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NECESSARY CONDITIONS FOR LOCAL CONTROLLABILITY OF A PARTICULAR CLASS OF SYSTEMS WITH TWO SCALAR CONTROLS

LAETITIA GIRALDI¹, PIERRE LISSY², CLÉMENT MOREAU^{2,3} AND
JEAN-BAPTISTE POMET¹

Abstract. We consider control-affine systems with two scalar controls, such that one control vector field vanishes at an equilibrium state. We state two necessary conditions for local controllability around this equilibrium, involving the iterated Lie brackets of the system vector fields, with controls that are either bounded, small in L^∞ or small in $W^{1,\infty}$. These results are illustrated with several examples.

Résumé. On s'intéresse à des systèmes de contrôle affines à deux contrôles scalaires, tels que l'un des champs de vecteurs associés aux contrôles s'annule en un état d'équilibre. On énonce deux conditions nécessaires de contrôlabilité locale au voisinage de cet équilibre, faisant rentrer en jeu des crochets de Lie itérés des champs de vecteurs du système, avec des contrôles pouvant être bornés, petits dans L^∞ ou petits dans $W^{1,\infty}$. Ces résultats sont illustrés à travers plusieurs exemples.

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1. INTRODUCTION

Let \mathbb{N} be the set of nonnegative integers. Let n, m in $\mathbb{N} \setminus \{0\}$ and consider a control-affine system with m controls:

$$\dot{z} = f_0(z) + \sum_{k=1}^m u_k(t) f_k(z), \quad (1)$$

where z is the state in \mathbb{R}^n , f_0, \dots, f_m are real analytic vector fields, and fixing the control $u = (u_1, \dots, u_m)$ to be an integrable function $[0, T] \rightarrow \mathbb{R}^m$, for some $T > 0$, defines a time-varying ordinary differential equation and hence, for each choice of initial condition $z(0)$, a unique solution $t \mapsto z(t)$ on $[0, T]$, or a smaller interval $[0, T')$ in case of finite time blow-up. This system is called *controllable* if, for any two points z_0 and z_1 in the state space \mathbb{R}^n , or possibly a subset of it, there exist a time T and an integrable control on $[0, T]$ such that the above-mentioned solution with $z(0) = z_0$ satisfies $z(T) = z_1$. See textbooks like [1–4] for further precisions on

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controllability. We are interested in *local* controllability around an equilibrium $(z^{\text{eq}}, u^{\text{eq}})$:

$$f_0(z^{\text{eq}}) + \sum_{k=1}^m u_k^{\text{eq}} f_k(z^{\text{eq}}) = 0. \quad (2)$$

System (1) is locally controllable around $(z^{\text{eq}}, u^{\text{eq}})$ if the above mentioned property occurs for z_0 and z_1 in arbitrarily small neighborhoods of z^{eq} , with solutions that remain in arbitrarily small neighborhoods of z^{eq} , in arbitrarily small time T , and with a control that takes values in a fixed neighborhood of u^{eq} or an arbitrarily small neighborhood of u^{eq} , or with a control that is arbitrarily close to the constant control $t \mapsto u^{\text{eq}}$ in some functional space. We detail in Section 2 these various notions of local controllability.

Some (now classical for some of them) sufficient conditions [5–7] and necessary conditions [7–11] are given in the literature in terms of finite jets of the vector fields, more precisely these conditions allow one to decide controllability or non-controllability based on the value of a finite number of Lie brackets at z^{eq} . It is intriguing that there is a significantly wide gap between the necessary and the sufficient conditions; as pointed out in [12], it is even not clear whether or not, for a general system in this gap, local controllability depends on a finite jet of the vector fields or not.

General sufficient conditions are available [7] for m larger than 1, but the literature on necessary conditions focuses on systems with a single scalar control ($m = 1$). This paper is specifically concerned with control systems with two scalar inputs, of the form

$$\dot{z} = f_0(z) + u_1 f_1(z) + u_2 f_2(z), \quad (3)$$

where the real analytic vector fields f_0, f_1, f_2 are such that f_0 and f_2 vanish at the equilibrium while f_1 does not:

$$f_0(z^{\text{eq}}) = 0, \quad f_2(z^{\text{eq}}) = 0, \quad f_1(z^{\text{eq}}) \neq 0. \quad (4)$$

Such systems have two controls but the effect of one of them vanishes at the point of interest. In a sense, the contribution of this paper is to study to what extent the second control helps controllability or, conversely, to what extent obstructions to controllability of the single input system $\dot{z} = f_0(z) + u_1 f_1(z)$ carry over when the second control u_2 is turned on.

Studying this very situation stemmed out of previous work from the authors on the controllability of magnetic micro-swimmers [13–15]. See these references for a description of these devices and their interest, for instance in micro-robotics and biomedical applications. The corresponding control systems are particular cases of (3)-(4), for which the authors have proved various controllability and non-controllability results. We believe that a more general treatment of systems of type (3)-(4), beyond the case of magnetic micro-swimmers, is of interest to the controllability problem in control theory. It is the purpose of the present paper.

The paper is structured as follows. Section 2 is devoted to precise definitions of various notions of local controllability and to recalling known controllability conditions for single-input systems. Our two main results are presented in Section 3. Section 4 illustrates the results with several examples. Section 5 is dedicated to the proofs. Finally, conclusions as well as some perspectives on further research are provided in Section 6.

2. PROBLEM STATEMENT

2.1. Various notions of local controllability

The following definition is taken from the textbook by Coron [3, Def. 3.2, p. 125]. $B(z, \eta)$ stands for the open ball with center z and radius $\eta > 0$, L^∞ is the usual functional space, the choice of a norm in \mathbb{R}^n and \mathbb{R}^m is indifferent.

Definition 2.1 (STLC). The control system (1) is STLC at the equilibrium $(z^{\text{eq}}, u^{\text{eq}})$ if, for every $\varepsilon > 0$, there exists $\eta > 0$ such that, for every z_0, z_1 in $B(z^{\text{eq}}, \eta)$, there exists a control $u(\cdot)$ in $L^\infty([0, \varepsilon], \mathbb{R}^m)$ such that the solution $z(\cdot) : [0, \varepsilon] \rightarrow \mathbb{R}^n$ of the control system (1) satisfies $z(0) = z_0$, $z(\varepsilon) = z_1$, and

$$\|u - u^{\text{eq}}\|_{L^\infty([0, \varepsilon], \mathbb{R}^m)} \leq \varepsilon.$$

STLC stands for *small time locally controllable*. Coron notes that it should rather be called small time locally controllable *with controls close to u^{eq}* . Historically, in the first classical papers on local controllability (e.g. [6, 16]), one fixes a bounded neighborhood of u^{eq} in \mathbb{R}^m and a system is locally controllable if, for any time T , there is a neighborhood of z^{eq} in which any two points may be joined in time T using controls with values in the fixed bounded neighborhood of u^{eq} , i.e. $\|u - u^{\text{eq}}\|_{L^\infty([0, \varepsilon], \mathbb{R}^m)}$ was not required to be small but only bounded; this notion was the one initially called STLC, while STLC in the sense of Definition 2.1 was rather called *Small Time Local Controllability with Small Controls* [17] (small refers to the distance to u^{eq}). That was a very natural terminology, but these terms drifted, and Definition 2.1 now prevails. In the present paper, we call B-STLC (the prefix B stands for bounded control) the “historical” STLC property:

Definition 2.2 (B-STLC). The control system (1) is *B-STLC* at $(z^{\text{eq}}, u^{\text{eq}})$ if there exists $\alpha > 0$ such that, for every $\varepsilon > 0$, there exists $\eta > 0$ such that, for every z_0, z_1 in $B(z^{\text{eq}}, \eta)$, there exists a control $u(\cdot)$ in $L^\infty([0, \varepsilon], \mathbb{R}^m)$ such that the solution $z(\cdot) : [0, \varepsilon] \rightarrow \mathbb{R}^n$ of the control system (1) satisfies $z(0) = z_0$, $z(\varepsilon) = z_1$, and

$$\|u - u^{\text{eq}}\|_{L^\infty([0, \varepsilon], \mathbb{R}^m)} \leq \alpha.$$

Remark 2.3. STLC implies B-STLC, choosing any $\alpha > 0$.

Remark 2.4. For a given control system that is B-STLC, let $\alpha_0 \geq 0$ be the infimum of the possible values of α in Definition 2.2. In [13, 14], the authors defined a quantitative version of B-STLC and called such a system “ α_0 -STLC”, and also α -STLC for all α larger than α_0 . Clearly the system is STLC (in the sense of Definition 2.1) if and only if $\alpha_0 = 0$; hence STLC is also referred to as 0-STLC in [13, 14]. We shall not use this notion of “ α -STLC” outside of the present remark.

More recently, a new notion has been introduced by Beauchard and Marbach in [11, Definition 4]. The idea is to ensure the smallness, not only of $u - u^{\text{eq}}$, but also of some derivatives, by requiring its norm to be bounded in Sobolev spaces. For I an interval and k a nonnegative integer, we recall that a function $f : I \rightarrow \mathbb{R}$ belongs to $W^{k, \infty}(I)$ if, for all $p \in \{0, \dots, k\}$, $f^{(p)} \in L^\infty(I)$. In this case, one can endow $W^{k, \infty}(I)$ with the following norm

$$\|f\|_{W^{k, \infty}} = \max_{p \in \{0, \dots, k\}} \|f^{(p)}\|_{L^\infty(I)},$$

that makes $W^{k, \infty}(I)$ a Banach space.

Definition 2.5 ($W^{k, \infty}$ -STLC). Let $k \in \mathbb{N}$. The control system (1) is $W^{k, \infty}$ -STLC at $(z^{\text{eq}}, u^{\text{eq}})$ if, for every $\varepsilon > 0$, there exists $\eta > 0$ such that, for every z_0, z_1 in $B(z^{\text{eq}}, \eta)$, there exists a control $u(\cdot)$ in $W^{k, \infty}([0, \varepsilon], \mathbb{R}^m)$ such that the solution $z(\cdot) : [0, \varepsilon] \rightarrow \mathbb{R}^n$ of the control system (1) satisfies $z(0) = z_0$, $z(\varepsilon) = z_1$, and

$$\|u - u^{\text{eq}}\|_{W^{k, \infty}([0, \varepsilon], \mathbb{R}^m)} \leq \varepsilon.$$

Remark 2.6. When $k = 0$, $W^{0, \infty}$ -STLC (i.e. L^∞ -STLC) is identical to STLC; when $k > 0$, $W^{k, \infty}$ -STLC is stronger than STLC, because it requires the control to be sufficiently smooth – notably, the control cannot be discontinuous.

2.2. Notions and notations

If f and g are real analytic vector fields, their Lie bracket is denoted by $[f, g]$ (its definition is recalled at the beginning of Section 5.1). The “ad” operator is defined by $(\text{ad}f)g = [f, g]$. The usual notation $\text{ad}_f^k g$ is defined

by induction with $\text{ad}_f^0 g = g$ and $\text{ad}_f^k g = [f, \text{ad}_f^{k-1} g] = [f, [f, \dots, [f, g], \dots]]$; given $k \geq 1$ different vector fields f_1, \dots, f_n , we use the following notation for iterated composition of $\text{ad} f_i$, where one must respect the order since these operators do not commute:

$$\left(\prod_{i=1}^k \text{ad} f_i \right) g = [f_1, [f_2, [f_3, \dots [f_k, g] \dots]]]. \quad (5)$$

The set of real analytic vector fields on \mathbb{R}^n (or on an open neighborhood of the equilibrium under consideration) is a Lie algebra over \mathbb{R} , with multiplication the Lie bracket. Given a family of real analytic vector fields $\mathcal{F} = \{g_1, \dots, g_N\}$, we denote by $\text{Lie}(\mathcal{F})$ the sub-algebra generated by these vector fields.

Remark 2.7. In many articles on local controllability, one also introduces the free Lie algebra of formal Lie brackets generated by N symbols $\hat{g}_1, \dots, \hat{g}_N$ and the (usually non injective) morphism into $\text{Lie}(\mathcal{F})$ defined as the substitution of these symbols by the actual vector fields. This is a precise and efficient tool to keep track of how brackets are constructed, independently of their realisation as a vector field: for instance, if g_1 and g_2 commute, $[\hat{g}_1, \hat{g}_2]$ is a nonzero element of the formal Lie algebra mapped to the zero vector field. Here, since we treat only brackets of order up to 5, we do not use these tools; we avoid ambiguous statements (like saying that the zero vector field is a bracket of order 2 in that situation).

Finally, we will use some condensed notations for some Lie brackets between the vector fields f_0, f_1 and f_2 defining the control systems (3) or (10) under examination: basically, f_{ij} stands for $[f_i, f_j]$, f_{ijk} for $[f_i, [f_j, f_k]]$, $f_{ij,kl}$ for $[f_{ij}, f_{kl}]$, and $f_{ij,klm}$ for $[f_{ij}, f_{klm}]$, and we will specifically use the following ones:

$$f_{01} = [f_0, f_1], f_{21} = [f_2, f_1]; \quad (6)$$

$$f_{101} = [f_1, [f_0, f_1]], f_{121} = [f_1, [f_2, f_1]]; \quad (7)$$

$$f_{21,01} = [[f_2, f_1], [f_0, f_1]]; \quad (8)$$

$$\text{for } (i, j, k) \in \{0, 2\}^3, f_{i1,jk1} = [[f_i, f_1], [f_j, [f_k, f_1]]]. \quad (9)$$

2.3. Known results for single-input systems

In this section, we set $m = 1$ in (1) and consider single-input control-affine systems:

$$\dot{z} = f_0(z) + u_1(t)f_1(z), \quad z \in \mathbb{R}^n \quad (10)$$

around the equilibrium $(z^{\text{eq}}, u^{\text{eq}}) = (0, 0)$, i.e. we assume $f_0(0) = 0$.

Definition 2.8. System (10) satisfies the Lie Algebra Rank Condition (LARC) at 0 if

$$\{g(0), g \in \text{Lie}(f_0, f_1)\} = \mathbb{R}^n. \quad (11)$$

It is well-known [18, 19] (see also [16, Proposition 6.2]) that the LARC is necessary for any form of STLC, but not sufficient. Stronger assumptions on the structure of Lie bracket spaces have to be made to obtain a sufficient condition. For $k \in \mathbb{N}$, let \mathcal{S}_k be the set of all iterated Lie brackets of f_0 and f_1 in which f_1 appears at most k times, and S_k the subspace of \mathbb{R}^n spanned by the value at 0 of the elements of \mathcal{S}_k evaluated at 0. The following is a translation of [16, Theorem 2.1], the main result in that reference. As discussed between Definition 2.1 and Definition 2.2, STLC meant B-STLC at the time where [16] was published, so that a faithful translation of [16, Theorem 2.1] should be about B-STLC. However, the reader may check that the proof by Sussmann in [16] indeed yields small controls, and therefore proves STLC, not only in [16]’s sense, but in our sense too, we therefore write a statement¹ that is formally stronger than the one in [16], but still correct and in the spirit of [16].

¹Coron does the same “abuse” when translating [7, Theorem 7.3] into [3, Theorem 3.29].

Proposition 2.9 ([16, Theorem 2.1, p. 688]). *If System (10) satisfies the LARC at 0 and, for all k in \mathbb{N} ,*

$$S_{2k+2} \subset S_{2k+1}, \quad (12)$$

then it is STLC.

Remark 2.10. Condition (12) is violated if for some $k \in \mathbb{N}$, some brackets in \mathcal{S}_{2k+2} , once evaluated at 0, do not belong to S_{2k+1} . Such brackets were historically called *bad* ones, since they appear in Proposition 2.9 as potential *obstructions* to local controllability. The bad brackets can be seen as directions towards which the system drifts (or a function of the state whose time-derivative always has the same sign, for all possible controls); they really have this effect only if one cannot find “good” brackets that cancel out the drift direction, “compensating” the bad ones and hence preventing them from obstructing local controllability.

However, as M. Kawski showed in [20, Example 2.5.1], this interpretation in terms of bad brackets is no longer relevant for higher values of k : goodness or badness of single higher order brackets must be replaced by more complex objects involving the image in $\text{Lie}(f_0, f_1)$ of the whole homogeneous component of the free Lie algebra, independently of the basis one chooses for it (see the first lines of Section 2.1 about the free Lie algebra, a language that we will not use any further). In the present paper and our own results, the bad/good bracket language is sufficient in section 3.1, but the introduction of the quadratic form D (see (17)) in Section 3.2 is an illustration of that fact.

