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On the (di)graphs with (directed) proper connection number two ¹

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Abstract

The (directed) *proper connection* number of a given (di)graph G is the least number of colors needed to edge-color G such that there exists a properly colored (di)path between every two vertices in G . There also exist vertex-coloring versions of the proper connection number in (di)graphs. We initiate the study of the *complexity* of computing the proper connection number and (two variants of) the proper vertex connection number, in undirected and directed graphs, respectively. First we disprove some conjectures of Magnan et al. (2016) on characterizing strong digraphs with *directed proper connection number* at most two. In particular, we prove that deciding whether a given digraph has directed proper connection number at most two is NP-complete. Furthermore, we show that there are infinitely many such digraphs without an even-length dicycle. To the best of our knowledge, the *proper vertex connection number* of digraphs has not been studied before. We initiate the study of proper vertex connectivity in digraphs and we prove similar results as for the arc version. Finally, on a more positive side we present polynomial-time recognition algorithms for *bounded-treewidth* graphs and *bipartite* graphs with *proper connection number* at most two.

Keywords: proper connection; directed graphs; even dicycles; NP-complete.

1. Introduction

There is a broad literature on frequency assignment problems in radio networks and their modeling by colorings problems on (di)graphs [11, 12,

¹Results of this paper were partially presented at the LAGOS'17 conference [19].

23]. We study a relaxed invariant, introduced by Borozan et al. [7], where we only impose for every two vertices u, v to have a properly edge-colored (di)path from u to v – the (di)graph itself may not be properly edge-colored. The latter concept is sometimes called *proper connectivity*. We motivate the study of proper connectivity next.

MOTIVATION. Properly colored paths have applications in many fields like genetics [16–18], social sciences [14], radio communication [31], etc. We present a concrete application of *proper connectivity* related to wireless networks [31]. Indeed in any network the natural requirement is to have all the parties connected and the goal is to maximize throughput. In wireless and communication networks, it is sometimes impossible to transmit a signal directly between, say, two communication towers A and B. Thus, the signal should go through some other towers. In order to maximize throughput, it is desirable to avoid interference by ensuring that the incoming and the outgoing signal from a tower should be on a different frequency. Suppose that we assign a vertex to each signal tower, an edge between two vertices if the corresponding signal towers are directly connected by a signal and a color to each edge corresponding to the frequency used for the communication. Then, the number of frequencies needed to assign the connections between towers so that there is always a path avoiding interference between each pair of towers is precisely the proper connection number of the corresponding graph.

Another application of proper connectivity is password sharing, where we aim at exchanging information between, say, secret agencies, while not repeating the same password for any two consecutive communications.

RELATED WORK. The proper connection number in undirected graphs was first defined in [7] by Borozan et al. In the same paper, the authors relate the proper connection number with the graph connectivity. Since then the problem was intensively studied from the combinatorial point of view [2, 7, 13, 22, 27–30, 32, 34]. In particular, bounds on the proper connection number of random graphs and bipartite graphs have been proved in [22] and [7, 26, 27], respectively. Relationships between the proper connection number and domination number can be found in [32]. Many generalizations of proper connectivity have been proposed [2, 7, 27, 33]. For instance, in this paper, we also study some notions of *proper vertex connection*, *i.e.*, vertex-coloring versions of the proper connection number (see [13, 30]). More recently, Magnant et al. studied the *directed* proper connection number for digraphs [34]. They proved that this number is always at most three and

they asked to characterize the digraphs with directed proper connection number at most two. In particular, they conjectured that all such digraphs must contain an even dicycle. For more details, see the following surveys on the proper connection number: [3, 31] and Chapter 16 of [4]. Nevertheless, no complexity results have been proved for this problem so far. Our goal is to fill in this gap in the literature, as discussed in Sections 3-5.

The proper connection number is related to the rainbow connection number, defined in [11]. The rainbow connection number of a given graph G is the least number of colors needed to edge-color G such that there exists a rainbow path (i.e., a path with all its edges colored with a distinct color) between every two vertices in G . Since computing the rainbow connection number is NP-hard and not FPT for any fixed $k \geq 2$ [1, 9] it is natural to study the complexity of computing the proper connection number – which can be seen as a relaxed variant of rainbow connectivity.

OUR RESULTS. In this paper we study the complexity of computing the (directed) proper connection number of a given (di)graph. This gives two parameters to study, that are denoted by $pc(G)$ for undirected graphs G and $\vec{pc}(D)$ for directed graphs D . We also study two variants of the proper *vertex* connection number, that are denoted by $pvc(G)$ and $vpc(G)$ for undirected graphs G ; $\vec{pvc}(D)$ and $\vec{vpc}(D)$ for directed graphs D . We will explain the slight difference between these two concepts of proper vertex connectivity in Section 2.

In Section 3, we study first the computational complexity of computing $\vec{pc}(D)$ for a given digraph D . In [34] Magnant et al. prove that $\vec{pc}(D) \leq 3$ for every strong digraph D . We prove here that deciding whether $\vec{pc}(D) \leq 2$ is NP-complete (Theorem 1). Furthermore, we disprove a conjecture of Magnant et al. from [34], claiming that if D is a strong digraph without an even dicycle, then $\vec{pc}(D) = 3$. Indeed we show, via an exhaustive search using a computer program, that D_7 , the only strongly biconnected graph without an even dicycle, has a properly connected 2-coloring. Based on the coloring of D_7 , we construct an infinite family of digraphs without an even dicycles that also have properly connected 2-colorings (Proposition 3).

Next, in Section 4, we turn our attention to the computational complexity of computing $\vec{pvc}(D)$ and $\vec{vpc}(D)$ for a given digraph D . To the best of our knowledge, the concept of directed proper vertex connectivity has not been studied before. We initiate the study of two proper vertex connection numbers for digraphs and we prove that every strong digraph admits a properly connected 3-coloring (Theorem 4). However, deciding whether $\vec{pvc}(D) \leq 2$, resp. whether $\vec{vpc}(D) \leq 2$, is also NP-complete (Theorem 6). Al-

though the proofs of Theorems 1 and 6 are similar, none of the two results is implied by the other. Furthermore, we stress that in contrast with our results for digraphs, we have $pvc(G) \leq vpc(G) \leq 2$ for any *undirected* graph G and both parameters can be trivially computed in polynomial time [13, 30].

Finally, in Section 5, we study the proper connection number of undirected graphs. This parameter is much more difficult to understand than the proper *vertex* connection numbers – though it has been extensively studied in the literature. We propose advances in the study of proper connectivity by presenting some tractable cases. In particular, Borozan et al. [7] show that any tree with proper connection number two is a path. We extend their result and prove that for some graph classes generalizing the trees, namely graphs with bounded treewidth and bipartite graphs, we can decide in polynomial time if the proper connection number is less than or equal to two. Our result for bipartite graphs is especially interesting since $pc(G)$ for bipartite graphs has been extensively studied in the literature and several sufficient conditions have been given for these graphs to have proper connection number equal to two [7, 10, 20, 26, 27]. We use in our proof the concept of bridge-block tree introduced by Tarjan [40]. We remark that independently from this work, Melville and Goddard also characterized bipartite graphs with proper connection number two [37].

The complexity of deciding whether $pc(G) \leq 2$ for general graphs is left as an interesting open question.

2. Preliminaries

We follow notations and terminology from [4, 6]. We only consider finite connected graphs and finite strong digraphs in the following. We write $[u = x_0, x_1, \dots, x_l = v]$ for an uv -(di)path.

