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THERE ARE NO DEVIATIONS FOR THE ERGODIC AVERAGES OF THE GIULIETTI–LIVERANI HOROCYCLE FLOWS ON THE TWO-TORUS

VIVIANE BALADI

ABSTRACT. We show that the ergodic integrals for the horocycle flow on the two-torus associated by Giulietti and Liverani to an Anosov diffeomorphism either grow linearly or are bounded, in other words there are no deviations. For this, we use topological invariance of the Artin–Mazur zeta function to exclude resonances outside of the open unit disc. Transfer operators acting on suitable spaces of anisotropic distributions and their Ruelle determinants are the key tools in the proof. As a bonus, we show that for any C^∞ Anosov diffeomorphism F on the two-torus, the correlations for the measure of maximal entropy and C^∞ observables decay with a rate strictly smaller than $e^{-h_{\text{top}}(F)}$. We compare our results with very recent related work of Forni.

1. INTRODUCTION

1.1. The results of Giulietti–Liverani. In a pioneering work, Giulietti and Liverani [7] introduced a “horocycle flow” on the torus, renormalizable by a given Anosov diffeomorphism: Fix $r > 1$ and let $F : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ be a C^r Anosov diffeomorphism on the two-torus. By Franks–Newhouse the stable bundle E^s of F (see [13, p. 805]) is orientable. Fixing an orientation E_+^s of E^s , Giulietti and Liverani assume that DF preserves this orientation and they introduce the flow $h^t = h_s^t$ on \mathbb{T}^2 obtained by solving $\partial_t h^t(f) = X(f)$, where $X(x) = v_{1,+}^s(x)$ is the unique vector of $E_+^s(x)$ of unit norm. We call h^t the (*unit¹ speed*) *Giulietti–Liverani (stable horocycle) flow (of F)*. (See Appendix A for basic facts about such flows.)

For any continuous function $f : \mathbb{T}^2 \rightarrow \mathbb{C}$, any $T > 0$, and any $x \in \mathbb{T}^2$, define the horocycle integral $H_{x,T}(f) = \int_0^T f(h^t(x)) dt$.

By unique ergodicity, we have for any such x and f

$$(1.1) \quad \lim_{T \rightarrow \infty} \frac{H_{x,T}(f)}{T} = \mu^s(f) := \int_{\mathbb{T}^2} f(x) d\mu^s,$$

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¹Giulietti and Liverani actually consider more general C^r time parametrizations but this is immaterial for the purposes of this work, as the resonances defined below are invariant under such time-changes since the transfer operator is modified by a C^r coboundary [7, (2.5)].

where μ^s is the unique invariant probability measure of the flow h^t .

Let $h_{top} = h_{top}(F) > 0$ be the topological entropy of F . Giulietti and Liverani [7] find $r_0 > 1$ such that, if $r \geq r_0$ there exists a bounded linear operator $\tilde{\mathcal{L}}$ associated with F (see (2.3) below) acting on a Banach space $\tilde{\mathcal{B}}_{GL}$ of distributions in a subset of the unit tangent bundle of \mathbb{T}^2 such that the following holds: The spectral radius of $\tilde{\mathcal{L}}$ on $\tilde{\mathcal{B}}_{GL}$ is $e^{h_{top}}$ and the essential spectral radius of $\tilde{\mathcal{L}}$ is strictly smaller than $\tilde{\rho}_{GL} = \tilde{\rho}_{GL}(r) < 1$. In addition, $\tilde{\rho}_0 := e^{h_{top}}$ is a simple eigenvalue and the only maximal eigenvalue. Restricting for simplicity to the case where $\tilde{\mathcal{L}}$ does not have any eigenvalue of modulus one and, in addition, all of its eigenvalues of modulus larger than $\tilde{\rho}_{GL}$ have trivial Jordan blocks (otherwise, additional factors $\log T^q$ with $q \geq 1$ can appear in the expansion (1.2) below), we let $\{\tilde{\rho}_j\}_{j=1}^{N_{GL}}$ be the $N_{GL} \geq 0$ eigenvalues of $\tilde{\mathcal{L}}$ of modulus $\tilde{\rho}_{GL} \leq |\tilde{\rho}_j| < e^{h_{top}}$ (ordered by decreasing modulus and repeated with multiplicity), and we let $\mathcal{O}_j \in \tilde{\mathcal{B}}_{GL}^*$ be the associated eigenvectors of the dual operator $\tilde{\mathcal{L}}^*$. We call $\{\tilde{\rho}_j\}_{j=0}^{N_{GL}}$ the *resonances*² of $\tilde{\mathcal{L}}$. We call the $N_{1,GL} \leq N_{GL}$ resonances such that $1 \leq |\tilde{\rho}_j| < e^{h_{top}}$ its *deviation resonances*. By [9] (see also [7, Lemma 2.11]), the maximal eigenvector $\tilde{\mathcal{L}}^* \tilde{\mu}^s = e^{h_{top}} \tilde{\mu}^s$ satisfies $\pi^*(\tilde{\mu}^s) = \mu^s$ (this fixes a normalisation of $\tilde{\mu}^s$), where π denotes the projection from the unit tangent bundle of \mathbb{T}^2 to \mathbb{T}^2 .

The first main result of Giulietti–Liverani [7, Theorem 2.8] gives the following expansion for the horocycle integral $H_{x,T}(f)$: For any $\delta \in (0, -\log \tilde{\rho}_{GL})$, there exists a constant $C < \infty$ such that, for any $T > 0$, and any $x \in \mathbb{T}^2$ there exist real numbers $\{C_j(x, T)\}_{j=0}^{N_{GL}}$ with $\sup_{x,T,j} |C_j(x, T)| \leq C$, such that for any C^r function $f : \mathbb{T}^2 \rightarrow \mathbb{C}$, we have³

$$(1.2) \quad H_{x,T}(f) = T \cdot C_0(x, T) \cdot \mu^s(f) + \sum_{j=1}^{N_{GL}} T^{\theta_j} \cdot C_j(x, T) \cdot \mathcal{O}_j(f \circ \pi) + \mathcal{R}_{x,T}(f),$$

where $\theta_j = \frac{\log |\tilde{\rho}_j|}{h_{top}} \in (-\infty, 1)$ and, for any $T > 0$,

$$\sup_x |\mathcal{R}_{x,T}(f)| \leq C(T^{\theta_{min}} \|f\|_{C^r} + \sup |f|), \quad \text{where } \theta_{min} = \frac{\log \tilde{\rho}_{GL} + \delta}{h_{top}} < 0.$$

Decomposing $f = \mu^s(f) + f - \mu^s(f)$, we get

$$(1.3) \quad T \cdot C_0(x, T) = H_{x,T}(1) = \int_0^T 1 \, dt = T,$$

up to replacing $\mathcal{O}_j(f \circ \pi)$ by $\mathcal{O}_j((f - \mu^s(f)) \circ \pi)$ in the right-hand-side of (1.2).

We refer to the introduction of [7] for the motivation arising from the well-known works of Forni and Flaminio giving expansions of ergodic averages along horocycles of (e.g.) geodesic flows on compact surfaces of constant negative curvature in terms of eigenvalues of the Laplacian and of horocycle-flow invariant distributions (see e.g. [5]). Some of the results of Flaminio and Forni have been extended to variable negative curvature geodesic flows [1] by adapting the ideas in [7].

²We shall see after (2.6) below that they do not depend on the Banach space.