The lowest-order possible obstruction to (12) occurs if the bracket f_{101} (defined in (7)) is such that $f_{101}(0)$ does not belong to S_1 . In that case, the following result has been shown by H. Sussmann:

Proposition 2.11 ([16, Proposition 6.3, p.707]). *If $f_{101}(0) \notin S_1$, then System (10) is not B-STLC.*

This result has been extended to higher values of the integer k by G. Stefani in [21]:

Proposition 2.12 ([21]). *Assume that there exists k in $\mathbb{N} \setminus \{0\}$ such that*

$$\text{ad}_{f_1}^{2k} f_0(0) \notin S_{2k-1}.$$

Then, System (10) is not B-STLC.

Returning to $k = 1$, assume now that the hypothesis of Proposition 2.11 is not satisfied, i.e. $f_{101}(0)$ does belong to S_1 . As explained for instance in [16], the next lowest-order bracket in \mathcal{S}_2 that can be an obstruction to controllability is $f_{01,001}$ (defined in (9)).

It is however noticed by H. Sussmann in [16, p.710] (the counter-example is recalled as (27) at the beginning of section 4.2) that $f_{01,001}(0) \notin S_1$ is *not* an obstruction to STLC as defined in Definition 2.1. In [9], M. Kawski obtained a new necessary condition by refining the space \mathcal{S}_1 :

Proposition 2.13 ([9]). *Let $\mathcal{S}' = \{\text{ad}_{f_0}^k(\text{ad}_{f_1}^3 f_0), k \in \mathbb{N}\}$ and S' the vector subspace of \mathbb{R}^n spanned by the value at 0 of the elements of \mathcal{S}' .*

If $f_{01,001}(0) \notin S_1 + S'$, then System (10) is not STLC.

More recently, K. Beauchard and F. Marbach showed that $f_{01,001}(0) \notin S_1$ is *indeed* an obstruction, but to a stronger notion of local controllability that we just recalled in Definition 2.5.

Proposition 2.14 ([11, Theorem 3]). *If $f_{01,001}(0) \notin S_1$, then System (10) is not $W^{1,\infty}$ -STLC.*

Concerning systems with control in \mathbb{R}^m , $m \geq 2$, a general sufficient condition for local controllability, in the vein of Proposition 2.9 but more complex, can be found in [7], but no necessary condition is known, to the best of our knowledge. The main results of this paper, stated in the next section, are a step in this direction in the sense that they give an extension of the necessary conditions contained in Propositions 2.11, 2.13 and 2.14 to the case where the system has two scalar controls, and the vector field associated to the second control vanishes at the equilibrium.

3. MAIN RESULTS

We now consider the control-affine system (3) (which is also system (1) with $m = 2$), assuming that (4) is satisfied at $z^{\text{eq}} = 0$, i.e. $f_0(0) = 0, f_2(0) = 0, f_1(0) \neq 0$, and we study local controllability for $(z, (u_1, u_2))$ close to the equilibria $(0, (0, u_2^{\text{eq}}))$, with u_2^{eq} arbitrary.

Remark 3.1. The fact that *any* value of u_2^{eq} is an equilibrium value of the control for the system (3) due to (4) is perhaps unusual. Of course, once some value of u_2^{eq} is set, one can always work around the null equilibrium $(0, (0, 0))$ by performing the following affine feedback transformation on the control u_2 : $\tilde{u}_2 = u_2 - u_2^{\text{eq}}$. With this transformed control, system (3) becomes

$$\dot{z} = \tilde{f}_0(z) + u_1 \tilde{f}_1(z) + \tilde{u}_2 \tilde{f}_2(z) \quad (13)$$

with $\tilde{f}_0 = f_0 + u_2^{\text{eq}} f_2$, $\tilde{f}_1 = f_1$ and $\tilde{f}_2 = f_2$. Assume that system (13) is STLC at $(0, (0, 0))$ and let ε be a positive real number, η be the associated parameter from Definition 2.1, and z_0, z_1 in $B(0, \eta)$. There exists controls u_1 and \tilde{u}_2 in $L^\infty([0, \varepsilon])$ such that the solution of (13) with $z(0) = z_0$ and these controls satisfy $z(\varepsilon) = z_1$, and

$$\|u_1\|_{L^\infty([0, \varepsilon], \mathbb{R})} \leq \varepsilon, \|\tilde{u}_2\|_{L^\infty([0, \varepsilon], \mathbb{R})} \leq \varepsilon.$$

Hence, the solution of system (3) with $z(0) = z_0$ and controls $u_2 = u_2^{\text{eq}} + \tilde{u}_2$ and u_1 satisfies $z(\varepsilon) = z_1$, and we also have $\|u_2 - u_2^{\text{eq}}\|_{L^\infty([0, \varepsilon], \mathbb{R})} \leq \varepsilon$ and $\|u_1\|_{L^\infty([0, \varepsilon], \mathbb{R})} \leq \varepsilon$. Therefore, system (13) is STLC at $(0, (0, 0))$ if and only if system (3) is STLC at $(0, (0, u_2^{\text{eq}}))$; this also holds for other types of local controllability.

Moreover, unlike STLC, the notion of B-STLC (Definition 2.2) is independent of u_2^{eq} : indeed, choosing α sufficiently large, it is easy to see that system (1) is B-STLC at *one* equilibrium $(0, (0, u_2^{\text{eq}}))$ if and only if it is B-STLC for *all* equilibria $(0, (0, u_2^{\text{eq}}))$.

Nevertheless, a key feature of our main results is that some particular values of u_2^{eq} (like the one called β in Theorem 3.2) play an important role for recovering controllability. Therefore, we choose to state Theorem 3.2 and 3.9 without readily taking $u_2^{\text{eq}} = 0$; while the proofs are conducted with $u_2^{\text{eq}} = 0$ using the feedback transformation described above.

3.1. Obstruction coming from brackets of order 3

We want to give a necessary condition that “generalizes” Proposition 2.11 for single-input systems to the class (1)-(4) of two-input systems. The bracket that carries the obstruction is again f_{101} but now, f_{121} also plays a role as it could compensate the bracket f_{101} .

The family \mathcal{S}_1 from Proposition 2.11 has to be replaced by a larger family \mathcal{R}_1 , made of all iterated Lie brackets of f_0, f_1 and f_2 where f_1 appears at most one time. We denote by R_1 the vector subspace of \mathbb{R}^n spanned by the value at 0 of all elements of \mathcal{R}_1 .

Let us now state our first main result.

Theorem 3.2. *Consider System (3) under Assumption (4). Assume $f_{101}(0) \notin R_1$.*

1. *If $f_{101}(0) \in R_1 + \text{Span}(f_{121}(0))$, let $\beta \in \mathbb{R}$ be such that*

$$f_{101}(0) + \beta f_{121}(0) \in R_1.$$

Then, for any $u_2^{\text{eq}} \in \mathbb{R}$ such that $u_2^{\text{eq}} \neq \beta$, system (3) is not STLC at $(0, (0, u_2^{\text{eq}}))$.

2. *If $f_{101}(0) \notin R_1 + \text{Span}(f_{121}(0))$, then, for any $u_2^{\text{eq}} \in \mathbb{R}$, system (3) is not B-STLC at $(0, (0, u_2^{\text{eq}}))$.*

Remark 3.3. In case 2., the second control does not improve controllability with respect to the single-input system obtained by taking $u_2 = 0$. In other words, the obstruction of Proposition 2.11 persists.

Remark 3.4. In case 1., the fact that the brackets $f_{121}(0)$ and $f_{101}(0)$ are colinear outside of R_1 is crucial. It allows the bracket f_{121} to possibly compensate the bracket f_{101} through the particular control $u_2^{\text{eq}} = \beta$. This critical value of the control is the only value around which system (3) may be STLC.

Remark 3.5. In case 1., a careful inspection of the proof leads to a more precise result : there is an obstruction to a weaker controllability property where the control u_1 is not required to be arbitrary small; this hybrid property could be called “B-STLC in u_1 – STLC in u_2 ” (or (B, L^∞) -STLC in the spirit of Definitions 3.6 and 3.7 below).

3.2. Obstruction coming from brackets of order 5

We now assume that the obstruction to STLC pointed out by Theorem 3.2 does not hold, i.e. we assume that $f_{101}(0) \in R_1$ and continue exploring possible obstructions to STLC. The result we give in this direction may be viewed as a “generalization” of Proposition 2.14, in the sense that it does not identify an obstruction to STLC in the sense of Definition 2.1 or 2.2, but rather to a stronger form of local controllability in the vein of the one from Definition 2.5, that has to be tailored to two-input systems (3)-(4):

Definition 3.6. Let $k \in \mathbb{N}$. The control system (3) is $(W^{k,\infty}, L^\infty)$ -STLC at $(z^{\text{eq}}, (u_1^{\text{eq}}, u_2^{\text{eq}}))$ if, for every $\varepsilon > 0$, there exists $\eta > 0$ such that, for every z_0, z_1 in $B(z^{\text{eq}}, \eta)$, there exists a control $(u_1(\cdot), u_2(\cdot))$ in $W^{k,\infty}([0, \varepsilon], \mathbb{R}) \times L^\infty([0, \varepsilon], \mathbb{R})$ such that the solution $z(\cdot) : [0, \varepsilon] \rightarrow \mathbb{R}^n$ of the control system (3) satisfies $z(0) = z_0$, $z(\varepsilon) = z_1$, and

$$\|u_1 - u_1^{\text{eq}}\|_{W^{k,\infty}([0,\varepsilon],\mathbb{R})} \leq \varepsilon, \quad \|u_2 - u_2^{\text{eq}}\|_{L^\infty([0,\varepsilon],\mathbb{R})} \leq \varepsilon. \quad (14)$$

Definition 3.7. Let $k \in \mathbb{N}$. The control system (3) is $(W^{k,\infty}, B)$ -STLC at $(z^{\text{eq}}, (u_1^{\text{eq}}, u_2^{\text{eq}}))$ if there exists $\alpha > 0$ such that, for every $\varepsilon > 0$, there exists $\eta > 0$ such that, for every z_0, z_1 in $B(z^{\text{eq}}, \eta)$, there exists a control $(u_1(\cdot), u_2(\cdot))$ in $W^{k,\infty}([0, \varepsilon], \mathbb{R}) \times L^\infty([0, \varepsilon], \mathbb{R})$ such that the solution $z(\cdot) : [0, \varepsilon] \rightarrow \mathbb{R}^n$ of the control system (3) satisfies $z(0) = z_0$, $z(\varepsilon) = z_1$, and

$$\|u_1 - u_1^{\text{eq}}\|_{W^{k,\infty}([0,\varepsilon],\mathbb{R})} \leq \varepsilon, \quad \|u_2 - u_2^{\text{eq}}\|_{L^\infty([0,\varepsilon],\mathbb{R})} \leq \alpha. \quad (15)$$

Remark 3.8. The norms used for each control are different in this controllability notion, making it a form of “hybrid” small-time local controllability. It fits the nature of system (3), where the second control plays a particular role due to the fact that f_2 vanishes at 0.

Let us now introduce the elements of the obstruction. This obstruction requires that

$$f_{101}(0) \in R_1, \quad f_{121}(0) \in R_1, \quad f_{21,01}(0) \in R_1. \quad (16)$$

The first point was introduced above to step out of the obstruction given by Theorem 3.2 and is fully justified; f_{121} also plays a role in Theorem 3.2, $f_{21,01}$, defined in (8), was not seen before; we do not think that an obstruction may be obtained in the case where the value at the origin of these two brackets is not in R_1 . The obstruction per se is more complex than the one in Proposition 2.14: instead of coming from a single “bad bracket” $f_{01,001}$, it comes from eight different brackets of order 5; for a given value of the equilibrium control u_2^{eq} , it is convenient to define a map $D_{u_2^{\text{eq}}} : \mathbb{R}^2 \rightarrow \mathbb{R}^n$ as follows:

$$D_{u_2^{\text{eq}}}(\lambda_1, \lambda_2) = \lambda_1^2(f_{01,001}(0) - u_2^{\text{eq}}f_{01,201}(0)) + \lambda_2^2(f_{21,021}(0) - u_2^{\text{eq}}f_{21,221}(0)) - \lambda_1\lambda_2(f_{21,001}(0) + f_{01,021}(0) - u_2^{\text{eq}}(f_{21,201}(0) + f_{01,221}(0))), \quad (17)$$

and there will be an obstruction if the vector $D_{u_2^{\text{eq}}}(\lambda_1, \lambda_2)$ stays, for all (λ_1, λ_2) different from $(0, 0)$, strictly on the same side of some hyperplane containing either R_1 or the larger subspace

$$R' = R_1 + \text{Span}(f_{01,201}(0), f_{21,221}(0), f_{21,201}(0), f_{01,221}(0)). \quad (18)$$

This illustrates the remark by Kawski in [20, Example 2.5.1], showing that obstructions to controllability rests on the behaviour of $D_{u_2^{\text{eq}}}(\lambda_1, \lambda_2)$ rather than only on the value of supposedly “bad” brackets taken individually.

We now state our main result.

Theorem 3.9. Consider system (3). Assume that (4) and (16) hold and let $u_2^{\text{eq}} \in \mathbb{R}$. Then :

1. If there exists a linear form φ on \mathbb{R}^n such that

$$R_1 \subset \ker \varphi, \tag{19}$$

$$\text{the quadratic form } (\lambda_1, \lambda_2) \mapsto \left\langle \varphi, D_{u_2^{\text{eq}}}(\lambda_1, \lambda_2) \right\rangle \text{ is positive definite,} \tag{20}$$

then system (3) is not $(W^{1,\infty}, L^\infty)$ -STLC at $(0, (0, u_2^{\text{eq}}))$.

2. If there exists a linear form ψ on \mathbb{R}^n such that

$$R' \subset \ker \psi, \tag{21}$$

$$\text{the quadratic form } (\lambda_1, \lambda_2) \mapsto \left\langle \psi, D_{u_2^{\text{eq}}}(\lambda_1, \lambda_2) \right\rangle \text{ is positive definite,} \tag{22}$$

then, system (3) is not $(W^{1,\infty}, B)$ -STLC at $(0, (0, u_2^{\text{eq}}))$.

The proof is given in section 5.3. Its structure mirrors the one of the proof of Theorem 3.2, but it features additional technical ideas, most of which are adapted from the proof of Proposition 2.13 conducted in [20, pp.40-72] to our two-input setting.

Remark 3.10. Notice that, since $R_1 \subset R'$, a linear form ψ that satisfies the requirements for Case 2., namely (21) and (22), would also satisfy the requirements (19) and (20) from Case 1..

Remark 3.11. As mentioned above when defining $D_{u_2^{\text{eq}}}(\lambda_1, \lambda_2)$ in Equation (17), the conditions (19)-(20) on φ (resp. (21)-(22) on ψ) can be understood in the following way: there exists an hyperplane H of \mathbb{R}^n , containing R_1 (resp. R'), such that $D_{u_2^{\text{eq}}}(\lambda_1, \lambda_2)$ strictly stays on the same side of H for all $(\lambda_1, \lambda_2) \in \mathbb{R}^2 \setminus \{(0, 0)\}$.

Remark 3.12. Similarly to Theorem 3.2, in Case 1., the ‘‘critical’’ values of u_2^{eq} for which (19) and (20) fail for all φ are the only ones for which one may overcome the obstruction and recover $(W^{1,\infty}, L^\infty)$ -STLC. However, while Theorem 3.2 shows that there exists a unique such value β , Theorem 3.9 does not guarantee existence or uniqueness of these critical values.

Remark 3.13. Conditions (19)-(20) in Case 1. (resp. (21)-(22) in Case 2.) exclude in particular the situation where $f_{01,001}(0)$ does not belong to R_1 (resp. R') but all the other brackets $f_{i1,jk1}$ do. In this situation, since none of the brackets of order 5 containing f_2 would play any role, one could believe that the potential effect of u_2 on the controllability properties of the system is erased, and that System (10) would not be $(W^{1,\infty}, B)$ -STLC, just like in the scalar-input case.

Nevertheless, it appears in the proof of Theorem 3.9 below that, in that situation, higher-order terms in the Chen-Fliess series involving f_2 cannot be easily dominated by the term associated to $f_{01,001}(0)$. Therefore, this particular case does not seem straightforwardly reducible to the scalar-input case, and dealing with would require a different strategy than the one used in our proof.

Some illustrating and enlightening examples concerning this theorem are presented in Section 4.2.

4. ILLUSTRATING EXAMPLES AND APPLICATIONS

4.1. Examples for Theorem 3.2

In case 2. of Theorem 3.2, the second control u_2 cannot compensate the obstruction to local controllability induced by f_{101} . The following example illustrates that case.

