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REAL SPECTRA AND ℓ-SPECTRA OF ALGEBRAS AND VECTOR LATTICES OVER COUNTABLE FIELDS

FRIEDRICH WEHRUNG

Dedicated to the memory of Klaus Keimel

Abstract. In an earlier paper we established that every second countable, completely normal spectral space is homeomorphic to the ℓ-spectrum of some Abelian ℓ-group. We extend that result to ℓ-spectra of vector lattices over any *countable* totally ordered division ring $\Bbbk$. Combining those methods with Baro's Normal Triangulation Theorem, we obtain the following result:

Theorem. *For every countable formally real field $\Bbbk$, every second countable, completely normal spectral space is homeomorphic to the real spectrum of some commutative unital $\Bbbk$-algebra.*

The countability assumption on $\Bbbk$ is necessary: there exists a second countable, completely normal spectral space that cannot be embedded, as a spectral subspace, into either the ℓ-spectrum of any right vector lattice over an uncountable directed partially ordered division ring, or the real spectrum of any commutative unital algebra over an uncountable field.

1. Introduction

The same way the usual Zariski spectrum of a commutative unital ring A topologizes the collection of all surjective homomorphisms from A onto a domain, the *real spectrum* of A topologizes the collection of all surjective homomorphisms from A onto a *totally ordered* domain. However, while the class of all topological spaces that arise as Zariski spectra of commutative unital rings has been well understood since Hochster [16] (we get exactly the so-called *spectral spaces*), the corresponding problem for real spectra has been open for decades, and dates back at least to Problem 12 in Klaus Keimel's 1995 survey paper [19]. Observe that the *order structure* of real spectra (without the topology) got completely elucidated in Dickmann, Gluschankof, and Lucas [13].

Building on Delzell and Madden's construction of a completely normal spectral space which cannot arise as a real spectrum [10], Mellor and Tressl [24] proved that the class $\mathcal{R}$ of all Stone duals of real spectra (which is thus a class of bounded distributive lattices) is not closed under $\mathscr{L}_{\infty\lambda}$-elementary equivalence, for any infinite cardinal λ. As usual, $\mathscr{L}_{\infty\lambda}$ denotes the extension of first-order logic allowing conjunctions and disjunctions of arbitrary length and universal or existential quantifiers over strings of variables of length less than λ. There is not much room for improvement of Mellor and Tressl's solution beyond $\mathscr{L}_{\infty\lambda}$: an infinite distributive

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lattice D belongs to $\mathcal{R}$ iff it carries a ring structure R and an isomorphism from the Stone dual of the real spectrum of R onto D; thus $\mathcal{R}$ is the class of lattice reducts of the class of all models of some $\mathscr{L}_{\omega_1\omega}$ sentence in a larger first-order language, and thus it is a so-called *projective class* (cf. the argument concluding the Introduction in Wehrung [37]).

Mellor and Tressl's solution involves (necessarily) the construction of lattices of arbitrarily large cardinality. The main result of the present paper is thus a positive counterpart of Mellor and Tressl's negative solution (cf. Corollary 8.7):

Theorem A. *Let $\Bbbk$ be a countable formally real field. Every second countable completely normal spectral space arises as the real spectrum of some commutative unital $\Bbbk$-algebra.*

Our road to Theorem A involves an extension of the tools introduced, in the author's paper [35], for proving an analogous result for the corresponding problem on *Abelian ℓ-groups*. Those tools are a blend of lattice theory and semilinear geometry, and they were originally focused on finite lattices of polyhedral cones in real vector spaces. One of the main underlying lattice-theoretical concepts of [35] is the one of *consonance* (cf. Section 2.1). We will need to introduce here the lattice-theoretical concept of *CN-purity* (cf. Definition 7.8) and relate it to Baro's Normal Triangulation Theorem from [2] (cf. Proposition 7.9).

The main result of [35] is a precursor of Theorem A for *ℓ-spectra* of Abelian ℓ-groups. The concept of ℓ-spectrum can in turn be extended to (right) vector lattices over totally ordered division rings $\Bbbk$, by allowing the ℓ-ideals to be closed under the action of $\Bbbk$. A byproduct of our study is the following version of Theorem A for vector lattices, which also extends the main result of [35] to those structures (cf. Corollary 6.3).

Theorem B. *Let $\Bbbk$ be a countable totally ordered division ring. Every second countable completely normal spectral space arises as the ℓ-spectrum of some right $\Bbbk$-vector lattice.*

Analogues of Mellor and Tressl's result, for spectra of various classes of ℓ-groups (including all Abelian ones), also answering related questions from Iberkleid *et al.* [17], were recently obtained by the author in [36, 37].

We also prove that the countability assumption on $\Bbbk$, in both Theorems A and B, cannot be dispensed with (cf. Corollary 9.7):

Theorem C. *There exists a second countable, completely normal spectral space that cannot be embedded, as a spectral subspace, into either the ℓ-spectrum of any right vector lattice over an uncountable directed partially ordered division ring or the real spectrum of any commutative unital algebra over an uncountable field.*

Theorem C follows from a more general result, Theorem 9.3, that states that if a homomorphic preimage of a distributive lattice D can be represented *via* a vector lattice on a directed partially ordered division ring of cardinality greater than the one of D, then every indecomposable element of D is join-irreducible.

We also prove (cf. Corollary 10.7) that a completely normal generalized spectral space X can be represented *via* vector lattices over arbitrary totally ordered division rings iff the Stone dual of X is a generalized dual Heyting algebra. The real spectrum analogue of that result fails (cf. Example 10.9).

2. Basic concepts

2.1. **Lattices.** A standard reference on lattice theory is Grätzer [15]. A nonzero element p in a lattice L with zero is

(1) *indecomposable* if ($p = x \vee y$ and $x \wedge y = 0$) $\Rightarrow$ ($x = 0$ or $y = 0$),
(2) *join-irreducible* if $p = x \vee y \Rightarrow$ ($p = x$ or $p = y$),
(3) *join-prime* if $p \leq x \vee y \Rightarrow$ ($p \leq x$ or $p \leq y$),

for all $x, y \in L$. In particular, (3)$\Rightarrow$(2)$\Rightarrow$(1), and (2)$\Leftrightarrow$(3) if L is distributive. We will denote by $\operatorname{Ji} L$ the set of all join-irreducible elements in L. If L is finite distributive, then for every $p \in \operatorname{Ji} L$ there exists a largest $p^\dagger \in L$ such that $p \nleq p^\dagger$. Furthermore, $p \wedge p^\dagger = p_*$, the unique lower cover of p in L, and the assignment $p \mapsto p^\dagger$ is order-preserving.

We say that a distributive lattice D is a *generalized dual Heyting algebra* if for all $a, b \in D$ there exists a smallest $x \in D$, then denoted $a \smallsetminus_D b$, such that $a \leq b \vee x$.

We denote by $\rightarrow_D$ the Heyting implication in any Heyting algebra D. That is, for $a, b \in D$, $a \rightarrow_D b$ is the largest $x \in D$ such that $a \wedge x \leq b$.

Two elements a and b in a distributive lattice D with zero are *consonant* (cf. Wehrung [35, Definition 2.2]) if there are $x, y \in D$ such that $a \leq b \vee x$, $b \leq a \vee y$, and $x \wedge y = 0$. A lattice map $f\colon D \to E$ is consonant if $f(x)$ and $f(y)$ are consonant whenever $x, y \in D$. We say that D is *completely normal* if every pair of elements in D is consonant.

For join-semilattices A and B, a join-homomorphism $f\colon A \to B$ is *closed* (cf. Wehrung [35, Definition 2.4]) if for all $a_0, a_1 \in A$ and $b \in B$, if $f(a_0) \leq f(a_1) \vee b$, then there exists $x \in A$ such that $a_0 \leq a_1 \vee x$ and $f(x) \leq b$.

2.2. **Stone duality for distributive lattices.** A topological space X is *generalized spectral* if it is sober and the set $\mathring{\mathcal{K}}(X)$ of all compact open subsets of X is a basis of the topology of X, closed under binary intersection. If, in addition, X is compact, we say that X is a *spectral space*. A map, between generalized spectral spaces, is *spectral* if the inverse image of any compact open subset is compact open. Borrowing notation from Rump and Yang [27], let us denote by $\mathbf{DL}_0$ the category of all distributive 0-lattices with 0-lattice homomorphisms with cofinal[1] range, and by $\mathbf{GSp}$ the category of all generalized spectral spaces with spectral maps. *Stone duality*, originating in Stone [32], states an equivalence of categories between $\mathbf{DL}_0$ and the opposite category of $\mathbf{GSp}$. That duality sends every distributive lattice D with zero to its *spectrum*, which is the space $\operatorname{Spec} D$ of all prime ideals of D with the hull-kernel topology; in the other direction, it sends every generalized spectral space X to the lattice $\mathring{\mathcal{K}}(X)$. The unit and counit of the duality are given by

$$\varepsilon_D\colon D \xrightarrow{\cong} \mathring{\mathcal{K}}(\operatorname{Spec} D)\,,\ a \mapsto \{P \in \operatorname{Spec} D \mid a \notin P\}\,, \qquad \text{whenever } D \in \mathbf{DL}_0\,;$$

$$\eta_X\colon X \xrightarrow{\cong} \operatorname{Spec} \mathring{\mathcal{K}}(X)\,,\ x \mapsto \Big\{V \in \mathring{\mathcal{K}}(X) \mid x \notin V\Big\}\,, \qquad \text{whenever } X \in \mathbf{GSp}\,.$$

For more detail about Stone duality, see Grätzer [15, § II.5] (where the functorial aspects are not treated), Johnstone [18, § II.3] (where the more standard case of bounded distributive lattices, and spectral spaces, is handled), and Rump and Yang [27, page 63] for the general case (in that paper generalized spectral spaces are called *schematic spaces*).

[1] A subset X in a poset P is *cofinal* if every element of P lies below some element of X.

A generalized spectral space X is *completely normal* if for any points x and y in the closure of a singleton $\{z\}$, either x belongs to the closure of $\{y\}$ or y belongs to the closure of $\{x\}$. It follows from Monteiro [25, Théorème V.3.1] that X is a completely normal space iff $\overset{\circ}{\mathcal{K}}(X)$ is a completely normal lattice.

2.3. **Vector lattices, ℓ-spectra.** Denote by G^+ the positive cone of any partially ordered group G. A division ring $\Bbbk$, endowed with a translation-invariant partial ordering, is a *partially ordered division ring* if $\Bbbk^{++} \overset{\text{def}}{=} \Bbbk^+ \setminus \{0\}$ is a (nonempty) multiplicative subsemigroup of $\Bbbk$, closed under $\xi \mapsto \xi^{-1}$. It is only in Sections 9 and 10 that the partially ordered division rings in question will not necessarily be totally ordered.

Let $\Bbbk$ be a partially ordered division ring. A right vector space $\mathbb{V}$ over $\Bbbk$, endowed with a translation-invariant lattice order, is a *vector lattice* if $\mathbb{V}^+\Bbbk^+ \subseteq \mathbb{V}^+$. Then

$$x\lambda \vee y\lambda = (x \vee y)\lambda \text{ and } x\lambda \wedge y\lambda = (x \wedge y)\lambda\,, \quad \text{for all } x, y \in \mathbb{V} \text{ and } \lambda \in \Bbbk^+\,. \tag{2.1}$$

Define ℓ-$\Bbbk$-*homomorphisms* as the maps preserving both the ℓ-group structure and the vector space structure. As customary in module theory, the structure of right vector space of $\mathbb{V}$ over $\Bbbk$ will be emphasized by writing $\mathbb{V}_\Bbbk$ instead of just $\mathbb{V}$. A linear subspace (*not only an additive subgroup*) of $\mathbb{V}$ is an ℓ-*ideal* of $\mathbb{V}_\Bbbk$ if it is both order-convex and closed under $x \mapsto |x|$. The lattice $\operatorname{Id}^\ell \mathbb{V}_\Bbbk$ of all ℓ-ideals of $\mathbb{V}_\Bbbk$ is algebraic (i.e., it is complete and every element is a join of compact elements; see Grätzer [15]).

Now assume that $\Bbbk$ is *directed* (equivalently, $\Bbbk = \Bbbk^+ + (-\Bbbk^+)$). Then the ℓ-ideals of $\mathbb{V}_\Bbbk$ are exactly the ℓ-ideals of the underlying Abelian ℓ-group of $\mathbb{V}$ which are closed under multiplication by positive scalars. Further, the usual argument, involving (2.1), shows that the collection $\operatorname{Id}^\ell_{\mathrm{c}} \mathbb{V}_\Bbbk$ of all finitely generated (equivalently, principal) ℓ-ideals of $\mathbb{V}_\Bbbk$ is a 0-sublattice of $\operatorname{Id}^\ell \mathbb{V}_\Bbbk$. The elements of $\operatorname{Id}^\ell_{\mathrm{c}} \mathbb{V}_\Bbbk$ are those of the form $\langle x \rangle^\ell_\Bbbk \overset{\text{def}}{=} \{y \in \mathbb{V} \mid (\exists \lambda \in \Bbbk^+)(|y| \leq |x|\lambda)\}$ for $x \in \mathbb{V}$. The lattice $\operatorname{Id}^\ell_{\mathrm{c}} \mathbb{V}_\Bbbk$ is a homomorphic image of the lattice of all convex additive ℓ-subgroups of $\mathbb{V}$ (via $\langle x \rangle^\ell \mapsto \langle x \rangle^\ell_\Bbbk$), thus it is completely normal (cf. Madden [20, § I.2]).

The ℓ-*spectrum* of $\mathbb{V}_\Bbbk$, denoted by $\operatorname{Spec}_\ell \mathbb{V}_\Bbbk$, can be defined as the Stone dual of the lattice $\operatorname{Id}^\ell_{\mathrm{c}} \mathbb{V}_\Bbbk$. It is a completely normal generalized spectral space.

2.4. ***f*-rings, $\Bbbk$-algebras, Brumfiel spectrum, and real spectrum.** Here we shall only set up some basic facts and notation over the structures in question, referring the reader to Delzell and Madden [11], Johnstone [18, Chapter 5], Keimel [19], Coste and Roy [9], Dickmann [12, Chapter 6], or Wehrung [34, § 4] for more information.

Recall (cf. Bigard *et al.* [4]) that an f-*ring* is a (not necessarily unital) ring A endowed with a translation-invariant lattice order for which $A^+ \cdot A^+ \subseteq A^+$ and $x \wedge y = 0$ implies that $x \wedge yz = x \wedge zy = 0$ whenever $x, y \in A$ and $z \in A^+$. A subset I of A is an ℓ-*ideal* if it is simultaneously an ℓ-ideal of the underlying ℓ-group of A and a two-sided ideal of the underlying ring of A. The lattice $\operatorname{Id}^{\mathrm{r}} A$ of all radical[2] ℓ-ideals of a commutative f-ring A is algebraic, and the collection $\operatorname{Id}^{\mathrm{r}}_{\mathrm{c}} A$ of all its finitely generated (equivalently, principal) members is a completely normal

[2]An ideal I of a commutative ring A is *radical* if $x^n \in I$ implies that $x \in I$ whenever $x \in A$ and n is a positive integer.

sublattice. The elements of that lattice are those of the form

$$\langle x\rangle^{\mathrm{r}} \stackrel{\mathrm{def}}{=} \big\{y \in A \mid |y|^n \leq |x|z \text{ for some } z \in A^+ \text{ and } n \text{ a positive integer}\big\} \quad \text{for } x \in A\,.$$

The Stone dual of $\mathrm{Id}^{\mathrm{r}}_{\mathrm{c}}\, A$ is the *Brumfiel spectrum* $\mathrm{Spec}_{\mathrm{B}}\, A$ of A.

A ring A, endowed with an additional structure of right vector space over a field $\Bbbk$, is a $\Bbbk$-*algebra* if $(xy)\cdot\lambda = (x\cdot\lambda)y = x(y\cdot\lambda)$ whenever $x, y \in A$ and $\lambda \in \Bbbk$. If A is unital then this amounts to a unital ring homomorphism from $\Bbbk$ to A with central range (viz. $\lambda \mapsto 1_A \cdot \lambda$). If, in addition, $\Bbbk$ is a partially ordered field and A is an f-ring with $A^+ \cdot \Bbbk^+ \subseteq A^+$, we say that A is an f-$\Bbbk$-*algebra* (cf. Delzell and Madden [11, § 6] for the unital case with $\Bbbk$ totally ordered).