³Giulietti–Liverani [7] limit the expansion to $|\tilde{\rho}_j| \geq 1$, i.e. $\theta_j \geq 0$, bounding the error term by $C\|f\|_{C^r}$. They write $\ell_j(x, T, f)$ instead of $C_j(x, T) \cdot \mathcal{O}_j(f \circ \pi)$. Their proof gives the slightly more precise result (1.2).

Remark 1.1 (Bound for r_0). An upper bound for r_0 can be read off from the bound for $\tilde{\rho}_{GL}(r)$ in [9], noting that the limit is r and not $r - 1$ in the case $\iota = 1$ there, and taking care of the fact that one of the regularity exponents is an integer. In [7], the authors mention that choosing $r_0 \geq 1$ large enough such that

$$e^{h_{top}(F)} / \lambda^{r_0/2} < 1$$

suffices (where $1 < \lambda < \min\{\lambda_u, 1/\lambda_s\}$, see (2.5), and λ_u, λ_s are defined in the beginning of §2). Compare this with (2.7) below.

1.2. Absence of deviation resonances and two consequences. The main technical result of the present work, Theorem 2.1, is that, if $r \geq r_1$ is large enough (r_1 depends on the expansion and contraction factors of F , see (2.7) and (2.8)) then $|\tilde{\rho}_j| < 1$ for all $j = 1, \dots, N_{GL}$. (In other⁴ words, *the transfer operator $\tilde{\mathcal{L}}$ does not have deviation resonances.*) As a consequence (Corollary 2.3), if $r \geq \max\{r_0, r_1\}$, the exponents θ_j in the expansion (1.2) all satisfy $\theta_j < 0$. In particular (even in the presence of Jordan blocks), there exists $C' < \infty$ and $\theta'_{min} < 0$ such that

$$(1.4) \quad \sup_x |H_{x,T}(f) - T \cdot \mu^s(f)| \leq C'(T^{\theta'_{min}} \|f\|_{C^r} + \sup |f|), \forall T > 0.$$

That is, *there are no deviations to the convergence of horocycle ergodic integrals.*

We next mention a consequence of Theorem 2.1 about the “cohomological equation.” Since $h^t(x)$ is minimal, Gottschalk–Hedlund’s theorem (see [14] for a recent account) implies, for any continuous f , that $\sup_{x,T} |H_{x,T}(f)| < \infty$ if and only if f is a continuous coboundary, i.e. there exists a continuous function \bar{f} such that for all x and all $T > 0$, the following cohomological equation holds:

$$\bar{f}(x) - \bar{f} \circ h^T(x) = \int_0^T f \circ h^t(x) dt.$$

The expansion (1.4) from Corollary 2.3 thus implies the following *dichotomy* for the Giulietti–Liverani flow of a C^r Anosov diffeomorphism with $r \geq \max\{r_0, r_1\}$: *If $f \in C^r$ then either $\mu^s(f) \neq 0$ (and $H_{x,T}(f)$ grows linearly) or f is a continuous coboundary.*

Another immediate consequence of Theorem 2.1 about the absence of deviation resonances for $\tilde{\mathcal{L}}$ is Corollary 2.5: For any C^∞ Anosov diffeomorphism F on the two-torus, the *correlations for the unique measure of maximal entropy μ_{top} and C^∞ observables decay with rate strictly smaller than $e^{-h_{top}(F)}$* . (In fact Corollary 2.5 only requires C^r for large enough r .)

Giovanni Forni [6] obtained (independently and simultaneously) related results⁵, in a more general setting: For any C^r ($r > 1$) pseudo Anosov diffeomorphism on a compact surface, the correlation spectrum of the Margulis measure is determined by the action of the diffeomorphism on the first cohomology, up to a power law error term (his results do not imply Corollary 2.5, even when the diffeomorphism is Anosov and r is large, see Remark 2.6).

We end this introduction by mentioning that if F is a C^r Anosov diffeomorphism on a compact connected C^∞ manifold M of dimension $d \geq 2$, and the dimension d_s of its stable bundle is equal to one (this is the situation when a horocycle flow

⁴“Men han har jo ikke noget paa,” sagde et lille Barn.

⁵We focus on the analogue of Corollary 2.5 but Forni’s main result is an analogue of Corollary 2.3 about equidistribution of stable or unstable curves.

of Giulietti–Liverani type can be constructed), then M is homeomorphic (and thus diffeomorphic) to the torus \mathbb{T}^d , and F is topologically conjugated to a linear toral automorphism A of \mathbb{T}^d (see [13] for a recent account of this result of Franks and Newhouse). See Remark 3.1 about the limits of our approach in dimensions $d \geq 3$.

Remark 1.2. After this paper was accepted for publication, Jérôme Carrand [4], following a suggestion of Selim Ghazouani, used Denjoy–Koksma to give a very short proof of a slightly weaker result (applying to a slightly larger class of flows): the ergodic integrals of a zero-average observable grow at most logarithmically.

2. STATEMENT OF THE MAIN THEOREM AND ITS TWO COROLLARIES

Fix $r > 1$ and let $F : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ be a C^r Anosov diffeomorphism on the two-torus. This means that there exist $C < \infty$ and $\lambda_u > 1$, $\lambda_s < 1$ such that the tangent space splits as $T\mathbb{T}^2 = E^s \oplus E^u$, with E^s and E^u preserved by DF , such that, for all x ,⁶

$$\|DF_x^n v\| \leq C \lambda_s^n \|v\|, \quad \forall v \in E^s(x), \quad \|DF_x^{-n} v\| \leq C \lambda_u^{-n} \|v\|, \quad \forall v \in E^u(x), \quad \forall n \geq 1.$$

By Franks–Newhouse (see e.g. [13]), since $d_s = 1$, the Anosov diffeomorphism F is topologically conjugated to a hyperbolic linear toral automorphism. In particular, F is topologically mixing. In fact, $d_s = 1$ also implies that E^s (see [13, p. 805]) is orientable. Fixing an orientation of E^s , we assume that DF preserves this orientation.

2.1. The Artin–Mazur zeta function of F . Let $A = A_F \in Gl_2(\mathbb{Z})$ be the hyperbolic matrix topologically conjugated to F (i.e. the unique linear map in the homotopy class of F). The eigenvalues⁷ of A are

$$\sigma e^{h_{\text{top}}(F)} \text{ and } e^{-h_{\text{top}}(F)} < 1, \text{ with } \sigma := \sigma_A = \det A \in \{\pm 1\}.$$

(The contracting eigenvalue is positive by our orientation preserving assumption.) Put $\lambda_A = e^{h_{\text{top}}(F)}$. An easy computation gives that the unweighted (Artin–Mazur) zeta function of A is just

$$(2.1) \quad \zeta_A(z) = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} \# \text{Fix } A^n = \begin{cases} \frac{(1-z)^2}{(1-\lambda_A z)(1-z/\lambda_A)} & \text{if } \sigma = +1 \\ \frac{1-z^2}{(1-\lambda_A z)(1+z/\lambda_A)} & \text{if } \sigma = -1. \end{cases}$$

(Use⁸ that $\# \text{Fix } A^n = |\det(1 - A^n)| = |1 - (\sigma \lambda_A)^n| \cdot (1 - \lambda_A^{-n})$ so that $\# \text{Fix } A^n = -(-\lambda_A)^n - \lambda_A^{-n} = \lambda_A^n + (-1)^n \lambda_A^{-n}$ if $\sigma = -1$ and n is odd, while $\# \text{Fix } A^n = -2 + \lambda_A^n + \lambda_A^{-n}$ otherwise, and handle separately even and odd n in this case, using that $\exp \sum_{j=1}^{\infty} 2 \cdot z^{2j}/2j = (1 - z^2)$.)