Example 4.1. Consider the system

$$\begin{cases} \dot{x} &= y^2 + yu_1, \\ \dot{y} &= 2y - u_1 + xu_2. \end{cases} \tag{23}$$

It is of the form (3) with

$$f_0 = \begin{pmatrix} y^2 \\ 2y \end{pmatrix}, \quad f_1 = \begin{pmatrix} y \\ -1 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 0 \\ x \end{pmatrix}.$$

Straightforward computations show that

$$R_1 = \text{Span}(\mathbf{e}_2), \quad [f_1, [f_0, f_1]](0) = -6\mathbf{e}_1, \quad f_{121}(0) = \mathbf{e}_2,$$

so we are in the case 2. of Theorem 3.2. Therefore, for any $u_2^{\text{eq}} \in \mathbb{R}$, system (23) is not B-STLC at $(0, (0, u_2^{\text{eq}}))$.

In case 1., Theorem 3.2 states that the system is not STLC around the equilibria $(0, (0, u_2^{\text{eq}}))$, unless u_2^{eq} is equal to a particular value β , that allows the bracket f_{121} to compensate the bracket f_{101} . Around the equilibrium $(0, (0, \beta))$, the system can then be STLC, like in the next example. The method used in the following example to show STLC was introduced in [15] to show local controllability of magnetically driven micro-swimming robots – as detailed below the following example. We reproduce it here on a simpler system.

Example 4.2. Consider the system

$$\begin{cases} \dot{x} &= y^2 + yu_1 - \frac{2}{\alpha}y^2u_2, \\ \dot{y} &= 2y - u_1 - \frac{1}{\alpha}yu_2, \end{cases} \quad (24)$$

for some $\alpha \neq 0$. Here we have

$$f_0 = \begin{pmatrix} y^2 \\ 2y \end{pmatrix}, \quad f_1 = \begin{pmatrix} y \\ -1 \end{pmatrix}, \quad f_2 = -\frac{1}{\alpha} \begin{pmatrix} 2y^2 \\ y \end{pmatrix}.$$

Straightforward computations show that

$$R_1 = \text{Span}(\mathbf{e}_2), \quad [f_1, [f_0, f_1]](0) = -6\mathbf{e}_1, \quad f_{121}(0) = \frac{6}{\alpha}\mathbf{e}_1,$$

so we are in the case 1. of Theorem 3.2. Therefore, the system (24) is not STLC at $(0, (0, u_2^{\text{eq}}))$ if $u_2^{\text{eq}} \neq \alpha$.

Let us now study controllability at $(0, (0, \alpha))$. As explained in Remark 3.1, we make the feedback transformation $\tilde{u}_2 = u_2 = \alpha + \tilde{u}_2$, which transforms system (24) into

$$\begin{cases} \dot{x} &= -y^2 + yu_1 - \frac{2}{\alpha}y^2\tilde{u}_2, \\ \dot{y} &= y - u_1 - \frac{1}{\alpha}y\tilde{u}_2, \end{cases} \quad \text{i.e. into (13) with } \tilde{f}_0 = \begin{pmatrix} -y^2 \\ y \end{pmatrix}, \quad (25)$$

and we show that this system is STLC at $(0, (0, 0))$ (in a sense the transformation “neutralizes” the bracket f_{101}). To this end, we use the sufficient Sussmann condition for controllability [7, Theorem 7.3] with $\theta = 1$ and the notation for G_η introduced in [13, Definition III.10]. Since $f_{121}(0) = \frac{6}{\alpha}\mathbf{e}_1$, the Lie brackets of order 3 generate the whole space, i.e. G_η is the whole tangent space if $\eta > 3$. The only Lie brackets of order at most 3 with an even number of 1 and 2 are $[f_1, [\tilde{f}_0, f_1]]$ and $[f_2, [\tilde{f}_0, f_2]]$, which are both zero and therefore belong trivially to G_3 .

Hence, the Sussmann condition from [7] is satisfied and system (25) is STLC at $(0, (0, 0))$, so system (24) is STLC at $(0, (0, \alpha))$.

Example 4.3 (Application to micro-swimmer robots). In addition to the previous examples, let us present a practical application of Theorem 3.2. The present paper was motivated by the work on controllability of micro-swimmer robot models made in [13–15]. The two swimmers studied in these papers are made of two (respectively

three) magnetized rigid segments, linked together with torsional springs, immersed in a low-Reynolds number fluid, and driven by a uniform in space, time-varying magnetic field \mathbf{H} . The swimmers' motion is assumed to be planar. The magnetic field \mathbf{H} belongs to the swimmers' plane and can therefore be decomposed, in the moving basis associated to the first segment, in two components called (H_\perp, H_\parallel) .

Seeing the magnetic field as a control function, the dynamics of both swimmers write as control systems that are exactly of type (3)-(4):

$$\dot{\mathbf{z}} = f_0(\mathbf{z}) + H_\perp f_1(\mathbf{z}) + H_\parallel f_2(\mathbf{z}), \quad (26)$$

with the state \mathbf{z} in \mathbb{R}^4 for the two-link swimmer (resp. \mathbb{R}^5 for the three-link swimmer). The detailed expressions of f_0 , f_1 and f_2 with respect to the system parameters are given in [13, Equations (12) to (16)] (resp. [15, Appendix]).

Moreover, assumptions (4) are satisfied. Hence, for all H_\parallel in \mathbb{R} , $(0, (0, H_\parallel))$ is an equilibrium point (the first zero is short for $(0, 0, 0, 0)$ in \mathbb{R}^4 (resp. $(0, 0, 0, 0, 0)$ in \mathbb{R}^5). One also has $R_1 = \text{Span}(\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$ (resp. $R_1 = \text{Span}(\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5)$) and the brackets of interest for Theorem 3.2 read:

$$[f_1, [f_0, f_1]](0) = (a_2, 0, 0, 0) \quad (\text{resp. } [f_1, [f_0, f_1]](0) = (a_3, 0, 0, 0, 0))$$

and

$$f_{121}(0) = (b_2, 0, 0, 0) \quad (\text{resp. } f_{121}(0) = (b_3, 0, 0, 0, 0)),$$

with a_2, a_3, b_2, b_3 constants that are nonzero under generic assumptions on the system parameters – see [13, Assumption III.2] (resp. [15, Assumption 1]).

We can therefore apply Theorem 3.2, Case 1. and conclude that the two-link swimmer (resp. three-link swimmer) is not STLC at $(0, (0, H_\parallel))$ for any H_\parallel such that $H_\parallel \neq a_2/b_2$ (resp. $H_\parallel \neq a_3/b_3$).

In [15], it is shown that the two-link swimmer (resp. the three-link swimmer) is indeed STLC at $(0, (0, a_2/b_2))$ (resp. $(0, (0, a_3/b_3))$), using the technique displayed in Example 4.2. However, the question of STLC at *other* equilibria of type $(0, (0, H_\parallel))$ was left open in [15, Remark 5]. Theorem 3.2 allows to answer that question: $(0, (0, a_2/b_2))$ (resp. $(0, (0, a_3/b_3))$) is the *only* equilibrium of this type for which the swimmer is STLC.

Remark 4.4. Former studies on the two-link swimmer had led to the following results: in [13], it is shown that the control system (26) associated to the 2-link swimmer is B-STLC at $(0, (0, 0))$ with controls bounded by $2a_2/b_2$; in [14], it is shown that it is moreover not STLC at $(0, (0, 0))$. The proof of this last result features an explicit construction of the function Φ that is used in the proof of Theorem 3.2 below.

4.2. Examples for Theorem 3.9

We start by considering a scalar-input system, inspired by the classical example given by Sussmann in [16, Equation (6.12), p. 711]:

$$\begin{cases} \dot{x} &= u_1, \\ \dot{y} &= x, \\ \dot{z} &= y, \\ \dot{w} &= x^3 + y^2 + z^2. \end{cases} \quad (27)$$

For this system, $f_{01,001}(0)$ is outside of S_1 . Yet it is shown (see [16]) that it is B-STLC (and STLC as well, as shown for instance in [11, Example 12]). Furthermore, Proposition 2.14 shows that it is *not* $W^{1,\infty}$ -STLC.

The following examples feature systems resembling System (27) with addition of a second control. Depending on the vector field f_2 associated to this second control, we will observe the role played by this second control in the two different cases of Theorem 3.9.

Example 4.5. Consider the control system

$$\begin{cases} \dot{x} &= u_1, \\ \dot{y} &= x, \\ \dot{z} &= y + xu_2, \\ \dot{w} &= x^3 + y^2 + z^2. \end{cases} \quad (28)$$

Straightforward computations show that $R_1 = R' = \text{Span}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ (in particular, one has $f_{101}(0) = f_{121}(0) = f_{01,21}(0) = 0$).

Moreover,

$$f_{01,001}(0) = f_{21,021}(0) = -2\mathbf{e}_4 \text{ and } f_{21,001}(0) = f_{01,021}(0) = 0.$$

Therefore, by choosing $\psi = -\mathbf{e}_4^*$, we are in case 2 of Theorem 3.9, and conclude that System (28) is not $(W^{1,\infty}, B)$ -STLC. Therefore, in this example, the second control does not help recover controllability, as in Example 4.1.

Example 4.6. Now, we consider the following system:

$$\begin{cases} \dot{x} &= u_1, \\ \dot{y} &= x, \\ \dot{z} &= y + xu_2, \\ \dot{w} &= x^3 + y^2 + z^2 + (y^2 + z^2)u_2. \end{cases} \quad (29)$$

All the relevant brackets are identical to those in the previous example, except that we now have

$$f_{01,201}(0) = -2\mathbf{e}_4 \text{ and } f_{21,221}(0) = -2\mathbf{e}_4.$$

Therefore, by choosing $\varphi = -\mathbf{e}_4^*$, we are in case 1 of Theorem 3.9, and conclude that System (28) is not $(W^{1,\infty}, L^\infty)$ -STLC at $(0, (0, 0))$.

However, setting $u_2 = -1$, System (29) becomes identical to the scalar-input system

$$\begin{cases} \dot{x} &= u_1, \\ \dot{y} &= x, \\ \dot{z} &= y - x, \\ \dot{w} &= x^3. \end{cases} \quad (30)$$

Let us show that this system is $W^{1,\infty}$ -STLC. In order to do this, we add the equation $\dot{u}_1 = v$ to the system and study controllability of this new extended system with state $(u_1, x, y, z, w) \in \mathbb{R}^5$ and control v . We check that, for this extended system, the Sussmann condition – already used above in Example 4.2 – is satisfied for $\theta = 1$ around the equilibrium $0_{\mathbb{R}^5}$. Indeed, the brackets $f_1, [f_0, f_1], [f_0, [f_0, f_1]], [f_0, [f_0, [f_0, f_1]]]$ and $[f_1, [f_0, [f_1, [f_0, [f_0, f_1]]]]$ span the whole state space at $0_{\mathbb{R}^5}$, and all the brackets of order at most 6 with an even number of times f_1 vanish at $0_{\mathbb{R}^5}$. Therefore, the extended system is STLC around $0_{\mathbb{R}^5}$. This means that both u_1 and $\dot{u}_1 = v$ can be arbitrary small in L^∞ -norm, and System (29) is indeed $(W^{1,\infty}, B)$ -STLC, and $(W^{1,\infty}, L^\infty)$ -STLC at $(0, (0, -1))$.

This second example illustrates the more interesting case of our result, in which the second control, when set to be around a specific value depending of the behaviour of brackets of order 5, helps recovering local controllability.

5. PROOFS OF THE THEOREMS

5.1. Notations and preliminaries

Solution associated to a control. Given a control $t \mapsto u(t) = (u_1(t), u_2(t))$ in $L^\infty([0, T], \mathbb{R}^2)$, we denote by

$$t \mapsto z_u(t)$$

the solution of (3) with control $u(\cdot)$ starting from the origin, i.e. $z_u(0) = 0$.

Vector fields and differential operators. A real analytic vector field f can indifferently be defined, in coordinates,

$$\text{either by } f(x) = \begin{pmatrix} a_1(x) \\ \vdots \\ a_n(x) \end{pmatrix} \text{ or by } f = \sum_{k=1}^n a_k \frac{\partial}{\partial x_k},$$

where a_1, \dots, a_n are real analytic functions. The first notation views f as assigning a tangent vector to each point while the second views f as a differential operator of order 1: $f\phi = \sum_{k=1}^n a_k \frac{\partial \phi}{\partial x_k}$ for any smooth function ϕ . For two vector fields f and g , we denote by fg the differential operator of order 2 obtained by composition: $(fg)\phi = f(g\phi)$. In coordinates, and if $g = \sum_{k=1}^n b_j \frac{\partial}{\partial x_j}$, one has

$$fg = \sum_{k=1}^n \sum_{j=1}^n a_k b_j \frac{\partial^2}{\partial x_k \partial x_j} + \sum_{j=1}^n \left(\sum_{k=1}^n a_k \frac{\partial b_j}{\partial x_k} \right) \frac{\partial}{\partial x_j}.$$

In that setting and as a differential operator, the Lie bracket (used so far in the paper, see beginning of section 2.2), is a commutator: $[f, g] = fg - gf$; one recovers the usual formula from fg and gf above.

Chen-Fliess series. Let $I = (i_1, \dots, i_k) \in \{0, 1, 2\}^k$ be a multi-index and f_0, f_1, f_2 the three vector fields defining the system in (3). One defines by iterated composition the k^{th} order differential operator

$$f_I = f_{i_1} f_{i_2} \dots f_{i_k},$$

or each $u = (u_1, u_2)$ in $L^\infty([0, T], \mathbb{R}^2)$, for a multi-index $I = (i_1, \dots, i_k) \in \{0, 1, 2\}^k$, the iterated integral $\int_0^T u_I$ is defined as follows :

$$\int_0^T u_I = \int_0^T \int_0^{\tau_k} \int_0^{\tau_{k-1}} \dots \int_0^{\tau_2} u_{i_k}(\tau_k) u_{i_{k-1}}(\tau_{k-1}) \dots u_{i_2}(\tau_2) u_{i_1}(\tau_1) d\tau_1 d\tau_2 \dots d\tau_k, \quad (31)$$

where the symbol u_0 is used with the convention $u_0 \equiv 1$.

Let $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}$ be a real analytic function defined in a neighbourhood of 0 in \mathbb{R}^n . The Chen-Fliess series associated to f_0, f_1, f_2, u and Φ at time T is defined as

$$\Sigma(u, f, \Phi, T) = \sum_I \left(\int_0^T u_I \right) (f_I \Phi)(0), \quad (32)$$

where the summation is made over all the multi-indices $I = (i_1, \dots, i_k)$ in $\{0, 1, 2\}^k$ with $k \in \mathbb{N}$. The Chen-Fliess series appears in a range of works in control theory and geometry (see [22–24]). It is shown in [16, Proposition 4.3, p. 698] that, due to the analyticity of the vector fields f_i , for all $A > 0$, there exists $T_0(A) > 0$ such that the series converges for any $T \leq T_0$ and any u such that $\|u\|_{L^\infty[0, T]} \leq A$, uniformly with respect to u and T , to $\Phi(z_u(T))$, i.e. one can write

$$\Phi(z_u(T)) = \Sigma(u, f, \Phi, T). \quad (33)$$

Finally, the analyticity of the f_i and Φ implies, as stated in [16, Lemma 4.2, p.697] the existence of some constant C independent of I such that

$$|(f_I \Phi)(0)| \leq C^k (k)!, \quad (34)$$

where k stands for the cardinal of I .

Constructing local coordinates from a collection of vector fields. The following lemma is classical in nature.

Lemma 5.1. *Given n smooth vector fields g_1, \dots, g_n such that $g_1(0), \dots, g_n(0)$ form a basis of \mathbb{R}^n , there is a set of local coordinates $z \mapsto s(z) = (s_1(z), \dots, s_n(z))$ defined on a neighborhood \mathcal{V} of 0 such that $s_1(0) = \dots = s_n(0) = 0$ and, for any i, j, k in $\{1, \dots, n\}$,*

$$\left. \begin{array}{l} g_1 s_1 = 1, \\ g_1 s_i = 0 \text{ if } i > 1 \end{array} \right\} \text{ identically on } \mathcal{V}, \quad (35)$$

$$g_j s_i(0) = \delta_{i,j} \quad (\text{i.e. } 1 \text{ if } i = j \text{ and } 0 \text{ otherwise}), \quad (36)$$

$$g_k g_j s_i(0) = 0 \quad \text{if } i \geq j \text{ and } k \geq j. \quad (37)$$

Proof. For any real number t and any smooth vector field X , let e^{tX} denote the flow of X at time t , defined $\mathbb{R}^n \rightarrow \mathbb{R}^n$ for all t if X is a complete vector field.

After possibly modifying the g_i 's far from 0 to achieve completeness (without changing the assumptions or conclusions of the lemma, which is local around 0), define the map $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by

$$T(t_1, \dots, t_n) = e^{t_1 g_1} \circ \dots \circ e^{t_n g_n}(0).$$

Clearly, $T(0) = 0$ and the columns of the Jacobian of T at 0 are the coordinates of the vectors $g_1(0), \dots, g_n(0)$, linearly independent, hence T defines a diffeomorphism from $(-\varepsilon, \varepsilon)^n$ onto $\mathcal{V} = T(-\varepsilon, \varepsilon)^n$ for $\varepsilon > 0$ small enough. Let $s : \mathcal{V} \rightarrow (-\varepsilon, \varepsilon)^n$ be the inverse of this diffeomorphism and let $s = (s_1, \dots, s_n)$.