Note 2.1. For a commutative f-$\Bbbk$-algebra A, the meaning of $\langle x\rangle^{\mathrm{r}}$, for $x \in A$, is unambiguous with respect to the action of $\Bbbk$, for whenever $x \in A$ and $\lambda \in \Bbbk$, $(x\cdot\lambda)^2 = (x\cdot\lambda^2)x \in \langle x\rangle^{\mathrm{r}}$ whence $x\cdot\lambda \in \langle x\rangle^{\mathrm{r}}$.

Let A be a commutative unital ring. A subset of A is a *cone* if it is both an additive and a multiplicative submonoid of A, containing all squares in A. A cone P of A is *prime* if $A = P \cup (-P)$ and $P \cap (-P)$ is a prime ideal of A. The *real spectrum of* A is the collection $\mathrm{Spec}_{\mathrm{r}}\, A$ of all prime cones of A, endowed with the topology generated by all subsets of the form $\{P \in \mathrm{Spec}_{\mathrm{r}}\, A \mid a \notin P\}$ for $a \in A$.

2.5. **Semi-algebraic sets and maps.** For more background, we refer the reader to Bochnak *et al* [5], van den Dries [33]. A field $\Bbbk$ is

- *formally real* if -1 is not a sum of squares in $\Bbbk$;
- *real-closed* if it is formally real, every polynomial of odd degree with coefficients in $\Bbbk$ has a root in $\Bbbk$, and every element of $\Bbbk$ is either x^2 or $-x^2$ for some $x \in \Bbbk$.

For a real-closed field $\Bbbk$ and a nonnegative integer d, a subset of $\Bbbk^d$ is *semi-algebraic* if it is a Boolean combination of subsets of the form $\big\{x \in \Bbbk^d \mid f(x) \geq 0\big\}$ for polynomials $f \in \Bbbk[x_1, \ldots, x_d]$. By the Tarski-Seidenberg quantifier elimination theorem, a subset of $\Bbbk^d$ is semi-algebraic iff it is first-order definable (with parameters) in the language of ring theory.

For semi-algebraic subsets $A \subseteq \Bbbk^m$ and $B \subseteq \Bbbk^n$, a map $f\colon A \to B$ is semi-algebraic if its graph is a semi-algebraic subset of $\Bbbk^{m+n}$.

2.6. **A final bit of notation.** The operators of closure and interior, relative to a subspace Ω in a given topological space, will be denoted by cl_Ω and int_Ω, respectively. If Ω is the ambient space we will omit the subscript Ω in that notation.

For any functions f, g, any subset Ω of the intersection of the domains of f and g, and any binary relation $\lhd$, we will set $[\![f \lhd g]\!]_\Omega \stackrel{\mathrm{def}}{=} \{x \in \Omega \mid f(x) \lhd g(x)\}$. Throughout the paper "countable" will mean "at most countable", and ω will denote the set of all nonnegative integers, also identified with the first infinite ordinal. We will denote by $\operatorname{card} X$ the cardinality of any set X.

3. Semilinear triangulation over totally ordered fields

Results from this section will be required in Sections 7 and 8. Our main sources will be Coppel [7], Rourke and Sanderson [26], Schrijver [28], van den Dries [33].

First some standard terminology. Let $\mathbb{V}$ be a vector space over a totally ordered field $\Bbbk$. A map $f\colon \mathbb{V} \to \Bbbk$ is an *affine functional* if $f - f(0)$ is a linear functional. A closed (resp., open) affine half-space of $\mathbb{V}$ is a subset of $\mathbb{V}$ of the form

$\{x \in \mathbb{V} \mid f(x) \geq 0\}$ (resp., $\{x \in \mathbb{V} \mid f(x) > 0\}$) for some nonconstant affine functional f on $\mathbb{V}$. A subset C of $\mathbb{V}$ is

- *semilinear* if it is a Boolean combination of half-spaces,
- *a polytope* if if it is the convex hull of a finite subset X of $\mathbb{V}$.

Proposition 3.1. *Let $\mathbb{V}$ be finite-dimensional. Then the polytopes of $\mathbb{V}$ are exactly the bounded intersections of finite collections of closed affine half-spaces of $\mathbb{V}$.*

Proposition 3.1 is usually stated in the literature over real vector spaces (see, e.g., Ziegler [38, Theorem 1.1]). Abstracting away some properties of the convex hull operator in $\mathbb{V}$ yields *linear geometries*, for which a generalization of Proposition 3.1, contained in Propositions IV.19 and V.60 in Coppel [7], is valid; that result implies that Proposition 3.1 extends to vector spaces over totally ordered division rings. On the other hand, the proof provided in Schrijver [28, Corollary 7.1(c)] carries over almost word for word to the latter context.

For affinely independent elements $a_0, \dots, a_n \in \mathbb{V}$, we define

- the *simplex* $[a_0, \dots, a_n] \overset{\text{def}}{=} \{\sum_{i=0}^n a_i\lambda_i \mid \text{all } \lambda_i \geq 0\,,\ \sum_i \lambda_i = 1\}$,
- the *open simplex* $\langle a_0, \dots, a_n\rangle \overset{\text{def}}{=} \{\sum_{i=0}^n a_i\lambda_i \mid \text{all } \lambda_i > 0\,,\ \sum_i \lambda_i = 1\}$, which is the *relative interior* (not to be confused with the topological interior) of $[a_0, \dots, a_n]$.

A *face* of a simplex (resp., open simplex) as above is the simplex (resp., open simplex) spanned by a nonempty subset of $\{a_0, \dots, a_n\}$. A set $\mathbb{K}$ of simplexes is a *simplicial complex* if any face of a member of $\mathbb{K}$ belongs to $\mathbb{K}$ and the nonempty intersection of any two members of $\mathbb{K}$ belongs to $\mathbb{K}$. The *geometric realization* of $\mathbb{K}$ is its union $|\mathbb{K}| \overset{\text{def}}{=} \bigcup \mathbb{K} = \bigcup\{S \mid S \in \mathbb{K}\}$. The relative interiors of the simplexes in $\mathbb{K}$ form a partition of $|\mathbb{K}|$; the unions of elements of that partition are the sets *partitioned by* $\mathbb{K}$. A simplicial complex $\mathbb{L}$ is a *subdivision of* $\mathbb{K}$ if every open simplex of $\mathbb{K}$ is partitioned by $\mathbb{L}$ and every open simplex of $\mathbb{L}$ is contained in an open simplex of $\mathbb{K}$; of course in that case $|\mathbb{K}| = |\mathbb{L}|$.

Proposition 3.2. *For all semilinear subsets $S, S_1, \dots, S_k$ of $\mathbb{V}$, with S closed bounded and $S_1, \dots, S_k \subseteq S$, there exists a simplicial complex $\mathbb{K}$ such that $S = |\mathbb{K}|$ and each S_i is partitioned by $\mathbb{K}$.*

Proof. By the result of Exercise (2.14)2, Chapter 8 in van den Dries [33] (stated there over any totally ordered field), there exists a "complex" $\mathbb{K}_0$, with union S, such that every S_i is a union of members of $\mathbb{K}_0$. By "complex" it is meant there that $\mathbb{K}_0$ is a finite set of open simplexes and for all $C_1, C_2 \in \mathbb{K}_0$, if $\operatorname{cl} C_1 \cap \operatorname{cl} C_2 \neq \varnothing$, then $\operatorname{cl} C_1 \cap \operatorname{cl} C_2 = \operatorname{cl} C$ for a common face C (not necessarily in $\mathbb{K}_0$) of C_1 and C_2. The "closure" $\mathbb{K}_1$ of $\mathbb{K}_0$, defined on page 121 of van den Dries [33] as the set of all faces of all open simplexes in $\mathbb{K}_0$, is also a complex, closed in the sense that every face of every member of $\mathbb{K}_1$ belongs to $\mathbb{K}_1$. It follows that $\mathbb{K} \overset{\text{def}}{=} \{\operatorname{cl} C \mid C \in \mathbb{K}_1\}$ is a simplicial complex. Since S is closed, $|\mathbb{K}| = |\mathbb{K}_1| = S$. Since every S_i is a union of members of $\mathbb{K}_0$ and every member of $\mathbb{K}_0$ is the relative interior or a member of $\mathbb{K}$ (namely its closure), every S_i is partitioned by $\mathbb{K}$. □

Alternatively, a proof of Proposition 3.2 can also be obtained by adapting the one of Rourke and Sanderson [26, Addendum 2.12] from the reals to arbitrary totally ordered division rings (once again commutativity is not needed). Numerous

changes are required in both definitions and arguments. The details, although not hard, are quite tedious.

4. Lattices of relatively open semilinear sets

Throughout this section we shall fix a totally ordered division ring $\Bbbk$ endowed with its interval topology, together with a topological right $\Bbbk$-vector space $\mathbb{V}$. For any set $\mathcal{F}$ of affine functionals on $\mathbb{V}$ and any subset Ω of $\mathbb{V}$, we define

- $\Sigma_{\mathcal{F},\Omega} \stackrel{\text{def}}{=} \{[\![f > 0]\!]_\Omega \mid f \in \mathcal{F}\} \cup \{[\![f < 0]\!]_\Omega \mid f \in \mathcal{F}\}$,
- $\overline{\Sigma}_{\mathcal{F},\Omega} \stackrel{\text{def}}{=} \{[\![f \geq 0]\!]_\Omega \mid f \in \mathcal{F}\} \cup \{[\![f \leq 0]\!]_\Omega \mid f \in \mathcal{F}\}$,
- $\mathrm{Bool}(\mathcal{F},\Omega)$, the Boolean subalgebra of the powerset of Ω generated by $\Sigma_{\mathcal{F},\Omega}$ (equivalenly, by $\overline{\Sigma}_{\mathcal{F},\Omega}$),
- $\mathrm{Clos}(\mathcal{F},\Omega)$ (resp., $\mathrm{Op}(\mathcal{F},\Omega)$), the $0,1$-sublattice of $\mathrm{Bool}(\mathcal{F},\Omega)$ consisting of all closed (resp., open) members of $\mathrm{Bool}(\mathcal{F},\Omega)$.

The notation $\mathrm{Bool}(\mathcal{F},\Omega)$, $\mathrm{Clos}(\mathcal{F},\Omega)$, $\mathrm{Op}(\mathcal{F},\Omega)$ thus differs from the one of the author's paper [35] in two aspects: $\mathcal{F}$ is now a set of affine functionals (as opposed to a set of hyperplanes), and the second parameter Ω stands for the ambient space. Aside from those differences, the corresponding material will be very similar to the one introduced in [35].

A subset of Ω is

- *basic*, with respect to $(\mathcal{F},\Omega)$, if it is the intersection of a finite collection of sets of the form $[\![f > 0]\!]_\Omega$, $[\![f < 0]\!]_\Omega$, or $[\![f = 0]\!]_\Omega$,
- *basic open*, with respect to $(\mathcal{F},\Omega)$, if it is the intersection of a finite collection of sets of the form $[\![f > 0]\!]_\Omega$ or $[\![f < 0]\!]_\Omega$.

In particular, Ω is basic open. In case the subset Ω is understood from the context, we will often use the shorthand

$$f^0 = \overline{f}^0 \stackrel{\text{def}}{=} [\![f = 0]\!]_\Omega\,; \tag{4.1}$$

$$f^+ \stackrel{\text{def}}{=} [\![f > 0]\!]_\Omega\,;\ f^- \stackrel{\text{def}}{=} [\![f < 0]\!]_\Omega\,;\ \overline{f}^+ \stackrel{\text{def}}{=} [\![f \geq 0]\!]_\Omega\,;\ \overline{f}^- \stackrel{\text{def}}{=} [\![f \leq 0]\!]_\Omega \tag{4.2}$$

for every $f\colon \mathbb{V} \to \Bbbk$.

The following lemma originates, in a more restrictive context, in Wehrung [35, Lemma 5.4]. We present an alternative, streamlined proof.

Lemma 4.1. *Let Ω be a convex subset of $\mathbb{V}$ and let $\mathcal{F}$ be a set of continuous affine functionals on $\mathbb{V}$. Then $\mathrm{Bool}(\mathcal{F},\Omega)$ is closed under both topological interior and closure operators relative to Ω, and $\mathrm{Op}(\mathcal{F},\Omega)$ is a Heyting subalgebra of the Heyting algebra of all relative open subsets of Ω. Furthermore, the 0-sublattice $\mathrm{Op}^-(\mathcal{F},\Omega)$ of $\mathrm{Op}(\mathcal{F},\Omega)$ generated by $\Sigma_{\mathcal{F},\Omega}$ satisfies $\mathrm{Op}(\mathcal{F},\Omega) = \mathrm{Op}^-(\mathcal{F},\Omega) \cup \{\Omega\}$.*

Proof. The first part of Lemma 4.1 boils down to establishing that whenever A is nonempty basic in $\mathrm{Bool}(\mathcal{F},\Omega)$, its closure $\mathrm{cl}_\Omega(A)$ belongs to $\mathrm{Bool}(\mathcal{F},\Omega)$ (for then, $\mathrm{Bool}(\mathcal{F},\Omega)$ is closed under topological closure). There are a finite subset $\mathcal{F}_0$ of $\mathcal{F}$ and an assignment $\varepsilon\colon \mathcal{F}_0 \to \{0,+,-\}$ such that (applying Notation (4.1), (4.2)) $A = \bigcap_{f\in\mathcal{F}_0} f^{\varepsilon(f)}$. The closure of A relative to Ω is contained in $B \stackrel{\text{def}}{=} \bigcap_{f\in\mathcal{F}_0} \overline{f}^{\varepsilon(f)}$. Pick $a \in A$. Since every member of $\mathcal{F}_0$ is an affine functional and since Ω is convex, for every $b \in B$ the half-open segment $[a,b)$ is contained in A; thus $b \in \mathrm{cl}_\Omega(A)$. This completes the proof that $B = \mathrm{cl}_\Omega(A)$.

For the last part of Lemma 4.1, it suffices to observe that every proper basic open subset of Ω belongs, by definition, to $\mathrm{Op}^-(\mathcal{F},\Omega)$. $\square$

Notation 4.2. For every $U \in \mathrm{Op}(\mathcal{F},\Omega)$, we set $\mathcal{F}_U \overset{\mathrm{def}}{=} \{f \in \mathcal{F} \mid U \cap [\![f=0]\!]_\Omega \neq \varnothing\}$. The set $\nabla U \overset{\mathrm{def}}{=} \bigcap([\![f=0]\!]_\Omega \mid f \in \mathcal{F}_U)$ is a closed affine subspace of $\mathbb{V}$.

The following result is an analogue of Wehrung [35, Lemma 6.4]. Since [35, Lemma 5.1] fails to relativize to arbitrary convex subsets of $\mathbb{V}$, we need to modify the original proof of [35, Lemma 6.4], taking the opportunity to slightly amplify its original statement.

Lemma 4.3. *Let Ω be a convex subset of $\mathbb{V}$ and let $\mathcal{F}$ be a finite set of continuous affine functionals on $\mathbb{V}$. Then a member P of $\mathrm{Op}(\mathcal{F},\Omega)$ is join-irreducible in $\mathrm{Op}(\mathcal{F},\Omega)$ iff it is basic open and $P \cap \nabla P \neq \varnothing$. Moreover, for each such P,*

(1) *the lower cover P_* of P in $\mathrm{Op}(\mathcal{F},\Omega)$ is equal to $P \setminus \nabla P$;*
(2) *there exists a unique $\varepsilon\colon \mathcal{F} \setminus \mathcal{F}_P \to \{+,-\}$ (the "sign function" of P) such that (cf. (4.2)) $P = \bigcap_{f\in\mathcal{F}\setminus\mathcal{F}_P} f^{\varepsilon(f)}$, and then $\mathrm{cl}_\Omega(P) = \bigcap_{f\in\mathcal{F}\setminus\mathcal{F}_P} \overline{f}^{\varepsilon(f)}$;*
(3) *the largest member $P^\dagger$ of $\mathrm{Op}(\mathcal{F},\Omega)$ not containing P is the complement in Ω of the set*

$$\mathrm{cl}_\Omega(P \cap \nabla P) = \mathrm{cl}_\Omega(P) \cap \nabla P = \bigcap_{f\in\mathcal{F}\setminus\mathcal{F}_P} \overline{f}^{\varepsilon(f)} \cap \nabla P\,. \tag{4.3}$$

Proof. We begin with a claim.