2.2. The extended dynamics \tilde{F} and the transfer operator $\tilde{\mathcal{L}}$. Identifying $T_x \mathbb{T}^2$ with \mathbb{R}^2 for any $x \in \mathbb{T}^2$, we let $x \mapsto \mathcal{C}^s(x) \subset T_x \mathbb{T}^2$ be a smooth stable cone field for F , that is, each $\mathcal{C}^s(x)$ is a strict subset of \mathbb{R}^2 of nonempty interior, with $\xi \cdot \mathcal{C}^s(x) \subset \mathcal{C}^s(x)$ for all $\xi \in \mathbb{R}$, and such that

$$\overline{DF^{-1}(\mathcal{C}^s(x))} \subset \mathcal{C}^s(F^{-1}(x)) \cup \{0\}, \quad \forall x \in \mathbb{T}^2,$$

⁶ $\|\cdot\|$ denotes the norm on $T_x \mathbb{T}^2$ induced by the Riemann metric on \mathbb{T}^2 .

⁷We can again reduce to the case $\sigma_A = 1$ by considering F^2 , but we find it instructive to write the zeta function in the general case.

⁸See e.g. the proof of [10, Lemma 18.6.2].

and there exists $\nu_s < 1$ such that

$$\|DF_x^{-1}(v)\| \geq \nu_s^{-1}\|v\|, \quad \forall x \in \mathbb{T}^2, \forall v \in C(F(x)).$$

We consider the following compact subspace of the unit tangent bundle of \mathbb{T}^2 :

$$\Omega^s = \{(x, v) \in \mathbb{T}^2 \times \mathbb{R}^2 \mid \|v\| = 1, \quad v \in \overline{C^s(x)}\}.$$

Orientability of E^s gives a decomposition $\Omega^s = \Omega_+^s \cup \Omega_-^s$. Since DF preserves the orientation of E^s , the C^{r-1} map defined on $\{(x, v) \in \Omega_+^s \mid DF_x(v) \in \overline{C^s(x)}\}$ by

$$\tilde{F}(x, v) = \left(F(x), \frac{D_x F(v)}{\|D_x F(v)\|} \right)$$

leaves invariant the set

$$E_{1,+}^s := \{(x, v) \in \Omega_+^s \mid v \in E^s(x)\} = \{(x, v_{1,+}^s(x)) \mid x \in \mathbb{T}^2\}.$$

It is easy to see that $(\tilde{F}, E_{1,+}^s)$ is a topologically mixing Axiom A repeller, with stable dimension one and unstable dimension two, and that the expansion in the new unstable direction is not smaller than $\lambda_u/\nu_s > \lambda_u$. For each $n \geq 1$, the periodic orbits of F^n are in bijection with the periodic orbits of \tilde{F}^n via the map $x \mapsto (x, v_{1,+}^s(x))$. Following [7], we set

$$(2.2) \quad \tilde{g}(x, v) = \frac{1}{|\text{Det } DF|_V \circ \tilde{F}^{-1}(x, v)},$$

where $V(\tilde{F}^{-1}(x, v)) = \mathbb{R} \cdot DF_x^{-1}v \subset T_{F^{-1}(x)}\mathbb{T}^2$ is the line generated by $DF_x^{-1}v$, and we consider the weighted transfer operator $\tilde{\mathcal{L}}$ of \tilde{F} defined by

$$(2.3) \quad (\tilde{\mathcal{L}}\tilde{\varphi})(x, v) = (\tilde{g}\tilde{\varphi}) \circ \tilde{F}^{-1}(x, v).$$

(As explained in [7], the operator $\tilde{\mathcal{L}}$ corresponds to the action of F on one-forms.)

2.3. The essential spectrum of $\tilde{\mathcal{L}}$ on the Banach space $\tilde{\mathcal{B}}$. Since $\lambda_u/\nu_s > \lambda_u$, and $\tilde{d}_s = 1$, if $r < \infty$ the results⁹ of [3, Theorem 1.1] (see also [2, Chap. 5] for a pedestrian account) imply that for any $\epsilon > 0$, there exists a Banach space $\tilde{\mathcal{B}} = \tilde{\mathcal{B}}^{t,s}(\tilde{F})$ of anisotropic distributions on Ω_+^s (supported in a neighbourhood \mathcal{E} of $E_{1,+}^s$) such that the essential spectral radius of $\tilde{\mathcal{L}}$ on $\tilde{\mathcal{B}}$ is strictly¹⁰ smaller than

$$(2.4) \quad \begin{aligned} \tilde{\rho}_{BT} := \tilde{\rho}_{BT}(r) &= \exp \max_{s,t \in \mathbb{R}} \sup_{t-(r-1-\epsilon) \leq s < 0 < t} \sup_{\tilde{\mu} \in \text{Erg}(\tilde{F})} \{h_{\tilde{\mu}} + \max\{-t\ell_{\tilde{\mu}}^u, |s|\ell_{\tilde{\mu}}^s\}\} \\ &= \exp \max_{s,t \in \mathbb{R}} \sup_{t-(r-1-\epsilon) \leq s < 0 < t} \sup_{\mu \in \text{Erg}(F)} \{h_{\mu} + \max\{-t\ell_{\mu}^u, |s|\ell_{\mu}^s\}\}, \end{aligned}$$

where $\ell_{\tilde{\mu}}^{u/s} = \int \log(\det DF|_{E^{u/s}}) d\tilde{\mu}$, and $\text{Erg}(\tilde{F})$ is the set of ergodic \tilde{F} -invariant probability measures, which is in bijection with $\text{Erg}(F)$, with $h_{\tilde{\mu}} = h_{\mu(\tilde{\mu})}$ and $\ell_{\tilde{\mu}}^{u/s} = \ell_{\mu(\tilde{\mu})}^{u/s}$, where $\ell_{\mu}^{u/s} = \int \log(\det DF|_{E^{u/s}}) d\mu$. (The ℓ^u are strictly positive and the ℓ^s strictly negative.)

Note that by [2, Chap. 5] we have $\tilde{\rho}_{BT} \leq \frac{e^{h_{\text{top}}(F)}}{\lambda^{(r-1)/2}}$ for any λ such that

$$(2.5) \quad 1 < \lambda < \min\{\lambda_u, 1/\lambda_s\}.$$

⁹Theorem 2.1 there shows that one does not need to replace $r-1$ by $r-2$.

¹⁰Take the Banach space $C^{t,s}(\tilde{F}, \mathcal{E})$ in [3, Theorem 1.1], for $\epsilon > 0$ and $t-(r-1-\epsilon) < s < 0 < t$.