By definition, for all i, j in $\{1, \dots, n\}$ and all t_1, \dots, t_n between $-\varepsilon$ and ε , one has $s_i(T(t_1, \dots, t_n)) = t_i$ and

$$e^{t g_j}(T(0, \dots, 0, t_j, \dots, t_n)) = T(0, \dots, 0, t_j + t, \dots, t_n).$$

This clearly translates, for z in \mathcal{V} , into

$$s_1(z) = \dots = s_{j-1}(z) = 0 \Rightarrow \begin{cases} g_j s_j(z) = 1, \\ g_j s_i(z) = 0 \text{ if } i \neq j. \end{cases}$$

If $j = 1$, the left-hand side imposes no condition on z and one gets (35). For general j , the above relation implies (36) (with $z = 0$, $s_j(z) = 0$ for all j), and also implies that $X g_j s_i(0) = 0$ if $i \geq j$ and X is a vector field such that $e^{tX}(0)$ remains, for t small, in $\{z \in \mathcal{V}, s_1(z) = \dots = s_{j-1}(z) = 0\}$. This yields (37). \square

5.2. Proof of Theorem 3.2

Most of the section consists in proving Proposition 5.2 below, more precise than Theorem 3.2. The proof relies on similar arguments than those used to prove Proposition 2.11 in [16, pp.707-710].

Proposition 5.2. *Assume that the assumptions of Case 1. (resp. Case 2.) of Theorem 3.2 are satisfied, with $u_2^{\text{eq}} = 0$, and fix and arbitrary $\alpha > 0$ (for case Case 2. only). There exists a neighborhood \mathcal{V} of the origin, a smooth function*

$$\Phi : \mathcal{V} \rightarrow \mathbb{R} \quad \text{satisfying} \quad \Phi(0) = 0, \quad d\Phi(0) \neq 0, \quad (38)$$

a constant $K > 0$, and some $\varepsilon^0 > 0$ such that for any T , $0 < T < \varepsilon^0$, any essentially bounded control $u : t \mapsto (u_1(t), u_2(t))$ defined on $[0, T]$ satisfying

$$\|u_1\|_{L^\infty([0, T])} \leq \varepsilon^0 \text{ and } \|u_2\|_{L^\infty([0, T])} \leq \varepsilon^0 \quad (\text{resp. } \|u_1\|_{L^\infty([0, T])} \leq \alpha \text{ and } \|u_2\|_{L^\infty([0, T])} \leq \alpha), \quad (39)$$

one has $z_u(t) \in \mathcal{V}$ for all t in $[0, T]$ and

$$\Phi(z_u(T)) \geq K \|v_1\|_{L^2([0, T])}^2 \quad (40)$$

with

$$v_1(t) = \int_0^t u_1(s) \, ds. \quad (41)$$

Proof of Theorem 3.2 assuming Proposition 5.2. Fix $\alpha > 0$ for case 2, choose ε smaller than the ε^0 given by Proposition 5.2. Then for any η , (38) implies that one may pick a point z_1 in $B(0, \eta)$ such that $\Phi(z_1) < 0$, and (40) then clearly implies that there may exist no control u satisfying (39) and such that $z_u(T) = z_1$. This contradicts STLC according to Definition 2.1 (resp. B-STLC according to Definition 2.2) and proves Theorem 3.2 in the case $u_2^{\text{eq}} = 0$. If $u_2^{\text{eq}} \neq 0$, just perform the feedback transformation suggested in Remark 3.1: $f_0 + u_1 f_1 + u_2 f_2 = \tilde{f}_0 + u_1 f_1 + \tilde{u}_2 f_2$ with $\tilde{f}_0 = f_0 + u_2^{\text{eq}} f_2$ and $\tilde{u}_2 = u_2 - u_2^{\text{eq}}$, apply the result to the system $\dot{z} = \tilde{f}_0 + u_1 f_1 + \tilde{u}_2 f_2$ at the equilibrium $z = 0$, $(u_1, \tilde{u}_2) = (0, 0)$, and deduce the result at the equilibrium $z = 0$, $(u_1, u_2) = (0, u_2^{\text{eq}})$ for the original system (noting that $f_{101}(0) + \beta f_{121}(0)$ is equal to $\tilde{f}_{101}(0) + (\beta - u_2^{\text{eq}}) \tilde{f}_{121}(0)$). \square

Proof of Proposition 5.2. The first step is to define a suitable coordinate function Φ .

Let $d_1 = \dim R_1$ and $d_2 = \dim(R_1 + \text{Span}(f_{121}(0)))$ (either $d_2 = d_1$ or $d_2 = d_1 + 1$). Let g_1, \dots, g_n be n vector fields such that:

- $g_1 = f_1$,
- $(g_1(0), \dots, g_n(0))$ is a basis of \mathbb{R}^n ,
- in Case 1., $(g_1(0), \dots, g_{d_1}(0))$ is a basis of R_1 and $g_{d_1+1} = f_{101}$,
- in Case 2., $(g_1(0), \dots, g_{d_2}(0))$ is a basis of $R_1 + \text{Span}(f_{121}(0))$ and $g_{d_2+1} = f_{101}$,

and define, from these vector fields, some local coordinates $z \mapsto (s_1(z), \dots, s_n(z))$ in a neighbourhood \mathcal{V} of 0 according to Lemma 5.1. Then, define Φ as follows:

- in case 1., $\Phi(\zeta) = s_{d_1+1}(\zeta)$,
- in case 2., $\Phi(\zeta) = s_{d_2+1}(\zeta)$.

The function Φ is real analytic on \mathcal{V} and has, by construction, the following properties:

$$\Phi(0) = 0, \quad (42)$$

$$\text{(Case 2.) } \forall g \in \mathcal{R}_1, (g\Phi)(0) = 0, \quad (43)$$

$$\text{(Case 1.) } \forall g \in \mathcal{R}_1 \cup \{f_{121}\}, (g\Phi)(0) = 0, \quad (44)$$

$$f_1\Phi = 0 \text{ on } \mathcal{V}, \quad (45)$$

$$(f_{101}\Phi)(0) = 1. \quad (46)$$

According to (45) and (4), one also has, with $I = (i_1, \dots, i_k)$,

$$f_I\Phi(0) = 0 \text{ if } i_1 = 0 \text{ or } i_1 = 2 \text{ or } i_k = 1. \quad (47)$$

In order to show that Φ satisfies (40), we then consider the Chen-Fliess series $\Sigma(u, f, \Phi, T)$ (see (32)) associated to the above constructed Φ and split its terms into six different types:

$$\Sigma(u, f, \Phi, T) = P_1 + P_2 + P_3 + P_4 + P_5 + P_6,$$

where each P_i contains the terms with multi-indices I defined as follows:

- $P_1 : I = (2, \dots)$, or $I = (0, \dots)$,
- $P_2 : I = (\dots, 1)$,
- $P_3 : I = (1, J)$ with J containing only 0's and 2's,
- $P_4 : I = (1, 1, 0)$,
- $P_5 : I = (1, 1, 2)$,
- $P_6 : \text{all the remaining terms.}$

The first two parts are easily dealt with: we have $P_1 = P_2 = 0$ because, according to (47), $f_I \Phi(0) = 0$ for I of these types. We also have $P_3 = 0$; indeed, let $I = (1, i_2, \dots, i_k)$ with $i_j = 0$ or 2 for all $j \in \{2, \dots, k\}$. Then, we can write that

$$f_1 f_{i_2} \dots f_{i_k} = [f_1, f_{i_2}] f_{i_3} \dots f_{i_k} + f_{i_2} f_1 \dots f_{i_k}, \quad (48)$$

and $(f_{i_2} f_1 \dots f_{i_k} \Phi)(0) = 0$ according to (47). Similarly, we have

$$[f_1, f_{i_2}] f_{i_3} \dots f_{i_k} = [[f_1, f_{i_2}], f_{i_3}] f_{i_4} \dots f_{i_k} + f_{i_3} [f_1, f_{i_2}] \dots f_{i_k}, \quad (49)$$

and $((f_{i_3} [f_1, f_{i_2}] \dots f_{i_k}) \Phi)(0) = 0$ because of assumption (4) and the fact that $i_3 = 0$ or 2 . Repeating this operation $k - 3$ more times, we eventually get that

$$(f_1 f_{i_2} \dots f_{i_k} \Phi)(0) = ([\dots [f_1, f_{i_2}], \dots, f_{i_k}] \Phi)(0). \quad (50)$$

But $[\dots [f_1, f_{i_2}], \dots, f_{i_k}]$ is in \mathcal{R}_1 , so $(f_1 f_{i_2} \dots f_{i_k} \Phi)(0) = 0$ because of (43). Therefore,

$$P_1 = P_2 = P_3 = 0. \quad (51)$$

The terms P_4 and P_5 , associated to the brackets f_{101} and f_{121} , are the key parts of the proof. Let us compute their value. For P_4 , we write

$$\begin{aligned} f_1 f_1 f_0 &= f_1 f_0 f_1 + f_1 [f_1, f_0] \\ &= f_1 f_0 f_1 - [f_1, f_0] f_1 - [[f_1, f_0], f_1]. \end{aligned}$$

The first two terms on the right-hand side vanish when evaluated at 0 against Φ because of (45), so $(f_{(1,1,0)} \Phi)(0) = -(f_{101} \Phi)(0) = 1$ by (46). Moreover, the control integral part is given by

$$\begin{aligned} \int_0^T u_{(1,1,0)} &= \int_0^T \int_0^s u_1(\sigma) \int_0^\sigma u_1(\tau) d\tau d\sigma ds \\ &= \int_0^T \int_0^s v_1'(\sigma) v_1(\sigma) d\sigma ds \end{aligned}$$

so, overall

$$P_4 = \frac{1}{2} \|v_1\|_{L^2}^2, \quad (52)$$

with v_1 defined in Equation (41).

Let us turn to P_5 . Expanding f_{121} into a sum of third order operators and applying (47) yields

$$(f_1 f_1 f_2 \Phi)(0) = (f_{121} \Phi)(0). \quad (53)$$

Here the two cases of the theorem differ. In case 2., (44) implies $f_{121} \Phi(0) = 0$, hence $P_5 = 0$. In case 1., this does not hold, but there is a real number β and a vector field g such that $f_{121} = -\beta f_{101} + g$, with $g \in \mathcal{R}_1$ and $\beta \neq 0$ because $f_{101}(0) \notin R_1$. Thanks to (53), (43) and (46), we conclude that $(f_1 f_1 f_2 \Phi)(0) = -\beta$. The control integral associated to P_5 reads

$$\int_0^T u_{(1,1,2)} = \int_0^T u_2(s) \int_0^s u_1(\sigma) \int_0^\sigma u_1(\tau) d\tau d\sigma ds \leq \frac{1}{2} \|u_2\|_{L^\infty} \|v_1\|_{L^2}^2,$$

and we deduce the following bound on P_5 , valid for both cases because it is obviously true if $P_5 = 0$:

$$|P_5| \leq \frac{1}{2} |\beta| \|u_2\|_{L^\infty} \|v_1\|_{L^2}^2. \quad (54)$$

Finally, we will show that the terms of P_6 add up to a small remainder:

Lemma 5.3. *For any $\alpha > 0$, there exists $T_0 > 0$ and a constant $D > 0$ such that, for all T in $[0, T_0]$, and all (u_1, u_2) satisfying $\|u_1\|_{L^\infty} \leq \alpha$ and $\|u_2\|_{L^\infty} \leq \alpha$, one has*

$$|P_6| \leq TD \|v_1\|_{L^2}^2. \quad (55)$$

Proof. Let I be a multi-index such that the associated term in the series is in P_6 . Then $I = (1, J, 1, K)$ with $K = (k_1, \dots, k_q)$ and $J = (j_1, \dots, j_r)$ such that $q \geq 1$, $q + r \geq 2$ and J contains only 0's and 2's. Let us denote by J_2 the number of 2's in J , and K_1 and K_2 respectively the number of 1's and 2's in K . The control integral associated to I reads

$$\int_0^T u_I = \int_0^T \int_0^{s_q} \int_0^{s_{q-1}} \cdots \int_0^{s_2} u_{k_q}(s_q) u_{k_{q-1}}(s_{q-1}) \cdots u_{k_1}(s_2) W_r(s_1) ds_1 \cdots ds_q, \quad (56)$$

with

$$W_r(s) = \int_0^s u_1(\tau_{r+1}) \int_0^{\tau_{r+1}} u_{j_r}(\tau_r) \cdots \int_0^{\tau_2} u_{j_1}(\tau_1) \int_0^{\tau_1} u_1(\tau_0) d\tau_0 \cdots d\tau_{r+1}.$$

Applying triangular inequality, bounding u_i by $\|u_i\|_{L^\infty}$ ($i = 1, 2$) and using $u_0 = 1$ yields

$$\left| \int_0^T u_I \right| \leq \|u_2\|_{L^\infty}^{K_2} \|u_1\|_{L^\infty}^{K_1} \int_0^T \int_0^{s_q} \int_0^{s_{q-1}} \cdots \int_0^{s_2} |W_r(s_1)| ds_1 \cdots ds_q, \quad (57)$$

which rewrites as

$$\left| \int_0^T u_I \right| \leq \|u_2\|_{L^\infty}^{K_2} \|u_1\|_{L^\infty}^{K_1} \frac{1}{(q-1)!} \int_0^T (T-s)^{q-1} |W_r(s)| ds. \quad (58)$$

The study of $W_r(s)$ splits into three cases.

- **Case $r=0$.** In this case, we have

$$W_0(s) = \int_0^s u_1(\tau_1) \int_0^{\tau_1} u_1(\tau_0) d\tau_0 d\tau_1 = \frac{1}{2} v_1^2(s). \quad (59)$$

- **Case $r=1$.** In this case, we integrate by parts to get

$$W_1(s) = v_1(s) \int_0^s u_{j_1}(\tau) v_1(\tau) d\tau + \int_0^s u_{j_1}(\tau) v_1^2(\tau) d\tau, \quad (60)$$

and we apply triangular inequality to bound the u_{j_1} term:

$$\|W_1(s)\| \leq \|u_2\|_{L^\infty}^{J_2} \left(|v_1(s)| \int_0^s |v_1(\tau)| d\tau + \int_0^s v_1^2(\tau) d\tau \right). \quad (61)$$

(recall that, here, $J_2 = 1$ if $j_1 = 2$ and $J_2 = 0$ if $j_1 = 0$.)

- **Case $r > 1$.** In this case, we integrate by parts to get

$$\begin{aligned} W_r(s) = & v_1(s) \int_0^s u_{j_r}(\tau_r) \int_0^{\tau_r} u_{j_{r-1}}(\tau_{r-1}) \cdots \int_0^{\tau_2} u_{j_1}(\tau_1) v(\tau_1) d\tau_1 \cdots d\tau_r \\ & - \int_0^s v_1(\tau_r) u_{j_r}(\tau_r) \int_0^{\tau_r} u_{j_{r-1}}(\tau_{r-1}) \cdots \int_0^{\tau_2} u_{j_1}(\tau_1) v(\tau_1) d\tau_1 \cdots d\tau_r, \end{aligned} \quad (62)$$

and we use triangular inequality to bound the u_{j_p} terms:

$$|W_r(s)| \leq \|u_2\|_{L^\infty}^{J_2} \left(|v_1(s)| \int_0^s \int_0^{\tau_r} \cdots \int_0^{\tau_1} |v_1(\tau_1)| d\tau_1 \cdots d\tau_r - \int_0^s |v_1(s)| \int_0^{\tau_r} \cdots \int_0^{\tau_1} |v_1(\tau_1)| d\tau_1 \cdots d\tau_r \right), \quad (63)$$

which rewrites as

$$|W_r(s)| \leq \|u_2\|_{L^\infty}^{J_2} \left(\frac{|v_1(s)|}{(r-1)!} \int_0^s (s-\sigma)^{r-1} |v_1(\sigma)| d\sigma - \frac{1}{(r-2)!} \int_0^s |v_1(\tau)| \int_0^\tau (\tau-\sigma)^{r-2} |v_1(\sigma)| d\sigma d\tau \right). \quad (64)$$

The Cauchy-Schwarz inequality yields

$$\int_0^s (s-\sigma)^{r-1} |v_1(\sigma)| d\sigma \leq \frac{s^{r-1/2}}{\sqrt{2r-1}} \|v_1\|_{L^2}. \quad (65)$$

We then apply (65) to (64), once to the first term and twice to the second term, which leads to the following upper bound:

$$|W_r(s)| \leq \|u_2\|_{L^\infty}^{J_2} \left(|v_1(s)| \frac{\|v_1\|_{L^2}}{(r-1)!} \frac{s^{r-1/2}}{\sqrt{2r-1}} + \frac{\|v_1\|_{L^2}^2}{(r-2)!} \frac{s^{r-1}}{\sqrt{(2r-3)(2r-2)}} \right). \quad (66)$$

Substituting (59), (61), and (66) in (58), and bounding $(T-s)$ by T , we have:

$$\left| \int_0^T u_I \right| \leq \|u_2\|_{L^\infty}^{K_2+J_2} \|u_1\|_{L^\infty}^{K_1} \frac{AT^{q+r-1}}{(q-1)!(r-2)!} \|v_1\|_{L^2}^2, \quad (67)$$

where $(r-2)!$ is replaced by 1 if $r \in \{0, 1\}$. Here and hereafter, A is a constant that may vary from line to line. Thanks to (34), we have, for some constant C independent of I ,

$$|(f_I \Phi)(0)| \leq C^{q+r+2} (q+r+2)!. \quad (68)$$

Combining (67) and (68), we find an upper bound for the whole term of index I from the series:

$$\left| \left(\int_0^T u_I \right) (f_I \Phi)(0) \right| \leq B(q, r) T \|u_2\|_{L^\infty}^{K_2+J_2} \|u_1\|_{L^\infty}^{K_1} \|v_1\|_{L^2}^2,$$

with

$$B(q, r) = AC^{q+r+2} T^{q+r-2} \frac{(q+r)!(q+r+2)^5}{q!r!}, \quad (69)$$

where we bounded $\frac{(q+r+2)}{(q-1)!(r-2)!}$ by $\frac{(q+r)!(q+r+2)^5}{q!r!}$ to encompass the cases $r=0$ and $r=1$.

