math. Preprint arXiv:1611.00696.

[16] R. Carlone, M. Malamud, A. Posilicano: *On the spectral theory of Gesztesy-Šeba realizations of 1-D Dirac operators with point interactions on a discrete set.* J. Differential Equations **254**(9) (2013) 3835–3902.

[17] V.A. Derkach, M.M. Malamud: *Generalized resolvents and the boundary value problems for Hermitian operators with gaps.* J. Funct. Anal. **95** (1991) 1–95.

[18] V.A. Derkach, M.M. Malamud: *The extension theory of Hermitian operators and the moment problem.* J. Math. Sciences **73** (1995) 141–242.

[19] V.A. Derkach, M.M. Malamud: *Extension Theory of Symmetric Operators and Boundary Value Problems.* Proceedings of Institute of Mathematics of NAS of Ukraine, Vol. 104, 2017.

[20] J. Dittrich, P. Exner, P. Šeba: *Dirac operators with a spherically symmetric $\delta$-shell interaction.* J. Math. Phys. **30**(12) (1989) 2875–2882.

[21] P. Exner: *Leaky quantum graphs: a review.* In: Analysis on graphs and its applications, Proc. Sympos. Pure Math. 77: 523–564. Amer. Math. Soc., Providence, RI, 2008.

[22] G. B. Folland: *Introduction to Partial Differential Equations*. Princeton University Press, Princeton, NJ, second edition, 1995.

[23] P. Freitas, P. Siegl: *Spectra of graphene nanoribbons with armchair and zigzag boundary conditions.* Rev. Math. Phys. **26** (2014) 1450018.

[24] F. Gesztesy, P. Šeba: *New analytically solvable models of relativistic point interactions*. Lett. Math. Phys. **13**(4) (1987) 345–358.

[25] M. Holzmann, T. Ourmières-Bonafos, K. Pankrashkin: *Dirac operators with Lorentz scalar shell interactions.* Rev. Math. Phys. **30** (2018) 1850013.

[26] W. McLean: *Strongly Elliptic Systems and Boundary Integral Equations*. Cambridge University Press, Cambridge, 2000.

[27] A. Moroianu, T. Ourmières-Bonafos, K. Pankrashkin: *Dirac operators on hypersurfaces as large mass limits.* Preprint arXiv:1811.03340.

[28] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, C. W. Clark: *NIST handbook of mathematical functions*. NIST & Cambridge University Press, 2010.

[29] T. Ourmières-Bonafos, F. Pizzichillo: *Dirac operators and shell interactions: a survey*. Preprint arXiv:1902.03901.

[30] T. Ourmières-Bonafos, L. Vega: *A strategy for self-adjointness of Dirac operators: application to the MIT bag model and $\delta$-shell interactions*. Publ. Mat. **62** (2018) 397–437.

[31] K. Pankrashkin, S. Richard: *One-dimensional Dirac operators with zero-range interactions: spectral, scattering, and topological results*. J. Math. Phys. **55** (2014) 062305.

[32] F. Pizzichillo, H. Van Den Bosch: *Self-Adjointness of two dimensional Dirac operators on corner domains*. Preprint arXiv:1902.05010.

[33] A. Posilicano: *Boundary triples and Weyl functions for singular perturbations of self-adjoint operators.* Meth. Funct. Anal. Topol. **10** (2004) 57–63.

[34] A. Posilicano: *Self-adjoint extensions of restrictions.* Oper. Matr. **2** (2008) 483–506.

[35] J. Saranen, G. Vainikko: *Periodic integral and pseudodifferential equations with numerical approximation.* Springer-Verlag, Berlin, 2002.

[36] K. M. Schmidt: *A remark on boundary value problems for the Dirac operator.* Quart. J. Math. Oxford Ser. (2) **46** (1995) 509–516.

[37] E. Stockmayer, S. Vugalter: *Infinite mass boundary conditions for Dirac operators.* J. Spect. Theory **9** (2019) 569–600.

[38] B. Thaller: *The Dirac equation*, Springer-Verlag, Berlin, 1992.

[39] C. Tretter: *Spectral theory of block operator matrices and applications.* Imperial College Press, 2008.