Since \tilde{F} is mixing, $\tilde{\rho}_0 := e^{h_{top}}$ is a simple eigenvalue and the only maximal eigenvalue. Set

$$(2.6) \quad \tilde{\rho}_{ess} := \max\{\tilde{\rho}_{GL}, \tilde{\rho}_{BT}\}.$$

The sets $\Sigma_{GL} := \{\tilde{\rho} \in \text{sp}(\tilde{\mathcal{L}}|_{\tilde{\mathcal{B}}_{GL}}) \mid |\tilde{\rho}| > \tilde{\rho}_{ess}\}$ and $\Sigma_{BT} := \{\tilde{\rho} \in \text{sp}(\tilde{\mathcal{L}}|_{\tilde{\mathcal{B}}}) \mid |\tilde{\rho}| > \tilde{\rho}_{ess}\}$ coincide (including multiplicities) by [3, App. A.2]. In addition, the finitely many corresponding generalised eigenvectors lie in $\tilde{\mathcal{B}}_{GL} \cap \tilde{\mathcal{B}}$. In view also of the results about the dynamical determinant (2.9) recalled below, it is legitimate to call the eigenvalues¹¹ $\Sigma_{BT} = \{\tilde{\rho}_j\}_{j=0}^N$ (repeated with multiplicity, ordered with decreasing modulus) of $\tilde{\mathcal{L}}$ of modulus $|\tilde{\rho}_j| > \tilde{\rho}_{ess}$ the *resonances* of F . We call¹² the $N_1 \leq N$ resonances $\{\tilde{\rho}_j\}_{j=1}^{N_1}$ with $1 \leq |\tilde{\rho}_j| < e^{h_{top}(F)}$ the *deviation resonances* of F .

Since $\sup_{\mu \in \text{Erg}(F)} \{h_\mu - \tau \ell_\mu^u\} < 0$ if $\tau > 1$, it is not hard to see that $\tilde{\rho}_{BT} < 1$ if

$$(2.7) \quad r > r_1 := 1 + \epsilon + \sup_{\mu \in \text{Erg}(F)} \frac{\ell_\mu^u - \ell_\mu^s}{-\ell_\mu^s}, .$$

(If F preserves area, (2.7) reduces to $r_1 = 3 + \epsilon$, taking t and $-s$ arbitrarily close to $(r-1)/2$.) Note that (2.7) implies

$$(2.8) \quad r_1 \leq 1 + \epsilon + \frac{-\log \Lambda_s + \log \Lambda_u}{-\log \lambda_s},$$

with $\Lambda_u \geq \lambda_u$ and $\Lambda_s \leq \lambda_s$ the strongest expansion, respectively contraction of F .

2.4. The determinant $\tilde{d}(z)$ of $\tilde{\mathcal{L}}$. Theorems 1.5 and §2 of [3] (see also [2, Chap. 6]) give that the dynamical determinant

$$(2.9) \quad d_{\tilde{F}, \tilde{g}}(z) = \exp - \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{(x,v) \in \text{Fix } \tilde{F}^n} \frac{\prod_{k=0}^{n-1} \tilde{g}(\tilde{F}^k(x,v))}{|\det(\text{Id} - D\tilde{F}^{-n}(x,v))|}$$

admits a holomorphic extension to the open disc of radius $\tilde{\rho}_{BT}^{-1}$, in which the zeroes of $d_{\tilde{F}, \tilde{g}}(z)$ are exactly the inverses of the eigenvalues of $\tilde{\mathcal{L}}$ of modulus $> \tilde{\rho}_{BT}$ (the order of the zero coincides with the algebraic multiplicity of the eigenvalue).

2.5. Statement of results. We now state our main result, using the notation from §2.3, in particular (2.3), (2.4), and (2.9):

Theorem 2.1 (Absence of deviation resonances). *Fix $r > 1$ and let F be a C^r Anosov diffeomorphism of the two-torus preserving the orientation of E^s .*

The only zero of the dynamical determinant $d_{\tilde{F}, \tilde{g}}(z)$ in the closed disc of radius $\min\{1, \tilde{\rho}_{BT}^{-1}\}$ is a simple zero at $e^{-h_{top}(F)}$. In particular the spectrum of the operator $\tilde{\mathcal{L}}$ (acting on $\tilde{\mathcal{B}}$) outside of the open disc of radius $\max\{1, \tilde{\rho}_{BT}\}$ consists in a simple eigenvalue at $e^{h_{top}(F)}$. (In the notation of §2.3, we have $N_1 = 0$.)

Remark 2.2. If F does not preserve the orientation of E^s , the same result holds up to introducing a non-mixing extension \tilde{F} of F such that \tilde{F}^{-1} exchanges Ω_+^s and Ω_-^s . The only difference is that there are two maximal eigenvalues, $\pm e^{h_{top}(F)}$.

Recalling r_0 from §1.1 (see also Remark 1.1), and using the notation from §2.3, in particular (2.6) and (2.7), we get:

¹¹The eigenvalues $\{\tilde{\rho}_j\}_{j=1}^N$ are in bijection with the possible poles v_j of the Fourier transform of the correlation function of the measure of maximal entropy of F , see also Corollary 2.5.

¹²Note that $N_1 = N_{1, GL}$.

Corollary 2.3 (No deviations for horocycle ergodic integrals). *Let $F : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ be a C^r Anosov diffeomorphism which preserves the orientation of E^s . If $r \geq \max\{r_0, r_1\}$, so that $\tilde{\rho}_{ess} < 1$, then for any $\delta > 0$ there¹³ exists $C' < \infty$ such that for any C^{r-1} function $f : \mathbb{T}^2 \rightarrow \mathbb{C}$*

$$(2.10) \quad \sup_x |H_{x,T}(f) - T \cdot \mu^s(f)| \leq C'(T^{\theta'_{min}} \|f\|_{C^{r-1}} + \sup |f|), \quad \forall T > 0,$$

where (taking small enough δ)

$$\theta'_{min} = \frac{\log \tilde{\rho}_{ess} + \delta}{h_{top}} < 0 \text{ if } N = 0, \quad \theta'_{min} = \frac{\delta + \log \max\{\tilde{\rho}_j\}_{j=1}^N}{h_{top}} < 0 \text{ if } N \geq 1.$$

In particular $\mu^s(f) = 0$ if and only if f is a continuous coboundary.

Remark 2.4. Our result does not exclude the existence of obstructions to Lipschitz (or Hölder) regularity of the solution \tilde{f} of the cohomological equation for f when $\mu^s(f) = 0$ [7, end of §5.1.2, Remark 5.10]. Such constructions involve a different transfer operator (see the proof of [7, Theorem 2.12]).

Proof of Corollary 2.3. Since we assume $\tilde{\rho}_{ess} < 1$, the corollary follows from the expansion (1.2) from [7, Theorem 2.8] combined with Theorem 2.1 and (1.3). (If $N \geq 1$, taking $\delta \in (0, -\log \max\{\tilde{\rho}_j\}_{j=1}^N)$ handles the possible Jordan blocks. This could be replaced by an appropriate power of $\log T$ in front of $T^{\theta'_{min}}$.) \square

Our final result is about the unique measure of maximal entropy μ_{top} of F :

Corollary 2.5 (Rates of mixing for the measure of maximal entropy). *Let $r > 1$ and let $F : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ be a C^r Anosov diffeomorphism on the two-torus. If $\tilde{\rho}_{BT} < 1$, then there exist $C < \infty$ and $\rho_{top} < e^{-h_{top}(F)}$ such that for any C^{r-1} functions¹⁴ $f_1, f_2 : \mathbb{T}^2 \rightarrow \mathbb{C}$*

$$\left| \int (f_1 \circ F^k) f_2 d\mu_{top} - \int f_1 d\mu_{top} \int f_2 d\mu_{top} \right| \leq C \rho_{top}^k \|f_1\|_{C^{r-1}} \|f_2\|_{C^{r-1}}, \quad \forall k \geq 0.$$

If $\tilde{\rho}_{BT} \geq 1$, then for any $\rho_{top} > e^{-h_{top}(F)} \tilde{\rho}_{BT}$ there exists C such that the above bound holds. (Note that $e^{-h_{top}(F)} \tilde{\rho}_{BT} < 1$.)