Claim. *If P is join-irreducible, then it is basic open and $P \cap \nabla P \neq \varnothing$.*

Proof of Claim. Since every element of $\mathrm{Op}(\mathcal{F},\Omega)$ is a finite union of basic open sets, every join-irreducible element of $\mathrm{Op}(\mathcal{F},\Omega)$ is basic open, thus convex. Now suppose that $P \cap \nabla P = \varnothing$, that is, $P \subseteq \bigcup_{f\in\mathcal{F}_P} [\![f \neq 0]\!]_\Omega$. Since P is join-irreducible in the distributive lattice $\mathrm{Op}(\mathcal{F},\Omega)$, $P \subseteq [\![f \neq 0]\!]_\Omega$ for some $f \in \mathcal{F}_P$, in contradiction with the definition of $\mathcal{F}_P$. $\square$ Claim.

Suppose, conversely, that P is basic open and $P \cap \nabla P \neq \varnothing$. For each $f \in \mathcal{F} \setminus \mathcal{F}_P$, P is contained in $[\![f \neq 0]\!]_\Omega = f^+ \cup f^-$ (cf. (4.2)), thus, since P is nonempty and convex, $P \subseteq f^{\varepsilon(f)}$ for a unique $\varepsilon(f) \in \{+,-\}$; whence $P \subseteq \bigcap_{f\in\mathcal{F}\setminus\mathcal{F}_P} f^{\varepsilon(f)}$. Since P is basic open, there are $\mathcal{X} \subseteq \mathcal{F}$ and $\eta\colon \mathcal{X} \to \{+,-\}$ such that $P = \bigcap_{f\in\mathcal{X}} f^{\eta(f)}$. For each $f \in \mathcal{X}$, P is contained in $f^{\eta(f)}$ thus it is disjoint from f^0, so $f \in \mathcal{F} \setminus \mathcal{F}_P$ and $\varepsilon(f) = \eta(f)$; whence $\mathcal{X} \subseteq \mathcal{F} \setminus \mathcal{F}_P$, and so $P \supseteq \bigcap_{f\in\mathcal{F}\setminus\mathcal{F}_P} f^{\varepsilon(f)}$. Therefore

$$P = \bigcap_{f\in\mathcal{F}\setminus\mathcal{F}_P} f^{\varepsilon(f)}\,. \tag{4.4}$$

Now the relative closure of P is obviously contained in $\overline{P} \overset{\mathrm{def}}{=} \bigcap_{f\in\mathcal{F}\setminus\mathcal{F}_P} \overline{f}^{\varepsilon(f)}$. Conversely, picking $p \in P$, any $x \in \overline{P}$ satisfies that the half-open segment $[p,x)$ is contained in P (this is again because Ω is convex and every member of $\mathcal{F}$ is an affine functional); thus $x \in \mathrm{cl}_\Omega(P)$. This completes the proof that $\overline{P} = \mathrm{cl}_\Omega(P)$, and thus of (2) above.

In order to prove that P is join-irreducible and that (1) holds, it suffices to prove that every join-irreducible element Q of $\mathrm{Op}(\mathcal{F},\Omega)$ properly contained in P is contained in $P \setminus \nabla P$. Applying (2), together with the Claim above, to Q, we get

a map $\eta\colon \mathcal{F}\setminus\mathcal{F}_Q \to \{+,-\}$ such that $Q = \bigcap_{f\in\mathcal{F}\setminus\mathcal{F}_Q} f^{\eta(f)}$. From $Q \subsetneqq P$ and (4.4) it thus follows that $\mathcal{F}_Q \subsetneqq \mathcal{F}_P$ and η extends ε. Pick $f \in \mathcal{F}_P \setminus \mathcal{F}_Q$. Then $Q \subseteq f^{\eta(f)}$ (because $f \notin \mathcal{F}_Q$) and $\nabla P \subseteq f^0$ (because $f \in \mathcal{F}_P$); whence $Q \cap \nabla P = \varnothing$, as required. The join-irreducibility of P, together with (1), follows.

Since the second equation in (4.3) follows from (2), establishing (4.3) reduces to proving the containment $\bigcap_{f\in\mathcal{F}\setminus\mathcal{F}_P} \overline{f}^{\varepsilon(f)} \cap \nabla P \subseteq \mathrm{cl}_\Omega(P\cap\nabla P)$. Letting x be any element of the left hand side of that containment, and picking $p \in P\cap\nabla P$, the half-open segment $[p,x)$ is contained in $P\cap\nabla P$; whence $x \in \mathrm{cl}_\Omega(P\cap\nabla P)$, as required. This completes the proof of (4.3). The statement that $P^\dagger$ is the complement in Ω of $\mathrm{cl}_\Omega(P\cap\nabla P)$ is then proved the same way as at the end of the proof of [35, Lemma 6.4]. □

The proof of the following easy corollary is, *mutatis mutandis*, identical to the one of Wehrung [35, Proposition 6.5], so we omit it.

Corollary 4.4. *Let Ω be a convex subset of $\mathbb{V}$, let $\mathcal{F}$ be a finite set of continuous affine functionals on $\mathbb{V}$, and let $P, Q \in \mathrm{Ji}\,\mathrm{Op}(\mathcal{F},\Omega)$. If $P \subsetneqq Q$, then $\nabla Q \subsetneqq \nabla P$.*

Although the following observation will not be needed in the remainder of the paper, it might find further uses so we record it here.

Proposition 4.5. *Let $\mathcal{F}$ be a finite set of continuous affine functionals on $\mathbb{V}$ and let P, Q be join-irreducible members of $\mathrm{Op}(\mathcal{F},\Omega)$. If $P\cap Q \neq \varnothing$, then $P\cap Q$ is join-irreducible in $\mathrm{Op}(\mathcal{F},\Omega)$ and $\mathcal{F}_{P\cap Q} = \mathcal{F}_P \cap \mathcal{F}_Q$. In particular, $\mathrm{Ji}\,\mathrm{Op}(\mathcal{F},\Omega)\cup\{\varnothing\}$ is a meet-semilattice under intersection.*

Proof. Pick $x \in P\cap\nabla P$ and $y \in Q\cap\nabla Q$, and set $z \overset{\mathrm{def}}{=} \frac{1}{2}(x+y)$. For each $f\in\mathcal{F}$, we shall determine the position of z with respect to $[\![f=0]\!]$. Denote by ε and η the sign functions of P and of Q, respectively. We separate cases.

- **Case 1.** $f \in \complement\mathcal{F}_P \cap \complement\mathcal{F}_Q$. Then $P \subseteq f^{\varepsilon(f)}$ and $Q \subseteq f^{\eta(f)}$. Since $P\cap Q \neq \varnothing$ it follows that $\varepsilon(f) = \eta(f)$, whence $x \in P \subseteq f^{\varepsilon(f)}$ and $y \in Q \subseteq f^{\varepsilon(f)}$, so $z \in f^{\varepsilon(f)} = f^{\eta(f)}$ as well.
- **Case 2.** $f \in \complement\mathcal{F}_P \cap \mathcal{F}_Q$. Then $\nabla Q \subseteq f^0$ (thus $f(y)=0$) and $P \subseteq f^{\varepsilon(f)}$ (thus $\varepsilon(f)f(x) > 0$). It follows that $\varepsilon(f)f(z) > 0$, that is, $z \in f^{\varepsilon(f)}$.
- **Case 3.** $f \in \mathcal{F}_P \cap \complement\mathcal{F}_Q$. In a similar fashion as in Case 2, we get $z \in f^{\eta(f)}$.
- **Case 4.** $f \in \mathcal{F}_P \cap \mathcal{F}_Q$. Then ∇P and ∇Q are both contained in f^0; whence $z \in f^0$.

Using Lemma 4.3, we get from Cases 1–4 above that

$$z \in P\cap Q \cap \bigcap \left(f^0 \mid f \in \mathcal{F}_P \cap \mathcal{F}_Q\right) . \tag{4.5}$$

In particular, $\mathcal{F}_P \cap \mathcal{F}_Q$ is contained in $\mathcal{F}_{P\cap Q}$. The converse containment being trivial, we get $\mathcal{F}_{P\cap Q} = \mathcal{F}_P \cap \mathcal{F}_Q$. By Lemma 4.3 together with (4.5), it follows that $P\cap Q$ is join-irreducible. □

The following Lemma 4.7 extends the author's [35, Lemma 6.6] from the linear case to the relativization of the affine case to a convex set. Its proof involves a lattice-theoretical homomorphism extension result established in [35, Lemma 4.2]. Let us first state a remark about the latter result.

Remark 4.6. In the original statement of [35, Lemma 4.2], the author had overlooked the fact that in the statement of that lemma, Condition (5) follows from

Conditions (2) and (3); thus it is redundant. Let us verify this, thus assuming that D is a Heyting subalgebra of a finite distributive lattice E (endowed with its unique structure of Heyting algebra) and $a, b \in E$ with $a \wedge b = 0$. We need to verify that for all join-irreducible $p, q \in D$, if $p \leq p_* \vee a$ and $q \leq q_* \vee b$, then p and q are incomparable. Suppose, to the contrary, that p and q are comparable, say $p \leq q$. We get $p \wedge b \leq (p_* \vee a) \wedge b = (p_* \wedge b) \vee (a \wedge b) \leq p_*$, thus, using the assumption that D is a Heyting subalgebra of E, we get $b \leq (p \to_E p_*) = (p \to_D p_*) = p^\dagger$. Now $p \leq q$ implies $p^\dagger \leq q^\dagger$, so $b \leq q^\dagger$. But then, $q \leq q_* \vee b \leq q^\dagger$, a contradiction.

Lemma 4.7. *Let Ω be a convex subset of $\mathbb{V}$, let $\mathcal{F} \cup \{g\}$ be a finite set of continuous affine functionals on $\mathbb{V}$, and let L be a generalized dual Heyting algebra. Every consonant 0-lattice homomorphism $\varphi\colon \operatorname{Op}(\mathcal{F}, \Omega) \to L$ extends to a unique lattice homomorphism $\psi\colon \operatorname{Op}(\mathcal{F} \cup \{g\}, \Omega) \to L$ such that for every $G \in \{[\![g > 0]\!]_\Omega, [\![g < 0]\!]_\Omega\}$,*

$$\psi(G) = \varphi_*(G) \overset{\text{def}}{=} \bigvee \left(\varphi(P) \smallsetminus_L \varphi(P_*) \mid P \in \operatorname{Ji} \operatorname{Op}(\mathcal{F}, \Omega),\ P \subseteq P_* \cup G\right) .$$

Proof. We verify that Conditions (1)–(4) of the statement of [35, Lemma 4.2] are satisfied (by Remark 4.6 above, Condition (5) is redundant). Conditions (1) and (2) follow from Lemma 4.1, whereas Condition (3) (i.e., $[\![g > 0]\!]_\Omega \cap [\![g < 0]\!]_\Omega = \varnothing$) is trivial. Since $P \cap \nabla P$ is convex, Condition (4) can be established the same way as in the proof of [35, Lemma 6.6]. Now apply [35, Lemma 4.2]. $\square$

Our next lemma extends the author's [35, Lemma 7.1] from the linear case to the relativization of the affine case to a convex set.

Lemma 4.8. *Let Ω be a convex subset of $\mathbb{V}$, let $\mathcal{F}$ be a finite set of continuous affine functionals on $\mathbb{V}$, and let $a, b \in \mathcal{F}$. We set $\mathcal{F}_\lambda \overset{\text{def}}{=} \mathcal{F} \cup \{a - \lambda b\}$ for evey $\lambda \in \Bbbk$. Then for all large enough positive $\lambda \in \Bbbk$, the following statement holds: for every generalized dual Heyting algebra L, every consonant 0-lattice homomorphism $\varphi\colon \operatorname{Op}(\mathcal{F}, \Omega) \to L$ can be extended to a lattice homomorphism $\psi\colon \operatorname{Op}(\mathcal{F}_\lambda, \Omega) \to L$ such that*

$$\psi\big([\![a > 0]\!]_\Omega \smallsetminus_{\operatorname{Op}(\mathcal{F}_\lambda, \Omega)} [\![b > 0]\!]_\Omega\big) = \varphi([\![a > 0]\!]_\Omega) \smallsetminus_L \varphi([\![b > 0]\!]_\Omega) . \tag{4.6}$$

Proof. The proof goes along the same lines as the one of [35, Lemma 7.1], with a trivialization of the proof of the corresponding Claim, as follows.

Let us state how large λ should be. Set $\mathcal{M} \overset{\text{def}}{=} \{U \in \operatorname{Op}(\mathcal{F}, \Omega) \mid [\![b > 0]\!]_\Omega \not\subseteq U\}$ and pick $x_U \in [\![b > 0]\!]_\Omega \setminus U$ for every $U \in \mathcal{M}$. Let λ be a strict upper bound in $\Bbbk$ of $\{0\} \cup \big\{a(x_U)b(x_U)^{-1} \mid U \in \mathcal{M}\big\}$. We claim that

$$[\![a < \lambda b]\!]_\Omega \subseteq U \ \Rightarrow\ [\![b > 0]\!]_\Omega \subseteq U\,, \qquad \text{whenever } U \in \operatorname{Op}(\mathcal{F}, \Omega)\,.$$

Indeed, suppose otherwise. Then $U \in \mathcal{M}$ by definition, thus $\lambda > a(x_U)b(x_U)^{-1}$ by assumption on λ, and thus (as $b(x_U) > 0$) $a(x_U) < \lambda b(x_U)$, and therefore $x_U \in [\![a < \lambda b]\!]_\Omega \setminus U$, a contradiction. This completes the proof of our claim.

The remainder of the proof goes exactly like the part of the proof of [35, Lemma 7.1] following its claim, with m replaced by λ and [35, Lemma 6.6] by its extension given in Lemma 4.7. Due to the crucial character of Lemma 4.8, we include a proof for convenience.

Let L be a generalized dual Heyting algebra and let $\varphi\colon \operatorname{Op}(\mathcal{F}, \Omega) \to L$ be a consonant 0-lattice homomorphism. We consider the extension ψ of φ, to a homomorphism from $\operatorname{Op}(\mathcal{F}_\lambda, \Omega)$ to L, given by Lemma 4.7, with $a - \lambda b$ in place of g. In

particular,

$$\psi([\![a > \lambda b]\!]_\Omega) = \bigvee \left(\varphi(P) \smallsetminus_L \varphi(P_*) \mid P \in \mathrm{Ji}\,\mathrm{Op}(\mathcal{F},\Omega)\,,\ P \subseteq P_* \cup [\![a > \lambda b]\!]_\Omega\right)\,. \tag{4.7}$$

We claim that the following inequality holds:

$$\varphi([\![a > 0]\!]_\Omega) \wedge \psi([\![a > \lambda b]\!]_\Omega) \leq \varphi([\![a > 0]\!]_\Omega) \smallsetminus_L \varphi([\![b > 0]\!]_\Omega)\,. \tag{4.8}$$

Since L is distributive, this amounts to proving the following statement:

$$\varphi([\![a > 0]\!]_\Omega) \wedge \big(\varphi(P) \smallsetminus_L \varphi(P_*)\big) \leq \varphi([\![a > 0]\!]_\Omega) \smallsetminus_L \varphi([\![b > 0]\!]_\Omega)\,,$$
$$\text{for every } P \in \mathrm{Ji}\,\mathrm{Op}(\mathcal{F},\Omega) \text{ such that } P \subseteq P_* \cup [\![a > \lambda b]\!]_\Omega\,. \tag{4.9}$$

Let P as in the premise of (4.9); so $P \cap \nabla P \subseteq [\![a > \lambda b]\!]_\Omega$. By Lemma 4.3, we get

$$\mathrm{cl}_\Omega(P) \cap \nabla P = \mathrm{cl}_\Omega(P \cap \nabla P) \subseteq [\![a \geq \lambda b]\!]_\Omega\,,$$

that is, $[\![a < \lambda b]\!]_\Omega \subseteq P^\dagger$. From the assumption on λ we thus get $[\![b > 0]\!]_\Omega \subseteq P^\dagger$, that is, $P \not\subseteq [\![b > 0]\!]_\Omega$. Since $[\![b > 0]\!]_\Omega \in \mathrm{Op}(\mathcal{F},\Omega)$, it follows that $P \cap [\![b > 0]\!]_\Omega \subseteq P_*$.