Given a graph $G = (V, E)$, an k -edge-coloring is a function $c : E \rightarrow \{1, \dots, k\}$. Such a coloring of G (and by extension, the edge-colored graph G_c itself) is called *proper* if every two incident edges differ in colors. The coloring c (and by extension, the edge-colored graph G_c) is called *properly connected* if there exists a properly colored path between every two vertices. The *proper connection number* of G , denoted $pc(G)$, is the least number of colors needed to find a properly connected edge-coloring of G .

Similarly, given a *digraph* $D = (V, A)$, an k -arc-coloring is a function $c : A \rightarrow \{1, \dots, k\}$. Such a coloring (and by extension, the arc-colored digraph D_c itself) is called *proper* if every two incident arcs differ in colors. The coloring c (and by extension, the arc-colored digraph D_c) is called *properly connected* if there exists a properly colored dipath from any vertex to any

other vertex. The *directed* proper connection number of D , denoted $\overrightarrow{pc}(D)$, is the least number of colors needed to find a properly connected arc-coloring of D .

These notions were adapted to *vertex-colorings*. Specifically, given a (di)graph $G = (V, E)$, an k -coloring is a function $c : V \rightarrow \{1, \dots, k\}$. Such a coloring (and by extension, the vertex-colored graph G_c) is called proper if every two adjacent vertices differ in colors. We define two different notions of proper *vertex* connectivity:

- Most authors call a coloring c (and by extension, G_c) properly vertex connected if, from any vertex to any other vertex, there exists a (di)path whose any two *internal* adjacent vertices differ in color (*e.g.*, see [30]). The (directed) *proper vertex connection number* of G , denoted $pvc(G)$ (or $\overrightarrow{pvc}(G)$ if G is a digraph), is the least number of colors needed to find a properly vertex connected coloring of G .
- Alternatively, say that a coloring c (a vertex-colored graph G_c , resp.) is vertex properly connected if, from any vertex to any other vertex, there exists a properly colored (di)path [13]. The (directed) *vertex proper connection number* of G , denoted $vpc(G)$ (or $\overrightarrow{vpc}(G)$ if G is a digraph), is the least number of colors needed to find a vertex properly connected coloring of G .

In order to better understand the difference between $pvc(G)$ and $vpc(G)$, let us consider a graph G with diameter two. Since any shortest-path in G has at most one internal vertex, we have that every 1-coloring is properly vertex connected, and so, $pvc(G) = 1$. However, $vpc(G) \geq 2$ for any graph G with at least two vertices.

Remark 1. *Deciding whether $pc(G) \leq k$ is in NP, Indeed, as a certificate for yes-instances, it suffices to give a properly connected k -edge-coloring and, for every pair u, v of vertices, a properly colored uv -path. Overall, we need to specify $\binom{n}{2}$ paths, each path having length at most $n - 1$. So, this is indeed a polynomial-size certificate. Furthermore, the correctness of this certificate can be checked in polynomial time, as follows. For every pair u, v of vertices, we check whether the corresponding path in the certificate is proper; since the coloring is also part of our certificate, this verification phase can be done in time linear in the size of the path. Overall, we can check whether this certificate is correct in time linear in its size, and so, in time polynomial in the order of the input graph G .*

We can prove similarly that deciding whether $\vec{pc}(D) \leq k$ (resp., $pvc(G) \leq k$, $\vec{pvc}(D) \leq k$, $vpc(G) \leq k$, $\vec{vpc}(D) \leq k$) is in NP.

It is open whether given a (di)graph G and a (vertex or edge) coloring c , we can check in polynomial time whether c is properly connected.

3. Directed proper connection number

This section is devoted to the complexity of computing $\vec{pc}(D)$ for a given digraph D . In Section 3.1, we prove that the corresponding decision problem is NP-complete. This further motivates the study of (polynomial-time computable) sufficient conditions for keeping this parameter as small as possible. In Section 3.2 we disprove a conjecture from [34]: namely, we show that there are infinitely many digraphs D such that $\vec{pc}(D) = 2$ and D has no even dicycle.

3.1. NP-hardness

As proved in [34], for every (strong) digraph D that is not a complete symmetric digraph we have $\vec{pc}(D) \in \{2, 3\}$. Therefore, computing $\vec{pc}(D)$ is equivalent to deciding whether $\vec{pc}(D) \leq 2$. In order to prove that this decision problem is NP-hard, we reduce POSITIVE NAE-SAT (Problem 1), which is a well-known variation of the Boolean Satisfiability problem and known to be NP-complete by Schaefer's dichotomy theorem [38], to it.

Problem 1 (POSITIVE NAE-SAT).

Input: A propositional formula Φ in conjunctive normal form, with unnegated variables.

Question: Does there exist a truth assignment satisfying Φ in which no clause has all its literals valued 1 ?

Theorem 1. Deciding whether $\vec{pc}(D) \leq 2$ for a given digraph D is NP-complete.

Proof. The problem is in NP (cf. Remark 1). In order to prove the NP-hardness, let Φ be an arbitrary formula in conjunctive normal form, with unnegated variables. The incidence graph of Φ is a bipartite graph B_Φ , with one partite set being the m clauses C_1, \dots, C_m of Φ and the other partite set being the n variables x_1, \dots, x_n that are contained in at least one clause; for every $1 \leq j \leq m$, there exists an edge between C_j and every variable x_i that

is contained in C_j . In what follows is a description of the transformation of B_Φ to a digraph D_Φ such that $\overline{pc}(D_\Phi) = 2$ if and only if Φ is a yes-instance of POSITIVE NAE-SAT. We illustrate the reduction in Figure 1.

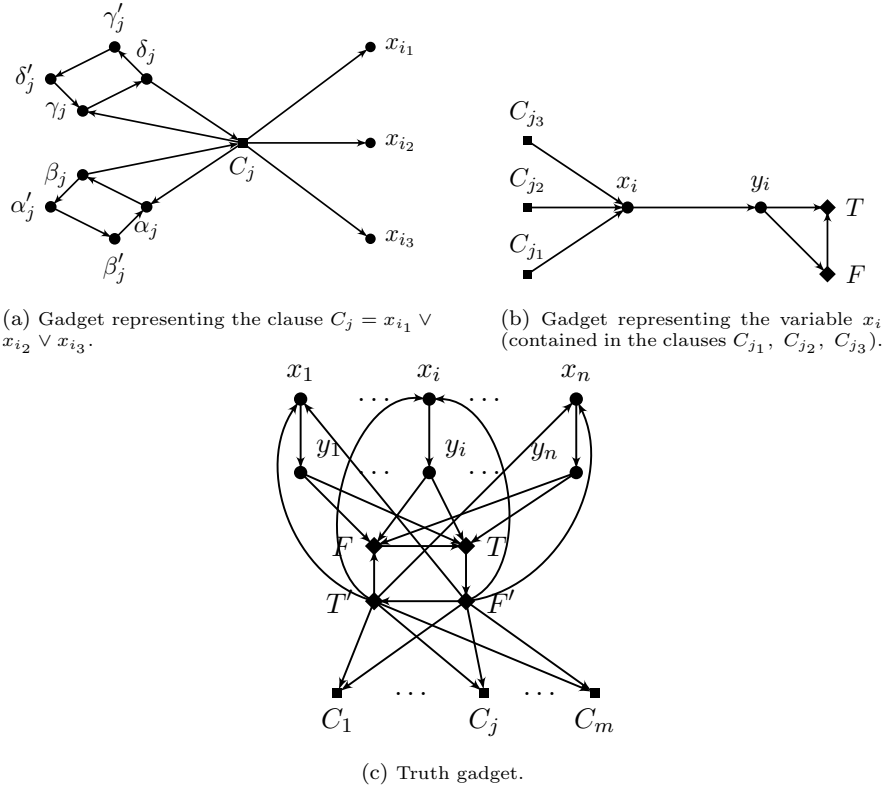


Figure 1: The reduction.

