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A Note on Graphs of Dichromatic Number 2

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Abstract

Neumann-Lara and Škrekovski conjectured that every planar digraph is 2-colourable. We show that this conjecture is equivalent to the more general statement that all oriented K_5 -minor-free graphs are 2-colourable.

1 Introduction

Digraphs and graphs considered here are loopless, and without parallel or anti-parallel arcs. A directed edge starting in u and ending in v is denoted by (u, v) , u is called its *tail* while v is its *head*. In a digraph D , a vertex set $X \subseteq V(D)$ is called *acyclic* if the induced subdigraph $D[X]$ is acyclic. An *acyclic colouring* of D with k colours is a mapping $c : V(D) \rightarrow [k]$ such that $c^{-1}(\{i\})$ is acyclic for all $i \in [k]$. The *dichromatic number* $\vec{\chi}(D)$ is defined as the minimal $k \geq 1$ for which such a colouring exists. For an undirected graph G , the dichromatic number $\vec{\chi}(G)$ is defined as the maximum dichromatic number an orientation of G can have.

This notion has been introduced in 1982 by Neumann-Lara [NL82], was rediscovered by Mohar [Moh03], and since then has received further attention, see [AH15, MW16, ACH⁺16, HM17, LM17, BHK18, HLTW19] for some recent results.

In analogy to the famous Four-Colour-Theorem, the following intriguing conjecture was proposed by Neumann-Lara [NL85] and independently by Škrekovski [BFJ⁺04].

Conjecture 1. *If G is a planar graph, then $\vec{\chi}(G) \leq 2$.*

The strongest partial result obtained so far is due to Li and Mohar who showed the following:

Theorem 1 ([LM17]). *Every planar digraph without directed triangles admits an acyclic 2-colouring.*

The purpose of this note is to show the following.

Theorem 2. *The following statements are equivalent:*

- *Every planar graph G has $\vec{\chi}(G) \leq 2$.*
- *Every K_5 -minor-free graph G fulfils $\vec{\chi}(G) \leq 2$. Moreover, any orientation of G admits an acyclic 2-colouring without monochromatic triangles.*

This strengthening is similar to the situation for undirected graph colourings, where it is known that all K_5 -minor-free graphs are 4-colourable [AH77, AHK77, Wag37].

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2 2-Colourings of Planar Digraphs

In the following, we will use the term *planar triangulation* when we mean a maximal planar graph on at least three vertices. It is well-known that the latter (up to the choice of the outer face and reflections) admit combinatorially unique crossing-free embeddings in the plane or on the sphere, in which every face is bounded by a triangle (from now on called facial triangles). A frequent tool in our proof will be the following Lemma, which has already been used in [LM17].

Lemma 3 ([LM17]). *Let D_1 and D_2 be digraphs which intersect in a tournament. Suppose that $c_1 : V(D_1) \rightarrow [k]$, $c_2 : V(D_2) \rightarrow [k]$ are acyclic colourings such that $c_1|_{V(D_1) \cap V(D_2)} = c_2|_{V(D_1) \cap V(D_2)}$. Then the common extension of c_1 and c_2 to $V(D_1) \cup V(D_2)$ defines an acyclic k -colouring of $D_1 \cup D_2$.*

In this section we prepare the proof of Theorem 2 with some strengthened but equivalent formulations of Neumann-Lara's Conjecture.

Proposition 1. *The following statements are equivalent:*

- (i) *Neumann-Lara's Conjecture, i.e., every planar digraph has an acyclic 2-colouring.*
- (ii) *Every oriented planar triangulation admits an acyclic 2-colouring without monochromatic facial triangles.*
- (iii) *For any planar triangulation T , any facial triangle $a_1a_2a_3$ in T , and any non-monochromatic pre-colouring $p : \{a_1, a_2, a_3\} \rightarrow \{1, 2\}$, every orientation \vec{T} of T admits an acyclic 2-colouring $c : V(\vec{T}) \rightarrow \{1, 2\}$ without monochromatic facial triangles such that $c(a_i) = p(a_i)$, $i \in \{1, 2, 3\}$.*
- (iv) *For any planar triangulation T , any triangle $a_1a_2a_3$ in T , and any non-monochromatic pre-colouring $p : \{a_1, a_2, a_3\} \rightarrow \{1, 2\}$, every orientation \vec{T} of T admits an acyclic 2-colouring $c : V(\vec{T}) \rightarrow \{1, 2\}$ without monochromatic triangles such that $c(a_i) = p(a_i)$, $i \in \{1, 2, 3\}$.*

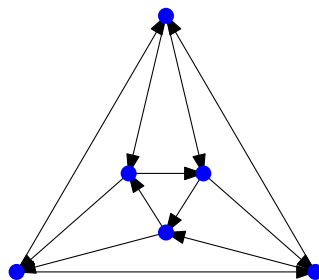


Figure 1: The octahedron-orientation \mathcal{O}_6 .