Now suppose that $P \subseteq [\![a > 0]\!]_\Omega$. Since $P \cap [\![b > 0]\!]_\Omega \subseteq P_*$, the inequalities $P \subseteq P_* \cup [\![a > 0]\!]_\Omega$ and $P \cap [\![b > 0]\!]_\Omega \subseteq P_*$ both hold, thus also

$$\varphi(P) \leq \varphi(P_*) \vee \varphi([\![a > 0]\!]_\Omega) \quad \text{and} \quad \varphi(P) \wedge \varphi([\![b > 0]\!]_\Omega) \leq \varphi(P_*)\,.$$

By Lemmas 3.5 and 3.6 in [35], we get $\varphi(P) \smallsetminus_L \varphi(P_*) \leq \varphi([\![a > 0]\!]_\Omega) \smallsetminus_L \varphi([\![b > 0]\!]_\Omega)$, which implies (4.9) right away.

It remains to handle the case where $P \not\subseteq [\![a > 0]\!]_\Omega$. From $\lambda > 0$ it follows that $[\![a > \lambda b]\!]_\Omega \subseteq [\![a > 0]\!]_\Omega \cup [\![b < 0]\!]_\Omega$, thus $P \subseteq P_* \cup [\![a > 0]\!]_\Omega \cup [\![b < 0]\!]_\Omega$, and thus, since P is join-prime in $\mathrm{Op}(\mathcal{F},\Omega)$, we get $P \subseteq [\![b < 0]\!]_\Omega$. It follows that $\varphi(P) \smallsetminus_L \varphi(P_*) \leq \varphi([\![b < 0]\!]_\Omega)$, thus, using the relations $\varphi([\![b > 0]\!]_\Omega) \wedge \varphi([\![b < 0]\!]_\Omega) = 0$ and $\varphi([\![a > 0]\!]_\Omega) \leq \varphi([\![b > 0]\!]_\Omega) \vee \big(\varphi([\![a > 0]\!]_\Omega) \smallsetminus_L \varphi([\![b > 0]\!]_\Omega)\big)$,

$$\begin{aligned}
&\varphi([\![a > 0]\!]_\Omega) \wedge \big(\varphi(P) \smallsetminus_L \varphi(P_*)\big) \leq \varphi([\![a > 0]\!]_\Omega) \wedge \varphi([\![b < 0]\!]_\Omega)\\
&\leq \big(\varphi([\![b > 0]\!]_\Omega) \wedge \varphi([\![b < 0]\!]_\Omega)\big) \vee \Big(\big(\varphi([\![a > 0]\!]_\Omega) \smallsetminus_L \varphi([\![b > 0]\!]_\Omega)\big) \wedge \varphi([\![b < 0]\!]_\Omega)\Big)\\
&= \big(\varphi([\![a > 0]\!]_\Omega) \smallsetminus_L \varphi([\![b > 0]\!]_\Omega)\big) \wedge \varphi([\![b < 0]\!]_\Omega)\\
&\leq \varphi([\![a > 0]\!]_\Omega) \smallsetminus_L \varphi([\![b > 0]\!]_\Omega)\,,
\end{aligned}$$

thus completing the proof of (4.9) in the general case, and therefore of (4.8).

Now $[\![a > 0]\!]_\Omega \subseteq [\![b > 0]\!]_\Omega \cup ([\![a > 0]\!]_\Omega \cap [\![a > \lambda b]\!]_\Omega)$, that is,

$$[\![a > 0]\!]_\Omega \smallsetminus_{\mathrm{Op}(\mathcal{F}_\lambda,\Omega)} [\![b > 0]\!]_\Omega \subseteq [\![a > 0]\!]_\Omega \cap [\![a > \lambda b]\!]_\Omega\,.$$

It follows that

$$\begin{aligned}
\psi([\![a > 0]\!]_\Omega \smallsetminus_{\mathrm{Op}(\mathcal{F}_\lambda,\Omega)} [\![b > 0]\!]_\Omega) &\leq \psi([\![a > 0]\!]_\Omega \cap [\![a > \lambda b]\!]_\Omega)\\
&= \varphi([\![a > 0]\!]_\Omega) \wedge \psi([\![a > \lambda b]\!]_\Omega) \leq \varphi([\![a > 0]\!]_\Omega) \smallsetminus_L \varphi([\![b > 0]\!]_\Omega)\,.
\end{aligned}$$

Since $\varphi([\![a > 0]\!]_\Omega) \leq \varphi([\![b > 0]\!]_\Omega) \vee \psi([\![a > 0]\!]_\Omega \smallsetminus_{\mathrm{Op}(\mathcal{F}_\lambda,\Omega)} [\![b > 0]\!]_\Omega)$, the converse inequality

$$\varphi([\![a > 0]\!]_\Omega) \smallsetminus_L \varphi([\![b > 0]\!]_\Omega) \leq \psi([\![a > 0]\!]_\Omega \smallsetminus_{\mathrm{Op}(\mathcal{F}_\lambda,\Omega)} [\![b > 0]\!]_\Omega)$$

holds, and therefore $\varphi([\![a > 0]\!]_\Omega) \smallsetminus_L \varphi([\![b > 0]\!]_\Omega) = \psi([\![a > 0]\!]_\Omega \smallsetminus_{\mathrm{Op}(\mathcal{F}_\lambda,\Omega)} [\![b > 0]\!]_\Omega)$. □

An iteration, of finite length $4 \cdot (\operatorname{card}\mathcal{F})^2$, of Lemma 4.8, involving a feature of consonance stated in Wehrung [35, Lemma 3.9], leads to the following extension, to our context, of [35, Lemma 7.2]. As Lemma 4.9 is, *mutatis mutandis*, obtained from Lemma 4.8 the same way as [35, Lemma 7.2] is obtained from [35, Lemma 7.1], we omit its proof.

Lemma 4.9. *Let Ω be a convex subset of $\mathbb{V}$, let $\mathcal{F}$ be a finite set of continuous affine functionals on $\mathbb{V}$, let L be a completely normal distributive lattice with zero, let $\varphi\colon \operatorname{Op}(\mathcal{F},\Omega) \to L$ be a 0-lattice homomorphism, and let $U, V \in \operatorname{Op}^-(\mathcal{F},\Omega)$ and $\gamma \in L$ such that $\varphi(U) \leq \varphi(V) \vee \gamma$. Then there exists a finite set $\mathcal{G}$, containing $\mathcal{F}$, of linear combinations of $\mathcal{F}$, together with $W \in \operatorname{Op}^-(\mathcal{G},\Omega)$ and a lattice homomorphism $\psi\colon \operatorname{Op}(\mathcal{G},\Omega) \to L$ extending φ, such that $U \subseteq V \cup W$ and $\psi(W) \leq \gamma$.*

5. Enlarging the range of a homomorphism: semilinear case

Throughout this short section $\Bbbk$ will be a totally ordered division ring endowed with its interval topology and I will be a set. We endow the right $\Bbbk$-vector space $\Bbbk^{(I)}$, with basis I, with the topology induced by the product topology on $\Bbbk^I$ where $\Bbbk$ is given the discrete topology.

This section will be devoted to stating the modifications required for adapting to our context the author's results from [35, § 8] regarding the enlargement of the range of a homomorphism from some $\operatorname{Op}(\mathcal{F},\Omega)$ to a bounded distributive lattice L.

Denote by $\mathcal{F}_I$ the set of all linear combinations of the form $f = \sum_{i\in S} \delta_i\xi_i + \lambda$, for a finite subset S of I and elements $\xi_i \in \Bbbk \setminus \{0\}$ and $\lambda \in \Bbbk$. In particular, f is a continuous affine functional on $\Bbbk^{(I)}$. Write $\operatorname{supp}(f) \overset{\text{def}}{=} S$, the *support* of f. Moreover, write $\operatorname{supp}(\mathcal{F}) \overset{\text{def}}{=} \bigcup(\operatorname{supp}(f) \mid f \in \mathcal{F})$ for every $\mathcal{F} \subseteq \mathcal{F}_I$.

Denote by J_2 the distributive lattice represented in Figure 5.1. As observed in [35, § 4], for every bounded distributive lattice D, the free product (i.e., coproduct) $D * \mathsf{J}_2$ of D and J_2 in the category of all bounded distributive lattices can be identified with the lattice of all triples $(x, y, z) \in D^3$ such that $z \leq x \wedge y$. *Via* that identification, (x, y, z) corresponds to $(x \wedge \mathsf{a}) \vee (y \wedge \mathsf{b}) \vee z$ (cf. [35, Lemma 4.1]). In particular, any largest element of D is also the largest element of $D * \mathsf{J}_2$.

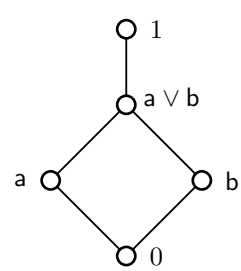

Figure 5.1. The lattice J_2

The proofs of the following lemmas are, *mutatis mutandis*, identical to those of Lemmas 8.2 and 8.3 of [35], respectively, and we thus omit them.

Lemma 5.1. *Let $\mathcal{F} \subseteq \mathcal{F}_I$, let $S \overset{\text{def}}{=} \operatorname{supp}(\mathcal{F})$, and let $i \in I \setminus \operatorname{supp}(\mathcal{F})$. We denote by $\psi\colon \operatorname{Op}(\mathcal{F},\Bbbk^{(I)}) \to \operatorname{Op}(\mathcal{F} \cup \{\delta_i\}, \Bbbk^{(I)})$ the inclusion map and $\varphi\colon \operatorname{Op}(\mathcal{F},\Bbbk^{(I)}) \to$*

$\mathrm{Op}(\mathcal{F}, \Bbbk^{(I)}) * \mathsf{J}_2$ *the diagonal map, respectively, and we set*

$$\varepsilon(X, Y, Z) \stackrel{\text{def}}{=} (X \cap [\![\delta_i > 0]\!]_{\Bbbk^{(I)}}) \cup (Y \cap [\![\delta_i < 0]\!]_{\Bbbk^{(I)}}) \cup Z$$

for all $(X, Y, Z) \in \mathrm{Op}(\mathcal{F}, \Bbbk^{(I)}) * \mathsf{J}_2$. *Then* ε *is an isomorphism and* $\psi = \varepsilon\varphi$.

Lemma 5.2. *Let* $\mathcal{F} \subseteq \mathcal{F}_I$ *and let* $i \in I \setminus \mathrm{supp}(\mathcal{F})$. *Let* L *be a bounded distributive lattice and let* $a, b \in L$ *such that* $a \wedge b = 0$. *Then every* $0,1$*-lattice homomorphism* $\varphi\colon \mathrm{Op}(\mathcal{F}, \Bbbk^{(I)}) \to L$ *extends to a unique lattice homomorphism* $\psi\colon \mathrm{Op}(\mathcal{F} \cup \{\delta_i\}, \Bbbk^{(I)}) \to L$ *such that* $\psi([\![\delta_i > 0]\!]_{\Bbbk^{(I)}}) = a$ *and* $\psi([\![\delta_i < 0]\!]_{\Bbbk^{(I)}}) = b$.

6. ℓ-spectra of countable vector lattices

The author proved in [35, Theorem 9.1] that every second countable completely normal generalized spectral space is homeomorphic to the ℓ-spectrum of some Abelian ℓ-group. In this section we shall show how to extend that result to ℓ-spectra of right vector lattices over arbitrary countable totally ordered division rings. As unproblematic as that extension may seem, the auxiliary results of Sections 4–5 will also play a crucial role in the proof of the extension of [35, Theorem 9.1] to f-rings stated in Theorem 8.4.

For a totally ordered division ring $\Bbbk$ and any set I, denote by $\mathrm{F}_\ell(I, \Bbbk)$ the free right $\Bbbk$-vector lattice on I, and by $\mathcal{O}^{\mathrm{pl}}(\Bbbk^{(I)})$ the sublattice of the powerset of $\Bbbk^{(I)}$ generated by all subsets $[\![f > 0]\!]_{\Bbbk^{(I)}}$ where $f\colon \Bbbk^{(I)} \to \Bbbk$ is a (finite) linear combination of coordinate projections. Madden's arguments in [20, Ch. III] (in particular Proposition III.1.3 and Lemma III.5.2 from that thesis), themselves extending results from Baker [1] and Bernau [3], from subfields of the reals to any totally ordered field, do not require the commutativity of $\Bbbk$. Hence,

Proposition 6.1. *For any totally ordered division ring* $\Bbbk$ *and any set* I, *the assignment* $\langle f \rangle^\ell_{\Bbbk} \mapsto [\![f \neq 0]\!]_{\Bbbk^{(I)}}$ *defines a lattice isomorphism from* $\mathrm{Id}^\ell_{\mathrm{c}}\, \mathrm{F}_\ell(I, \Bbbk)_{\Bbbk}$ *onto* $\mathcal{O}^{\mathrm{pl}}(\Bbbk^{(I)})$.

Theorem 6.2. *Let* $\Bbbk$ *be a countable totally ordered division ring. Then every countable, completely normal, distributive lattice* L *with zero is isomorphic to* $\mathrm{Id}^\ell_{\mathrm{c}}\, \mathbb{V}_{\Bbbk}$ *for some right* $\Bbbk$*-vector lattice* $\mathbb{V}$.

Outline of proof. Our proof follows exactly the same lines as the one of Wehrung [35, Theorem 9.1], thus we will just show the required changes:

- The additive subgroup Λ generated by all coordinate projection maps $\mathbb{R}^{(\omega)} \twoheadrightarrow \mathbb{R}$ is to be replaced by the subspace $\Lambda_{\Bbbk}$ consisting of all linear combinations of coordinate projections $\Bbbk^{(\omega)} \twoheadrightarrow \Bbbk$.
- The notation $\mathrm{Op}(\mathcal{H})$, used in [35] with $\mathcal{H}$ a set of (linear) hyperplanes, is replaced here by $\mathrm{Op}(\mathcal{A}, \Bbbk^{(\omega)})$, for a finite subset $\mathcal{A}$ of $\Lambda_{\Bbbk}$.
- The step $n = 3m$ involves Lemma 5.2 (linear case, $\Omega \stackrel{\text{def}}{=} \Bbbk^{(\omega)}$) instead of [35, Lemma 8.3].
- The step $n = 3m + 1$ involves Lemma 4.7 (linear case, $\Omega \stackrel{\text{def}}{=} \Bbbk^{(\omega)}$) instead of [35, Lemma 6.6].
- The step $n = 3m + 2$ involves Lemma 4.9 (linear case, $\Omega \stackrel{\text{def}}{=} \Bbbk^{(\omega)}$) instead of [35, Lemma 7.2].
- The final step, invoking Baker's representation of the ℓ-ideals of the free Abelian ℓ-group on a countably infinite set, is to be replaced by Proposition 6.1. $\square$

By applying Stone duality, we thus obtain:

Corollary 6.3. *Let $\Bbbk$ be a countable totally ordered division ring. Then every second countable, completely normal generalized spectral space is homeomorphic to the ℓ-spectrum of some right $\Bbbk$-vector lattice.*

7. Extending homomorphisms of lattices of open semi-algebraic sets

Now we move to the context of semi-algebraic sets and real spectra. Throughout this section we shall fix a real-closed field $\Bbbk$ endowed with its interval topology, together with a positive integer d. We shall endow $\Bbbk^d$ with its natural structure of topological vector space over $\Bbbk$, and establish a few extension theorems for lattice embeddings between finite lattices of semi-algebraic relatively open subsets of finite powers of the unit interval of $\Bbbk$. Those finite lattices will resemble our previously introduced lattices $\mathrm{Op}(\mathcal{A},\Omega)$, modulo a semi-algebraic homeomorphism whose existence will be ensured by a strong form of triangulation, due to Baro, stated in Proposition 7.3.