Clause gadget. (see Fig. 1a). Let $1 \leq j \leq m$ be fixed. We replace every edge $\{C_j, x_i\}$ in B_Φ by an arc (C_j, x_i) (oriented from C_j to x_i). Furthermore, we add eight vertices $\alpha_j, \beta_j, \gamma_j, \delta_j$ and $\alpha'_j, \beta'_j, \gamma'_j, \delta'_j$ such that: the triples $[C_j, \alpha_j, \beta_j, C_j]$ and $[C_j, \gamma_j, \delta_j, C_j]$ induce dicycles of length three, while the 4-tuples $[\alpha_j, \beta_j, \alpha'_j, \beta'_j, \alpha_j]$ and $[\gamma_j, \delta_j, \gamma'_j, \delta'_j, \gamma_j]$ induce dicycles of length four.

Our construction will ensure that there is no other arc incident to either $\alpha_j, \beta_j, \gamma_j, \delta_j$ or $\alpha'_j, \beta'_j, \gamma'_j, \delta'_j$. Note that it will imply that the only $\alpha_j \delta_j$ -dipath is $[\alpha_j, \beta_j, C_j, \gamma_j, \delta_j]$, and similarly the only $\gamma_j \beta_j$ -dipath is $[\gamma_j, \delta_j, C_j, \alpha_j, \beta_j]$. Hence, in order to obtain proper dipaths with two colors, we need to color the arcs (C_j, α_j) , (β_j, C_j) and (γ_j, δ_j) with the same color and to color the arcs (α_j, β_j) , (C_j, γ_j) and (δ_j, C_j) with the other color.

Variable gadget. (see Fig. 1b). We add two vertices F and T , along with an arc (F, T) . We show later how to use these two vertices in order to obtain a truth assignment satisfying Φ . Let $1 \leq i \leq n$ be fixed. We add a vertex y_i , along with an arc (x_i, y_i) and two more arcs (y_i, F) and (y_i, T) .

Our construction will ensure that y_i is the only out-neighbour of x_i . This will imply that any dipath ending at x_i can be made longer only by passing through the arc (x_i, y_i) . The color of this arc (x_i, y_i) will intuitively represent the truth assignment of the variable x_i .

Truth gadget. (see Fig. 1c). We add two vertices T', F' such that $[F, T, F', T', F]$ induces a dicycle of length four. Finally, for every $1 \leq i \leq n$ we add the arcs (T', x_i) and (F', x_i) , and for every $1 \leq j \leq m$, we add the arcs (T', C_j) and (F', C_j) .

Observe that by construction, for every vertex $v \in V(D_\Phi) \setminus \{T', F'\}$, any vF' -dipath (any vT' -dipath) must pass through the arc (T, F') (the dipath $[T, F', T']$). The color of this arc (T, F') will intuitively represent the value “True”. Furthermore, the arcs (T', x_i) and (F', x_i) , for every $1 \leq i \leq n$, and the arcs (T', C_j) and (F', C_j) , for every $1 \leq j \leq m$, are added to the construction simply to ensure the existence of a properly connected arc-coloring of D_Φ if Φ is a yes-instance of POSITIVE NAE-SAT (this will be made more precise in the following).

The resulting digraph D_Φ can be constructed from Φ in $\mathcal{O}(\sum_{j=1}^m |C_j|)$ -time, that is linear in the size of the formula. In what follows, we prove $\vec{pc}(D_\Phi) = 2$ if and only if Φ is a yes-instance of POSITIVE NAE-SAT. Since POSITIVE NAE-SAT is NP-complete, it proves the theorem.

In one direction, assume $\vec{pc}(D_\Phi) = 2$. We call Blue and Red the two colors used in a properly connected arc-coloring of D_Φ . Without loss of generality, the arc (T, F') is colored Blue. Furthermore, recall that by construction, for every $1 \leq j \leq m$, the three arcs (C_j, α_j) , (β_j, C_j) and (γ_j, δ_j) must have the same color, while the three arcs (α_j, β_j) , (C_j, γ_j) and (δ_j, C_j) must have the other color (otherwise, the arc-coloring could not be properly connected). By symmetry, let us assume that (C_j, α_j) , (β_j, C_j) and (γ_j, δ_j) are colored Red (and so, (α_j, β_j) , (C_j, γ_j) and (δ_j, C_j) are colored Blue).

We claim that there must exist a variable x_i contained in C_j such that the arc (x_i, y_i) is colored Red. Indeed, we have on one hand that any $\beta_j T'$ -dipath must start with a subdipath $[\beta_j, C_j, x_i, y_i]$, for some x_i contained in C_j . In particular, since the arc-coloring is assumed to be properly connected

there must exist a proper $\beta_j T$ -dipath, and so, there must exist a dipath $[\beta_j, C_j, x_i, y_i]$ that is proper. Since on the other hand, the arc (β_j, C_j) is colored Red, therefore the arc (x_i, y_i) in this proper dipath must be also colored Red. In the same way (up to replacing β_j by δ_j), there must exist a variable x_i contained in C_j such that the arc (x_i, y_i) is colored Blue. Therefore, by assigning 1 to all the variables x_i such the arc (x_i, y_i) is colored Blue, and by assigning 0 to all the remaining variables (variables $x_{i'}$ such the arc $(x_{i'}, y_{i'})$ is colored Red), one obtains a truth assignment satisfying Φ in which no clause has all its literals valued 1. This implies that Φ is a yes-instance of POSITIVE NAE-SAT.

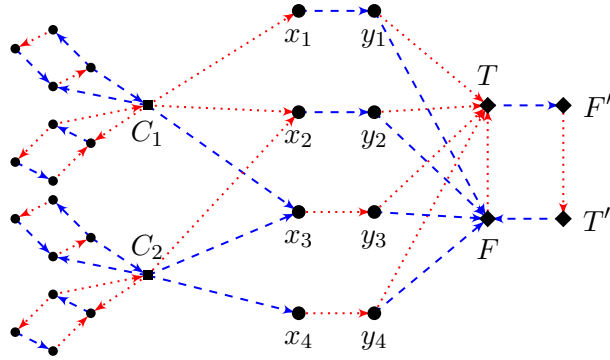


Figure 2: A properly connected arc-coloring of D_Φ , with $\Phi = (x_1 \vee x_2 \vee x_3) \wedge (x_2 \vee x_3 \vee x_4)$. For ease of readability, the arcs (T', x_i) and (F', x_i) , for every $1 \leq i \leq n$, and the arcs (T', C_j) and (F', C_j) , for every $1 \leq j \leq m$, are not drawn.

In the other direction, assume that Φ is a yes-instance of POSITIVE NAE-SAT. Let us consider a truth assignment satisfying Φ in which no clause has all its literals valued 1. We color the arcs of D_Φ as follows (see Figure 2 for an illustration).

- For every $1 \leq i \leq n$, the arc (x_i, y_i) is colored Blue if the variable x_i is valued 1, and it is colored Red otherwise. Furthermore, the arcs (y_i, T) and (F', x_i) are both colored Red while the arcs (y_i, F) and (T', x_i) are both colored Blue.
- For every $1 \leq j \leq m$, the arcs (C_j, α_j) , and (β_j, C_j) are colored Red and the arcs (C_j, γ_j) and (δ_j, C_j) are colored Blue. The arcs (α_j, β_j) , (α'_j, β'_j) , (δ_j, γ'_j) and (δ'_j, γ_j) are colored Blue, whereas the arcs (β_j, α'_j) , (β'_j, α_j) , (γ_j, δ_j) and (γ'_j, δ'_j) are colored Red. Furthermore, for every variable x_i contained in C_j , the arc (C_j, x_i) is colored

Red if x_i is valued 1 and it is colored Blue otherwise. The arcs (T', C_j) and (F', C_j) are colored Blue and Red, respectively.