Proof. (i) \implies (ii): Suppose that every planar digraph is 2-colourable and let \vec{T} be an arbitrary orientation of a planar triangulation T . Looking at the orientation \mathcal{O}_6 of the octahedron graph depicted in Figure 1, it is easily observed that in any acyclic 2-colouring the triangle bounding the outer face cannot be monochromatic. Now consider a crossing-free spherical embedding of \vec{T} . For every facial triangle in this embedding whose orientation is transitive, we take a copy of \mathcal{O}_6 and glue this copy into the face in such a way that the outer three edges of \mathcal{O}_6 are identified with the three edges of the facial triangle (to make the orientations of the identified edges compatible,

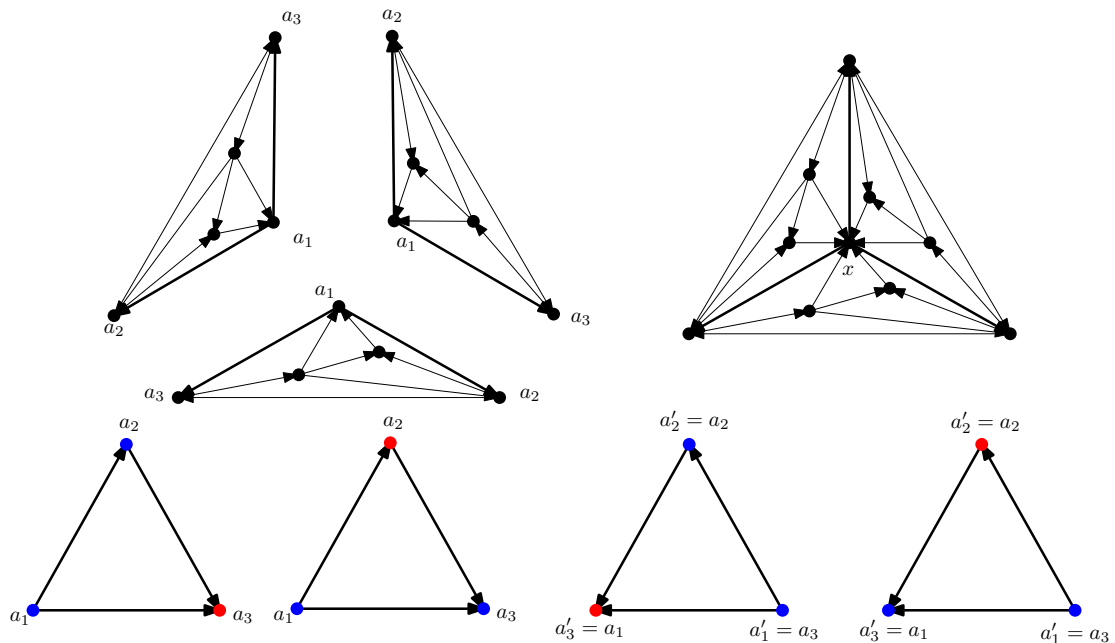


Figure 2: Top: Illustration of the construction of \vec{T}^* from three copies of \vec{T} . Bottom: The restriction of the colourings c_1, c_2, c'_1, c'_2 to the triangle t . Blue vertices have colour 1, red corresponds to colour 2.

it might be necessary to reflect and rotate the embedding of \mathcal{O}_6 shown in Figure 1). This creates a crossing-free embedding of a planar oriented triangulation \vec{T}^Δ . By assumption, \vec{T}^Δ admits an acyclic 2-colouring. This colouring restricted to the vertices of the subdigraph \vec{T} clearly is still valid. Furthermore, no triangle in \vec{T} can be monochromatic: This follows by definition if the triangle forms a directed cycle. If the orientation is transitive, by definition of \vec{T}^Δ , a monochromatic colouring would contradict the fact that \mathcal{O}_6 has no acyclic 2-colouring with the outer three vertices being coloured the same.

(ii) \implies (iii): Suppose that (ii) holds. Let \vec{T} be an orientation of a planar triangulation T and let $a_1 a_2 a_3$ be the vertices of a facial triangle t of T , equipped with a non-monochromatic pre-colouring $p : \{a_1, a_2, a_3\} \rightarrow \{1, 2\}$.