For any given q and r , there are $2^r 3^q$ corresponding indices I . More precisely:

- for any given q , K_2 , K_1 , there are $\binom{q}{K_1}$ choices to place the 2's and then $\binom{q-K_1}{K_2}$ choices to place the 1's.
- for any given r , J_2 , there are $\binom{r}{J_2}$ choices to place the 1's.

Therefore, summing all the terms in P_6 , we obtain an upper bound:

$$|P_6| \leq T \|v_1\|_{L^2}^2 \sum_{\substack{r \geq 0, q \geq 1 \\ r+q \geq 2}} B(q, r) \sum_{J_2=0}^r \binom{r}{J_2} \|u_2\|_{L^\infty}^{J_2} \sum_{K_1=0}^q \binom{q}{K_1} \|u_1\|_{L^\infty}^{K_1} \sum_{K_2=0}^{q-K_1} \binom{q-K_1}{K_2} \|u_2\|_{L^\infty}^{K_2}. \quad (70)$$

This rewrites as

$$|P_6| \leq T \|v_1\|_{L^2}^2 \sum_{\substack{r \geq 0, q \geq 1 \\ r+q \geq 2}} B(q, r) (1 + \|u_2\|_{L^\infty})^r (1 + \|u_2\|_{L^\infty} + \|u_1\|_{L^\infty})^q. \quad (71)$$

Using $(1 + \|u_2\|_{L^\infty}) \leq (1 + \|u_2\|_{L^\infty} + \|u_1\|_{L^\infty})$ and renumbering the terms of the sum for $p \geq 2$ and $0 \leq r \leq p$ (such that $p = r + q$ in equations (70) and (71)), one obtains

$$|P_6| \leq T \|v_1\|_{L^2}^2 \sum_{p \geq 2} \sum_{r=0}^p B(p-r, r) (1 + \|u_2\|_{L^\infty} + \|u_1\|_{L^\infty})^p. \quad (72)$$

From (69), we have

$$\sum_{r=0}^p B(p-r, r) = \sum_{r=0}^p AC^{p+2} T^{p-2} \frac{(p)!(p+2)^5}{(p-r)!r!}.$$

Using Stirling's formula, we deduce that for any $\alpha > 0$, there exists $T_0 > 0$ and a constant $D > 0$ such that, for all T in $[0, T_0]$, and all (u_1, u_2) satisfying $\|u_1\|_{L^\infty} \leq \alpha$ and $\|u_2\|_{L^\infty} \leq \alpha$, the series in (72) converges for all T in $[0, T_0]$.

Including its limit for $T = T_0$ and the other constants in a new constant D , we finally obtain (55) for any $T \in [0, T_0]$. \square

We now end the proof in both cases. Let T_0 and D be as defined in Lemma 5.3.

In case 1., using (51) and (52), we have

$$\Sigma(u, f, \Phi, T) = \frac{1}{2} \|v_1\|_{L^2}^2 + P_5 + P_6, \quad (73)$$

knowing, thanks to (54) and (55) that

$$|P_5 + P_6| \leq \frac{1}{2} |\beta| \|u_2\|_{L^\infty} \|v_1\|_{L^2}^2 + TD \|v_1\|_{L^2}^2. \quad (74)$$

Let ε be a real positive number such that $\varepsilon(\frac{1}{2}|\beta| + D) \leq \frac{1}{4}$. Let $\varepsilon_0 = \min(T_0, \varepsilon)$. Taking a smaller T if necessary, we can assume that $z_u(t) \in \mathcal{V}$ for all t in $[0, T]$. Assume that $T \leq \varepsilon_0$ and $\|u_2\|_{L^\infty} \leq \varepsilon_0$. Using (74) into (73), we obtain that $\Sigma(u, f, \Phi, T) \geq \frac{1}{4} \|v_1\|_{L^2}^2$ for all $T \leq \varepsilon_0$ and $\|u_2\|_{L^\infty} \leq \varepsilon_0$, i.e. we have proven (40) with $K = \frac{1}{4}$.

In case 2., we obtain that

$$\Sigma(u, f, \Phi, T) = \frac{1}{2} \|v_1\|_{L^2}^2 + P_6. \quad (75)$$

Let ε be a real positive number such that $\varepsilon D \leq \frac{1}{4}$. Let $\varepsilon_0 = \min(T_0, \varepsilon)$. Taking a smaller T if necessary, we can assume that $z_u(t) \in \mathcal{V}$ for all t in $[0, T]$. Assume that $T \leq \varepsilon_0$, $\|u_1\|_{L^\infty} \leq \alpha$ and $\|u_2\|_{L^\infty} \leq \alpha$ (for an arbitrary $\alpha > 0$). Using (55) into (75), we obtain that $\Sigma(u, f, \Phi, T) \geq \frac{1}{4} \|v_1\|_{L^2}^2$ for all $T \leq \varepsilon_0$, i.e. we have proven (40) with $K = \frac{1}{4}$.

This concludes the proof of Proposition 5.2. \square

5.3. Proof of Theorem 3.9

Most of this section consists in proving Proposition 5.4 below, more precise than Theorem 3.9.

Proposition 5.4. *Assume that the assumptions of Case 1. (resp. Case 2.) of Theorem 3.9 are satisfied, and fix and arbitrary $\alpha > 0$ (for case Case 2. only). There exists a neighborhood \mathcal{V} of the origin, a smooth function*

$$\Phi : \mathcal{V} \rightarrow \mathbb{R} \quad \text{satisfying} \quad \Phi(0) = 0, \quad d\Phi(0) \neq 0, \quad (76)$$

a constant $K > 0$, and some $\varepsilon^0 > 0$ such that, for any T , $0 < T < \varepsilon^0$, any essentially bounded control $u : t \mapsto (u_1(t), u_2(t))$ defined on $[0, T]$ satisfying

$$\|u_1\|_{W^{1,\infty}([0,T])} \leq \varepsilon^0 \quad \text{and} \quad \|u_2\|_{L^\infty([0,T])} \leq \varepsilon^0 \quad (\text{resp. } \|u_2\|_{L^\infty([0,T])} \leq \alpha), \quad (77)$$

one has $z_u(t) \in \mathcal{V}$ for all t in $[0, T]$ and

$$\Phi(z_u(t)) \geq K \left(\|w_1\|_{L^2([0,t])}^2 + \|w_2\|_{L^2([0,t])}^2 \right), \quad 0 \leq t \leq T \quad (78)$$

with

$$w_1(t) = \iint_{0 \leq s_2 \leq s_1 \leq t} u_1(s_1) \, ds_1 \, ds_2, \quad w_2(t) = \iint_{0 \leq s_2 \leq s_1 \leq t} u_1(s_1) u_2(s_2) \, ds_1 \, ds_2. \quad (79)$$

Remark 5.5. Notice that u_1 and u_2 are not treated symmetrically in defining w_1 and w_2 above: w_1 depends on u_1 only, while w_2 depends on u_2 and u_1 .

Proof of Theorem 3.9 assuming Proposition 5.4. (Very similar to the proof of Theorem 3.2 assuming Proposition 5.2 in section 5.2.)

Fix $\alpha > 0$ for case 2, choose ε smaller than the ε^0 given by Proposition 5.4. Then for any η , (76) implies that one may pick a point z_1 in $B(0, \eta)$ such that $\Phi(z_1) < 0$, and (78) then clearly implies that there may exist no control u satisfying (77) and such that $z_u(T) = z_1$. This contradicts $(W^{1,\infty}, L^\infty)$ -STLC according to Definition 3.6 (resp. $(W^{1,\infty}, B)$ -STLC according to Definition 3.7) and proves Theorem 3.2 in the case $u_2^{\text{eq}} = 0$. If $u_2^{\text{eq}} \neq 0$, just perform the same feedback transformation as in the proof of Proposition 5.2 and deduce the result at the equilibrium $z = 0$, $(u_1, u_2) = (0, u_2^{\text{eq}})$ for the original system. \square

Proof of Proposition 5.4. Let $T > 0$, $u_1 \in W^{1,\infty}([0, T])$ and $u_2 \in L^\infty([0, T])$.

We will prove (78) in several steps :

Step 1. We define a suitable function Φ .

Step 2. We study the associated Chen-Fliess series and, following [20, pp.41-56], we perform a ‘‘change of basis’’ over the differential operators f_I that are associated with an index I containing two or three 1’s.

Step 3. We isolate and calculate the sum of four dominating terms, called \mathcal{P}_{dom} ,

Step 4. We deal with the remaining terms in the series, by showing that they are small compared to \mathcal{P}_{dom} ,

Step 5. Finally we sum all the terms and prove (78).

Step 1. Construction of Φ .

We start with the following lemma, stating that the assumptions made in Theorem 3.9 imply that $f_1(0), f_{01}(0), f_{21}(0)$ are linearly independent.

Lemma 5.6. *If the vector fields f_0, f_1 and f_2 are such that (4) (with $z^{\text{eq}} = 0$) and (16) hold and, for some $u_2^{\text{eq}} \in \mathbb{R}$ and for all $(\lambda_1, \lambda_2) \in \mathbb{R}^2 \setminus \{(0, 0)\}$, $D_{u_2^{\text{eq}}}(\lambda_1, \lambda_2) \notin R_1$, then*

$$\text{Rank}\{f_1(0), f_{01}(0), f_{21}(0)\} = 3. \quad (80)$$

Proof. First, notice that (80) is preserved by the feedback transformation $\tilde{u}_2 = u_2 - u_2^{\text{eq}}$ described in Remark 3.1, so we can assume without loss of generality that $u_2^{\text{eq}} = 0$ in the statement of the Lemma.

Assume that (80) fails. Since $f_1(0) \neq 0$ (see (4)), we have three possible cases:

- (i) $\text{Rank}\{f_1(0), f_{21}(0)\} = 2$,
- (ii) $\text{Rank}\{f_1(0), f_{21}(0)\} = 1$ and $\text{Rank}\{f_1(0), f_{01}(0)\} = 2$,
- (iii) $\text{Rank}\{f_1(0), f_{01}(0), f_{21}(0)\} = 1$.

In case (i), there exists two real numbers μ, ν such that $f_{01}(0) = \mu f_{21}(0) + \nu f_1(0)$; introducing $H = f_{01} - \mu f_{21} - \nu f_1$, we have

$$f_{01} = \mu f_{21} + \nu f_1 + H, \quad H(0) = 0. \quad (81)$$

Taking the Lie brackets of both sides by f_0 yields

$$f_{001} = \mu f_{021} + \nu f_{01} + [f_0, H]. \quad (82)$$

We take the Lie bracket by f_{01} , and we use (81), (82), and the Jacobi identity in the last line, so that

$$\begin{aligned}
f_{01,001} &= \mu f_{01,021} + [f_{01}, [f_0, H]] \\
&= \mu f_{01,021} + [\mu f_{21} + \nu f_1 + H, [f_0, H]] \\
&= \mu f_{01,021} + \mu [f_{21}, [f_0, H]] + \nu [f_1, [f_0, H]] + [H, [f_0, H]] \\
&= \mu f_{01,021} + \mu f_{21001} - \mu^2 f_{21021} - \mu\nu f_{2101} + \nu f_{1001} - \mu\nu f_{1021} - \nu^2 f_{101} \\
&= -\mu^2 f_{21,021} + \mu(f_{01,021} + f_{21,001}) - \mu\nu([f_0, f_{121}] + 2f_{21,01}) + \nu[f_0, f_{101}] - \nu^2 f_{101} + [H, [f_0, H]].
\end{aligned} \tag{83}$$

With D_0 the map defined in (17), this implies

$$D_0(1, \mu) = -\mu\nu([f_0, f_{121}](0) + 2f_{21,01}(0)) + \nu[f_0, f_{101}](0) - \nu^2 f_{101}(0) + [H, [f_0, H]](0), \tag{84}$$

where the right-hand side is in R_1 from (16) and the fact that $f_0(0) = H(0) = 0$.

In case (ii), we repeat the above calculations while swapping the roles of f_{01} and f_{21} , yielding an expression for $f_{21,021}$ instead of $f_{01,001}$ in equation (83) and obtaining that $D_0(\mu, 1) \in R_1$. Finally, case (iii) is actually contained in case (i) by taking $\mu = 0$ in equation (81).

Overall, we have proven in each case the existence of $(\lambda_1, \lambda_2) \neq (0, 0)$ such that $D_0(\lambda_1, \lambda_2) \in R_1$ if (80) fails, whence the Lemma. \square

Now, let $d_1 = \dim R_1$ and $d_2 = \dim R'$ and construct n real analytic vector fields g_1, \dots, g_n enjoying the following properties:

- (1) $g_1 = f_1$, $g_2 = [f_0, f_1]$, $g_3 = [f_2, f_1]$, and $(g_1(0), \dots, g_n(0))$ is an orthonormal basis of \mathbb{R}^n for the canonical scalar product,
- (2) in Case 1., $(g_1(0), \dots, g_{d_1}(0))$ is a basis of R_1 and g_{d_1+1} is such that

$$\langle \varphi, g_{d_1+1}(0) \rangle = 1 \quad \text{and} \quad \langle \varphi, g_j(0) \rangle = 0, \quad j \neq d_1 + 1, \tag{85}$$

- (3) in Case 2., $(g_1(0), \dots, g_{d_2}(0))$ is a basis of R' and g_{d_2+1} is such that

$$\langle \varphi, g_{d_2+1}(0) \rangle = 1 \quad \text{and} \quad \langle \varphi, g_j(0) \rangle = 0, \quad j \neq d_2 + 1, \tag{86}$$

The first point is possible thanks to Lemma 5.6 and the two other points thanks to the properties (19), (20), (21), (22) of the covectors φ and ψ . Then define, from these g_1, \dots, g_n , some local coordinates $x \mapsto (s_1(x), \dots, s_n(x))$ in a neighbourhood \mathcal{V} of 0 according to Lemma 5.1 and define Φ as follows:

- in Case 1., $\Phi(\zeta) = s_{d_1+1}(\zeta)$,
- in Case 2., $\Phi(\zeta) = s_{d_2+1}(\zeta)$.

By virtue of Lemma 5.1 and (4), the function Φ satisfies:

$$\Phi(0) = 0, \tag{87}$$

$$f_1 \Phi = 0 \text{ on } \mathcal{V}, \tag{88}$$

$$f_I \Phi(0) = 0 \text{ if } i_1 = 0 \text{ or } i_1 = 2 \text{ or } i_k = 1. \tag{89}$$

Moreover, in Case 1., one has, thanks to (36) and (85) for the first point, and (37) for the second point,

$$d\Phi(0) = \varphi, \tag{90}$$

$$\forall k, \ell \in \{1, \dots, d_1\}, k \geq \ell \Rightarrow (g_k g_\ell \Phi)(0) = 0, \tag{91}$$

and, similarly in Case 2., one has, thanks to (36) and (86) for the first point, and (37) for the second point,

$$d\Phi(0) = \psi, \tag{92}$$

$$\forall k, \ell \in \{1, \dots, d_2\}, k \geq \ell \Rightarrow (g_k g_\ell \Phi)(0) = 0. \tag{93}$$

Step 2. Change of basis.