Definition 7.1. Given semi-algebraic subsets S, S_1, $\dots$, S_l of $\Bbbk^d$ with all $S_i \subseteq S$, a *triangulation* of $(S; S_1, \dots, S_l)$ is a pair $(\mathbb{K}, \tau)$ where $\mathbb{K}$ is a simplicial complex in $\Bbbk^d$ and $\tau\colon S \to |\mathbb{K}|$ is a semi-algebraic homeomorphism such that each $\tau[S_i]$ is partitioned by $\mathbb{K}$.

Note 7.2. Triangulations are often defined by replacing τ by τ^{-1} in Definition 7.1.

The Triangulation Theorem for semi-algebraic sets (cf. Bochnak *et al.* [5, Theorem 9.2.1]) states that every finite sequence $(S; S_1, \dots, S_l)$, where S and all S_i are semi-algebraic, S is closed bounded, and each $S_i \subseteq S$, has a triangulation. Baro strengthens this result in [2] with his "Normal Triangulation Theorem" , valid in any o-minimal structure over a real-closed field. Baro's [2, Theorem 1.4], together with the remark on top of [2, p. 277], can be formulated as follows.

Proposition 7.3. *Let $\mathbb{K}$ be a simplicial complex of $\Bbbk^d$ and let S_1, $\dots$, S_l be semi-algebraic subsets of $|\mathbb{K}|$. Then there exists a triangulation $(\mathbb{L}, \psi)$ of $(|\mathbb{K}|; S_1, \dots, S_l)$ such that $\mathbb{L}$ is a subdivision of $\mathbb{K}$ and $\psi[S] = S$ for every open simplex S of $\mathbb{K}$.*

Remark 7.4. Shiota establishes in [31, Proposition 3.9] a version of Proposition 7.3 for $\Bbbk = \mathbb{R}$. It is conceivable that a suitable application of Tarski-Seidenberg's transfer principle, similar to the argument of Coste [8], might turn Shiota's result into the general form of Proposition 7.3. The author did not investigate this.

Definition 7.5. For any semi-algebraic subset Ω of $\Bbbk^d$, we denote by $\mathcal{O}^{\mathrm{sa}}(\Omega)$ the lattice of all semi-algebraic subsets of Ω that are open with respect to the relative topology on Ω. A (necessarily finite) sublattice $\mathbb{D}$ of $\mathcal{O}^{\mathrm{sa}}([-1,1]^d)$ is *flatly triangulable* if there are a polytope Ω of $\Bbbk^d$, a semi-algebraic homeomorphism $\tau\colon [-1,1]^d \to \Omega$, and a finite set $\mathcal{A}$ of affine functionals on $\Bbbk^d$ such that $\tau\mathbb{D} \stackrel{\mathrm{def}}{=} \{\tau[U] \mid U \in \mathbb{D}\}$ is equal to $\mathrm{Op}(\mathcal{A},\Omega)$.

Observe that the definition of a flatly triangulable lattice involves not only the algebraic structure of the lattice, but also the way it is embedded as a lattice of semi-algebraic open sets. Our next lemma states that the polytope Ω of Definition 7.5 can be taken as the geometric realization of a simplicial complex partitioning all elements of $\mathrm{Op}(\mathcal{A},\Omega)$.

Lemma 7.6. *Let $\mathbb{D}$ be a flatly triangulable sublattice of $\mathcal{O}^{\mathrm{sa}}([-1,1]^d)$. Then there are a simplicial complex $\mathbb{K}$ of $\Bbbk^d$ with $|\mathbb{K}|$ convex, a semi-algebraic homeomorphism $\tau\colon [-1,1]^d \to |\mathbb{K}|$, and a finite set $\mathcal{A}$ of affine functionals on $\Bbbk^d$ such that $\tau\mathbb{D} = \mathrm{Op}(\mathcal{A}, |\mathbb{K}|) = \mathrm{Op}^-(\mathcal{A}, |\mathbb{K}|)$ and every member of $\mathrm{Bool}(\mathcal{A}, |\mathbb{K}|)$ is partitioned by $\mathbb{K}$.*

Proof. Let Ω, $\mathcal{A}$, τ witness the flat triangulability of $\mathbb{D}$. By Proposition 3.2, there exists a simplicial complex $\mathbb{K}$ such that $|\mathbb{K}| = \Omega$ and every member of $\mathrm{Bool}(\mathcal{A}, \Omega)$ is partitioned by $\mathbb{K}$. Adjoining to $\mathcal{A}$ the constant function with value 1 ensures that $\mathrm{Op}(\mathcal{A}, \Omega) = \mathrm{Op}^-(\mathcal{A}, \Omega)$. □

Lemma 7.7. *Every finite subset $\mathbb{X}$ of $\mathcal{O}^{\mathrm{sa}}([-1,1]^d)$ is contained in a flatly triangulable sublattice of $\mathcal{O}^{\mathrm{sa}}([-1,1]^d)$.*

Proof. By the Triangulation Theorem for semi-algebraic sets (cf. Bochnak *et al.* [5, Theorem 9.2.1]), there exists a triangulation $(\mathbb{K}, \tau)$ of $([-1,1]^d; (X \mid X \in \mathbb{X}))$. Now by Proposition 3.1, there exists a finite set $\mathcal{A}$ of nonconstant affine functionals on $\Bbbk^d$ such that every simplex in $\mathbb{K}$ belongs to $\mathrm{Clos}(\mathcal{A}, \Bbbk^d)$. For every $X \in \mathbb{X}$, $\tau[X]$ is partitioned by $\mathbb{K}$, thus it belongs to $\mathrm{Bool}(\mathcal{A}, \Bbbk^d)$; it is also a relative open subset of $|\mathbb{K}|$, thus it in fact belongs to $\mathrm{Op}(\mathcal{A}, |\mathbb{K}|)$. Therefore, $\mathbb{X}$ is contained in the flatly triangulable lattice $\tau^{-1}\,\mathrm{Op}(\mathcal{A}, |\mathbb{K}|)$. □

The following lattice-theoretical concept will be repeatedly used throughout the remainder of the paper.

Definition 7.8. Let A be a 0-sublattice of a distributive lattice B with zero. We say that A is *CN-pure in B* if every 0-lattice homomorphism from A to a completely normal distributive lattice L with zero extends to a lattice homomorphism from B to L.

Most of our interest in flat triangulability arises from the following result.

Proposition 7.9. *Let $\mathbb{D}$ and $\mathbb{E}$ be finite sublattices of $\mathcal{O}^{\mathrm{sa}}([-1,1]^d)$ with $\mathbb{D} \subseteq \mathbb{E}$. If $\mathbb{D}$ is flatly triangulable, then $\mathbb{D}$ is CN-pure in $\mathbb{E}$.*

Proof. Let $\mathbb{K}$, $\mathcal{A}$, τ be as in Lemma 7.6. By Proposition 7.3, there are a subdivision $\mathbb{L}$ of $\mathbb{K}$ and a semi-algebraic self-homeomorphism ψ of $|\mathbb{K}| = |\mathbb{L}|$ such that

(1) for each $Y \in \mathbb{E}$, $\psi[Y]$ is partitioned by $\mathbb{L}$;
(2) $\psi[S] = S$ for every open simplex S of $\mathbb{K}$.

By Proposition 3.1, there exists a finite set $\mathcal{B}$ of nonconstant affine functionals on $\Bbbk^d$, containing $\mathcal{A}$, such that every simplex in $\mathbb{L}$ belongs to $\mathrm{Bool}(\mathcal{B}, \Bbbk^d)$ (thus to $\mathrm{Clos}(\mathcal{B}, \Bbbk^d)$). Now for every $X \in \mathbb{D}$, $\tau[X]$ belongs to $\mathrm{Op}(\mathcal{A}, |\mathbb{K}|)$ thus it is a union of open simplexes of $\mathbb{K}$, and thus, by (2) above, $\psi\tau[X] = \tau[X]$. This means that

$$\begin{tikzcd}
\mathbb{D} \arrow[r, "\overline{\tau}", "\cong"'] \arrow[d, hook, "\text{inclusion}"'] & \mathrm{Op}(\mathcal{A}, |\mathbb{K}|) \arrow[d, hook, "\text{inclusion}"] \\
\mathbb{E} \arrow[r, hook, "\overline{\psi}\circ\overline{\tau}"] & \mathrm{Op}(\mathcal{B}, |\mathbb{K}|)
\end{tikzcd}$$

FIGURE 7.1. A commutative diagram of finite distributive lattices

the diagram represented in Figure 7.1 (in which $\overline{\tau}\colon X \mapsto \tau[X]$ and $\overline{\psi}\colon Y \mapsto \psi[Y]$)

is commutative. Now by (a finite iteration of) Lemma 4.7, $\mathrm{Op}(\mathcal{A},|\mathbb{K}|)$ is CN-pure in $\mathrm{Op}(\mathcal{B},|\mathbb{K}|)$. Since $\overline{\tau}$ is an isomorphism, every 0-lattice homomorphism ξ from $\mathbb{D}$ to a completely normal distributive lattice L with zero induces the 0-lattice homomorphism $\xi\circ\overline{\tau}^{-1}\colon \mathrm{Op}(\mathcal{A},|\mathbb{K}|)\to L$, which extends to a lattice homomorphism $\eta\colon \mathrm{Op}(\mathcal{B},|\mathbb{K}|)\to L$, which induces the lattice homomorphism $\eta\circ\overline{\psi}\circ\overline{\tau}\colon \mathbb{E}\to L$, which extends ξ. Hence $\mathbb{D}$ is CN-pure in $\mathbb{E}$. $\square$

We conclude this section with a variant of Lemma 5.1 tailored to the semi-algebraic case.

Lemma 7.10. *Let $I\cup\{j\}$ be a finite set, with $j\notin I$, and let $\mathbb{D}$ be a $0,1$-sublattice of $\mathcal{O}^{\mathrm{sa}}([-1,1]^I)$. We set*

$$\varepsilon(X,Y,Z)\overset{\mathrm{def}}{=}(X\times(0,1])\cup(Y\times[-1,0))\cup(Z\times[-1,1])\quad\textit{whenever }(X,Y,Z)\in\mathbb{D}*\mathsf{J}_2\,.$$

Then ε defines a $0,1$-lattice embedding from $\mathbb{D}*\mathsf{J}_2$ into $\mathcal{O}^{\mathrm{sa}}([-1,1]^{I\cup\{j\}})$. Furthermore, if $\mathbb{D}$ is flatly triangulable, then so is the range of ε.

Proof. The set $T\overset{\mathrm{def}}{=}\varepsilon(X,Y,Z)$ determines the triple $(X,Y,Z)\in\mathbb{D}*\mathsf{J}_2$, because

$$\begin{aligned}X&=\bigl\{t\in[-1,1]^I\mid(t,1)\in T\bigr\}\,,\\ Y&=\bigl\{t\in[-1,1]^I\mid(t,-1)\in T\bigr\}\,,\\ Z&=\bigl\{t\in[-1,1]^I\mid(t,0)\in T\bigr\}\,.\end{aligned}$$

It follows that ε is a $0,1$-lattice embedding. Denote by $\overline{\mathbb{D}}$ the range of ε.

Now suppose that $\mathbb{D}$ is flatly triangulable. Let Ω be a polytope, let $\mathcal{A}$ be a finite set of affine functionals on $\Bbbk^I$, and let $\tau\colon[-1,1]^I\to\Omega$ be a semi-algebraic homeomorphism such that $\tau\mathbb{D}=\mathrm{Op}(\mathcal{A},\Omega)$. Then $\Omega\times[-1,1]$ is a polytope of $\Bbbk^{I\cup\{j\}}$ and $\overline{\tau}\colon[-1,1]^{I\cup\{j\}}=[-1,1]^I\times[-1,1]\to\Omega\times[-1,1]$, $(x,t)\mapsto(\tau(x),t)$ is a semi-algebraic homeomorphism. Furthermore, denoting by $p\colon\Bbbk^{I\cup\{j\}}\twoheadrightarrow\Bbbk$ the jth coordinate projection, defining $\overline{a}\colon(x,t)\mapsto a(x)$ for $a\in\mathcal{A}$, and setting $\overline{\mathcal{A}}\overset{\mathrm{def}}{=}\{\overline{a}\mid a\in\mathcal{A}\}$, it is not hard to verify that $\overline{\tau}\overline{\mathbb{D}}=\mathrm{Op}\bigl(\overline{\mathcal{A}}\cup\{p\}\,,\Omega\times[-1,1]\bigr)$. $\square$

8. Real spectra of countable algebras

Now we are about to reach our main result, Theorem 8.4. We will state that result in the language of f-rings, radical ℓ-ideals, and Brumfiel spectra. Translations to real spectra will follow right away.

Definition 8.1. For any real-closed field $\Bbbk$ and any set I, we set

$$[-1,1]^{(I)}\overset{\mathrm{def}}{=}\Bigl\{x\in\Bbbk^{(I)}\mid-1\leq x_i\leq1\text{ for every }i\in I\Bigr\}\,,\text{ the }\textit{cube of dimension }I\text{ in }\Bbbk\,.$$

We endow the $\Bbbk$-algebra $\Bbbk^{[-1,1]^{(I)}}$, of all maps $[-1,1]^{(I)}\to\Bbbk$, with its componentwise ordering and we denote by $\Bbbk\langle I\rangle$ the f-subring of $\Bbbk^{[-1,1]^{(I)}}$ generated by the constant map with value 1 together with all projection maps ($[-1,1]^{(I)}\to\Bbbk$, $p\mapsto p(i)$) for $i\in I$.

A subset X of $[-1,1]^{(I)}$ is *semi-algebraic* if there are a finite subset J of I and a semi-algebraic subset Y of $[-1,1]^{(J)}$ such that $X=\bigl\{p\in[-1,1]^{(I)}\mid p\restriction_J\in Y\bigr\}$ — from now on let us write this as $X=Y\times[-1,1]^{(I\setminus J)}$. Accordingly, we extend the notation $\mathcal{O}^{\mathrm{sa}}(\Omega)$ to the infinite-dimensional case (cf. Definition 7.5), and we denote by $\mathcal{O}^{\mathrm{sa}}(\Omega)$ the lattice of all semi-algebraic relatively open subsets (with respect to

the topology induced by the product topology on $[-1,1]^{(I)}$) of a semi-algebraic subset Ω of $[-1,1]^{(I)}$.

In particular, $\Bbbk\langle I\rangle$ is a commutative unital f-ring. For all $x \in \Bbbk\langle I\rangle$, the *cozero set of* x, $\mathrm{COZ}(x) \overset{\mathrm{def}}{=} \bigl\{p \in [-1,1]^{(I)} \mid x(p) \neq 0\bigr\}$ is a semi-algebraic relatively open subset of $[-1,1]^{(I)}$.

Proposition 8.2. *For any real-closed field $\Bbbk$ and any set I, the assignment $\langle a\rangle^{\mathrm{r}} \mapsto \mathrm{COZ}(a)$ defines a lattice isomorphism from $\mathrm{Id}^{\mathrm{r}}_{\mathrm{c}}\,\Bbbk\langle I\rangle$ onto $\mathcal{O}^{\mathrm{sa}}([-1,1]^{(I)})$.*

Proof. Since every element of $\Bbbk\langle I\rangle$ or $\mathcal{O}^{\mathrm{sa}}([-1,1]^{(I)})$, respectively, depends only on a finite set of coordinates in I, it is easy to reduce the problem to the case where I is finite. Now that case is covered by Delzell and Madden [11, Proposition 6.2]. $\square$

The following easy lemma was already stated in Wehrung [34, Lemma 4.9].