Recalling (2.7), note that if $r \geq r_1$ then $\tilde{\rho}_{BT} < 1$. Clearly, if $\tilde{\rho}_{BT} \geq 1$, then the bound ρ_{top} given by Corollary 2.5 is $\geq e^{-h_{top}(F)}$.

Remark 2.6 (Comparing with the results of [6]). In our setting, Forni's results imply in particular¹⁵ that if F is C^r for $r > 2$, there exists $C < \infty$ such that for any C^1 functions f_1 and f_2 , we have

$$(2.11) \quad \left| \int (f_1 \circ F^k) f_2 d\mu_{top} - \int f_1 d\mu_{top} \int f_2 d\mu_{top} \right| \leq C k^2 e^{-kh_{top}} \|f_1\|_{C^1} \|f_2\|_{C^1}, \quad \forall k \geq 0.$$

It suffices to assume that $r > 2$ and f_1 and f_2 are C^1 to get (2.11), but the error term there is not as good as in Corollary 2.5.

¹³ $\|f\|_{C^{r-1}}$ can be replaced by a weaker norm. In the area preserving case, $C^{(r-1)/2}$.

¹⁴The C^{r-1} norms can be replaced by weaker norms. In the area preserving case, $C^{(r-1)/2}$.

¹⁵We do not discuss Forni's analog of Corollary 2.3, which contains an additional logarithmic factor $(\log T)^2$ in the error term.

Proof of Corollary 2.5. We may assume that F preserves the orientation of E^s , since otherwise we can replace F by F^2 (the two maps have the same measure of maximal entropy and $h_{top}(F^2) = 2h_{top}(F)$).

Let $\mu^u = \pi^*(\tilde{\mu}^u)$ where $\tilde{\mathcal{L}}(\tilde{\mu}^u) = e^{h_{top}(F)}\tilde{\mu}^u$, normalised by $\mu^s(\mu^u) = \tilde{\mu}^s(\tilde{\mu}^u) = 1$. By [9], the distribution $\varphi \mapsto \mu^s(\varphi\mu^u)$ is in fact the unique measure of maximal entropy μ_{top} of F . (The fact that [9] use another Banach space does not matter by [3, App. A.2].) Then using the spectral decomposition for $\tilde{\mathcal{L}}$ and the information from Theorem 2.1 gives the claim, just like in [9]. (For example, if $\tilde{\rho}_{BT} < 1$, we may choose ρ_{top} as follows: If $\tilde{\mathcal{L}}$ has no eigenvalue except $e^{h_{top}(F)}$ of modulus strictly larger than $\tilde{\rho}_{BT}$, then we can take any $\rho_{top} > \tilde{\rho}_{BT}/e^{h_{top}(F)}$ since $\tilde{\rho}_{BT} < 1$. Otherwise, letting $\tilde{\rho}_1$ be the eigenvalue of $\tilde{\mathcal{L}}$ of largest modulus $\neq e^{h_{top}(F)}$, we have $|\tilde{\rho}_1| < 1$ by Theorem 2.1, and for any fixed integer $q \geq 0$, taking $\rho_{top} > |\tilde{\rho}_1|/e^{h_{top}(F)}$, we have $k^q|\tilde{\rho}_1|^k e^{-kh_{top}(F)} < C\rho_{top}^k$ for all k .) \square

3. PROOF OF THEOREM 2.1

In order to prove Theorem 2.1, we introduce the following notation: If $\Phi : M \rightarrow M$ is an Axiom A diffeomorphism with a basic set X and $\phi : X \rightarrow \mathbb{C}$ admits a Hölder extension to a neighbourhood of X , we set $\phi^{(n)} = \prod_{k=0}^{n-1} (\phi \circ \Phi^k)$, and

$$d_{\Phi, \phi}(z) = \exp - \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{y \in \text{Fix } \Phi^n \cap X} \frac{\phi^{(n)}(y)}{|\det(\text{Id} - D\Phi^{-n}(y))|}.$$

For such a fixed Φ and X , we say that $\phi_1 <_{Per} \phi_2$ if $\phi_1, \phi_2 : X \rightarrow \mathbb{R}_*^+$ satisfy

$$\limsup_{n \geq 1} \sup_{y \in \text{Fix } \Phi^n \cap X} \frac{(\phi_1^{(n)}(y))^{1/n}}{(\phi_2^{(n)}(y))^{1/n}} < 1.$$

Finally, the following two elementary facts will be used in the proofs: For a 2×2 matrix Q , we have $\text{Det}(\text{Id} - Q) = 1 + \text{Det } Q - \text{tr } Q$. If \tilde{Q} is a 3×3 matrix of the form $\tilde{Q} = \begin{pmatrix} Q & q \\ 0 & \eta \end{pmatrix}$, with q an arbitrary column vector in \mathbb{R}^2 and $\eta \in \mathbb{R}$, we have

$$\text{Det}(\text{Id} - \tilde{Q}) = (1 + \text{Det } Q - \text{tr } Q)(1 - \eta).$$

If $F = A$ (so that $r = \infty$) and $\sigma_A = 1$ then the above facts for $Q = A^{-n}$ give¹⁶

$$(3.1) \quad d_{\tilde{F}, \tilde{g}}(z) = \prod_{j=0}^{\infty} (1 - z\lambda_A^{1-2j}).$$

(We shall not use the above expression.)

Proof of Theorem 2.1. As a warmup, we assume that $r > 3$, we fix $\epsilon > 0$ small, and we consider the transfer operator

$$(3.2) \quad \mathcal{L}\varphi(x) = \frac{\varphi(F^{-1}(x))}{|\text{Det } DF|_{E^s} \circ F^{-1}(x)}$$

acting on the Banach space of distributions $\mathcal{B} = C^{(1+\alpha)/2, -(1+\alpha)/2}(F, \mathbb{T}^2)$ defined in [3, §4], where its spectral radius is $e^{h_{top}(F)}$, while its essential spectral radius is

¹⁶Formula (3.1) recovers the eigenvalues of the operator associated to A obtained in [7, §5.2].

strictly smaller than (recalling $\lambda > 1$ from (2.5))

$$\rho_{ess,\alpha} := \frac{e^{h_{top}(F)}}{\lambda^{(1+\alpha)/2}},$$

where the regularity¹⁷ of the bundle E^s is $C^{1+\alpha}$ for some $\alpha > 0$. By mixing, $e^{h_{top}(F)}$ is simple and is the only eigenvalue of \mathcal{L} of maximal modulus. Finally [3, Theorem 1.5] also implies that the dynamical determinant

$$d_{F,g^s}(z) = \exp - \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{x \in \text{Fix } F^n} \frac{(g^s)^{(n)}(x)}{|\det(\text{Id} - DF^{-n}(x))|},$$

where $g^s = (|\text{Det } DF|_{E^s})^{-1}$, is holomorphic in the disc of radius $1/\rho_{ess,\alpha}$ where its zeroes are the inverses of the eigenvalues of \mathcal{L} .