Consider the Chen-Fliess series $\Sigma(u, f, \Phi, T)$ associated to Φ , given by (32). We first part the series in the sum of the terms where derivation along f_1 appears at most three times and these where it appears at least four times:

$$\Sigma(u, f, \Phi, T) = \mathcal{P}_0 + \mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3 + \mathcal{P}_{\geq 4} \quad (94)$$

where the subscript refers to the number of times the index 1 appears in the multi-index $I \in \{0, 1, 2\}^k$.

As already noticed in the proof of Theorem 3.2 (equations (48) to (51)), we have $\mathcal{P}_0 = \mathcal{P}_1 = 0$, using that f_0 and f_2 vanish at 0 and the properties of Φ .

Now, we turn to \mathcal{P}_2 and \mathcal{P}_3 . The purpose of this step is the same as the classification in different types (P_1 to P_6) made in the proof of Theorem 3.2: classify the terms into convenient categories, in order to study them more easily. However, here, instead of simply categorizing the multi-indices I , we follow [20] and perform a change of basis over the operators f_I , combined with successive integrations by parts of the iterated integrals $\int u_I$.

This suitable change of basis is performed in detail in [20, pp.41-56] for a scalar-input system and can be adapted to the two-control system (3), yielding the following expressions:

$$\mathcal{P}_2 = \sum_{\ell \in \mathbb{N}} \sum_{L \in \{0,2\}^\ell} \sum_{\mu \in \{0, \dots, \ell\}} b_{\mu}^{2,L}(W_{\mu}^{2,L}\Phi)(0), \quad (95)$$

$$\mathcal{P}_3 = \sum_{\ell \in \mathbb{N}} \sum_{L \in \{0,2\}^\ell} \sum_{\xi=1}^3 \sum_{(\mu, \nu) \in \mathcal{J}(\xi, \ell)} b_{\xi, \mu, \nu}^{3,L}(W_{\xi, \mu, \nu}^{3,L}\Phi)(0), \quad (96)$$

where

$$\begin{aligned} \mathcal{J}(1, \ell) &= \{(\mu, \nu) \in \mathbb{N}^2, \mu \leq \nu, \mu + 2\nu \leq \ell - 1\}, \\ \mathcal{J}(2, \ell) &= \{(\mu, \nu) \in \mathbb{N}^2, 2\mu + \nu \leq \ell - 1\}, \\ \mathcal{J}(3, \ell) &= \{(\mu, \nu) \in \mathbb{N}^2, \nu \leq 2\mu \leq \ell - 1\}, \end{aligned} \quad (97)$$

Let $\ell \in \mathbb{N}$ and $L = (i_1, \dots, i_\ell)$ in $\{0, 2\}^\ell$. The explicit expressions of the differential operators W read

$$W_{\mu}^{2,L} = \left(\prod_{j=1}^{\ell-2\mu-1} \text{adf}_{i_{\ell-j+1}} \right) \left[\left(\prod_{j=\ell-2\mu}^{\ell-\mu-1} \text{adf}_{i_{\ell-j+1}} \right) f_1, \left[\left(\prod_{j=\ell-\mu}^{\ell-1} \text{adf}_{i_{\ell-j+1}} \right) f_1, f_{i_1} \right] \right] \text{ if } 0 \leq 2\mu \leq \ell - 1, \quad (98)$$

$$W_{\mu}^{2,L} = \left(\left(\prod_{j=1}^{\mu} \text{adf}_{i_{\ell-j+1}} \right) f_1 \right) \left(\left(\prod_{j=\mu+1}^{\ell} \text{adf}_{i_{\ell-j+1}} \right) f_1 \right) \text{ if } \ell \leq 2\mu \leq 2\ell, \quad (99)$$

$$W_{1, \mu, \nu}^{3,L} = - \left(\prod_{j=1}^{\ell-2\nu-\mu-1} \text{adf}_{i_{\ell-j+1}} \right) \left[\left(\prod_{j=\ell-2\nu-\mu}^{\ell-2\nu-1} \text{adf}_{i_{\ell-j+1}} \right) f_1, \left[\left(\prod_{j=\ell-2\nu}^{\ell-\nu-1} \text{adf}_{i_{\ell-j+1}} \right) f_1, \left[\left(\prod_{j=\ell-\nu}^{\ell-1} \text{adf}_{i_{\ell-j+1}} \right) f_1, f_{i_1} \right] \right] \right], \quad (100)$$

$$W_{2, \mu, \nu}^{3,L} = \left(\left(\prod_{j=1}^{\ell-2\mu-\nu-1} \text{adf}_{i_{\ell-j+1}} \right) \left[\left(\prod_{j=\ell-2\mu-\nu}^{\ell-\mu-\nu-1} \text{adf}_{i_{\ell-j+1}} \right) f_1, \left[\left(\prod_{j=\ell-\mu-\nu}^{\ell-\nu-1} \text{adf}_{i_{\ell-j+1}} \right) f_1, f_{i_\nu} \right] \right] \right) \left(\left(\prod_{j=\ell-\nu+1}^{\ell} \text{adf}_{i_{\ell-j+1}} \right) f_1 \right), \quad (101)$$

$$W_{3, \mu, \nu}^{3,L} = \left(\left(\prod_{j=1}^{\ell-\mu-\nu} \text{adf}_{i_{\ell-j+1}} \right) f_1 \right) \left(\left(\prod_{j=\ell-\mu-\nu+1}^{\ell-\mu} \text{adf}_{i_{\ell-j+1}} \right) f_1 \right) \left(\left(\prod_{j=\ell-\mu+1}^{\ell} \text{adf}_{i_{\ell-j+1}} \right) f_1 \right), \quad (102)$$

and the associated iterated integrals are respectively given by

$$b_{\mu}^{2,L} = -\frac{1}{2} \int u_{i_1} \int u_{i_2} \cdots \int u_{i_{\ell-2\mu}} (\int u_{i_{\ell-2\mu+1}} \cdots \int u_{i_{\ell-\mu}} \int u_1) (\int u_{i_{\ell-\mu+1}} \cdots \int u_{i_{\ell}} \int u_1) \text{ if } 0 \leq 2\mu \leq \ell - 1, \quad (103)$$

$$b_{\mu}^{2,L} = (\int u_{i_1} \cdots \int u_{i_{\mu}} \int u_1) (\int u_{i_{\mu+1}} \cdots \int u_{i_{\ell}} \int u_1) \text{ if } \ell \leq 2\mu \leq 2\ell, \quad (104)$$

$$b_{1,\mu,\nu}^{3,L} = \delta \int u_{i_1} \cdots \int u_{i_{\ell-2\nu-\mu}} (\int u_{i_{\ell-2\nu-\mu+1}} \cdots \int u_{i_{\ell-2\nu}} \int u_1) (\int u_{i_{\ell-2\nu+1}} \cdots \int u_{i_{\ell-\nu}} \int u_1) (\int u_{i_{\ell-\nu+1}} \cdots \int u_{i_{\ell}} \int u_1), \quad (105)$$

$$b_{2,\mu,\nu}^{3,L} = \frac{1}{2} \left(\int u_{i_1} \cdots \int u_{i_{\ell-2\mu-\nu}} (\int u_{i_{\ell-2\mu-\nu+1}} \cdots \int u_{i_{\ell-\mu-\nu}} \int u_1) (\int u_{i_{\ell-\mu-\nu+1}} \cdots \int u_{i_{\ell-\nu}} \int u_1) \right) (\int u_{i_{\ell-\nu+1}} \cdots \int u_{i_{\ell}} \int u_1), \quad (106)$$

$$b_{3,\mu,\nu}^{3,L} = \delta (\int u_{i_1} \cdots \int u_{i_{\ell-\mu-\nu}} \int u_1) (\int u_{i_{\ell-\mu-\nu+1}} \cdots \int u_{i_{\ell-\mu}} \int u_1) (\int u_{i_{\ell-\mu+1}} \cdots \int u_{i_{\ell}} \int u_1). \quad (107)$$

The bounds on the iterated integrals have been removed to lighten notations. To avoid any notational ambiguity, let us specify that the products of two or three integrals appearing in expressions (103), (105) and in the first term of (106) denote a single integrand, whereas expressions (104) and (107) denote products of two or three independent integrals. The constant δ that appears in (105) and (107) depends on ℓ , μ and ν and take only values 1 , $\frac{1}{2}$ or $\frac{1}{6}$.

Step 3. Dominating terms. The essence of the whole proof is to show that the four terms in (95) in \mathcal{P}_2 , for which $\ell = 3$, $\mu = 1$ and L in $\{(0, 0, 0), (0, 2, 2), (0, 0, 2), (0, 2, 0)\}$ dominate the rest of the series. We call \mathcal{P}_{dom} the sum of these four terms.

The differential operators in these four terms read, with notations from (9),

$$W_1^{2,(0,0,0)} = -f_{01,001}, \quad W_1^{2,(0,2,2)} = -f_{21,021}, \quad W_1^{2,(0,0,2)} = -f_{21,001}, \quad W_1^{2,(0,2,0)} = -f_{01,021}, \quad (108)$$

and the associated integrals read

$$b_1^{2,(0,0,0)} = -\frac{1}{2} \|w_1\|_{L^2}^2, \quad b_1^{2,(0,2,2)} = -\frac{1}{2} \|w_2\|_{L^2}^2, \quad b_1^{2,(0,0,2)} = b_1^{2,(0,2,0)} = -\frac{1}{2} \int w_1 w_2, \quad (109)$$

where w_1 and w_2 are defined in (79). Now, let $\alpha = f_{01,001}\Phi(0)$, $\beta = f_{21,021}\Phi(0)$ and $\gamma = \frac{1}{2}(f_{21,001}\Phi(0) + f_{01,021}\Phi(0))$. Gathering (108) and (109) into the expression of \mathcal{P}_{dom} , we get

$$\mathcal{P}_{\text{dom}} = \frac{1}{2} \alpha \|w_1\|_{L^2}^2 + \frac{1}{2} \beta \|w_2\|_{L^2}^2 + \gamma \left(\int w_1 w_2 \right). \quad (110)$$

from which we get the lower bound

$$\mathcal{P}_{\text{dom}} \geq \frac{1}{2} \alpha \|w_1\|_{L^2}^2 + \frac{1}{2} \beta \|w_2\|_{L^2}^2 - |\gamma| \|w_1\|_{L^2} \|w_2\|_{L^2}, \quad (111)$$

which means that:

- in Case 1. of Theorem 3.9, $\mathcal{P}_{\text{dom}} \geq \frac{1}{2} \langle \varphi, D_0(\|w_1\|_{L^2}, \|w_2\|_{L^2}) \rangle$,
- in Case 2. of Theorem 3.9, $\mathcal{P}_{\text{dom}} \geq \frac{1}{2} \langle \psi, D_0(\|w_1\|_{L^2}, \|w_2\|_{L^2}) \rangle$,

where D_0 is defined in (17)

Assumption (20) (resp. (22)) in Theorem 3.9 ensures the existence of $K > 0$ such that $\langle \varphi, D_0(\lambda_1, \lambda_2) \rangle \geq 4K(\lambda_1^2 + \lambda_2^2)$ (resp. $\langle \psi, D_0(\lambda_1, \lambda_2) \rangle \geq 4K(\lambda_1^2 + \lambda_2^2)$). In both cases, we obtain the following bound on \mathcal{P}_{dom} :

$$\mathcal{P}_{\text{dom}} \geq 2K(\|w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2). \quad (112)$$

Step 4.

The goal of this step is to show that, under the right assumptions on t and the controls, the sum of the Chen-Fliess series without the dominating terms studied in the previous step is smaller in absolute value than $K(\|w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2)$, where K is defined in Equation (112).

First of all, let us deal with the four remaining terms in \mathcal{P}_2 for which $\ell = 3$ and $\mu = 1$ (once removed the dominating ones from **Step 3.**), namely for indices L in $\{(2, 0, 0), (2, 2, 2), (2, 0, 2), (2, 2, 0)\}$. We denote by $\mathcal{P}_{2,1}$ the sum of these four terms. They require special attention because their behaviour depends on which case of the theorem is considered. We have

$$W_1^{2,(2,0,0)} = -f_{01,201}, \quad W_1^{2,(2,2,2)} = -f_{21,221}, \quad W_1^{2,(2,0,2)} = -f_{21,201}, \quad W_1^{2,(2,2,0)} = -f_{01,221}. \quad (113)$$

Now, notice that in Case 2. of Theorem 3.9, these four brackets belong to R' when evaluated at 0; therefore, by construction of Φ , one has

$$W_1^{2,(0,0,2)}\Phi(0) = W_1^{2,(2,2,2)}\Phi(0) = W_1^{2,(2,0,2)}\Phi(0) = W_1^{2,(0,2,2)}\Phi(0) = 0,$$

and $\mathcal{P}_{2,1} = 0$. On the other hand, in Case 1. of the theorem, these terms do not vanish anymore, and we need to look at the associated integrals $b_1^{2,(2,0,0)}$, $b_1^{2,(2,2,2)}$, $b_1^{2,(2,0,2)}$, $b_1^{2,(2,2,0)}$, that read

$$b_1^{2,(2,0,0)} = -\frac{1}{2} \int u_2 w_1^2, \quad b_1^{2,(2,2,2)} = -\frac{1}{2} \int u_2 w_2^2, \quad b_1^{2,(2,0,2)} = b_1^{2,(2,2,0)} = -\frac{1}{2} \int u_2 w_1 w_2, \quad (114)$$

Let $A = \max(|W_1^{2,(0,0,2)}\Phi(0)|, |W_1^{2,(2,2,2)}\Phi(0)|, |W_1^{2,(2,0,2)}\Phi(0)|, |W_1^{2,(0,2,2)}\Phi(0)|)$; bounding u_2 by $\|u_2\|_{L^\infty}$ in (114), we obtain that $\mathcal{P}_{2,1}$ is bounded as follows:

$$|\mathcal{P}_{2,1}| \leq A \|u_2\|_{L^\infty} (\|w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2) \quad (115)$$

Therefore, letting $T_1 = \frac{K}{6\beta}$, for all u_2 such that $\|u_2\|_{L^\infty} \leq T_1$ and for all $T \in [0, T_1]$, one has

$$|\mathcal{P}_{2,1}| \leq \frac{K}{6} (\|w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2). \quad (116)$$

The rest of this step, dealing with all the remaining terms in the series, is valid for both cases of the theorem. We will make use of the following lemma, that states useful estimations on some control integrals.

Lemma 5.7. *Let $T > 0$ and $u_1 \in W^{1,\infty}([0, T])$. There exists M_1 and M_2 independent of u_1 and T such that:*

$$\|v_1\|_{L^4}^4 \leq M_1 \|u_1\|_{L^\infty}^2 \|w_1\|_{L^2}^2, \quad (117)$$

$$\|v_1\|_{L^3}^3 \leq M_2 \|u_1\|_{W^{1,\infty}} \|w_1\|_{L^2}^2. \quad (118)$$

Proof. Both estimations are consequences of the Gagliardo-Nirenberg inequality (see for example [25, Theorem 9.3]), that we recall here : let p, q, r, α real numbers and j, m integers satisfying $1 \leq q, r \leq +\infty$, $\frac{j}{m} \leq \alpha \leq 1$ and

$$\frac{1}{p} = j + \left(\frac{1}{r} - m\right)\alpha + \frac{1 - \alpha}{q}. \quad (119)$$

Then, for u a function on $[0, T]$ such that $u \in L^q([0, T])$ and its weak derivative $u^{(m)}$ is in $L^r([0, T])$, then $u^{(j)}$ is in $L^p([0, T])$, and there exists $C > 0$ independent of u such that

$$\|u^{(j)}\|_{L^p} \leq C \|u^{(m)}\|_{L^r}^\alpha \|u\|_{L^q}^{1-\alpha}. \quad (120)$$

Taking $u = w_1$ (which belongs to $W^{3,\infty}([0, T])$) and applying (120) for $p = 4, q = 2, r = \infty, \alpha = \frac{1}{2}, j = 1$, and $m = 2$ yields (117), and applying it again for $p = 3, q = 2, r = \infty, \alpha = \frac{1}{3}, j = 1$, and $m = 3$ yields (118). \square

The following calculations can be summarized as follows:

- i) Remaining terms in \mathcal{P}_2 .
 - a. Vanishing terms.
 - b. Case $1 \leq \mu \leq \frac{\ell-1}{2}$.
 - c. Case $\frac{\ell}{2} \leq \mu \leq \ell - 2$.
 - d. Sum of all the remaining terms in \mathcal{P}_2 , called $\mathcal{P}_{2,2}$.
- ii) Terms in \mathcal{P}_3 .
 - a. Vanishing terms.
 - b. Case $\xi = 1$.
 - c. Case $\xi = 2, \mu = 0$.
 - d. Case $\xi = 2, \mu \geq 1$.
 - e. Case $\xi = 3$.
- iii) Terms in $\mathcal{P}_{\geq 4}$.
- i) Remaining terms in \mathcal{P}_2 .
 - a. Vanishing terms.