Case 1: t is not directed in \vec{T} . By relabelling, we may assume the transitive orientation $(a_1, a_2), (a_1, a_3), (a_2, a_3) \in E(\vec{T})$. Now consider a plane embedding of \vec{T} in which $a_1 a_2 a_3$ forms the bounding triangle of the outer face and where a_1, a_2, a_3 appear in clockwise order. We now define a new oriented planar triangulation \vec{T}^* as follows: We consider the embedding and orientation of the K_4 as shown in Figure 2, in which the outer face has a clockwise direction and the central vertex is a source. Into each of the three inner faces of this embedding, we can now glue a copy of \vec{T} with the described embedding in such a way that all orientations of identified edges agree. The vertex a_1 from each copy now is identified with the central vertex, which we call x . This oriented planar triangulation \vec{T}^* according to assumption admits an acyclic 2-colouring $c^* : V(\vec{T}^*) \rightarrow \{1, 2\}$ without monochromatic facial triangles. By relabelling the colours, we may assume that $c^*(x) = 1$. Because the outer triangle is not monochromatic, there have to be edges $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$ on the outer triangle such that $c^*(u_1) = c^*(v_2) = 1, c^*(u_2) =$

$c^*(v_1) = 2$. To both e_1, e_2 we have a corresponding copy of \vec{T} , and the 2-colourings c_1, c_2 of \vec{T} which c^* induces on these copies are still valid acyclic colourings. Furthermore, there are no monochromatic facial triangles in \vec{T} with respect to c_1, c_2 : Every bounded facial triangle of \vec{T} in the corresponding copy also forms a bounded facial triangle of \vec{T}^* , while the outer triangles of the copies contain e_1 respectively e_2 and are thus not monochromatic under c_1 respectively c_2 . From the way we glued the three copies of \vec{T} we conclude that $c_1(a_1) = c^*(x) = 1, c_1(a_2) = c^*(u_1) = 1, c_1(a_3) = c^*(v_1) = 2$ and $c_2(a_1) = c^*(x) = 1, c_2(a_2) = c^*(u_2) = 2, c_2(a_3) = c^*(v_2) = 1$.

Now consider the oriented planar triangulation \vec{T} which is obtained from \vec{T} by reversing the orientations of all edges. Let us rename the vertices of t according to $a'_1 := a_3, a'_3 := a_1, a'_2 := a_2$. We have $(a'_1, a'_2), (a'_1, a'_3), (a'_2, a'_3) \in E(\vec{T})$. We can therefore apply the same arguments as above to \vec{T} with the transitive labeling $a'_1 a'_2 a'_3$ of the triangle t . Hence, we obtain acyclic 2-colourings $c'_1, c'_2 : V(\vec{T}) \rightarrow \{1, 2\}$ of \vec{T} without monochromatic facial triangles such that $c'_1(a'_1) = c'_1(a'_2) = 1, c'_1(a'_3) = 2$ and $c'_2(a'_1) = 1, c'_2(a'_2) = 2, c'_2(a'_3) = 1$.

Finally, because the acyclic 2-colourings of \vec{T} and \vec{T} coincide, we conclude that $c_1, c_2, c'_1, c'_2 : V(\vec{T}) \rightarrow \{1, 2\}$ all form acyclic 2-colourings of \vec{T} without monochromatic facial triangles. It is easy to see that the colourings c_1, c_2, c'_1 together extend all possible non-monochromatic pre-colourings p of the triangle t with exactly one vertex of colour 2 (see the illustration in Figure 2). Hence, after flipping all colours from 1 to 2 and 2 to 1 we have found extending colourings for all possible pre-colourings p without monochromatic facial triangles, as desired.

Case 2: t is directed in \vec{T} . After relabelling (and possibly exchanging the colours 1 and 2), we may suppose that $(a_1, a_2), (a_2, a_3), (a_3, a_1) \in E(\vec{T})$ and $p(a_1) = 1, p(a_2) = 1, p(a_3) = 2$. Consider the orientation \vec{T}_e obtained from \vec{T} by reversing the edge $e = (a_3, a_1)$. By the first case, we know that \vec{T}_e admits an acyclic 2-colouring c_e without monochromatic facial triangles which extends p . Because the endpoints of e receive different colours, it follows directly that c_e also defines an acyclic 2-colouring of \vec{T} with the required properties, and the claim follows also in this case.