Lemma 8.3. *Let A be a commutative* f*-ring, let L be a distributive lattice with zero, and let $\varphi\colon \mathrm{Id}^{\mathrm{r}}_{\mathrm{c}}\,A \twoheadrightarrow L$ be a closed surjective lattice homomorphism. Then the subset $I \overset{\mathrm{def}}{=} \{x \in A \mid \varphi(\langle x\rangle^{\mathrm{r}}) = 0\}$ is a radical ℓ-ideal of A and there is a unique isomorphism $\psi\colon \mathrm{Id}^{\mathrm{r}}_{\mathrm{c}}(A/I) \to L$ such that $\psi(\langle x+I\rangle^{\mathrm{r}}) = \varphi(\langle x\rangle^{\mathrm{r}})$ for every $x \in A$.*

Theorem 8.4. *Let $\Bbbk$ be a countable totally ordered field. Then every countable, completely normal, bounded distributive lattice L is isomorphic to $\mathrm{Id}^{\mathrm{r}}_{\mathrm{c}}\,A$ for some commutative unital* f*-$\Bbbk$-algebra A.*

Proof. If $\overline{\Bbbk}$ denotes the real closure of $\Bbbk$, then any f-$\overline{\Bbbk}$-algebra is also a f-$\Bbbk$-algebra. Since $\mathrm{Id}^{\mathrm{r}}_{\mathrm{c}}\,A$ is not affected by the change of field (cf. Note 2.1), we may thus replace $\Bbbk$ by its real closure, and thus assume that $\Bbbk$ is real-closed. Now we follow the lines of the proof of Wehrung [35, Theorem 9.1], replacing the original auxiliary results by those obtained in the present paper's Sections 4–7.

We fix a generating subset $\{a_m \mid m < \omega\}$ of L and a sequence $((S_m, A_m) \mid m < \omega)$, where each S_m is a finite subset of ω and each $A_m \in \mathcal{O}^{\mathrm{sa}}([-1,1]^{(S_m)})$, such that $A_0 = \varnothing$ and

$$\mathcal{O}^{\mathrm{sa}}([-1,1]^{(\omega)}) = \Bigl\{A_m \times [-1,1]^{(\omega\setminus S_m)} \mid m < \omega\Bigr\}\,.$$

Further, we fix an enumeration $((X_k, Y_k, \gamma_k) \mid k < \omega)$ of all triples (X, Y, γ) where $X, Y \in \mathcal{O}^{\mathrm{sa}}([-1,1]^{(\omega)})$ and $\gamma \in L$.

We shall construct inductively a sequence $((I_n, \mathbb{D}_n, \varphi_n) \mid n < \omega)$, where for every $n < \omega$, I_n is a finite subset of ω, $\mathbb{D}_n$ is a flatly triangulable $0,1$-sublattice of $\mathcal{O}^{\mathrm{sa}}([-1,1]^{(I_n)})$, and $\varphi_n\colon \mathbb{D}_n \to L$ is a $0,1$-lattice homomorphism, subjected to the following conditions:

(1) every a_m belongs to the range of some φ_n,
(2) for all m there exists n such that $S_m \subseteq I_n$ and $A_m \times [-1,1]^{(I_n\setminus S_m)} \in \mathbb{D}_n$,
(3) for all $k \leq n$, if $X_k = X \times [-1,1]^{(\omega\setminus I_n)}$ and $Y_k = Y \times [-1,1]^{(\omega\setminus I_n)}$ for some $X, Y \in \mathbb{D}_n$ such that $\varphi_n(X) \leq \varphi_n(Y) \vee \gamma_k$, then there exists $W \in \mathbb{D}_{n+1}$ such that $X \times [-1,1]^{(I_{n+1}\setminus I_n)} \subseteq \bigl(Y \times [-1,1]^{(I_{n+1}\setminus I_n)}\bigr) \cup W$ and $\varphi_{n+1}(W) \leq \gamma_k$,
(4) for all $n < \omega$ and every $U \in \mathbb{D}_n$, $U \times [-1,1]^{(I_{n+1}\setminus I_n)} \in \mathbb{D}_{n+1}$ and the equality
$\varphi_n(U) = \varphi_{n+1}(U \times [-1,1]^{(I_{n+1}\setminus I_n)})$ holds.

For $n = 0$ we just set $I_0 \stackrel{\text{def}}{=} \{0\}$ (thus $[-1,1]^{(I_0)} = [-1,1]^{\{0\}} \cong [-1,1]$), $\mathbb{D}_0 \stackrel{\text{def}}{=} \{\varnothing, [-1,1]\}$, and $\varphi_0 \colon \mathbb{D}_0 \to \{0_L, 1_L\}$ is the unique isomorphism.

Suppose $(I_n, \mathbb{D}_n, \varphi_n)$ already constructed.

If $n = 3m$ for an integer m, denote by k the least element of $\omega \setminus I_n$, set $I_{n+1} \stackrel{\text{def}}{=} I_n \cup \{k\}$, and let $\varepsilon \colon \mathbb{D}_n * \mathsf{J}_2 \hookrightarrow \mathcal{O}^{\text{sa}}([-1,1]^{(I_n \cup \{k\})})$ be the $0,1$-lattice embedding defined in Lemma 7.10. By that lemma, the range $\mathbb{D}_{n+1}$ of ε is flatly triangulable and there exists a unique $0,1$-lattice homomorphism $\varphi_{n+1} \colon \mathbb{D}_{n+1} \to L$ extending φ_n in the sense of (4) above whereas sending $[-1,1]^{(I_n)} \times (0,1]$ to a_m and $[-1,1]^{(I_n)} \times [-1,0)$ to 0. This will take care of (1) above.

If $n = 3m+1$ for an integer m, we set $I_{n+1} \stackrel{\text{def}}{=} I_n \cup S_m$. Setting $\mathbb{D}'_n \stackrel{\text{def}}{=} \bigl\{U \times [-1,1]^{(S_m \setminus I_n)} \mid U \in \mathbb{D}_n\bigr\}$, it follows from Lemma 7.7 that $\mathbb{D}'_n \cup \bigl\{A_m \times [-1,1]^{(I_n \setminus S_m)}\bigr\}$ is contained in a flatly triangulable sublattice $\mathbb{D}_{n+1}$ of $\mathcal{O}^{\text{sa}}([-1,1]^{(I_{n+1})})$. Now $\mathbb{D}'_n$ is easily seen to be flatly triangulable. Hence, by Proposition 7.9, the $0,1$-lattice homomorphism $\mathbb{D}'_n \to L$, $U \times [-1,1]^{(S_m \setminus I_n)} \mapsto \varphi_n(U)$ extends to a lattice homomorphism $\varphi_{n+1} \colon \mathbb{D}_{n+1} \to L$. This will take care of (2) above.

Let finally $n = 3m+2$ for some integer m. Now set $I_{n+1} \stackrel{\text{def}}{=} I_n$. Since $\mathbb{D}_n$ is flatly triangulable, there are a polytope Ω, a finite set $\mathcal{A}$ of affine functionals on $\Bbbk^{I_n}$, and a semi-algebraic homeomorphism $\tau \colon [-1,1]^{(I_n)} \to \Omega$ such that $\tau \mathbb{D}_n = \operatorname{Op}(\mathcal{A}, \Omega)$. We may add to $\mathcal{A}$ the constant function with value 1 and thus assume that $\operatorname{Op}(\mathcal{A}, \Omega) = \operatorname{Op}^-(\mathcal{A}, \Omega)$. By a finite iteration of Lemma 4.9, initially applied to the homomorphism $\varphi_n \tau^{-1} \colon \operatorname{Op}(\mathcal{A}, \Omega) \to L$, there exists a finite set $\mathcal{B}$ of linear combinations of $\mathcal{A}$, containing $\mathcal{A}$ (so $\operatorname{Op}(\mathcal{B}, \Omega) = \operatorname{Op}^-(\mathcal{B}, \Omega)$), together with a homomorphism $\psi \colon \operatorname{Op}(\mathcal{B}, \Omega) \to L$ extending $\varphi_n \tau^{-1}$, such that for every $k \leq n$, if $X_k = X \times [-1,1]^{(\omega \setminus I_n)}$ and $Y_k = Y \times [-1,1]^{(\omega \setminus I_n)}$ for some $X, Y \in \mathbb{D}_n$ with $\varphi_n(X) \leq \varphi_n(Y) \vee \gamma_k$, there exists $W \in \operatorname{Op}(\mathcal{B}, \Omega)$ such that $\tau[X] \subseteq \tau[Y] \cup W$ and $\psi(W) \leq \gamma_k$. The lattice $\mathbb{D}_{n+1} \stackrel{\text{def}}{=} \tau^{-1} \operatorname{Op}(\mathcal{B}, \Omega)$ is flatly triangulable and contains $\mathbb{D}_n$, and the homomorphism $\varphi_{n+1} \stackrel{\text{def}}{=} \psi\tau$ extends φ_n. This will take care of (3) above.

The sequence $((I_n, \mathbb{D}_n, \varphi_n) \mid n < \omega)$ satisfies Conditions (1)–(4). For each $n < \omega$, $\mathbb{E}_n \stackrel{\text{def}}{=} \bigl\{U \times [-1,1]^{(\omega \setminus I_n)} \mid U \in \mathbb{D}_n\bigr\}$ is an isomorphic copy of $\mathbb{D}_n$ in $\mathcal{O}^{\text{sa}}([-1,1]^{(\omega)})$ and the assignment $U \times [-1,1]^{(\omega \setminus I_n)} \mapsto \varphi_n(U)$ defines a $0,1$-lattice homomorphism $\psi_n \colon \mathbb{E}_n \to L$. It follows from Condition (4) that both sequences $(\mathbb{E}_n \mid n < \omega)$ and $(\psi_n \mid n < \omega)$ are ascending; thus the union ψ of all ψ_n is a $0,1$-lattice homomorphism from $\mathbb{E} \stackrel{\text{def}}{=} \bigcup (\mathbb{E}_n \mid n < \omega)$ to L. Moreover, it follows from Condition (1) that ψ is surjective, from Condition (2) that $\mathbb{E} = \mathcal{O}^{\text{sa}}([-1,1]^{(\omega)})$, and from Condition (3) that ψ is a closed map. By Proposition 8.2, ψ thus induces a closed surjective lattice homomorphism $\psi' \colon \operatorname{Id}^{\text{r}}_{\text{c}} \Bbbk\langle\omega\rangle \twoheadrightarrow L$. Now by Lemma 8.3, ψ' in turn induces an isomorphism $\operatorname{Id}^{\text{r}}_{\text{c}} A \to L$ where A is the quotient of $\Bbbk\langle\omega\rangle$ by a suitable radical ℓ-ideal. $\square$

Corollary 8.5. *Let $\Bbbk$ be a countable totally ordered field. Then every countable, completely normal distributive lattice L with zero is isomorphic to $\operatorname{Id}^{\text{r}}_{\text{c}} A$ for some commutative* f-$\Bbbk$-*algebra A.*

Proof. Observe that L is an ideal of the bounded distributive lattice $L^\infty \stackrel{\text{def}}{=} L \cup \{\infty\}$ for a new top element ∞. Since L^∞ is a countable completely normal bounded

distributive lattice, it follows from Theorem 8.4 that $L^\infty \cong \operatorname{Id}_{\mathrm{c}}^{\mathrm{r}} A$ for some commutative unital f-$\Bbbk$-algebra A. Therefore, $J \stackrel{\mathrm{def}}{=} \{x \in A \mid \langle x\rangle^{\mathrm{r}} \neq \infty\}$ is a radical ℓ-ideal of A, thus a commutative f-$\Bbbk$-algebra, and $\operatorname{Id}_{\mathrm{c}}^{\mathrm{r}} J \cong L$. □

Since $\operatorname{Id}_{\mathrm{c}}^{\mathrm{r}} A$ is the Stone dual of the Brumfiel spectrum of A, we thus obtain the following topological version of Theorem 8.4 and Corollary 8.5.

Corollary 8.6. *Let $\Bbbk$ be a countable totally ordered field. Every second countable, completely normal generalized spectral space X is homeomorphic to the Brumfiel spectrum of some commutative* f-$\Bbbk$-*algebra A. If, in addition, X is compact* (*i.e., it is a spectral space*)*, then A can be taken unital.*

In Wehrung [34, § 4], we observe, invoking results by Delzell and Madden [11], Schwartz and Madden [30], and Schwartz [29], that *the class of real spectra of all commutative unital rings* (*resp., so-called* real-closed rings) *and the class of Brumfiel spectra of all commutative unital* f-*rings coincide*. The corresponding constructions carry over to algebras over totally ordered fields. This is used in our next result.

Corollary 8.7. *Let $\Bbbk$ be a countable formally real field. Every second countable completely normal spectral space is homeomorphic to the real spectrum of some countable commutative unital $\Bbbk$-algebra.*

Proof. Since $\Bbbk$ is formally real, it has a compatible total ordering, that we shall fix through the remainder of the proof. By Corollary 8.6, the given space is homeomorphic to the Brumfiel spectrum of some commutative unital f-$\Bbbk$-algebra A. Now we argue as in the proof of [34, Corollary 4.17], reducing the Brumfiel spectrum to the real spectrum. In some more detail: first observe that the assignment $P \mapsto A^+ + P$ defines a homeomorphism from $\operatorname{Spec}_{\mathrm{B}} A$ onto $\operatorname{Spec}_{\mathrm{r}}(A, A^+) \stackrel{\mathrm{def}}{=} \{Q \in \operatorname{Spec}_{\mathrm{r}} A \mid A^+ \subseteq Q\}$. Now $\operatorname{Spec}_{\mathrm{r}}(A, A^+)$ is a closed subspace of $\operatorname{Spec}_{\mathrm{r}} A$, which is homeomorphic to the Zariski spectrum $\operatorname{Spec} \mathrm{C}_{\mathrm{r}}(A)$ of the real closure $\mathrm{C}_{\mathrm{r}}(A)$ of A (cf. Schwartz [29]); thus $\operatorname{Spec}_{\mathrm{r}}(A, A^+) \cong \operatorname{Spec}\bigl(\mathrm{C}_{\mathrm{r}}(A)/I\bigr)$ for some radical ideal I of $\mathrm{C}_{\mathrm{r}}(A)$. Now the ring $\mathrm{C}_{\mathrm{r}}(A)/I$ is real-closed, thus $\operatorname{Spec}\bigl(\mathrm{C}_{\mathrm{r}}(A)/I\bigr) \cong \operatorname{Spec}_{\mathrm{r}}\bigl(\mathrm{C}_{\mathrm{r}}(A)/I\bigr)$. Therefore, $\operatorname{Spec}_{\mathrm{B}} A \cong \operatorname{Spec}_{\mathrm{r}}\bigl(\mathrm{C}_{\mathrm{r}}(A)/I\bigr)$. Now since A is a $\Bbbk$-algebra, so is $\mathrm{C}_{\mathrm{r}}(A)/I$. □

9. The case of uncountable partially ordered division rings

This section will be mostly devoted to the discussion of an example, showing that the countability assumption on $\Bbbk$ cannot be dispensed with in our positive representability results, most notably Theorems 6.2 and 8.4. This will be done by isolating an additional property satisfied by all homomorphic images of lattices that can be represented over division rings of larger cardinality.

Definition 9.1. Let $\Bbbk$ be a partially ordered division ring, let $\mathbb{V}$ be a right $\Bbbk$-vector lattice, and let D be a distributive lattice with zero. A map $\mu\colon \mathbb{V} \to D$ is a *premeasure* if $\mu\restriction_{\mathbb{V}^+}$ is a 0-lattice homomorphism, $\mu(x) = \mu(|x|)$, and $\mu(x\lambda) \leq \mu(x)$ whenever $(x, \lambda) \in \mathbb{V} \times \Bbbk$. If, in addition, $\mu^{-1}\{0_D\} = \{0_{\mathbb{V}}\}$, then we say that μ is a *measure*.