We start by ruling out the vanishing operators, through the following lemma:

Lemma 5.8. *One has, for $\ell \geq 0$ and for $L \in \{0, 2\}^\ell$:*

- (1) $(W_0^{2,L}\Phi)(0) = 0$,
- (2) $(W_\ell^{2,L}\Phi)(0) = 0$,
- (3) $(W_{\ell-1}^{2,L}\Phi)(0) = 0$.

Proof. Let $\ell \geq 0$ and $L \in \{0, 2\}^\ell$.

(1). In Case 1. (resp. Case 2.), $W_0^{2,L}(0)$ belongs to R_1 (resp. R'), so we use property (90) (resp. (92)) of Φ to conclude.

(2). The operator $W_\ell^{2,L}$ takes the form hf_1 with h some vector field, so we use property (88) of Φ to conclude.

(3). Assume we are in Case 1. of Theorem 3.9. First, we consider the case $i_1 = 0$. Then, $W_{\ell-1}^{2,L} = hg_2$ with $h \in \mathcal{R}_1$ and g_2 defined in Step 1.. Because h belongs to \mathcal{R}_1 and $(g_1(0), \dots, g_{d_1}(0))$ is a basis of R_1 , there exists $\lambda_1, \dots, \lambda_{d_1}$ in \mathbb{R} such that $h(0) = \sum_{j=1}^{d_1} \lambda_j g_j(0)$. Introducing $g_0 = h - \sum_{j=1}^{d_1} \lambda_j g_j$, we have $g_0(0) = 0$ and $h = g_0 + \sum_{j=1}^{d_1} \lambda_j g_j$. Since $g_0(0) = 0$, we immediately have $(g_0 g_2 \Phi)(0) = 0$. Moreover, for $j \in \{2, \dots, d_1\}$, $(g_j g_2 \Phi)(0) = 0$ by virtue of property (91) of Φ . Finally, noticing that $g_1 g_2 = [g_1, g_2] + g_2 g_1 = f_{101} + g_2 g_1$, and using again (91) as well as point (1) of the lemma (since $f_{101}(0) \in R_1$), we have $(g_1 g_2 \Phi)(0) = 0$. Overall, $W_{\ell-1}^{2,L}\Phi(0) = (hg_2\Phi)(0) = 0$.

In the case $i_\ell = 2$, $W_{\ell-1}^{2,L} = hg_3$ with $h \in \mathcal{R}_1$. Again, we write $h = g_0 + \sum_{j=1}^{d_1} \lambda_j g_j$ with $g_0(0) = 0$, and proceed as above to show that $(g_j g_3 \Phi)(0) = 0$ for all $j = 0, \dots, d_1$, using (91) and the fact that $[g_1, g_3](0) = f_{121}(0)$ and $[g_2, g_3](0) = f_{01,21}(0)$ belong to R_1 .

Case 2. of the Theorem is strictly analogous, with d_2 appearing instead of d_1 . \square

Remaining terms.

According to Lemma 5.8, and since we already ruled out the terms for which $\ell = 3$ through \mathcal{P}_{dom} and $\mathcal{P}_{2,1}$, the remaining terms in \mathcal{P}_2 satisfy $\ell \geq 4$. Let $\ell \geq 4$ and $L \in \{0, 2\}^\ell$. Let L_2 be the number of 2's in L . There are two cases to study depending on the index μ .

- b. Case $1 \leq \mu \leq \frac{\ell-1}{2}$.

According to (103), we have

$$b_\mu^{2,L} = \frac{1}{2} \int u_{i_1} \int u_{i_2} \cdots \int u_{i_{\ell-2\mu}} \left(\int u_{i_{\ell-2\mu+1}} \cdots \int u_{i_{\ell-\mu}} \int u_1 \right) \left(\int u_{i_{\ell-\mu+1}} \cdots \int u_{i_\ell} \int u_1 \right). \quad (121)$$

There are three cases to study, depending on the value of $i_{\ell-\mu}$ and i_ℓ .

(1) $i_{\ell-\mu} = 0$ and $i_\ell = 0$. Bounding u_2 by $\|u_2\|_{L^\infty}$ gives

$$|b_\mu^{2,L}| \leq \frac{1}{2} \|u_2\|_{L^\infty}^2 \int_0^T \frac{(T-s)^{\ell-3}}{(\ell-3)!} |w_1(s)|^2 ds \quad (122)$$

if $\mu = 1$ and

$$|b_\mu^{2,L}| \leq \frac{1}{2} \|u_2\|_{L^\infty}^2 \int_0^T \frac{(T-s)^{\ell-2\mu-1}}{(\ell-2\mu-1)!} \left(\int_0^s \frac{(s-\sigma)^{\mu-2}}{(\mu-2)!} |w_1(\sigma)| d\sigma \right)^2 ds \quad (123)$$

if $\mu > 1$. We use Cauchy-Schwarz inequality and bound $(T-s)$ by T in (123) to get

$$|b_\mu^{2,L}| \leq \frac{1}{2} \|u_2\|_{L^\infty}^2 \frac{T^{\ell-2\mu-1}}{(\ell-2\mu-1)!} \int_0^T \frac{(T-s)^{2\mu-3}}{(2\mu-3)(\mu-2)!^2} \int_0^s w_1^2(\sigma) d\sigma ds. \quad (124)$$

Bounding $(T-s)$ by T and the integral between 0 and s by the integral between 0 and T in (124) yields

$$|b_\mu^{2,L}| \leq \frac{1}{2} \|u_2\|_{L^\infty}^2 \frac{T^{\ell-3}}{(2\mu-3)(\ell-2\mu-1)!(\mu-2)!^2} \|w_1\|_{L^2}^2. \quad (125)$$

Finally, we bound $\frac{1}{(\mu-2)!}$ by $\frac{\mu}{(\mu-1)!}$ in (125) to obtain a bound that encompasses all values of μ :

$$|b_\mu^{2,L}| \leq \frac{1}{2} \|u_2\|_{L^\infty}^2 \frac{\mu^2 T^{\ell-3}}{(\ell-2\mu-1)!(\mu-1)!^2} \|w_1\|_{L^2}^2. \quad (126)$$

(2) $i_{\ell-\mu} = 2$ and $i_\ell = 2$. Bounding u_2 by $\|u_2\|_{L^\infty}$ gives

$$|b_\mu^{2,L}| \leq \frac{1}{2} \|u_2\|_{L^\infty}^2 \int_0^T \frac{(T-s)^{\ell-3}}{(\ell-3)!} |w_2(s)|^2 ds \quad (127)$$

if $\mu = 1$ and

$$|b_\mu^{2,L}| \leq \frac{1}{2} \|u_2\|_{L^\infty}^{L_2-2} \int_0^T \frac{(T-s)^{\ell-2\mu-1}}{(\ell-2\mu-1)!} \left(\int_0^s \frac{(s-\sigma)^{\mu-2}}{(\mu-2)!} |w_2(\sigma)| d\sigma \right)^2 ds \quad (128)$$

if $\mu > 1$, and we repeat steps (124) and (125) in (128) to obtain the upper bound

$$|b_\mu^{2,L}| \leq \frac{1}{2} \|u_2\|_{L^\infty}^{L_2-2} \frac{\mu^2 T^{\ell-3}}{(\ell-2\mu-1)!(\mu-1)!^2} \|w_2\|_{L^2}^2. \quad (129)$$

(3) $i_{\ell-\mu} = 2$ and $i_\ell = 0$, or $i_{\ell-\mu} = 0$ and $i_\ell = 2$. Bounding u_2 by $\|u_2\|_{L^\infty}$ gives

$$|b_\mu^{2,L}| \leq \frac{1}{2} \|u_2\|_{L^\infty}^2 \int_0^T \frac{(T-s)^{\ell-3}}{(\ell-3)!} |w_1(s)w_2(s)| ds \quad (130)$$

if $\mu = 1$ and

$$|b_\mu^{2,L}| \leq \frac{1}{2} \|u_2\|_{L^\infty}^{L_2-1} \int_0^T \frac{(T-s)^{\ell-2\mu-1}}{(\ell-2\mu-1)!} \left(\int_0^s \frac{(s-\sigma)^{\mu-2}}{(\mu-2)!} |w_1(\sigma)| d\sigma \right) \left(\int_0^s \frac{(s-\sigma)^{\mu-2}}{(\mu-2)!} |w_2(\sigma)| d\sigma \right) ds \quad (131)$$

if $\mu > 1$. Then, we repeat steps (124) and (125) in (131) and use the fact that $2xy \leq x^2 + y^2$ to obtain

$$|b_\mu^{2,L}| \leq \frac{1}{4} \|u_2\|_{L^\infty}^{L_2-1} \frac{\mu^2 T^{\ell-3}}{(l-2\mu-1)!(\mu-1)!^2} (\|w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2). \quad (132)$$

c. Case $\frac{\ell}{2} \leq \mu \leq \ell - 2$.

According to (104), we have

$$b_\mu^{2,L} = \left(\int u_{i_1} \cdots \int u_{i_\mu} \int u_1 \right) \left(\int u_{i_{\mu+1}} \cdots \int u_{i_\ell} \int u_1 \right).$$

Similarly to the case $2 \leq 2\mu \leq l-1$, the study of $b_\mu^{2,L}$ then splits in three cases depending on the value of $i_{\ell-\mu}$ and i_ℓ . Applying Cauchy-Schwarz inequality again leads to the following upper bounds:

(1) if $i_\mu = 0$ and $i_\ell = 0$,

$$|b_\mu^{2,L}| \leq \|u_2\|_{L^\infty}^{L_2} \frac{T^{\ell-3}}{\sqrt{(2\mu-1)(\ell-\mu-1)(l-\mu-1)!}(\mu-2)!} \|w_1\|_{L^2}^2. \quad (133)$$

(2) if $i_\mu = 2$ and $i_\ell = 2$,

$$|b_\mu^{2,L}| \leq \|u_2\|_{L^\infty}^{L_2-2} \frac{T^{\ell-3}}{\sqrt{(2\mu-1)(\ell-\mu-1)(l-\mu-1)!}(\mu-2)!} \|w_2\|_{L^2}^2. \quad (134)$$

(3) if $i_\mu = 2$ and $i_\ell = 0$, or $i_\mu = 0$ and $i_\ell = 2$,

$$|b_\mu^{2,L}| \leq \frac{1}{2} \|u_2\|_{L^\infty}^{L_2-1} \frac{T^{\ell-3}}{\sqrt{(2\mu-1)(\ell-\mu-1)(l-\mu-1)!}(\mu-2)!} (\|w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2). \quad (135)$$

d. Sum of all the remaining terms in \mathcal{P}_2 .

We call $\mathcal{P}_{2,2}$ the sum of all the terms in \mathcal{P}_2 for which $\ell \geq 4$, namely

$$\mathcal{P}_{2,2} = \sum_{\ell \geq 4} \sum_{\mu=0}^{\ell} \sum_{L \in \{0,2\}^\ell} b_\mu^{2,L} W_\mu^{2,L} \Phi(0). \quad (136)$$

Using (34), we bound the $(W_\mu^{2,L} \Phi)(0)$ with $C^{\ell+2}(\ell+2)!$, and use the upper bounds obtained in (126), (129), (132), (133), (134) and (135):

$$|\mathcal{P}_{2,2}| \leq \sum_{\ell \geq 4} C^{\ell+2} T^{\ell-3} \sum_{\mu=0}^{\ell} B(\ell, \mu) \left(\|w_1\|_{L^2}^2 \sum_{\substack{L \in \{0,2\}^\ell \\ i_{\ell-\mu}=0, i_\ell=0}} \|u_2\|_{L^\infty}^{L_2} + \|w_2\|_{L^2}^2 \sum_{\substack{L \in \{0,2\}^\ell \\ i_{\ell-\mu}=2, i_\ell=2}} \|u_2\|_{L^\infty}^{L_2-2} + \frac{\|w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2}{2} \sum_{\substack{L \in \{0,2\}^\ell \\ i_{\ell-\mu}+i_\ell=2}} \|u_2\|_{L^\infty}^{L_2-1} \right), \quad (137)$$

with $B(\ell, \mu) = \frac{\mu^2(\ell+2)!}{(\ell-2\mu-1)!(\mu-1)!^2}$ if $0 \leq 2\mu \leq \ell-1$ and $\frac{(\ell+2)!}{(\ell-\mu-1)!(\mu-2)!}$ if $\ell/2 \leq \mu \leq \ell$.

For given ℓ and μ , straightforward combinatorial calculation yields to

$$\sum_{\substack{L \in \{0,2\}^\ell \\ i_{\ell-\mu}=0, i_\ell=0}} \|u_2\|_{L^\infty}^{L_2} = \sum_{\substack{L \in \{0,2\}^\ell \\ i_{\ell-\mu}=2, i_\ell=2}} \|u_2\|_{L^\infty}^{L_2-2} = \frac{1}{2} \sum_{\substack{L \in \{0,2\}^\ell \\ i_{\ell-\mu}+i_\ell=2}} \|u_2\|_{L^\infty}^{L_2-1} = (1 + \|u_2\|_{L^\infty})^{\ell-2}. \quad (138)$$

Substituting (138) in (137) yields

$$\begin{aligned} |\mathcal{P}_{2,2}| &\leq 2(\|w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2) \sum_{\ell \geq 4} C^{\ell+2} (1 + \|u_2\|_{L^\infty})^{\ell-2} T^{\ell-3} \sum_{\mu=0}^{\ell} B(\ell, \mu) \\ &\leq 2C^6 T (1 + \|u_2\|_{L^\infty})^2 (\|w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2) \sum_{\ell \geq 0} (C(1 + \|u_2\|_{L^\infty})T)^\ell \sum_{\mu=0}^{\ell+4} B(\ell+4, \mu). \end{aligned}$$

For T small enough, the series above converges and therefore there exists T_2 such that for all $T \in [0, T_2]$,

$$|\mathcal{P}_{2,2}| \leq \frac{K}{6} (\|w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2). \quad (139)$$

ii) Terms in \mathcal{P}_3 .

a. Vanishing terms.

We start by ruling out the vanishing terms through the following lemma:

Lemma 5.9. *One has, for $\ell \geq 0$ and for $L \in \{0, 2\}^\ell$:*

- (1) $(W_{2,0,0}^{3,L} \Phi)(0) = 0$,
- (2) $(W_{2,0,1}^{3,L} \Phi)(0) = 0$,
- (3) $(W_{3,1,1}^{3,L} \Phi)(0) = 0$,
- (4) $(W_{3,\mu,0}^{3,L} \Phi)(0) = 0$ for all admissible μ .

The proof is based on the same arguments as the proof of Lemma 5.8.

Now, let $\ell \geq 1$, $L \in \{0, 2\}^\ell$ and L_2 the number of 2's in L .

b. Case $\xi = 1$.