We will show that

$$\text{spec}(\mathcal{L}|_{\mathcal{B}}) \cap \{\max(\rho_{ess,\alpha}, 1) < |\rho| \leq e^{h_{top}(F)}\} = \{e^{h_{top}(F)}\},$$

and if, in addition $\rho_{ess,\alpha} < 1$ (since α is generically smaller than 1, this is a very strong assumption, even for large r), then $\mathcal{L} : \mathcal{B} \rightarrow \mathcal{B}$ does not have eigenvalues outside the open unit disc except for $e^{h_{top}}$. The starting point is the fact that the Artin–Mazur zeta function of F coincides with the zeta function (2.1) of its linear model:

$$(3.3) \quad \zeta_F(z) = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} \left(\sum_{x \in \text{Fix } F^n} 1 \right) = \zeta_A(z).$$

Next, we use that for all n and any $x \in \text{Fix } F^n$,

$$\begin{aligned} 1 &= \frac{|\text{Det}(\text{Id} - DF^{-n}(x))|}{|\text{Det}(\text{Id} - DF^{-n}(x))|} = \frac{|1 + \text{Det } DF^{-n}(x) - \text{tr}(DF^{-n}(x))|}{|\text{Det}(\text{Id} - DF^{-n}(x))|} \\ &= \frac{(\sigma_A)^n (-1 - \text{Det } DF^{-n}(x) + \text{tr}(DF^{-n}(x)))}{|\text{Det}(\text{Id} - DF^{-n}(x))|}. \end{aligned}$$

Since for any $x \in \text{Fix } F^n$ we have (using orientation preserving on E^s)

$$\text{tr}(DF^{-n}(x)) = (|\text{Det } DF^n|_{E^s}(x))^{-1} + (\text{Det } DF^n|_{E^u}(x))^{-1},$$

setting $w = \sigma_A \cdot z$, it follows that

$$(3.4) \quad \zeta_A(z) = \frac{d_{F,1}(w) \cdot d_{F,1/\text{Det } DF}(w)}{d_{F,g^u}(w) \cdot d_{F,g^s}(w)}$$

where $g^u = (\text{Det } DF|_{E^u})^{-1}$. Let $\beta > 0$ be¹⁸ such that $E^u(F)$ (and thus g^u) is $C^{1+\beta}$ and assume that $\epsilon < \beta < 2$. Then there exists [3, Theorem 1.1] a Banach space $\mathcal{B}_\beta = C^{(\beta-\epsilon)/2, -(\beta-\epsilon)/2}(F, \mathbb{T}^2)$ such that the transfer operator $\mathcal{L}_{F,g^u}(\varphi) = (g^u\varphi) \circ F^{-1}$ acting on \mathcal{B}_β has spectral radius < 1 (because $|g^u| <_{P_{er}} |\det DF|^{-1}$) and essential spectral radius strictly smaller than its spectral radius, and, in addition, $d_{F,g^u}(w)$ is holomorphic¹⁹ in a disc of radius > 1 .

We will see below that the two determinants in the numerator of (3.4) are holomorphic in the disc of radius $\rho_{SRB,ess}^{-1} > \max\{\rho_{ess,\alpha}^{-1}, 1\}$. Since $d_{F,g^s}(w)$ is the dynamical determinant of \mathcal{L} which is holomorphic in the disc of radius $1/\rho_{ess,\alpha}$,

¹⁷Of course, $\alpha = \infty$ in the linear case $F = A$, but Anosov proved that, generically, $\alpha < 1$.

¹⁸In fact, $\beta = \alpha$ in the present setting.

¹⁹We also have that $d_{F,g^u}(w)$ cannot vanish inside a disc of radius > 1 . Since this determinant appears in the denominator, its zeroes do not matter to us.

the domains²⁰ of holomorphy (and of spectral interpretation of zeroes) of the four determinants in the right hand side of (3.4) include the disc of radius $1/\rho_{ess,\alpha}$.

Applying [3, Theorem 1.1] again, the transfer operator $\mathcal{L}_{F,SRB}(\varphi) = \frac{\varphi \circ F^{-1}}{|\text{Det } DF| \circ F^{-1}}$ acting on \mathcal{B} has essential spectral radius bounded by $\rho_{SRB,ess} < \rho_{ess,\alpha}$ (we have $\rho_{SRB,ess} \leq \lambda^{-(r-1)/2}$) and spectral radius one, with a simple eigenvalue at 1 as only eigenvalue on the unit circle. Therefore, if $\sigma_A = +1$ then $d_{F,1/|\text{Det } DF}(w) = d_{F,1/|\text{Det } DF}(z)$ is holomorphic in the disc of radius $\rho_{SRB,ess}^{-1}$, with $d_{F,1/|\text{Det } DF}(1) = 0$, and this determinant has no other zeroes in the closed unit disc. If $\sigma_A = -1$ then $d_{F,1/|\text{Det } DF}(w) = d_{F,1/|\text{Det } DF}(-z)$ is holomorphic in the disc of radius $\rho_{SRB,ess}^{-1}$, with $d_{F,1/|\text{Det } DF}(-1) = 0$, and this determinant has no other zeroes in the closed unit disc.

Finally, we claim that $d_{F,1}(w)$ is holomorphic in the disc of radius $\rho_{SRB^*,ess}^{-1} > 1$, with $d_{F,1}(1) = 0$, where 1 is simple zero and the only zero of $d_{F,1}(w)$ in the closed unit disc: Indeed, $d_{F,1}(w)$ can be viewed as the determinant of the operator

$$\mathcal{L}_{F^{-1},SRB}\varphi = |\text{Det } DF| \cdot (\varphi \circ F)$$

acting on the Banach space $\hat{\mathcal{B}} = C^{(t,s)}(F^{-1})(F^{-1}, \mathbb{T}^2)$, for suitable $t - (r-1) < s < 0$, associated to F^{-1} , on which its essential spectral radius is bounded by $\rho_{SRB^*,ess} < \tilde{\rho}_{ess,\alpha}$. The spectral radius of $\mathcal{L}_{F^{-1},SRB}$ is equal to 1, and, since F^{-1} is mixing, the eigenvalue 1 is simple and it is the only eigenvalue of modulus equal to one. (The respective simple zeroes of $d_{F,1}(z)$ and $d_{F,1/|\text{Det } DF}$ at $z = 1$ account for the double zero of $\zeta_F(z)$ at $z = 1$ when $\sigma_A = +1$. Otherwise, the simple zeroes at $z = \pm 1$ account for the zeroes of $(1 - z^2)$.)

Assume by contradiction that \mathcal{L} has an eigenvalue $\rho \neq e^{h_{top}(F)}$ with $|\rho| \geq 1$, then $d_{F,g}(1/\rho) = 0$. This would imply that $d_{F,1}(1/\rho) = 0$ or $d_{F,1/|\text{Det } DF}(1/\rho) = 0$ (to get a cancellation), and both claims are impossible.