(iii) \implies (iv): We prove the statement by induction on the number of vertices. In the base case, where \vec{T} is an oriented triangle, the statement clearly holds true. Now let $n \geq 4$, and assume that the statement holds for all triangulations with less than n vertices. Let \vec{T} be an arbitrary orientation of some planar triangulation T with n vertices. If T is 4-connected, then the only triangles in T are the facial triangles and therefore the claim follows from (iii). Therefore we may suppose that T is not 4-connected, i.e., there exists a *separating triangle* $x_1 x_2 x_3$ in T . Consider some plane crossing-free embedding of T . Here, $x_1 x_2 x_3$ separates the vertices in its interior ($V_{\text{in}} \subseteq V(T)$) from those in its exterior ($V_{\text{out}} \subseteq V(T)$). Let $\vec{T}_{\text{out}} := \vec{T}[V_{\text{out}} \cup \{x_1, x_2, x_3\}]$ and $\vec{T}_{\text{in}} := \vec{T}[V_{\text{in}} \cup \{x_1, x_2, x_3\}]$. Both form oriented planar triangulations on less than n vertices. To prove that \vec{T} satisfies the inductive claim, let $t = a_1 a_2 a_3$ be a given triangle in T equipped with a non-monochromatic pre-colouring $p : \{a_1, a_2, a_3\} \rightarrow \{1, 2\}$. We must either have $\{a_1, a_2, a_3\} \subseteq V_{\text{out}} \cup \{x_1, x_2, x_3\}$ or $\{a_1, a_2, a_3\} \subseteq V_{\text{in}} \cup \{x_1, x_2, x_3\}$. Assume that we are in the first case, the second case is completely analogous. Then, by the induction hypothesis, there exists an acyclic colouring $c_{\text{out}} : V(\vec{T}_{\text{out}}) \rightarrow \{1, 2\}$ without monochromatic triangles such that $c_{\text{out}}(a_i) = p(a_i), i \in \{1, 2, 3\}$. The restriction of c_{out} to $x_1 x_2 x_3$ now defines a non-monochromatic pre-colouring for \vec{T}_{in} , and it follows from the induction hypothesis that there exists an acyclic 2-colouring c_{in} of \vec{T}_{in} without monochromatic triangles which agrees with c_{out} on $V(\vec{T}_{\text{out}}) \cap V(\vec{T}_{\text{in}}) = \{x_1, x_2, x_3\}$. By Lemma 3, the common extension of $c_{\text{out}}, c_{\text{in}}$ to $V(\vec{T})$

now defines an acyclic 2-colouring of \vec{T} , extending the given pre-colouring p of t and without monochromatic triangles. This verifies the inductive claim.

(iv) \Rightarrow (i): This follows since every planar graph is a subgraph of a planar triangulation. \square

Because any edge in a planar triangulation lies on a triangle, we directly obtain the following.

Corollary 4. *Under the assumption of Neumann-Lara's Conjecture, every orientation of a planar graph admits an acyclic 2-colouring without monochromatic triangles which can be chosen to extend any given pre-colouring of an edge or any non-monochromatic pre-colouring of a triangle.*

3 K_5 -Minor-Free Graphs

Given a pair G_1, G_2 of undirected graphs such that $V(G_1) \cap V(G_2)$ forms a clique of size i in both G_1 and G_2 , and such that $|V(G_1)|, |V(G_2)| > i$, the graph G with $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$ is called the *proper i -sum* of G_1 and G_2 . A graph obtained from G by deleting some (possibly all or none) of the edges in $E(G_1) \cap E(G_2)$ is said to be an *i -sum* of G_1 and G_2 . The central tool in proving Theorem 2 is the following classical result due to Wagner. By V_8 we denote the so-called *Wagner graph*, that is the graph obtained from C_8 by joining any two diagonally opposite vertices by an edge.

Theorem 5 ([Wag37]). *A simple graph is K_5 -minor-free if and only if it can be obtained from planar graphs and copies of V_8 by means of repeated i -sums with $i \in \{0, 1, 2, 3\}$.*

V_8 is triangle-free and admits an acyclic 2-colouring for any orientation. Moreover, it can be easily checked that such a colouring can be chosen to extend any given pre-colouring of two adjacent vertices. We are now in the position to prove Theorem 2.

Proof of Theorem 2. Assume that Neumann-Lara's Conjecture holds true. We have to prove that every oriented K_5 -minor-free graph admits an acyclic 2-colouring without monochromatic triangles. In fact, we prove the following slightly stronger statement:

Every orientation of a K_5 -minor-free graph admits an acyclic 2-colouring without monochromatic triangles which can be chosen to extend any given pre-colouring of an edge or any non-monochromatic pre-colouring of a triangle.

Assume towards a contradiction that there exists a K_5 -minor-free graph G which does not satisfy this claim and choose G minimal with respect to the number of vertices, and among all such graphs maximal with respect to the number of edges.