Lemma 9.2. *The following statements hold, for any premeasure $\mu\colon \mathbb{V} \to D$:*

(1) $\mu(x + y) \leq \mu(x) \vee \mu(y)$ *for all $x, y \in \mathbb{V}$, with equality holding if $x, y \in \mathbb{V}^+$;*

(2) $\mu(x\lambda) = \mu(x)$ *for all* $x \in \mathbb{V}$ *and all* $\lambda \in \Bbbk \setminus \{0\}$*;*
(3) *the subset* $I \overset{\text{def}}{=} \mu^{-1}\{0\}$ *is an* ℓ*-ideal of* $\mathbb{V}_\Bbbk$ *and the assignment* $x + I \mapsto \mu(x)$ *defines a measure from* $(\mathbb{V}/I)_\Bbbk$ *to* D*.*

Proof. Ad (1). If $x, y \in \mathbb{V}^+$, then, since $x \vee y \leq x + y \leq 2(x \vee y)$ within $\mathbb{V}^+$ and $\mu(2(x \vee y)) = \mu((x \vee y) \cdot 2) \leq \mu(x \vee y)$ we get $\mu(x + y) = \mu(x \vee y) = \mu(x) \vee \mu(y)$. In the general case, $\mu(x+y) = \mu(|x+y|) \leq \mu(|x|+|y|) = \mu(|x|) \vee \mu(|y|) = \mu(x) \vee \mu(y)$.

Ad (2). Just observe that $\mu(x) = \mu(x\lambda\lambda^{-1}) \leq \mu(x\lambda) \leq \mu(x)$.

Ad (3) is straightforward. □

Theorem 9.3. *Let* D *be a distributive lattice with zero. If there are a partially ordered division ring* $\Bbbk$ *with* $\operatorname{card} D < \operatorname{card} \Bbbk^+$*, a right* $\Bbbk$*-vector lattice* $\mathbb{V}$*, and a surjective premeasure* $\mu\colon \mathbb{V}_\Bbbk \twoheadrightarrow D$*, then every indecomposable element of* D *is join-irreducible.*

Proof. It follows from Lemma 9.2(3) that μ can be taken a *measure*. Suppose that D has an indecomposable element $\boldsymbol{e}$ which is not join-irreducible. Since μ is surjective, there are $a, b \in \mathbb{V}^+$ such that $\boldsymbol{e} = \mu(a) \vee \mu(b)$ whereas $\mu(a) < \boldsymbol{e}$ and $\mu(b) < \boldsymbol{e}$. Setting $e \overset{\text{def}}{=} a + b$, we get $\mu(e) = \boldsymbol{e}$.

Set $x \smallsetminus y \overset{\text{def}}{=} x - (x \wedge y)$, for all $x, y \in \mathbb{V}$.

Claim 1. *Let* $x, y \in \mathbb{V}$ *be incomparable. Then* $\mu(x - y) \neq \boldsymbol{e}$*.*

Proof of Claim. From $(x \smallsetminus y) \wedge (y \smallsetminus x) = 0$ we get $\mu(x \smallsetminus y) \wedge \mu(y \smallsetminus x) = 0$. Since x and y are incomparable and $\mu^{-1}\{0\} = \{0\}$, $\mu(x \smallsetminus y)$ and $\mu(y \smallsetminus x)$ are both nonzero. Since $\boldsymbol{e}$ is indecomposable in D, it follows that $\mu(x \smallsetminus y) \vee \mu(y \smallsetminus x) \neq \boldsymbol{e}$, so $\mu(x - y) = \mu(|x - y|) = \mu(x \smallsetminus y) \vee \mu(y \smallsetminus x) \neq \boldsymbol{e}$. □ Claim 1.

Until the end of the proof of Theorem 9.3 we will set $E \overset{\text{def}}{=} \{x \in \mathbb{V} \mid \mu(x) \leq \boldsymbol{e}\}$. Observe that E is an ℓ-ideal of $\mathbb{V}_\Bbbk$ containing e as an element. Set

$$r(x) \overset{\text{def}}{=} \{\lambda \in \Bbbk \mid \mu(x - e\lambda) \neq \boldsymbol{e}\} = \{\lambda \in \Bbbk \mid \mu(x - e\lambda) < \boldsymbol{e}\}\,, \quad \text{for all } x \in E\,.$$

Claim 2. *For every* $x \in E$*, the set* $r(x)$ *is order-convex in* $\Bbbk$*.*

Proof of Claim. Let $\alpha < \gamma < \beta$ in $\Bbbk$ with $\{\alpha, \beta\} \subseteq r(x)$. If $\gamma \notin r(x)$, then, by Claim 1, either $x \leq e\gamma$ or $e\gamma \leq x$. In the first case, $e\beta - x \geq e(\beta - \gamma) \geq 0$ with $\beta - \gamma \in \Bbbk^{++}$, thus, using Lemma 9.2, $\mu(e\beta - x) \geq \mu(e(\beta - \gamma)) = \mu(e) = \boldsymbol{e}$, in contradiction with $\beta \in r(x)$. In the second case, $x - e\alpha \geq e(\gamma - \alpha) \geq 0$ with $\gamma - \alpha \in \Bbbk^{++}$, thus $\mu(x - e\alpha) \geq \mu(e(\gamma - \alpha)) = \mu(e) = \boldsymbol{e}$, in contradiction with $\alpha \in r(x)$. □ Claim 2.

Claim 3. *For all* $x \in E$ *and all distinct* $\alpha, \beta \in \Bbbk$*,* $\mu(x - e\alpha) \vee \mu(x - e\beta) = \boldsymbol{e}$*.*

Proof of Claim. From $e(\alpha - \beta) = (x - e\beta) - (x - e\alpha)$ it follows, using Lemma 9.2, that $\boldsymbol{e} = \mu(e) = \mu(e(\alpha - \beta)) \leq \mu(x - e\alpha) \vee \mu(x - e\beta)$. The converse inequality follows from $x \in E$. □ Claim 3.

Claim 4. *For every* $x \in E$*, any two distinct elements of* $r(x)$ *are incomparable.*

Proof of Claim. Suppose, to the contrary, that there are $\alpha < \beta$ in $r(x)$. By Claim 2, the interval $[\alpha, \beta]$ is contained in $r(x)$. By Claim 3, it follows that the assignment $\gamma \mapsto \mu(x - e\gamma)$ defines a one-to-one map from $[\alpha, \beta]$ into D; whence $\operatorname{card}[\alpha, \beta] \leq \operatorname{card} D$. Since $\xi \mapsto \alpha + (1 + \xi)^{-1}(\beta - \alpha)$ defines a one-to-one mapping from $\Bbbk^+$ into $(\alpha, \beta]$, it follows that $\operatorname{card} \Bbbk^+ \leq \operatorname{card} D$, a contradiction. □ Claim 4.

Claim 5. *Let $x \in E$. If $0 \in r(x)$, then $x\lambda \leq e$ for every $\lambda \in \Bbbk^{++}$.*

Proof of Claim. By Claim 4 and since $0 < \lambda^{-1}$, we get $\lambda^{-1} \notin r(x)$. By Claim 1, it follows that either $e\lambda^{-1} \leq x$ or $x \leq e\lambda^{-1}$. In the first case, $\mu(x) \geq \mu(e\lambda^{-1}) = \mu(e) = \boldsymbol{e}$, in contradiction with $0 \in r(x)$. Therefore the second case holds. □ Claim 5.

Now we can finish the proof of Theorem 9.3. Since $\{a, b\} \subseteq E$ and $\mu(a)$ and $\mu(b)$ are both distinct from $\boldsymbol{e}$, we get $0 \in r(a)$ and $0 \in r(b)$. By Claim 5, it follows that $3a \leq e$ and $3b \leq e$. Adding those inequalities together yields $3e \leq 2e$, so $e = 0$, and so $\boldsymbol{e} = 0$, a contradiction. □

Throughout the remainder of this section we shall denote by $[0, 1]$ the unit interval of the rational ordered field $\mathbb{Q}$. The lattice $\mathbb{D}$ of all finite unions of relatively open intervals of $[0, 1]$ (with rational endpoints) is obviously bounded and distributive.

Proposition 9.4. *The lattice $\mathbb{D}$ is completely normal.*

Proof. Denote by $d\colon (x, y) \mapsto |x - y|$ the canonical distance function on $[0, 1]$. For $A, B \in \mathbb{D}$, with respective complements A' and B', the subsets X and Y defined by

$$X \stackrel{\text{def}}{=} \{z \in [0, 1] \mid d(z, B') < d(z, A')\} \text{ and } Y \stackrel{\text{def}}{=} \{z \in [0, 1] \mid d(z, A') < d(z, B')\}$$

both belong to $\mathbb{D}$. Moreover, $A \subseteq B \cup X$, $B \subseteq A \cup Y$, and $X \cap Y = \varnothing$. □

Now we can start reaping the consequences of Theorem 9.3.

Corollary 9.5. *Let $\Bbbk$ be a directed partially ordered division ring. If $\Bbbk$ is uncountable, then there are no right vector lattice $\mathbb{V}$ over $\Bbbk$ and no surjective lattice homomorphism from $\operatorname{Id}^{\ell}_{\mathrm{c}} \mathbb{V}_{\Bbbk}$ onto $\mathbb{D}$.*

Proof. The assumption that $\Bbbk$ is directed ensures that $\langle x \wedge y\rangle^{\ell}_{\Bbbk} = \langle x\rangle^{\ell}_{\Bbbk} \wedge \langle y\rangle^{\ell}_{\Bbbk}$ for all $x, y \in \mathbb{V}^{+}$ (use (2.1)); whence $\operatorname{Id}^{\ell}_{\mathrm{c}} \mathbb{V}_{\Bbbk}$ is a sublattice of $\operatorname{Id}^{\ell} \mathbb{V}_{\Bbbk}$. Now suppose that the conclusion of Corollary 9.5 fails. By precomposing the given homomorphism with the map $x \mapsto \langle x\rangle^{\ell}_{\Bbbk}$, we obtain a surjective premeasure $\mu\colon \mathbb{V} \twoheadrightarrow \mathbb{D}$. The top element of $\mathbb{D}$, namely $[0, 1]$, is indecomposable in $\mathbb{D}$, but not join-irreducible (e.g., $[0, 1] = [0, 1) \cup (0, 1]$). Apply Theorem 9.3. □

Corollary 9.6. *Let $\Bbbk$ be a directed partially ordered field. If $\Bbbk$ is uncountable, then there are no commutative* f-$\Bbbk$*-algebra A and no lattice homomorphism from $\operatorname{Id}^{\mathrm{r}}_{\mathrm{c}} A$ onto $\mathbb{D}$.*

Proof. Denote by $\mathbb{V}$ the underlying right $\Bbbk$-vector lattice of A. As in Wehrung [34, Proposition 5.6] and using Note 2.1, observe that the assignment $\langle x\rangle^{\ell}_{\Bbbk} \mapsto \langle x\rangle^{\mathrm{r}}$ defines a surjective lattice homomorphism from $\operatorname{Id}^{\ell}_{\mathrm{c}} \mathbb{V}_{\Bbbk}$ onto $\operatorname{Id}^{\mathrm{r}}_{\mathrm{c}} A$. Apply Corollary 9.5. □

By Proposition 9.4, $\operatorname{Spec} \mathbb{D}$ is a completely normal spectral space. By applying Stone duality to our previous results, we obtain:

Corollary 9.7. *The space $\operatorname{Spec} \mathbb{D}$ cannot be embedded, as a spectral subspace, into either following kind of spectral space:*

(1) $\operatorname{Spec}_{\ell} \mathbb{V}_{\Bbbk}$*, for any uncountable directed partially ordered division ring $\Bbbk$ and any right $\Bbbk$-vector lattice $\mathbb{V}$;*
(2) $\operatorname{Spec}_{\mathrm{r}} A$*, for any uncountable field $\Bbbk$ and any commutative unital $\Bbbk$-algebra A.*

Proof. *Ad* (1). Apply Stone duality to Corollary 9.5.

Ad (2). Suppose that $\operatorname{Spec}\mathbb{D}$ embeds, as a spectral subspace, into $\operatorname{Spec}_{\mathrm{r}} A$ for a commutative unital algebra A over an uncountable field $\Bbbk$. Since $\operatorname{Spec}_{\mathrm{r}} A$ is nonempty, $\Bbbk$ is formally real, which means that it can be partially ordered *via* the positive cone $\Bbbk^+$ consisting of all sums of squares in $\Bbbk$. Observe that this ordering makes $\Bbbk$ directed, for

$$x = \left(\frac{x+1}{2}\right)^2 - \left(\frac{x-1}{2}\right)^2 \qquad \text{for all } x \in \Bbbk\,.$$

Since $\operatorname{Spec}_{\mathrm{r}} A$ is homeomorphic to the Brumfiel spectrum of the universal commutative unital f-ring $\widetilde{A}$ of A (cf. Delzell and Madden [11, Proposition 3.3]), $\operatorname{Spec}\mathbb{D}$ embeds as a spectral subspace into $\operatorname{Spec}_{\mathrm{B}} \widetilde{A}$. Applying Stone duality, it follows that $\mathbb{D}$ is a homomorphic image of $\operatorname{Id}^{\mathrm{r}}_{\mathrm{c}} \widetilde{A}$. Since every square of $\Bbbk$ belongs to $\widetilde{A}^+$ and by definition of the ordering on $\Bbbk$, the canonical map from $\Bbbk$ to $\widetilde{A}$ is order-preserving, so $\widetilde{A}$ is also an f-$\Bbbk$-algebra; in contradiction with Corollary 9.6. $\square$

Let us point that analogues of Corollary 9.5, established in a completely different, module-theoretical context, can be found in Goodearl and Menal's paper [14]. A key point is [14, Lemma 2.1], which can be viewed as an analogue of Claim 3 in the proof of Theorem 9.3.

10. Representability over arbitrary totally ordered division ring

It is well known that every completely normal generalized dual Heyting algebra is isomorphic to the principal ℓ-ideal lattice of some Abelian ℓ-group (cf. Cignoil *et al.* [6, Theorem 3.3], Iberkleid *et al.* [17, Theorem 3.1.1]). In this section we shall amplify that result, in that process noticeably modifying the presentation of the above works, enabling us to extend it to vector lattices over arbitrary totally ordered division rings and to state it in *functorial* terms. Functoriality will be achieved by a construction *via* generators and relations (cf. Definition 10.1).

Following the notation of Iberkleid *et al.* [17], for any elements a and b in a distributive lattice D, let $a \ll b$ hold if $a \leq b$ and for all $x \in D$, $b \leq a \vee x$ implies that $b \leq x$. Equivalently, $a \leq b$ and $b \smallsetminus_D a = b$.

Definition 10.1. Let D be a distributive lattice with zero. For any totally ordered division ring $\Bbbk$ and any right $\Bbbk$-vector lattice $\mathbb{V}$, a 0-lattice homomorphism $\nu\colon D \to \mathbb{V}^+$ is a *pre-scale* if $a \ll b$ implies that $\nu(a)\lambda \leq \nu(b)$ whenever $a, b \in D$ and $\lambda \in \Bbbk$. If, in addition, $\nu^{-1}\{0_{\mathbb{V}}\} = \{0_D\}$, we say that ν is a *scale*.

Given D and $\Bbbk$, we denote by $\Bbbk\lfloor D\rfloor$ the (image of the) universal pre-scale on D. In more detail, $\Bbbk\lfloor D\rfloor$ is thus the right $\Bbbk$-vector lattice defined by generators $\overline{a}$, for $a \in D$, and relations $\overline{0} = 0$, $\overline{a \vee b} = \overline{a} \vee \overline{b}$, and $\overline{a \wedge b} = \overline{a} \wedge \overline{b}$, for $a, b \in D$, together with $\overline{a}\lambda \leq \overline{b}$ whenever $a \ll b$ in D and $\lambda \in \Bbbk$.