We now prove the theorem: The only differences are that we will need 3×3 matrices instead of 2×2 matrices, and six determinants instead of four. Also we only need that E^u and E^s are C^γ for some $\gamma > 0$. To fix ideas, assume first that $\tilde{\rho}_{BT} < 1$ so that $\min\{1, \tilde{\rho}_{BT}^{-1}\} = 1$. First

$$\zeta_{\tilde{F}}(z) = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} \left(\sum_{x \in \text{Fix } \tilde{F}^n} 1 \right) = \zeta_A(z).$$

(Recall (3.3) and (2.1).) Second, at any $\tilde{x} = (x, v) = (x, v_{1,+}^s(x)) \in \text{Fix } \tilde{F}^n$, we have

$$D\tilde{F}_{\tilde{x}}^n = \begin{pmatrix} DF_x^n & a \\ 0 & \tilde{\lambda}^n(x) \end{pmatrix} \quad \text{with } \tilde{\lambda}^n(x) = \frac{\text{Det } DF^n|_{E^u(x)}}{|\text{Det } DF^n|_{E^s(x)}},$$

where $|\tilde{\lambda}^n(x)| \geq \frac{\lambda_u^n}{\lambda_s^n}$. Therefore, we may use for such \tilde{x} the decomposition

$$\begin{aligned} 1 &= \frac{|\text{Det}(\text{Id} - D\tilde{F}^{-n}(\tilde{x}))|}{|\text{Det}(\text{Id} - \tilde{D}F^{-n}(\tilde{x}))|} \\ &= \frac{(|1 + \text{Det } DF^{-n}(x) - \text{tr}(DF^{-n}(x))|) \left(1 - \frac{|\text{Det } DF^n|_{E^s(x)}}{|\text{Det } DF^n|_{E^u(x)}}\right)}{|\text{Det}(\text{Id} - D\tilde{F}^{-n}(\tilde{x}))|}. \end{aligned}$$

²⁰Note that $d_{F,g^u}(z)$ is holomorphic in a disc of radius > 1 with no zeroes in the closed unit disc even if $\rho_{ess,\alpha} > 1$. The argument for $\tilde{\mathcal{L}}$ will exploit a similar feature.

Third, since for all n and all $x \in \text{Fix } F^n$,

$$\begin{aligned} & (|1 + \text{Det } DF^{-n}(x) - \text{tr}(DF^{-n}(x))| \\ &= (\sigma_A)^n (-1 - \text{Det } DF^{-n}(x) + (|\text{Det } DF^n|_{E^s(x)})^{-1} + (\text{Det } DF^n|_{E^u(x)})^{-1}), \end{aligned}$$

setting $w = \sigma_A \cdot z$, it follows that

$$\begin{aligned} \zeta_A(z) &= \frac{d_{\tilde{F},1}(w) \cdot d_{\tilde{F},1/\text{Det } DF}(w) \cdot d_{\tilde{F},\tilde{g}^u}(w) \cdot d_{\tilde{F},(\tilde{g}^u)^2/g^s}(w)}{d_{\tilde{F},g^s}(w) \cdot d_{\tilde{F},\tilde{g}^u}(w) \cdot d_{\tilde{F},\tilde{g}^u/\tilde{g}}(w) \cdot d_{\tilde{F},(\tilde{g}^u)^2}(w)} \\ (3.5) \quad &= \frac{d_{\tilde{F},1}(w) \cdot d_{\tilde{F},1,\text{Det } DF}(w) \cdot d_{\tilde{F},(\tilde{g}^u)^2/\tilde{g}}(w)}{d_{\tilde{F},\tilde{g}}(w) \cdot d_{\tilde{F},\tilde{g}^u/\tilde{g}}(w) \cdot d_{\tilde{F},(\tilde{g}^u)^2}(w)}, \end{aligned}$$

where \tilde{g} was defined²¹ in (2.2) and $\tilde{g}^u(x, v) := g^u(x)$ is C^γ .

As mentioned above, $d_{\tilde{F},\tilde{g}}(z)$ is the dynamical determinant of $\tilde{\mathcal{L}}$ acting on $\tilde{\mathcal{B}}$, which is holomorphic in the disc of radius $1/\tilde{\rho}_{BT}$.

It is easy to see that $d_{\tilde{F},1}(w)$ is the dynamical determinant of the transfer operator associated to the SRB measure of the mixing attractor \tilde{F}^{-1} acting on the Banach space $\tilde{\mathcal{B}}_{\tilde{F}^{-1}} = \mathcal{C}^{t,s}(\tilde{F}^{-1}, \mathcal{E})$ for suitable $t - (r-1) < s < 0$, associated to \tilde{F}^{-1} , so, by [3, Theorem 1.5 and §2], this factor is holomorphic in a disc of radius $\tilde{\rho}_{SRB^*,ess}^{-1} > \tilde{\rho}_{BT}^{-1} > 1$, and its only zero in the closed unit disc is a simple zero at $w = 1$. (If $\sigma_A = -1$ then this gives a zero at $z = -1$.)

Next, $d_{\tilde{F},1/|\text{Det } DF|}(w)$ is the dynamical determinant of the transfer operator \mathcal{M} of the mixing repeller \tilde{F} weighted by $1/|\text{Det } DF|$, acting on the Banach space $\tilde{\mathcal{B}}$. By the Pesin entropy formula the pressure²² of $-\log |\text{Det } DF|_{E^u}$ for the attractor F is equal to zero. Since this pressure coincides with the pressure of $-\log |\text{Det } DF|_{E^u}$ for the repeller \tilde{F} , applying [3, Theorem 1.5] the determinant $d_{\tilde{F},1/|\text{Det } DF|}(w)$ is holomorphic in a disc of radius $\tilde{\rho}_{SRB,ess}^{-1} > \tilde{\rho}_{BT}^{-1} > 1$ and its only zero in the closed unit disc is a simple zero at $w = 1$. (The fixed point of \mathcal{M} is the SRB measure of F in x multiplied by a Dirac mass at $v_{x,+}^1$.) If $\sigma_A = -1$, replacing \mathcal{M} by $-\mathcal{M}$, we get that the determinant $d_{\tilde{F},1/\text{Det } DF}(w)$ is holomorphic in a disc of radius $\tilde{\rho}_{SRB,ess}^{-1} > \tilde{\rho}_{BT}^{-1} > 1$ and its only zero in the closed unit disc is a simple zero at $w = -1$, giving a zero at $z = 1$.

The last determinant in the numerator of (3.5) is $d_{\tilde{F},(\tilde{g}^u)^2/\tilde{g}}(w)$, which is associated to a transfer operator \mathcal{N} of spectral radius < 1 on the Banach space $\tilde{\mathcal{B}}_\gamma = C^{\gamma/2, -\gamma/2}(\tilde{F}, \mathcal{E})$ using $(\tilde{g}^u)^2/\tilde{g}(z) <_{Per} |\det DF|^{-1}$ (the essential spectral radius of \mathcal{N} is < 1 , but we do not claim that this essential spectral radius is smaller than $\tilde{\rho}_{BT}$). Therefore, $d_{\tilde{F},(\tilde{g}^u)^2/\tilde{g}}(w)$ cannot vanish on the closed unit disc.

Finally, the spectral radii of the two remaining operators in the denominator of (3.5) acting on $\tilde{\mathcal{B}}_\gamma$ are smaller than 1 since

$$|\tilde{g}^u|/\tilde{g} <_{Per} |\det DF|^{-1}, \quad (\tilde{g}^u)^2 <_{Per} |\det DF|^{-1}.$$

So the right-hand-side of (3.5) is a quotient of two holomorphic functions in a disc of radius > 1 , such that the only zeroes of the numerator in the closed unit disc are a double zero at $z = 1$ when $\sigma_A = 1$, and simple zeroes at $z = \pm 1$ otherwise.

²¹Note that $\tilde{g}(x, v_{1,+}^s(x)) = g^s(x)$.