- Finally, the arcs (F, T) and (F', T') are both colored Red, while the arcs (T, F') and (T', F) are both colored Blue.

Let us prove that the resulting arc-coloring of D_Φ is properly connected. In order to do so, we first notice that all the dipaths in the following collection \mathcal{P}^- (ending in T) are proper:

- $[F', T', F, T]$;
- $[\alpha'_j, \beta'_j, \alpha_j, \beta_j, C_j, x_i, y_i, F, T]$, if x_i is contained in C_j and valued 0;
- $[\gamma'_j, \delta'_j, \gamma_j, \delta_j, C_j, x_i, y_i, T]$, if x_i is contained in C_j and valued 1.

Similarly, we notice that all the dipaths in the following collection \mathcal{P}^+ (starting in T) are proper:

- $[T, F', T', F]$;
- $[T, F', C_j, \gamma_j, \delta_j, \gamma'_j, \delta'_j]$ and $[T, F', T', C_j, \alpha_j, \beta_j, \alpha'_j, \beta'_j]$, for every $1 \leq j \leq m$;
- $[T, F', x_i, y_i]$ with x_i being valued 1;
- $[T, F', T', x_i, y_i]$ with x_i being valued 0.

Let us finally prove that for every $u, v \in V(D_\Phi)$, there exists a proper uv -dipath. We consider three different cases.

1. If some dipath of $\mathcal{P}^- \cup \mathcal{P}^+$ contains a uv -subdipath then the statement holds true.
2. Otherwise, if there exist a proper uT -dipath (subdipath of a dipath in \mathcal{P}^-) and a proper Tv -dipath (subdipath of a dipath in \mathcal{P}^+) that only intersect in T , then we claim that the statement is also true. Indeed, both proper dipaths can be merged into one since by construction, every proper uT -dipath ends with an arc colored Red whereas every proper Tv -dipath starts with an arc colored Blue.
3. Furthermore, it can be checked that any two vertices u, v fall in one of the two above cases, unless $u, v \in \{\alpha_j, \beta_j, \gamma_j, \delta_j, C_j, \alpha'_j, \beta'_j, \gamma'_j, \delta'_j\}$ for some $1 \leq j \leq m$. In this case, there is a proper uv -dipath that is a subdipath of either: $[\alpha'_j, \beta'_j, \alpha_j, \beta_j, C_j, \gamma_j, \delta_j, \gamma'_j, \delta'_j]$;

or $[\gamma'_j, \delta'_j, \gamma_j, \delta_j, C_j, \alpha_j, \beta_j, \alpha'_j, \beta'_j]$;
or of one of the two dicycles $[\alpha_j, \beta_j, \alpha'_j, \beta'_j, \alpha_j]$ or $[\gamma_j, \delta_j, \gamma'_j, \delta'_j, \gamma_j]$.
So, a path also exists in this final case.

□

3.2. Strong digraphs without an even dicycle

In [34], Magnant et al. conjecture that if D is a strong digraph without an even dicycle, then $\vec{p}\vec{e}(D) = 3$. In the light of our hardness result of Theorem 1, it is even more interesting to study their conjecture since graphs without an even dicycle can be recognized in polynomial time [36]. However, we disprove the conjecture and we construct an infinite collection of digraphs that disprove their claim.

For that, we first consider the case of strongly biconnected digraphs. There is only one strongly biconnected digraph without an even dicycle (up to an isomorphism), namely the digraph D_7 from Figure 3a [35]. The following Lemma can be deduced from the (properly connected) arc-coloring indicated in Figure 3b.

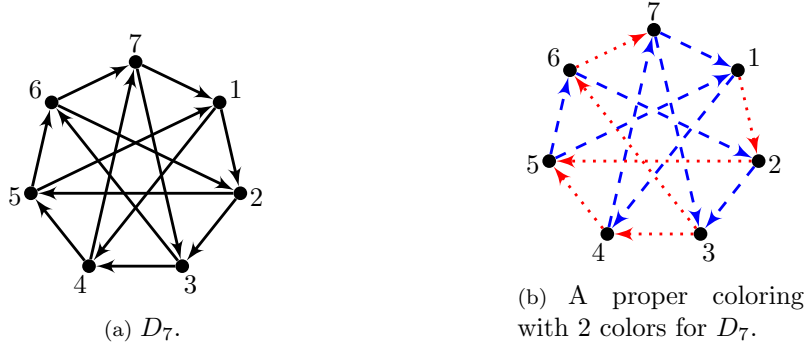


Figure 3: Digraph D_7 .

Lemma 2. $\vec{p}\vec{e}(D_7) = 2$. Moreover, there exists a properly connected arc-coloring with the property that both outgoing arcs from vertex 3 and both ingoing arcs to vertex 5 have the same color c_1 . Furthermore, both ingoing arcs to vertex 3 and both outgoing arcs from vertex 5 have the same color c_2 , with $c_1 \neq c_2$.

Observe that there exist at least two such colorings (as stated in Lemma 2) since we can always exchange the colors of all the arcs while keeping the property to be properly connected.

Furthermore, based on Lemma 2, we can construct infinitely many other counterexamples to the conjecture in [34]. In order to do that, let H_1, H_2 be two digraphs and v_1, v_2 two fixed vertices from H_1 and H_2 , respectively. We denote by $(H_1, v_1) \odot (H_2, v_2)$ the digraph obtained from the union of graphs H_1 and H_2 by identifying vertices v_1 and v_2 . We recursively define an infinite family $(SD_n)_{n \in \mathbb{N}_{\geq 1}}$ of strong digraphs and fix a vertex v_n in every digraph SD_n as follows, starting from D_7 and using vertices $x_1 = 3$ and $x_2 = 5$ from Lemma 2.

1. $SD_1 = D_7$; we fix in SD_1 vertex $v_1 = x_2$;
2. $SD_n = (SD_{n-1}, v_{n-1}) \odot (D_7, x_1)$ and we set $v_n = x_2$.

Proposition 3. *For every $n \geq 1$, SD_n is a strong digraph without an even dicycle having $\vec{pc}(SD_n) = 2$. Moreover, there is a properly connected arc-coloring of SD_n with two colors $c_1 \neq c_2$ where all ingoing arcs to v_n have color c_1 and all outgoing arcs from v_n have color c_2 .*

Proof. We prove the result by induction on n . If $n = 1$, then the result follows from Lemma 2. Otherwise $n \geq 2$ and we assume for our induction hypothesis that SD_{n-1} has the stated properties. By construction, SD_n is a strong digraph. Furthermore, it has no even dicycle since every dicycle of SD_n is either contained in SD_{n-1} or in D_7 .

Finally, by the induction hypothesis, there exists a (properly connected) 2-arc coloring of SD_{n-1} with two colors Red and Blue such that all ingoing arcs to v_{n-1} are colored Red, and all outgoing arcs from v_{n-1} are colored Blue. By Lemma 2, there exists a (properly connected) 2-arc coloring of D_7 such that both outgoing arcs from x_1 and both ingoing arcs in x_2 are colored Blue and all outgoing arcs from $v_n = x_2$ and ingoing arcs in x_1 are colored Red (Figure 4).