Lemma 10.2. *The assignment $D \mapsto \Bbbk\lfloor D\rfloor$ can be extended to a functor, from the category of all distributive lattices with zero and closed homomorphisms, to right $\Bbbk$-vector lattices with ℓ-$\Bbbk$-homomorphisms.*

Proof. Any closed 0-lattice homomorphism $f\colon D \to E$, between distributive lattices with zero, preserves the relations defining the construction $\Bbbk\lfloor_\rfloor$. This means that there exists a unique ℓ-$\Bbbk$-homomorphism $\Bbbk\lfloor f\rfloor\colon \Bbbk\lfloor D\rfloor \to \Bbbk\lfloor E\rfloor$ sending each $\overline{a}$ to $\overline{f(a)}$. $\square$

We will be interested only in the case where D is a completely normal generalized dual Heyting algebra and $\Bbbk$ is a totally ordered division ring. As we will see shortly, in that case everything boils down to the case where D is finite, that is, the poset $P \stackrel{\text{def}}{=} \operatorname{Ji} D$ is a root system (i.e., every principal upper subset of P is a chain). For $x \in \Bbbk^P$, denote by $\Gamma(x)$ the set of all maximal elements of the *support* $\{p \in P \mid x(p) \neq 0\}$ of x. Letting $x \leq y$ hold iff $x(p) < y(p)$ for every $p \in \Gamma(y - x)$, the Hahn power $(\Bbbk^P, +, 0, \leq)$ is a right $\Bbbk$-vector lattice. The argument of the proof in Iberkleid *et al.* [17, Theorem 3.1.1(3)] yields the following.

Lemma 10.3. *Let D be finite. The map $\nu_D \colon D \to (\Bbbk^P)^+$, sending every a to the characteristic function of the set of all maximal elements of $P{\downarrow}a \stackrel{\text{def}}{=} \{p \in P \mid p \leq a\}$, is a scale on D.*

Our next lemma shows that $(\Bbbk^P, \nu_D)$ is the universal (pre-)scale on D.

Lemma 10.4. *Let D be a finite completely normal dual Heyting algebra, set $P \stackrel{\text{def}}{=} \operatorname{Ji} D$, let $\Bbbk$ be a totally ordered division ring, and let $\mathbb{V}$ be a right $\Bbbk$-vector lattice. Then for every pre-scale $\nu \colon D \to \mathbb{V}^+$, there exists a unique ℓ-$\Bbbk$-homomorphism $\varphi \colon \Bbbk^P \to \mathbb{V}$ such that $\nu = \varphi \circ \nu_D$. Moreover, if ν is a scale, then φ is an ℓ-$\Bbbk$-embedding.*

Proof. Necessarily, $\varphi \colon \Bbbk^P \to \mathbb{V}$, $x \mapsto \sum (\nu(p)x(p) \mid p \in P)$.

Let $x \in \Bbbk^P$. Since P is a root system, the lower subsets $P \downarrow p$, for $p \in \Gamma(x)$, are pairwise disjoint, thus the elements $x_p \stackrel{\text{def}}{=} x{\restriction}_{P\downarrow p}$ (extended by 0 outside $P \downarrow p$) generate pairwise orthogonal ℓ-ideals in $\Bbbk^P$ and $x = \sum (x_p \mid p \in \Gamma(x))$. Now each x_p is either positive or negative according to whether $x(p) > 0$ or $x(p) < 0$, respectively. In particular, setting $\Gamma^+(x) \stackrel{\text{def}}{=} \{p \in \Gamma(x) \mid x(p) > 0\}$, we get

$$x \vee 0 = \sum \big(x_p \mid p \in \Gamma^+(x)\big) \text{ and } \varphi(x) \vee 0 = \sum \big(\varphi(x_p) \mid p \in \Gamma^+(x)\big) \ .$$

Therefore, φ is an ℓ-$\Bbbk$-homomorphism.

Now suppose that ν is a scale. We must prove that $\varphi(x) \geq 0$ implies that $x \geq 0$, for every $x \in \Bbbk^P$. Using the decomposition $x = \sum (x_p \mid p \in \Gamma(x))$ above, it suffices to consider the case where the support of x has a largest element p. In that case, $\nu(p) > 0$, together with the assumption that ν is a pre-scale, entails that $\varphi(x)$ is either positive or negative according to whether $x(p) > 0$ or $x(p) < 0$, that is, $x > 0$ or $x < 0$, respectively; so we are done. $\square$

Theorem 10.5. *Let $\Bbbk$ be a totally ordered division ring. Then for every completely normal generalized dual Heyting algebra D, the assignment $a \mapsto \langle \overline{a} \rangle^{\ell}_{\Bbbk}$ defines an isomorphism from D onto $\operatorname{Id}^{\ell}_{\mathrm{c}} \Bbbk\lfloor D \rfloor_{\Bbbk}$. This homomorphism is natural in D with respect to closed 0-lattice homomorphisms.*

Proof. Owing to Maksimova [21] (where the result is stated in terms of Heyting algebras), the variety **CNdHA** of all completely normal generalized dual Heyting algebras (in the signature $(\vee, \wedge, \smallsetminus)$) is generated by all finite chains. Now for every positive integer n, every totally ordered member of **CNdHA** generated by an n-element chain has at most $n+1$ elements. Using McKenzie *et al.* [23, Lemma 4.98], it follows that every n-generated member of **CNdHA** has at most $(n+1)^{(n+1)^n}$ elements; so every member of **CNdHA** is the directed union of its finite subalgebras. This reduces the verification of Theorem 10.5 to the finite case.

Now if D is finite, all required proofs are straightforward except perhaps the one of the surjectivity of $a \mapsto \langle \overline{a} \rangle^{\ell}_{\Bbbk}$. However, the decomposition $x = \sum (x_p \mid p \in \Gamma(x))$ introduced in the proof of Lemma 10.4 reduces the problem to elements x whose support has a largest element p, in which case $\langle x \rangle^{\ell}_{\Bbbk} = \langle \overline{p} \rangle^{\ell}_{\Bbbk}$.

Finally, the required naturality statement follows from Lemma 10.2. □

Theorem 10.6. *For any completely normal distributive lattice D with zero, the following are equivalent:*

(i) *There are a partially ordered division ring $\Bbbk$ and a right $\Bbbk$-vector lattice $\mathbb{V}$, such that every subset of $\Bbbk$ with at most* $\operatorname{card} D$ *elements has an upper bound, and $D \cong \operatorname{Id}^{\ell}_{\mathrm{c}} \mathbb{V}_{\Bbbk}$.*

(ii) *For every totally ordered division ring $\Bbbk$, there exists a right $\Bbbk$-vector lattice $\mathbb{V}$ such that $D \cong \operatorname{Id}^{\ell}_{\mathrm{c}} \mathbb{V}_{\Bbbk}$.*

(iii) *D is a generalized dual Heyting algebra.*

Proof. (i)⇒(iii). Let $D = \operatorname{Id}^{\ell}_{\mathrm{c}} \mathbb{V}_{\Bbbk}$ and set, as in the proof of Theorem 9.3, $x \smallsetminus y \overset{\mathrm{def}}{=} x - (x \wedge y)$ for all $x, y \in \mathbb{V}$. For all $a \in D$ pick $\overline{a} \in \mathbb{V}^{+}$ such that $a = \langle \overline{a} \rangle^{\ell}_{\Bbbk}$. For all $a, b \in D$, the set $\left\{ \langle \overline{a} \smallsetminus \overline{b}\lambda \rangle^{\ell}_{\Bbbk} \mid \lambda \in \Bbbk^{++} \right\}$ is contained in D, thus there exists a subset $S_{a,b}$ of $\Bbbk^{++}$, with at most $\operatorname{card} D$ elements, such that

$$\left\{ \langle \overline{a} \smallsetminus \overline{b}\lambda \rangle^{\ell}_{\Bbbk} \mid \lambda \in \Bbbk^{++} \right\} = \left\{ \langle \overline{a} \smallsetminus \overline{b}\lambda \rangle^{\ell}_{\Bbbk} \mid \lambda \in S_{a,b} \right\} .$$

The set $S \overset{\mathrm{def}}{=} \bigcup (S_{a,b} \mid a, b \in D)$ has thus at most $(\operatorname{card} D)^3$ elements, and

$$\left\{ \langle \overline{a} \smallsetminus \overline{b}\lambda \rangle^{\ell}_{\Bbbk} \mid \lambda \in \Bbbk^{++} \right\} = \left\{ \langle \overline{a} \smallsetminus \overline{b}\lambda \rangle^{\ell}_{\Bbbk} \mid \lambda \in S \right\} \quad \text{whenever } a, b \in D .$$

The set S has an upper bound ξ in $\Bbbk^{++}$ (this is obvious if D is finite, and follows from our assumption otherwise). Hence $\langle \overline{a} \smallsetminus \overline{b}\xi \rangle^{\ell}_{\Bbbk} \subseteq \langle \overline{a} \smallsetminus \overline{b}\lambda \rangle^{\ell}_{\Bbbk}$ for all $a, b \in D$ and all $\lambda \in \Bbbk^{++}$. We claim that for all $a, b \in D$, $a \smallsetminus_D b$ is defined and equal to $\langle \overline{a} \smallsetminus \overline{b}\xi \rangle^{\ell}_{\Bbbk}$. Indeed, $\langle \overline{a} \rangle^{\ell}_{\Bbbk} \subseteq \langle \overline{b} \rangle^{\ell}_{\Bbbk} \vee \langle \overline{a} \smallsetminus \overline{b}\xi \rangle^{\ell}_{\Bbbk}$. Further, for all $x \in D$ such that $a \leq b \vee x$, there exists $\lambda \in \Bbbk^{++}$ such that $\overline{a} \leq (\overline{b} + \overline{x})\lambda$, which implies that $\overline{a} \smallsetminus \overline{b}\lambda \leq \overline{x}\lambda$, whence $\langle \overline{a} \smallsetminus \overline{b}\xi \rangle^{\ell}_{\Bbbk} \subseteq \langle \overline{a} \smallsetminus \overline{b}\lambda \rangle^{\ell}_{\Bbbk} \subseteq \langle \overline{x} \rangle^{\ell}_{\Bbbk} = x$, thus completing the proof of our claim.

(iii)⇒(ii) follows from Theorem 10.5.

(ii)⇒(i). There are totally ordered fields as in (i). □

By Stone duality, we obtain:

Corollary 10.7. *A completely normal generalized spectral space X is the ℓ-spectrum of a right vector lattice over any totally ordered division ring iff the Stone dual of X is a generalized dual Heyting algebra.*

We shall now relate Theorem 10.6 to the results of Section 9.

Theorem 9.3 is not the first occurrence of the property "indecomposable ⇒ join-irreducible" in the literature. In particular, a slight modification of the argument of Martínez [22, Lemma 2.2.3] yields the following:

Proposition 10.8. *Let D be a completely normal generalized dual Heyting algebra. Then every indecomposable element of D is join-irreducible.*

Proof. Let $e \in D$ and let $a, b \in D$ such that $e = a \vee b$. From the complete normality of D we get the disjoint join $e = a' \vee b'$ with $a' \overset{\text{def}}{=} a \smallsetminus_D (b \smallsetminus_D a)$ and $b' \overset{\text{def}}{=} (b \smallsetminus_D a) \smallsetminus_D a = b \smallsetminus_D a$. If e is indecomposable, then $e \in \{a', b'\}$, thus, since $a' \leq a$ and $b' \leq b$, we get $e \in \{a, b\}$. Therefore, e is join-irreducible. $\square$

Although Theorem 9.3 and Proposition 10.8 may raise the hope that the representability of a distributive lattice D, as the range of premeasures on vector lattices over arbitrary totally ordered fields, implies that D is a generalized dual Heyting algebra, the following example shows that this is not the case; and a bit more.

Example 10.9. *A countable, bounded, completely normal distributive lattice D (necessarily not a generalized dual Heyting algebra) such that:*

(1) *For every totally ordered division ring $\Bbbk$, there are a right $\Bbbk$-vector lattice $\mathbb{V}$ such that $\operatorname{Spec} D$ embeds into $\operatorname{Spec}_\ell \mathbb{V}_\Bbbk$ as a spectral subspace.*
(2) *For every formally real field $\Bbbk$, there exists a commutative unital $\Bbbk$-algebra A such that $\operatorname{Spec}_\mathrm{r} A \cong \operatorname{Spec} D$.*
(3) *There exists a totally ordered field $\Bbbk$ such that there is no right $\Bbbk$-vector lattice $\mathbb{V}$ with $\operatorname{Spec}_\ell \mathbb{V}_\Bbbk \cong \operatorname{Spec} D$.*

Proof. Let D be the lattice denoted by $\boldsymbol{D}_\omega$ in Wehrung [35, Example 10.5]. That is, D consists of all pairs (X, k), where X is a subset of ω that is either finite or cofinite, $k \in \{0, 1, 2\}$, $k = 0$ implies that X is finite, and $k > 0$ implies that X is cofinite. The ordering on D is defined componentwise.

Setting $\mathbf{3} \overset{\text{def}}{=} \{0, 1, 2\}$, the sublattice E of $\mathbf{3}^\omega$, consisting of all eventually constant sequences, is a dual Heyting algebra. For each $x \in E$, the subset $\operatorname{supp}(x) \overset{\text{def}}{=} \{n < \omega \mid x(n) \neq 0\}$ is either finite or cofinite in ω and $x(\infty)$, defined as $x(n)$ for large enough n, belongs to $\mathbf{3}$. The map $\pi\colon E \to D$, $x \mapsto (\operatorname{supp}(x), x(\infty))$ is a surjective lattice homomorphism (cf. Wehrung [35, Example 10.6]).

Let $\Bbbk$ be a totally ordered division ring and consider the lexicographical product K of $\Bbbk$ by itself, endowed with its canonical structure of (totally ordered) right $\Bbbk$-vector lattice. The map $\rho\colon K \to \mathbf{3}$ defined by the rule

$$\rho(x, y) \overset{\text{def}}{=} \begin{cases} 0\,, & \text{if } x = y = 0\,, \\ 1\,, & \text{if } x = 0 \text{ and } y \neq 0\,, \\ 2\,, & \text{if } x \neq 0\,, \end{cases} \qquad \text{for all } (x, y) \in K$$

is a surjective measure from $K_\Bbbk$ onto $\mathbf{3}$. Denoting by $\mathbb{V}$ the right $\Bbbk$-vector lattice consisting of all eventually constant sequences of elements of K, the assignment $((x_n, y_n) \mid n < \omega) \mapsto (\rho(x_n, y_n) \mid n < \omega)$ defines a surjective measure $\overline{\rho}\colon \mathbb{V}_\Bbbk \twoheadrightarrow E$. Therefore, $\pi \circ \overline{\rho}$ is a surjective measure from $\mathbb{V}_\Bbbk$ onto D, so it defines a surjective homomorphism from $\operatorname{Id}_\mathrm{c}^\ell \mathbb{V}_\Bbbk$ onto D. By Stone duality, (1) follows.

Now let us prove (2). We may assume that $\Bbbk$ is a real-closed field. Replacing the real field $\mathbb{R}$ by $\Bbbk$ in the argument of the proof, relative to $\mathbb{R}[[G]]$, of Dickmann *et al.* [13, Proposition 1.1], then taking G a divisible totally ordered Abelian group with exactly two nonzero Archimedean classes, we get a totally ordered real-closed $\Bbbk$-algebra A, without zero divisors, such that the real spectrum of A is the two-element chain. Denoting by K any real-closed field extension of A, it follows that the map $\varepsilon\colon \mathbf{3} \twoheadrightarrow \mathbf{2}$, defined by $\varepsilon(0) = 0$ and $\varepsilon(1) = \varepsilon(2) = 1$, is represented, with respect to the Stone dual of the functor $\operatorname{Spec}_\mathrm{r}$, by the inclusion map from A into K. Therefore, mimicking part of the argument of [34, Theorem 5.4], $\operatorname{Spec} D \cong \operatorname{Spec}_\mathrm{r} R$

where the "condensate" R is defined as the $\Bbbk$-algebra of all eventually constant sequences of elements of K with limit in A.

Finally, there is no least $x \in D$ such that $(\omega, 2) \leq (\omega, 1) \vee x$; whence D is not a generalized dual Heyting algebra. Using Corollary 10.7, Item (3) follows. $\square$

This suggests the following problem.

Problem. Characterize the spectral spaces that can be represented as real spectra of algebras over any formally real field.

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