By Corollary 4, we know that the claim is fulfilled by all planar graphs and by V_8 , and so it follows from Theorem 5 that G is the i -sum of two K_5 -minor-free graphs G_1, G_2 with fewer vertices, where $0 \leq i \leq 3$. By the minimality assumption, we therefore know that G_1, G_2 satisfy the claim. Clearly, every super-graph of G does not satisfy the assertion as well. Therefore, by the assumed edge-maximality, G must in fact be the proper i -sum of G_1 and G_2 .

Now choose some orientation \vec{G} of G for which the above claim fails. Denote by \vec{G}_1, \vec{G}_2 the induced orientations on the subgraphs G_1, G_2 .

Let $e = uv \in E(G) = E(G_1) \cup E(G_2)$ be an arbitrary edge with a given pre-colouring $p : \{u, v\} \rightarrow \{1, 2\}$. W.l.o.g. assume that $e \in E(G_1)$. Let $c_1 : V(G_1) \rightarrow \{1, 2\}$ be an acyclic 2-colouring of \vec{G}_1 without monochromatic triangles which extends p . The clique $C = V(G_1) \cap V(G_2)$ in G_2 is either empty, a single vertex, and edge or a triangle. In each case, the restriction $p' = c_1|_C$ (if non-empty) can be considered as a pre-colouring of a vertex, an edge or a triangle in G_2 with two colours. In the case where C is a triangle, by the choice of c_1 , we furthermore know that

the pre-colouring p is not monochromatic. We therefore conclude that in any case, there is an acyclic 2-colouring $c_2 : V(G_2) \rightarrow \{1, 2\}$ of \vec{G}_2 without monochromatic triangles which extends p' . Therefore c_1 and c_2 agree on $V(G_1) \cap V(G_2) = C$ and it follows from Lemma 3 that the common extension of c_1, c_2 to $V(G)$ defines an acyclic 2-colouring of \vec{G} which extends p . Because every triangle of G is fully contained in G_1 or G_2 , we also have that there are no monochromatic triangles under c .

Similarly, for any triangle $t = a_1a_2a_3$ in G equipped with a non-monochromatic pre-colouring $p : \{a_1, a_2, a_3\} \rightarrow \{1, 2\}$, we may assume w.l.o.g. that t is fully contained in G_1 . Again, we find a pair of acyclic 2-colourings c_1, c_2 of \vec{G}_1, \vec{G}_2 such that c_1 extends p , c_2 coincides with c_1 on the clique $V(G_1) \cap V(G_2)$, and there are no monochromatic triangles in G_j under c_j for $j = 1, 2$. Finally, the common extension of c_1, c_2 to $V(G)$ by Lemma 3 defines an acyclic 2-colouring of \vec{G} with the desired properties.

From this we conclude that \vec{G} admits an extending acyclic 2-colouring without monochromatic triangles for any pre-colouring of an edge and for any non-monochromatic pre-colouring of a triangle. This is a contradiction to our choice of \vec{G} . This shows that the initial assumption was false and concludes the proof of the Theorem. \square

4 Conclusion

A natural question that comes out from the discussion in this paper is the following.

Question 1. *What is the largest minor-closed class \mathcal{G}_2 of undirected graphs with dichromatic number at most 2?*

While $\vec{\chi}(K_6) = 2$, it is known that $\vec{\chi}(K_7) = 3$. Therefore, \mathcal{G}_2 is a subclass of the K_7 -minor-free graphs. However, \mathcal{G}_2 seems to be a lot smaller than this class. In fact, there are $K_{3,3}$ -minor-free graphs with dichromatic number greater than 2, see Figure 3 for a simple example.

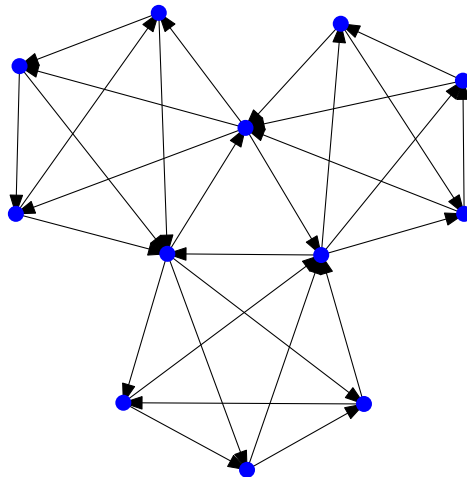


Figure 3: An oriented $K_{3,3}$ -minor-free graph without an acyclic 2-colouring.

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