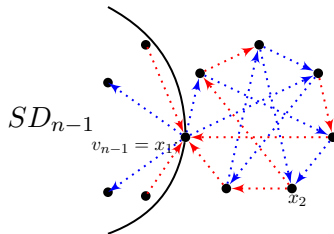


Figure 4: A proper coloring for SD_n

Since $E(SD_n) = E(SD_{n-1}) \dot{\cup} E(D_7)$, the two colorings of SD_{n-1} and D_7 determine an arc-coloring of SD_n . To prove that this coloring is properly

connected it suffices to consider pairs of vertices u, v with $u \in V(SD_{n-1})$ and $v \in V(D_7)$ and to prove that there exist a proper uv -dipath and a proper vu -dipath. Indeed, since the coloring restricted to SD_{n-1} is properly connected, there exists a proper uv_{n-1} -dipath P in SD_{n-1} . Similarly, there exists a proper x_1v -dipath Q in D_7 . By merging these two dipaths identifying $v_{n-1} = x_1$ we obtain an uv -dipath in SD_n . The obtained dipath is proper since the ingoing arc to v_{n-1} in P has color Red and the outgoing arc from x_1 contained in Q has color Blue. Note that by using a similar argument, we can obtain a proper vu -dipath. Therefore, this coloring of SD_n is properly connected and $v_n = x_2$ verifies the stated properties. \square

4. (Directed) vertex proper connection number

In this section, we initiate the study of the (directed) proper vertex connection number and the (directed) vertex proper connection number in (di)graphs.

Since $pvc(G) \leq vpc(G)$ for any graph G , and in the same way $\overrightarrow{pvc}(D) \leq \overrightarrow{vpc}(D)$ for any digraph D , we mostly focus on the (directed) vertex proper connection number. We observed that this number must be at least two for graphs with more than one vertex. The bound is tight for *undirected* graphs: indeed, every connected graph has a spanning tree, and we can choose its two partite sets as our two color classes². However, the vertex proper connection number is more difficult to study in directed graphs than in undirected graphs.

We prove in Section 4.1 that for every strong digraph D , $\overrightarrow{pvc}(D) \leq \overrightarrow{vpc}(D) \leq 3$. Next, we prove that computing $\overrightarrow{pvc}(D)$, resp. $\overrightarrow{vpc}(D)$, for a given digraph D is NP-hard (Section 4.2). Interestingly, we so obtain similar results for the directed *vertex* proper connection number and the directed proper *vertex* connection number as for the directed proper connection number in digraphs.

4.1. Upper-bound

In what follows is a proof for an upper bound on the vertex proper connection number of a digraph. Unlike the proof for the arc version from [34], our proof is not based on the existence of an ear decomposition of a strong digraph.

²For the proper vertex connection number, we have $pvc(G) = \min\{\text{diam}(G) - 1, 2\}$ for any graph G .

Theorem 4. *If D is a strong digraph, then we have $\overrightarrow{pv\check{c}}(D) \leq \overrightarrow{vp\check{c}}(D) \leq 3$.*

Proof. The inequality $\overrightarrow{pv\check{c}}(D) \leq \overrightarrow{vp\check{c}}(D)$ is trivial. In what follows, we prove $\overrightarrow{vp\check{c}}(D) \leq 3$. We will proceed by induction on the number of vertices in D . If D is a strong digraph with at most three vertices, then any coloring for which all vertices have different colors is properly connected, hence $\overrightarrow{vp\check{c}}(D) \leq 3$. Otherwise let $n \geq 4$. Assume by induction hypothesis that for every strong digraph with $n - 1$ vertices, there exists a vertex-coloring with at most 3 colors that is properly connected. Let D be a strong digraph with n vertices. It suffices to prove that D has a spanning strong digraph D' with $\overrightarrow{vp\check{c}}(D') \leq 3$.

In particular, let D' be a minimally strong spanning digraph of D (*i.e.*, the removal of any arc in D' leaves a smaller spanning digraph which is not strong). Since D' is minimally strong, it has a vertex with in-degree and out-degree equal to 1 [21]. Let v be such a vertex, with u and w being its unique in-neighbor and out-neighbor, respectively. We remove v from D' and we add an arc from u to w if this arc is not already present. Clearly, the resulting digraph D'' is still strong and it has order $n - 1$. Furthermore by induction hypothesis, $\overrightarrow{vp\check{c}}(D'') \leq 3$. Consider such a (properly connected) coloring for D'' . We can obtain from this coloring a properly connected coloring for D' as follows. We color all vertices of $D' \setminus v$ as in D'' , then finally we color v with any color that is not already used for u nor w . Hence $\overrightarrow{vp\check{c}}(D) \leq \overrightarrow{vp\check{c}}(D') \leq 3$. \square

4.2. NP-completeness

Since, by Theorem 4, we have $2 \leq \overrightarrow{vp\check{c}}(D) \leq 3$ for every (strong) digraph D , one important problem is to decide whether $\overrightarrow{vp\check{c}}(D) = 2$. We show that this decision problem is in fact NP-hard by reducing it from 3-SAT (Theorem 6). Our hardness result also holds for the directed proper vertex connection number, due to the following lemma:

Lemma 5. *Computing $\overrightarrow{vp\check{c}}(D)$ for a given strong digraph D can be reduced in polynomial-time to computing $\overrightarrow{pv\check{c}}(D')$ for some other strong digraph D' .*

Proof. Let $D = (V, A)$ be an arbitrary strong digraph. Consider the strong digraph D' obtained from D by adding, for any $v \in V$, a new vertex v' and the two arcs (v, v') and (v', v) . We claim that $\overrightarrow{pv\check{c}}(D') = \overrightarrow{vp\check{c}}(D)$, that will prove the lemma.

In one direction, assume $\overrightarrow{vp\check{c}}(D) = k$ and let c be a corresponding (vertex properly connected) k -coloring. We extend c to a k -coloring c' of D' , simply

by coloring v' with $c(v)$. For every $u, v \in V$, let $x \in \{u, u'\}$ and $y \in \{v, v'\}$ be arbitrary. By construction any uv -dipath P in D is a subdipath of some xy -dipath P' in D' . Furthermore, all the internal vertices of P' are contained in P . Since by the hypothesis, at least one such path P must be proper, c' is properly vertex connected, and so, $\overrightarrow{pvc}(D') \leq \overrightarrow{pvc}(D)$.

In the other direction, assume $\overrightarrow{pvc}(D') = k$ and let c' be a corresponding (properly vertex connected) k -coloring. Let c be the restriction of c' to D . By construction, for every $u, v \in V$, we have that the internal vertices on any $u'v'$ -dipath must induce a uv -dipath P in D . Since c' is properly vertex connected (by the hypothesis), there is an $u'v'$ -dipath whose internal vertices induce a proper dipath P from u to v . Hence, c is vertex properly connected, and so, $\overrightarrow{pvc}(D) \leq \overrightarrow{pvc}(D')$. □

We are now ready to prove the main result of this section.

Theorem 6. *Deciding whether $\overrightarrow{pvc}(D) = 2$, resp. whether $\overrightarrow{pvc}(D) = 2$, for a given digraph D is NP-complete.*

Proof. Both problems are in NP (cf. Remark 1). Furthermore, by Lemma 5, it suffices to prove the result for $\overrightarrow{pvc}(D)$. As in the proof of Theorem 1, let Φ be an arbitrary formula in 3-conjunctive normal form and B_Φ the incidence graph of Φ . Assume that there is no clause in Φ containing both a variable and its negation. Next we describe the transformation of the incidence graph B_Φ to a digraph D_Φ such that $\overrightarrow{pvc}(D_\Phi) = 2$ if and only if Φ is a yes-instance of 3-SAT. This will prove the NP-hardness, and so, the result.