According to (105), we have

$$b_{1,\mu,\nu}^{3,L} = \delta \int u_{i_1} \cdots \int u_{i_{\ell-2\nu-\mu}} \left(\int u_{i_{\ell-2\nu-\mu+1}} \cdots \int u_{i_{\ell-2\nu}} \int u_1 \right) \left(\int u_{i_{\ell-2\nu+1}} \cdots \int u_{i_{\ell-\nu}} \int u_1 \right) \left(\int u_{i_{\ell-\nu+1}} \cdots \int u_{i_\ell} \int u_1 \right). \quad (140)$$

We bound u_2 with $\|u_2\|_{L^\infty}$ in (140) to obtain:

$$|b_{1,\mu,\nu}^{3,L}| \leq \delta \|u_2\|_{L^\infty}^{L_2} \underbrace{\int \cdots \int}_{\ell-2\nu-\mu \text{ times}} \left(\underbrace{\int \cdots \int}_{\mu \text{ times}} |v_1| \right) \left(\underbrace{\int \cdots \int}_{\nu \text{ times}} |v_1| \right)^2. \quad (141)$$

Hence, if $\mu = \nu = 0$, (141) becomes

$$|b_{1,0,0}^{3,L}| \leq \delta \|u_2\|_{L^\infty}^{L_2} \int_0^T \frac{(T-s)^{\ell-1}}{(\ell-1)!} |v_1(s)|^3 ds, \quad (142)$$

if $\nu = 0$ and $\mu \geq 1$, (141) becomes

$$|b_{1,\mu,0}^{3,L}| \leq \delta \|u_2\|_{L^\infty}^{L_2} \int_0^T \frac{(T-s)^{\ell-\mu-1}}{(\ell-\mu-1)!} v_1(s)^2 \int_0^s \frac{(s-\sigma)^{\mu-1}}{(\mu-1)!} |v_1(\sigma)| d\sigma ds, \quad (143)$$

and if $\nu \geq 1$, (141) becomes

$$|b_{1,\mu,\nu}^{3,L}| \leq \delta \|u_2\|_{L^\infty}^{L_2} \int_0^T \frac{(T-s)^{\ell-2\nu-\mu-1}}{(\ell-2\nu-\mu-1)!} \left(\int_0^s \frac{(s-\sigma)^{\nu-1}}{(\nu-1)!} |v_1(\sigma)| d\sigma \right)^2 \left(\int_0^s \frac{(s-\sigma)^{\mu-1}}{(\mu-1)!} |v_1(\sigma)| d\sigma \right) ds. \quad (144)$$

In (142), (143) and (144), we bound $(t - s)$ by t , $(s - \sigma)$ by s and s by t where necessary, which yields the following bound encompassing the three cases (142), (143) and (144):

$$|b_{1,\mu,\nu}^{3,L}| \leq A(\mu, \nu) T^{\ell-1} \|u_2\|_{L^\infty}^{L_2} \int_0^T |v_1(s)|^3 ds, \quad (145)$$

where $A(\mu, \nu)$ is equal to $\frac{\delta}{(\ell-1)!}$, $\frac{\delta}{(\ell-\mu-1)! (\mu-1)!}$ or $\frac{\delta}{(\ell-2\nu-\mu-1)! (\nu-1)!^2 (\mu-1)!}$ depending on the cases enumerated above. Then, we use estimation (118) in Lemma 5.7 to obtain:

$$|b_{1,\mu,\nu}^{3,L}| \leq A(\mu, \nu) K T^{\ell-1} \|u_2\|_{L^\infty}^{L_2} \|u_1\|_{W^{1,\infty}} \|w_1\|_{L^2}^2. \quad (146)$$

Let us call $\mathcal{P}_{3,1}$ the sum of all the terms of the Chen-Fliess series for which $k = 3$ and $\xi = 1$. Bounding the operators $W_{1,\mu,\nu}^{3,L}$ by $C^{\ell+2}(\ell+2)!$ thanks to (34) and using (138) the upper bound obtained in (146), we obtain

$$\begin{aligned} |\mathcal{P}_{3,1}| &\leq K \|u_1\|_{W^{1,\infty}} \|w_1\|_{L^2}^2 \sum_{\ell \geq 2} C^{\ell+2}(\ell+2)! T^{\ell-1} \sum_{\mu,\nu} A(\mu, \nu) \sum_{L \in \{0,2\}^\ell} \|u_2\|_{L^\infty}^{L_2} \\ &\leq K C^3 (1 + \|u_2\|_{L^\infty}) \|u_1\|_{W^{1,\infty}} \|w_1\|_{L^2}^2 \sum_{\ell \geq 0} (C(1 + \|u_2\|_{L^\infty}) T)^\ell \sum_{\mu,\nu} (\ell+2)! A(\mu, \nu). \end{aligned}$$

For T small enough, the series above converges and therefore, there exists $T_{3,1}$ such that, for $\|u_1\|_{W^{1,\infty}} \leq T_{3,1}$, and for all $T \in [0, T_{3,1}]$,

$$|\mathcal{P}_{3,1}| \leq \frac{K}{6} \|w_1\|_{L^2}^2. \quad (147)$$

Remark 5.10. The terms in $\mathcal{P}_{3,1}$ are the only ones for which the smallness of $\|u_1\|_{W^{1,\infty}}$ is needed.

c. Case $\xi = 2, \mu = 0$.

Considering only the nonvanishing terms, we know that $\ell \geq 3$ and $\nu \geq 2$. According to (106), we have

$$b_{2,0,\nu}^{3,L} = \frac{1}{2} \left(\int u_{i_1} \cdots \int u_{i_{\ell-\nu}} v_1^2 \right) \left(\int u_{\ell-\nu+1} \cdots \int u_\ell \int u_1 \right). \quad (148)$$

There is two cases to study, depending on the value of i_ℓ .

(1) $i_\ell = 0$. We bound u_2 by $\|u_2\|_{L^\infty}$ in (148):

$$|b_{2,0,\nu}^{3,L}| \leq \frac{1}{2} \|u_2\|_{L^\infty}^{L_2} \left(\int_0^T \frac{(T-s)^{\ell-\nu-1}}{(\ell-\nu-1)!} v_1(s)^2 ds \right) \left(\int_0^T \frac{(T-s)^{\nu-2}}{(\nu-2)!} w_1(s) ds \right). \quad (149)$$

Then, Cauchy-Schwarz inequality yields

$$|b_{2,0,\nu}^{3,L}| \leq \frac{1}{2} \|u_2\|_{L^\infty}^{L_2} \frac{T^{\ell-\nu-\frac{1}{2}}}{\sqrt{2\ell-2\nu-1}(\ell-\nu-1)!} \|v_1\|_{L^4}^2 \frac{T^{\nu-\frac{3}{2}}}{\sqrt{2\nu-3}(\nu-2)!} \|w_1\|_{L^2}. \quad (150)$$

Finally, we use estimation (117) in Lemma 5.7 to obtain:

$$|b_{2,0,\nu}^{3,L}| \leq \frac{1}{2} K \frac{T^{\ell-2}}{\sqrt{(2\ell-2\nu-1)(2\nu-3)(\ell-\nu-1)(\nu-2)!}} \|u_2\|_{L^\infty}^{L_2} \|u_1\|_{L^\infty}^2 \|w_1\|_{L^2}^2. \quad (151)$$

(2) $i_\ell = 2$. We repeat steps (149) to (151) and obtain

$$|b_{2,0,\nu}^{3,L}| \leq \frac{1}{2} K \frac{T^{\ell-2}}{\sqrt{(2\ell-2\nu-1)(2\nu-3)(\ell-\nu-1)(\nu-2)!}} \|u_2\|_{L^\infty}^{L_2-1} \|u_1\|_{L^\infty}^2 \|w_1\|_{L^2} \|w_2\|_{L^2}, \quad (152)$$

hence

$$|b_{2,0,\nu}^{3,L}| \leq \frac{1}{4} K \frac{T^{\ell-2}}{\sqrt{(2\ell-2\nu-1)(2\nu-3)(\ell-\nu-1)(\nu-2)!}} \|u_2\|_{L^\infty}^{L_2-1} \|u_1\|_{L^\infty}^2 (\|w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2). \quad (153)$$

d. Case $\xi = 2, \mu \geq 1$.

In this case, we notice that

$$b_{2,\mu,\nu}^{3,L} = b_\mu^{2,(i_1, \dots, i_{\ell-\nu})} b_1^{1,(i_{\ell-\nu+1}, \dots, i_\ell)}. \quad (154)$$

Bounds for $|b_\mu^{2,(i_1, \dots, i_{\ell-\nu})}|$ were already found in (126), (129) and (132). Moreover, $|b_1^{1,(i_{\ell-\nu+1}, \dots, i_\ell)}|$ is bounded as follows:

$$|b_1^{1,(i_{\ell-\nu+1}, \dots, i_\ell)}| \leq \frac{T^{\nu+1}}{(\nu+1)!} \|u_2\|_{L^\infty}^{J_2} \|u_1\|_{L^\infty}, \quad (155)$$

where J_2 designates the number of 2's amongst $(i_{\ell-\nu+1}, \dots, i_\ell)$.

Using (155) as well as (126), (129) and (132) in (154), we obtain the following bounds:

(1) if $i_{\ell-\mu-\nu} = 0$ and $i_{\ell-\nu} = 0$,

$$|b_{2,\mu,\nu}^{3,L}| \leq \frac{1}{2} \|u_2\|_{L^\infty}^{L_2} \|u_1\|_{L^\infty} \frac{\mu^2 T^{\ell-2}}{(l-2\mu-\nu-1)!(\mu-1)!2\nu!} \|w_1\|_{L^2}^2. \quad (156)$$

(2) if $i_{\ell-\mu-\nu} = 2$ and $i_{\ell-\nu} = 2$,

$$|b_{2,\mu,\nu}^{3,L}| \leq \frac{1}{2} \|u_2\|_{L^\infty}^{L_2-2} \|u_1\|_{L^\infty} \frac{\mu^2 T^{\ell-2}}{(l-2\mu-\nu-1)!(\mu-1)!2\nu!} \|w_2\|_{L^2}^2. \quad (157)$$

(3) if $i_{\ell-\mu-\nu} = 0$ and $i_{\ell-\nu} = 2$ or $i_{\ell-\mu-\nu} = 2$ and $i_{\ell-\nu} = 0$,

$$|b_{2,\mu,\nu}^{3,L}| \leq \frac{1}{4} \|u_2\|_{L^\infty}^{L_2-1} \|u_1\|_{L^\infty} \frac{\mu^2 T^{\ell-2}}{(l-2\mu-\nu-1)!(\mu-1)!2\nu!} (\|w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2). \quad (158)$$

Let us call $\mathcal{P}_{3,2}$ the sum of all the terms of the Chen-Fliess series for which $k = 3$ and $\xi = 2$. We proceed as for $\xi = 1$: bounding the operators $W_{2,\mu,\nu}^{3,L}$ by $C^{l+2}(l+2)!$ thanks to (34) and using the upper bounds obtained in (151), (153), and (156) to (158), as well as (138), we obtain that there exists $T_{3,2}$ such that for all $T \in [0, T_{3,2}]$,

$$|\mathcal{P}_{3,2}| \leq \frac{K}{6} (\|w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2). \quad (159)$$

e. Case $\xi = 3$.

Considering only the nonvanishing terms, we know that $\mu \geq 2, \nu \geq 1$ and $\ell - \mu - \nu \geq 2$. There is again three cases to study, depending on the value of $i_{\ell-\mu-\nu}$ and i_ℓ .

(1) $i_{\ell-\mu-\nu} = 0$ and $i_\ell = 0$. Bounding u_2 with $\|u_2\|_{L^\infty}$ yields

$$|b_{3,\mu,\nu}^{3,L}| \leq \delta \|u_2\|_{L^\infty}^{L_2} \left(\underbrace{\int \cdots \int}_{\ell-\mu-\nu-1 \text{ times}} w_1 \right) \left(\underbrace{\int \cdots \int}_{\mu-1 \text{ times}} w_1 \right) \left(\underbrace{\int \cdots \int}_{\nu+1 \text{ times}} u_1 \right). \quad (160)$$

Using Cauchy-Schwarz inequality on the first two terms and equation (155) on the last term of (160), we obtain

$$|b_{3,\mu,\nu}^{3,L}| \leq \delta \|u_2\|_{L^\infty}^{L_2} \|u_1\|_{L^\infty} \frac{T^{\ell-2}}{\sqrt{(2\ell-2\mu-2\nu-3)(2\mu-3)(\ell-2\mu-\nu-2)!(\mu-2)!2\nu!}} \|w_1\|_{L^2}^2. \quad (161)$$

(2) $i_{\ell-\mu-\nu} = 2$ and $i_\ell = 2$. Repeating steps (160) and (161) yields

$$|b_{3,\mu,\nu}^{3,L}| \leq \delta \|u_2\|_{L^\infty}^{L_2-2} \|u_1\|_{L^\infty} \frac{T^{\ell-2}}{\sqrt{(2\ell-2\mu-2\nu-3)(2\mu-3)(\ell-2\mu-\nu-2)! (\mu-2)! \nu!}} \|w_2\|_{L^2}^2. \quad (162)$$

(3) $i_{\ell-\mu-\nu} = 2$ and $i_\ell = 0$ or $i_{\ell-\mu-\nu} = 0$ and $i_\ell = 2$. Repeating steps (160) and (161) yields

$$|b_{3,\mu,\nu}^{3,L}| \leq \frac{1}{2} \delta \|u_2\|_{L^\infty}^{L_2-1} \|u_1\|_{L^\infty} \frac{T^{\ell-2}}{\sqrt{(2\ell-2\mu-2\nu-3)(2\mu-3)(\ell-2\mu-\nu-2)! (\mu-2)! \nu!}} (\|w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2). \quad (163)$$

Let us call $\mathcal{P}_{3,3}$ the sum of all the terms of the Chen-Fliess series for which $k = 3$ and $\xi = 3$. We proceed as before : bounding the operators $W_{3,\mu,\nu}^{3,L}$ by $C^{\ell+2}(\ell+2)!$ thanks to (34) and using the upper bounds obtained in (161), (162), and (163), we obtain that there exists $T_{3,3}$ such that for all $T \in [0, T_{3,3}]$,

$$|\mathcal{P}_{3,3}| \leq \frac{K}{6} (\|w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2). \quad (164)$$

iii) Terms in $\mathcal{P}_{\geq 4}$.

It is shown in [8, proof of Property (P)] that, in the scalar-input case the ‘‘high-order’’ remaining terms in the series, associated with indices containing more than four 1’s, are such that there exists $T_4 > 0$ such that for $T \in [0, T_4]$,

$$|\mathcal{P}_{\geq 4}| \leq (1 + DT) \|v_1\|_{L^4}^4. \quad (165)$$

for some $D > 0$ that only depends on f_0 and Φ . The computations made in [8, proof of Property (P)] are straightforwardly adaptable to the two-control case, by systematically bounding $u_2(t)$ with $\|u_2\|_{L^\infty}$ in the iterated integrals.

Applying (117) from Lemma 5.7 in (165), we obtain that

$$|\mathcal{P}_{\geq 4}| \leq M(1 + DT) \|u_1\|_{L^\infty}^2 \|w_1\|_{L^2}^2, \quad (166)$$

$$\leq \frac{K}{6} \|w_1\|_{L^2}^2, \quad (167)$$

for $\|u_1\|_{L^\infty}$ smaller than $T_5 = \left(\frac{K}{12M(1+DT_4)}\right)^{1/2}$.

Step 5. Sum of all the terms.

So far, we have obtained bounds for parts of the Chen-Fliess series in:

- (116) (only in Case 1.) and (139) for terms in \mathcal{P}_2 ,
- (147), (159), and (164) for terms in \mathcal{P}_3 ,
- (167) for terms in $\mathcal{P}_{\geq 4}$,

under conditions on T and occasionally upper bounds on $\|u_1\|_{W^{1,\infty}}$ and $\|u_2\|_{L^\infty}$, depending on the constants $T_1, T_2, T_{3,1}, T_{3,2}, T_{3,3}, T_4, T_5$. We are now ready to sum all the terms and conclude the proof.

Let $\varepsilon^0 = \min(T_1, T_2, T_{3,1}, T_{3,2}, T_{3,3}, T_4, T_5)$ and $T \in [0, \varepsilon^0]$. Let $\alpha > 0$ and u_1 and u_2 controls on $(0, T)$ such that $\|u_1\|_{W^{1,\infty}} \leq \varepsilon^0$ and $\|u_2\|_{L^\infty} \leq \varepsilon^0$ (resp. $\|u_2\|_{L^\infty} \leq \alpha$). Gathering all the bounds, and using the lower bound for P obtained in (112), we deduce that the Chen-Fliess series $\Sigma(u, f, \Phi, T)$ satisfies

$$\Sigma(u, f, \Phi, T) \geq \mathcal{P}_{\text{dom}} - |\mathcal{P}_{2,1}| - |\mathcal{P}_{2,2}| - |\mathcal{P}_{3,1}| - |\mathcal{P}_{3,2}| - |\mathcal{P}_{3,3}| - |\mathcal{P}_{\geq 4}|, \quad (168)$$

$$\geq \mathcal{P}_{\text{dom}} - 6 \cdot \frac{K}{6} (\|w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2), \quad (169)$$

$$\geq K(\|w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2). \quad (170)$$

In both cases, we have proven (78). This concludes the proof of Proposition 5.4. \square

6. CONCLUSION

In this paper, we explored the controllability properties of systems with two controls (3), satisfying the assumption (4). We have stated two results on these systems, Theorem 3.2 and Theorem 3.9, providing necessary conditions for local controllability around equilibria. These results extend the classical necessary conditions stated for scalar-control systems in [16]. Moreover, they are, to the best of our knowledge, the first results of this nature for non-scalar-input systems.

This work does not only present a theoretical interest for control theory. Using Theorem 3.2, we were able to completely solve an open question concerning the local controllability of magnetically controlled micro-swimming robots (see Example 4.3). One can use our results to easily and systematically address local controllability issues in similar applied situations.

Our necessary conditions are only based on brackets of order three and five, but there are higher-order brackets that may prevent S_2 to be contained in S_1 (in the single-input case see for instance [11]). Giving necessary conditions based on these brackets, for instance adapting the results from [11] to the situation (3)-(4), is a possible continuation of the present work. The complexity of the higher-order terms structure in the Chen-Fliess series however makes the analysis very intricate.

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