²²The pressure of $-\log |\text{Det } D\tilde{F}|_{E^u}$ for the repeller \tilde{F} is strictly smaller than zero.

It follows that the only possible zero of $d_{\tilde{F},\tilde{g}}(z)$ in the closed unit disc is a simple zero at $\lambda_A^{-1} = e^{-h_{top}(F)} < 1$. This ends the proof of the theorem if $\tilde{\rho}_{BT} < 1$.

If $\tilde{\rho}_{BT} \geq 1$, the same argument works, replacing the unit disc by the disc of radius $\tilde{\rho}_{BT}$ for spectral claims and by the disc of radius $\tilde{\rho}_{BT}^{-1}$ for determinants. \square

Remark 3.1 (Higher dimension). Consider an Anosov diffeomorphism F with $d_s = 1$ and $d_u = 2$. Then, assuming that DF preserves the orientation of E^s , and that the eigenvalues of the hyperbolic linear matrix A conjugated to F satisfy $\lambda_s < 1 < \lambda_{u,min} < \lambda_{u,max}$, the unweighted Artin–Mazur zeta function of F is (note that $\lambda_s \lambda_{u,min} \lambda_{u,max} = 1$, and, since d is odd, $+1$ and $(-1)^d \det(A^{-n})$ cancel for all n)

$$\zeta_A(z) = \frac{(1 - \lambda_s z)(1 - \lambda_{u,min} z)(1 - \lambda_{u,max} z)}{(1 - z/\lambda_s)(1 - z/\lambda_{u,min})(1 - z/\lambda_{u,max})}.$$

Factoring the zeta function as a product of dynamical determinants for an extended dynamics \tilde{F} as in the proof of Theorem 2.1, one cannot exclude a priori eigenvalues of modulus ≥ 1 for the operator (in the denominator) weighted by $1/|\det DF|_{E^s}|$, since the corresponding zeroes could be cancelled by the determinants of the transfer operators in the numerator weighted by $\det DF|_{E_{max}^u}$ or even $\det DF|_{E_{min}^u}$, which also have spectral radius strictly larger than 1.

APPENDIX A. PROPERTIES OF GIULIETTI–LIVERANI FLOWS

Let h^t be a (not necessarily unit speed) Giulietti–Liverani flow of a C^r Anosov diffeomorphism F . If $r > 2$, since the stable bundle of F is $C^{1+\alpha}$ for some $\alpha > 0$ (see e.g. [11] and references therein), the map $x \mapsto h^t(x)$ is $C^{1+\alpha}$. Clearly, h^t cannot have fixed points or periodic orbits. It is well known that a periodic orbit-free flow on \mathbb{T}^2 has a global transversal, with the corresponding Poincaré map topologically conjugated to an irrational rigid rotation R_ω , and that the flow is uniquely ergodic (and thus minimal, using the conjugacy with the linear model). By [7, Lemma 1.1], the rotation number ω of the rotation thus associated to a Giulietti–Liverani flow satisfies²³

$$(A.1) \quad b\omega^2 + (a - d)\omega - c = 0,$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{Z})$ is the hyperbolic linear toral automorphism topologically conjugated to F . In particular, $b \neq 0$, and the continued fraction of ω is periodic and thus of constant type. (The orientation preserving assumption also implies that the contracting eigenvalue of A is positive.)

Finally, by [7, Lemma 1.1], if $r \geq 3$, any C^r periodic orbit-free flow h_0^t on \mathbb{T}^2 whose Poincaré map P_0 has rotation number ω satisfying (A.1) for some hyperbolic matrix $A \in \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{Z})$, with positive contracting eigenvalue, is the (not necessarily unit speed) Giulietti–Liverani flow of an Anosov diffeomorphism F_0 , which is $C^{r'}$ for all $r' < r - 1$: Indeed, the C^r circle diffeomorphism P_0 has rotation number ω of constant type and is thus conjugated with the rigid rotation R_ω via a $C^{r'}$ circle diffeomorphism ϕ [12, Théorème fondamental, p. 9], for any $r' < r - 1$. Let h_1^t be the unit speed C^r reparametrisation of h_0^t . Then the (unit speed) linear flow h_ω^t over R_ω is the Giulietti–Liverani flow of the hyperbolic automorphism A .

²³As pointed out by the Editor, in the first statement of [7, Lemma 1.1] the sentence “it is topologically conjugated to a rigid rotation with rotation number ω such that...” should be replaced by “it is topologically orbit equivalent with a flow whose Poincaré map on a global transversal has rotation number ω such that...”

In addition, the flow h_ω^t is conjugated with h_1^t via a $C^{r'}$ toral diffeomorphism Φ which coincides with ϕ on the transversal and maps the stable lines of h_ω^t to the orbits of h_i^t , $i = 0, 1$. Finally, h_1^t is the (unit speed) Giulietti–Liverani flow of the $C^{r'}$ Anosov diffeomorphism $F_0 = \Phi \circ A \circ \Phi^{-1}$. (Use that the stable manifolds of F_0 coincide with the orbits of h_i^t , $i = 0, 1$.)

REFERENCES

- [1] A. Adam, Horocycle averages on closed manifolds and transfer operators, arXiv:1809.04062 (2018)
- [2] V. Baladi *Dynamical Zeta Functions and Dynamical Determinants for Hyperbolic Maps*, Ergebnisse der Mathematik und ihrer Grenzgebiete, **68** Springer, Cham (2018)
- [3] V. Baladi and M. Tsujii, *Dynamical determinants and spectrum for hyperbolic diffeomorphisms*, in Geometric and Probabilistic Structures in Dynamics, 29–68, Contemp. Math. **469**, Amer. Math. Soc. (2008)
- [4] J. Carrand, *Logarithmic bounds for ergodic averages of constant type rotation number flows on the torus: a short proof*, arXiv:2012.07481 (2020)
- [5] L. Flaminio and G. Forni, *Invariant distributions and time averages for horocycle flows*, Duke Math. J. **119** 465–526 (2003)
- [6] G. Forni, *On the equidistribution of unstable curves for pseudo-Anosov diffeomorphisms of compact surfaces*, arXiv:2007.03144
- [7] P. Giulietti and C. Liverani, *Parabolic dynamics and anisotropic Banach spaces*, JEMS **21** 2793–2858 (2019)
- [8] S. Gouëzel and C. Liverani, *Banach spaces adapted to Anosov systems*, Ergodic Theory Dynam. Systems **26** 189–217 (2006)
- [9] S. Gouëzel and C. Liverani, *Compact locally maximal hyperbolic sets for smooth maps: fine statistical properties*, J. Diff. Geom. **79** (2008)
- [10] A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Encyclopedia Math. and Applications **54** Cambridge University Press (1995)
- [11] B. Hasselblatt, *Regularity of the Anosov splitting and of horospheric foliations*, Ergodic Theory Dynam. Systems **14** 645–666 (1994)
- [12] M.-R. Herman, *Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations*, Inst. Hautes Études Sci. Publ. Math. **49** (1979) 5–233.
- [13] K. Hiraide, *A simple proof of the Franks–Newhouse theorem on codimension-one Anosov diffeomorphisms*, Ergod. Th. Dynam. Sys. **21** 801–806 (2001)
- [14] R. McCutcheon, *The Gottschalk–Hedlund theorem*, Amer. Math. Monthly **106** 670–672 (1999)

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