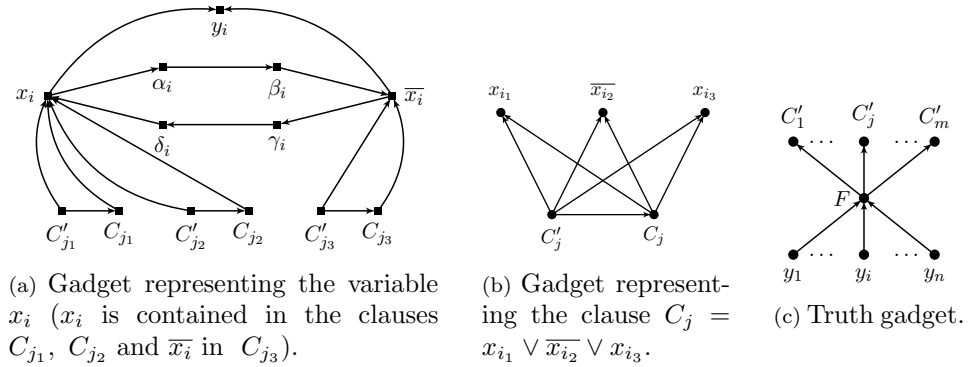


Figure 5: The reduction.

Variable gadget. (see Fig. 5a). Let $1 \leq i \leq n$ be fixed. We add a vertex \bar{x}_i corresponding to the negation of variable x_i . Furthermore, we add four new vertices: $\alpha_i, \beta_i, \gamma_i, \delta_i$ and the necessary arcs such that $[x_i, \alpha_i, \beta_i, \bar{x}_i, \gamma_i, \delta_i, x_i]$ is a dicycle. This dicycle is used to ensure that the vertices x_i and \bar{x}_i have different colors in any vertex properly connected 2-coloring of D_Φ . We also add a vertex y_i , along with the arcs $(x_i, y_i), (\bar{x}_i, y_i)$.

Clause gadget. (see Fig. 5b). Let $1 \leq j \leq m$ be fixed. We replace every edge $\{C_j, x_i\}$ in B_Φ by an arc (C_j, l_i) (directed from C_j to l_i), where l_i is the literal corresponding to variable x_i contained in clause C_j (l_i is either x_i or \bar{x}_i). Furthermore we add a new vertex C'_j , an arc from this vertex to C_j and arcs from C'_j to all the out-neighbors of C_j , that is to all the literals contained in clause C_j . Thus, the vertex C'_j can reach the dicycle corresponding to x_i through dipaths $[C'_j, l_i]$ or $[C'_j, C_j, l_i]$. Our construction will ensure that (C'_j, C_j) is the only in-arc for C_j . Therefore, the vertices C_j and C'_j must have distinct colors in any vertex properly connected coloring of D_Φ . In particular, at least one of these two above $C'_j l_i$ -dipaths must be proper.

Truth gadget. (see Fig. 5c). We add a new vertex F . For every $1 \leq j \leq m$ we add an arc (F, C'_j) . In the same way, for every $1 \leq i \leq n$ we add an arc (y_i, F) .

For every $1 \leq j \leq m$, note that since we have that $[F, C'_j, C_j]$ is the only FC_j -dipath, the vertices F and C_j must have the same color in any vertex properly connected 2-coloring of D_Φ . In particular, there must be, in such a coloring, an out-neighbor of C_j with a different color than F , that will correspond to a literal $l_i \in C_j$ valued by "True".

Furthermore for every $1 \leq i \leq n$, the vertices y_i together with the dicycle containing the vertices x_i and \bar{x}_i will ensure that if any of the two vertices x_i or \bar{x}_i can be reached from a vertex corresponding to a clause by a proper dipath, then F can be reached.

The digraph D_Φ can be constructed from Φ in $\mathcal{O}(m)$ -time, hence linear in the number of clauses. Next we prove that $\overrightarrow{vpc}(D_\Phi) = 2$ if and only if Φ is a yes-instance of 3-SAT.

First assume $\overrightarrow{vpc}(D_\Phi) = 2$. Consider such a (vertex properly connected) coloring c of D_Φ with two colors, which we will call Blue and Red. W.l.o.g. assume that $c(F) = Red$. On the one hand, let $1 \leq j \leq m$ be fixed. Since the only FC_j -dipath is $[F, C'_j, C_j]$, we must have $c(C'_j) = Blue$ and $c(C_j) = Red$. In this situation, at least one of the outgoing arcs from C_j must have its other end colored with Blue. The latter corresponds to a literal l in C_j . On

the other hand, for every $1 \leq i \leq n$, since $[x_i, \alpha_i, \beta_i]$ and $[\alpha_i, \beta_i, \bar{x}_i]$ are the only $x_i\beta_i$ -dipath and $\alpha_i\bar{x}_i$ -dipath, respectively, we have that $c(x_i) \neq c(\bar{x}_i)$. Then we can assign value 1 to all literals whose corresponding vertex has color Blue, and value 0 to the other literals. Since every vertex C_j is adjacent to a color Blue vertex, we obtain a truth assignment satisfying Φ .

Assume now that Φ is a yes-instance of 3-SAT. Consider a truth assignment satisfying Φ . We define a coloring c for the vertices of D_Φ as follows (see Fig. 6).

- $c(F) = Red$.
- For every $1 \leq j \leq m$, $c(C_j) = Red$ and $c(C'_j) = Blue$.
- $c(y_i) = Blue$.
 Furthermore if x_i is true, then $c(x_i) = c(\beta_i) = c(\gamma_i) = Blue$ and $c(\bar{x}_i) = c(\delta_i) = c(\alpha_i) = Red$.
 Otherwise $c(x_i) = c(\beta_i) = c(\gamma_i) = Red$ and $c(\bar{x}_i) = c(\delta_i) = c(\alpha_i) = Blue$.

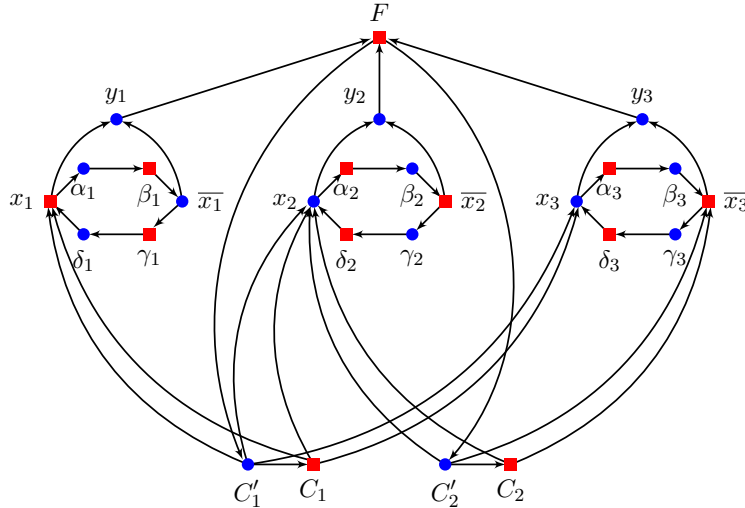


Figure 6: A vertex properly connected coloring of D_Φ , with $\Phi = (x_1 \vee x_2 \vee x_3) \wedge (x_2 \vee \bar{x}_3)$ corresponding to the truth assignment $x_1 = 0, x_2 = 1, x_3 = 1$

We shall prove that c is properly connected. First note that there exists a proper dipath from every vertex to F . More precisely the following dipaths are proper:

- $[\gamma_i, \delta_i, x_i, \alpha_i, \beta_i, \bar{x}_i, y_i, F]$ if x_i is a variable valued 1;
- $[\alpha_i, \beta_i, \bar{x}_i, \gamma_i, \delta_i, x_i, y_i, F]$ if x_i is a variable valued 0;
- $[C'_j, C_j, x_i, \alpha_i, \beta_i, \bar{x}_i, y_i, F]$ if x_i is valued 1 and is contained in C_j ;
- $[C'_j, C_j, \bar{x}_i, \gamma_i, \delta_i, x_i, y_i, F]$ if \bar{x}_i is valued 1 and is contained in C_j ;
- $[C'_j, x_i, y_i, F]$ if x_i is valued 0 and is contained in C_j ;
- $[C'_j, \bar{x}_i, y_i, F]$ if \bar{x}_i is valued 0 and is contained in C_j .

Let us denote by \mathcal{P}^- this above collection of dipaths. Also, the following dipaths from vertex F are proper:

- $[F, C'_j, C_j, x_i, \alpha_i, \beta_i, \bar{x}_i, y_i]$ and $[F, C'_j, C_j, x_i, \alpha_i, \beta_i, \bar{x}_i, \gamma_i, \delta_i]$ if x_i is a variable valued 1 in clause C_j ;
- $[F, C'_j, C_j, \bar{x}_i, \gamma_i, \delta_i, x_i, y_i]$ and $[F, C'_j, C_j, \bar{x}_i, \gamma_i, \delta_i, x_i, \alpha_i, \beta_i]$ if \bar{x}_i is valued 1 and is contained in clause C_j ;
- $[F, C'_j, x_i, y_i]$ and $[F, C'_j, x_i, \alpha_i, \beta_i, \bar{x}_i, \gamma_i, \delta_i]$ if x_i is a variable valued 0 in clause C_j ;
- $[F, C'_j, \bar{x}_i, y_i]$ and $[F, C'_j, \bar{x}_i, \gamma_i, \delta_i, x_i, \alpha_i, \beta_i]$ if \bar{x}_i is valued 0 and is contained in clause C_j .

Let us denote by \mathcal{P}^+ this above collection of dipaths.

We shall prove that for every two distinct (ordered) vertices u and v there exists a proper dipath of which they are the respective head and tail. This follows from the following case analysis.

1. If there exists a dipath in $\mathcal{P}^- \cup \mathcal{P}^+$ that contains an uv -subdipath, then we are done;
2. In the same way, we claim that we are also done if there exist a proper uF -dipath (subdipath of a dipath in \mathcal{P}^-) and a proper Fv -dipath (subdipath of a dipath in \mathcal{P}^+) that only intersect in F . Indeed, by merging these two dipaths in F , we obtain a proper uv -dipath.
3. Furthermore, it can be checked that any two vertices u, v fall in one of the two above cases, unless u and v are contained in a dicycle from variables gadgets $[x_i, \alpha_i, \beta_i, \bar{x}_i, \gamma_i, \delta_i, x_i]$. In this case, since the above dicycle is proper we are done.

Hence the coloring c is properly connected. □

Remark 2. *It is not hard to see that for any digraph D , $\overrightarrow{pvc}(D) \leq t$ if and only if $\text{diam}(D) \leq t + 1$ for $t \in \{0, 1\}$. Therefore, deciding whether $\overrightarrow{pvc}(D) = 0$, resp. whether $\overrightarrow{pvc}(D) = 1$ can be done in polynomial time.*

5. Proper connection number

The last section is devoted to partial results on the proper connection number of graphs. The complexity of computing this parameter remains open, but we propose advances in this direction by presenting some tractable cases. In particular, we present graph classes, generalizing that of trees, in which we can decide in polynomial time if the proper connection number of a graph is two.

Bounded-treewidth graphs. We recall that a chordal graph is a graph with no induced cycles of length at least four. A graph G has treewidth at most k if there exists a supergraph of G that is chordal and has no clique of size at least $k + 2$ [5]. In particular, trees are exactly the graphs of treewidth one.

The MSO_2 logic is a fragment of second-order logic where the second-order quantification is limited to quantification over sets (*i.e.*, quantifications over binary relations is not allowed). The formulas that are written in MSO_2 logic for graphs use individual variables $u, v, e \dots$ to represent vertices and edges, set variables X, Y, F, \dots to represent vertex-subsets and edge-subsets, and atomic formulas such as set containment – denoted $X \subseteq Y$ – and incidence between a vertex and an edge – *i.e.*, $\text{inc}(v, e)$ is true if and only if vertex v is an end of edge e . By Courcelle’s Theorem [15], every problem expressible in MSO_2 logic can be solved in linear-time on bounded-treewidth graphs.

Lemma 7. *If $G = (V, E)$ has bounded treewidth, then it can be decided in linear-time whether $pc(G) \leq 2$.*

Proof. By Courcelle’s Theorem [15], it suffices to prove that the problem can be written as a formula in MSO_2 logic. Such a formula can be obtained from a slight variation of the corresponding formula for the HAMILTONIAN PATH problem. More precisely, $pc(G) \leq 2$ if and only if there exists a bipartition E_0, E_1 of the edge-set such that, for every $u, v \in V$, there is some set of edges F that induces a connected subgraph, with the properties that $u, v \in V(G[F])$ and every vertex is incident to at most one edge in

$F \cap E_i$ for every $i \in \{0, 1\}$. The latter can be written as an MSO_2 formula, namely:

$$\begin{aligned} \exists E_0 \subseteq E, E_1 \subseteq E \cdot (\forall u \in V, v \in V \exists F \subseteq E, V_F \subseteq V \cdot (\\ & [\forall x \in V x \in V_F \iff \exists e \in F \cdot \text{inc}(x, e)] \\ & \wedge [\forall X, Y \subseteq V_F X \cup Y = V_F \implies \exists e \in F, x \in X, y \in Y \cdot \text{inc}(x, e) \wedge \text{inc}(y, e)] \\ & \wedge [u, v \in V_F] \\ & \wedge [\forall x \in V_F, e \in E_0 \cap F, e' \in E_0 \cap F \text{ inc}(x, e) \wedge \text{inc}(x, e') \implies e = e'] \\ & \wedge [\forall x \in V_F, e \in E_1 \cap F, e' \in E_1 \cap F \text{ inc}(x, e) \wedge \text{inc}(x, e') \implies e = e'])). \end{aligned}$$

The second line defines V_F as the vertex-set of $G[F]$. The third line ensures that $G[F]$ is connected (*i.e.*, for every bipartition of V_F , there is an edge of F with an end in each side). Finally, the two last lines ensure that every vertex of V_F can be incident to at most one edge of $F \cap E_0$, resp. of $F \cap E_1$. \square

We think that our approach can be generalized in order to prove that for any bounded-treewidth graph G , and for any constant k , it can be decided in linear-time whether $pc(G) \leq k$. However, we leave open the complexity of computing the proper connection number on bounded-treewidth graphs.

Bipartite graphs. The special case of bipartite graphs has been extensively studied in the literature [7, 10, 20, 26, 27]. Several sufficient conditions have been given for these graphs to have proper connection number equal to two. Before concluding this section, we base on one of these conditions in order to provide a complete characterization of bipartite graphs with proper connection number two.

We introduce additional terminology. Let $G = (V, E)$ be a connected graph. We remind that a *bridge* of G is an edge $e \in E$ such that $G \setminus e = (V, E \setminus \{e\})$ is disconnected. A *bridge-block* of G is a connected component of $G \setminus F = (V, E \setminus F)$, with F being the set of bridges of G . It is well-known that if we take the bridge-blocks of G as vertices, and for every bridge we add an edge between the two bridge-blocks that contain its ends, the resulting graph is a tree, sometimes called the *bridge-block tree* of G , that is linear-time computable [40].

Theorem 8. *Let $G = (V, E)$ be a connected bipartite graph. We have $pc(G) \leq 2$ if and only if the bridge-block tree of G is a path. Furthermore, if $pc(G) \leq 2$, then such a coloring can be computed in linear-time.*

As a byproduct of Theorem 8, we retrieve a known characterization of trees with proper connection number at most two.

Corollary 9 ([7]). *Let T be a tree. We have $pc(T) \leq 2$ if and only if T is a path.*

The remaining of the section is devoted to the proof of Theorem 8. There are two main ingredients in the proof of Theorem 8 that will be stated in Lemmas 10 and 11. For that we need to introduce additional notions. We say that an edge-coloring of G has the *strong property* if, for every two vertices u and v , there are at least two proper uv -paths (not necessarily disjoint) with the two edges being incident to u (resp., to v) in these two paths being of different colors.

Lemma 10 ([27]). *If G is a connected bipartite bridgeless graph, then $pc(G) \leq 2$. Furthermore, such a coloring can be produced with the strong property.*

The proof of Lemma 10 is constructive. However, the computational complexity of this construction is not explicitly given in [27]. A careful analysis shows that it can be done in linear time. Indeed, this construction scans all the biconnected components of G exactly once; for every biconnected component of G , we use an earlier construction of Borozan et al. for *biconnected* bipartite graphs [7]. Since computing the biconnected components of a given graph takes linear time [25], it suffices to prove that this construction of Borozan et al. can be implemented to run in linear time for every biconnected bipartite graph G . For the latter construction, we first compute a minimal biconnected spanning subgraph H of G , then we scan all the ears in an ear decomposition of H exactly once. Since both computing a minimal biconnected spanning subgraph and computing an ear decomposition can be done in linear time (*e.g.*, see [24, 39]), this construction can also be implemented to run in linear time, hence the following remark holds.

Remark 3. *The construction from Lemma 10 can be implemented in linear time.*

Hence, by Lemma 10, it remains to study bipartite graphs with bridges. Furthermore, it has been proved in [2] that if a vertex in G is incident to at least k bridges then $pc(G) \geq k$. So, we can deduce from this lower-bound that there cannot be more than 2 bridges that are incident to every vertex. However, the latter is not sufficient, even for bipartite graphs, in order to ensure that $pc(G) \leq 2$. The following stronger lower-bound can be seen as a generalization of [8, Lemma 2.1].

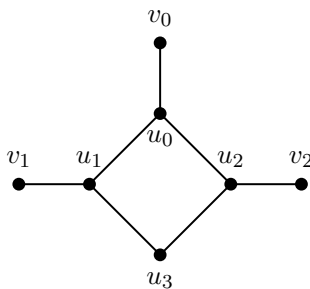


Figure 7: A bipartite graph G with $pc(G) = 3$.

Lemma 11. *Let $G = (V, E)$ be a connected graph, B be a bridge-block of G that is bipartite. If B is incident to at least three bridges then $pc(G) \geq 3$.*

Proof. Suppose, for the sake of a contradiction, that c is a properly connected 2-edge-coloring of G . Assume that $\{u_0, v_0\}$, $\{u_1, v_1\}$, $\{u_2, v_2\}$ are three bridges in G and $u_0, u_1, u_2 \in V(B)$ (possibly, $u_i = u_j$ for some $i, j \in \{0, 1, 2\}$). If, for $i, j \in \{0, 1, 2\}$, u_i and u_j are in the same partite set of B , then any path connecting u_i and u_j in B is of even length. Thus, since in any proper path with 2 colors the colors of the edges must alternate, $c(\{u_i, v_i\}) \neq c(\{u_j, v_j\})$. This implies that not all vertices of $\{u_0, u_1, u_2\}$ are in the same partite set. Similarly, if u_i and u_j are in different partite sets, then any path connecting u_i and u_j in B is of odd length, and so, $c(\{u_i, v_i\}) = c(\{u_j, v_j\})$. Without loss of generality, let u_0 and u_1 be in the same partite set, and hence u_2 is in the other one. It follows, $c(\{u_0, v_0\}) = c(\{u_1, v_1\}) = c(\{u_2, v_2\})$ and $c(\{u_0, v_0\}) \neq c(\{u_1, v_1\})$, a contradiction. \square

Proof of Theorem 8. On the one direction, assume that the bridge-block tree of G is not a path. Edges in this tree are in a one-to-one correspondence with bridges in G . So, there is a bridge-block of G that is incident to at least three bridges. Since such a component must be bipartite (because G is), it follows $pc(G) \geq 3$ by Lemma 11, a contradiction.

On the other direction, assume that the bridge-block tree of G is a path. This gives us a linear ordering B_0, B_1, \dots, B_l of the bridge-blocks, that can be computed in linear time [40].

We claim that for every $0 \leq i \leq l$ there exists a (linear-time computable) coloring of $G[\bigcup_{j \leq i} V(B_j)]$ with the following two properties:

1. $G[\bigcup_{j \leq i} V(B_j)]$ is properly connected;
2. the coloring restricted to B_i has the strong property.

We divide the proof in three cases.

Assume for the first base case $i = 0$. By Lemma 10 we have $pc(B_0) \leq 2$ and such a coloring can be produced with the strong property. Recall that such a coloring can be computed in $\mathcal{O}(|V(B_0)| + |E(B_0)|)$ -time (see Remark 3). So, the claim is proved for this base case.

Consider now $i = 1$ for the second base case. Again, by Lemma 10, we can color the edges of B_0 , resp. the edges of B_1 , with 2 colors, such that B_0 , resp. B_1 , is properly connected and the coloring for every bridge-block has the strong property. This takes $\mathcal{O}(|V(B_0) \cup V(B_1)| + |E(B_0) \cup E(B_1)|)$ -time (Remark 3). Then coloring the unique bridge between B_0 and B_1 arbitrarily results in having $G[V(B_0) \cup V(B_1)]$ properly connected. This takes constant-time, and so, this proves the claim for this base case.

Finally, assume $i > 1$. Assume by induction hypothesis that the claim holds for $i - 1$. In this situation let us call c the color of the unique bridge between B_{i-2} and B_{i-1} . We first color the edges of B_i with two colors so that the coloring is properly connected and it has the strong property; such a coloring is guaranteed to exist by Lemma 10, and it can be computed in $\mathcal{O}(|V(B_i)| + |E(B_i)|)$ -time. Then there are two subcases to be distinguished. Either all the paths between the two bridges incident in B_{i-1} have even length, in which case we color the bridge between B_{i-1} and B_i with a color different than c . Or all these paths have odd length, in which case we color the bridge between B_{i-1} and B_i with color c . Note that it can be easily checked in which case we are by looking for the end-vertices of the two bridges that are on the same partite set of the bipartition of B_i . Overall, since we can compute the two partite sets of B_i in $\mathcal{O}(|V(B_i)| + |E(B_i)|)$ -time, we can also choose the color for the bridge between B_{i-1} and B_i within the same amount of time. In doing so, we extend the coloring in such a way that $G[\bigcup_{j \leq i} V(B_j)]$ is properly connected. Therefore, the claim holds for i .

In particular, by taking $i = l$, we obtain that $pc(G) \leq 2$. Such a coloring can be computed in $\mathcal{O}(\sum_{i=0}^l (|V(B_i)| + |E(B_i)|))$ -time, that is linear in the size of G since the bridge-block tree of G can be computed in linear time. \square

6. Conclusions

We give the first known complexity results on proper connectivity, in undirected graphs and in directed graphs. In particular, computing the directed proper connection numbers of digraphs is NP-hard. Computing the proper *vertex* connection number of undirected graphs (resp., their vertex proper connection number) is trivial. The only remaining case is the proper connection number of undirected graphs. We leave the complexity of computing the proper connection number of a given graph as an interesting open question. An intermediate problem will be to characterize the graphs with proper connection number at most two. We also intend to study the complexity of computing the proper connection number of *bipartite* graphs, thereby generalizing the positive result of Theorem 8.

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