Infinite dimensional semiclassical analysis and applications to a model in nuclear magnetic resonance

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Abstract

We are concerned in this paper with the connection between the dynamics of a model related to Nuclear Magnetic Resonance (NMR) in Quantum Field Theory (QFT) and its classical counterpart known as the Maxwell-Bloch equations. The model in QFT is a model of Quantum Electrodynamics (QED) considering fixed spins interacting with the quantized electromagnetic field in an external constant magnetic field. This model is close to the common spin-boson model. The classical model goes back to F. Bloch, Physical Review, vol. 70, 460 (1946). Our goal is not only to study the derivation of the Maxwell-Bloch equations but to also establish a semiclassical asymptotic expansion of arbitrary high order with control of the error terms of this standard nonlinear classical motion equations. This provides therefore quantum corrections of any order in powers of the semiclassical parameter of the Bloch equations. Besides, the asymptotic expansion for the photon number is also analyzed and a law describing the photon number time evolution is written down involving the radiation field polarization. Since the quantum photon state Hilbert space (radiation field) is infinite dimensional we are thus concerned in this article with the issue of semiclassical calculus in an infinite dimensional setting. In this regard, we are studying standard notions as Wick and anti-Wick quantizations, heat operator, Beals characterization theorem and compositions of symbols in the infinite dimensional context which can have their own interest.

Keywords: Semiclassical analysis, infinite dimensional analysis, composition of operators, Wick quantization, anti-Wick quantization, Wick symbol, Husimi function, Wiener spaces, Heat operator, symbolic calculus, QED, quantum electrodynamics, Maxwell-Bloch equations, Bloch equations, NMR, Nuclear Magnetic Resonance, photon emission, photon number.

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1 Statement of the results.

The aim of this work is to carry on the study of an infinite dimensional symbolic calculus begun in [4] and to apply it to the semiclassical limit of the evolution for a quantum field model in Nuclear Magnetic Resonance (NMR).

This model is introduced in Section 4.11 in Reuse [50], (see also [20] and [39]). It is devoted to the interaction between a finite number of fixed (heavy) spin-$\frac{1}{2}$ particles, for instance atomic nuclei, with the quantized electromagnetic field together with a constant external magnetic field. This interaction model is closely related to the spin-boson model (for example, see [28], [29], [36], [11], [54] , [24]). The Hamiltonian of the system studied here is also similar to the more complicated Pauli Fierz Hamiltonian (see [23], [12]), where the terms concerning the spin particles motion are deleted.

NMR can also be modelled by earlier equations due to F.Bloch [15] (1946). We prove here that they are the semiclassical limit of the QED model. In this early model, spins are viewed as vectors $\mathbf{S}^{\lambda}(t) \in \mathbb{R}^3$, $\lambda = 1, \ldots, P$.

If the particle λ is fixed at the point x_λ of $\mathbb{R}^3$, time evolutions of these spin vectors follow the Bloch equations:

$$\frac{d}{dt}\mathbf{S}^\lambda(t) = 2\mathbf{B}^{ext}(x_\lambda, t) \times \mathbf{S}^\lambda(t)$$

where $\mathbf{B}^{ext}(x,t)$ is a non quantized external magnetic field. One handicap of this model is the long time behavior of the solutions. Indeed, the behavior of the solutions is not always physically consistent for large time t. To overcome that difficulty, one usually inserts ad hoc additional terms in the Bloch equations [15], which are no more useful in the QED model.

In the QED model, the spins are average values of evolving quantum observables. We aim at studying the semiclassical asymptotic of the average at time $t > 0$, of some observables related to the QED model. Indeed the Hamiltonian depends on the semiclassical parameter $h > 0$. In particular, concerning the spin observables, we prove that the Bloch model is a semiclassical limit of the QED model (see (6.11)(6.12) and also (6.17)(6.18)). This can therefore be viewed as the derivation of the Bloch equations. Note that a full asymptotic expansion with control of the error terms is in addition obtained. We point out that [39] adresses a similar issue (see also [20, 52]).

In the same way, we study the average values at time $t > 0$ and at each point of $\mathbb{R}^3$ of the electric and magnetic fields. This evolution is consistent with the Maxwell equations, provided each particle with spin has a current density explicitly expressed in terms of the averages of the spins (at the same t) and with the help of an ultraviolet cutoff, as in formulas (1.22), (1.23).

We finally give an asymptotic estimate of the average number of photons created in a time unit. Note that time dynamic evolutions for some other models in infinite dimension have been studied for example in [18, 22, 25, 44, 45].

The QED model. The Hilbert space describing the states of the whole system, consisting of the quantum field and of the P particles, in the presence of a constant magnetic field β, is the completed tensor product $\mathcal{H}_{ph} \otimes \mathcal{H}_{sp}$ of two Hilbert spaces that we shall define now.

The Hilbert space $\mathcal{H}_{sp}$ describing the states of P fixed particles with spin-$\frac{1}{2}$, without interaction, at a given time, is $\mathcal{H}_{sp} = (\mathbb{C}^2)^{\otimes P}$. The fermionic property of the spin-$\frac{1}{2}$ fixed particles is not taken into account here. In the space $\mathcal{H}_{sp}$, we shall use the operators related to the spins of the different particles. Let σ_j $(1 \leq j \leq 3)$ be the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.1}$$

For all $\lambda \leq P$ and all $m \leq 3$, we denote by $\sigma_m^{[\lambda]}$ the operator in $\mathcal{H}_{sp}$ defined by:

$$\sigma_m^{[\lambda]} = I \otimes \cdots \otimes I \otimes \sigma_m \otimes I \otimes \cdots \otimes I, \tag{1.2}$$

where σ_m is located at the λ^{th} position.

The one photon configuration Hilbert space H is the set of mappings $f \in L^2(\mathbb{R}^3, \mathbb{R}^3)$ satisfying $k \cdot f(k) = 0$ almost everywhere in $k \in \mathbb{R}^3$ (see [46]) where $|f|^2 = \int_{\mathbb{R}^3} |f(k)|^2 dk$. The Hilbert space $\mathcal{H}_{ph}$ of photon quantum states is the symmetrized Fock space $\mathcal{F}_s(H_{\mathbf{C}})$ over the complexified space of H. We follow [49], see also [53], for Fock spaces considerations and notations, in particular, for the usual operators in these spaces: the Segal field $\Phi_S(V)$ associated with an element V in H^2, the $\Gamma(T)$ and $\mathrm{d}\Gamma(T)$ operators associated with some operators T acting in H^2. Note that, throughout this paper, the space H^2 is sometimes identified with the complexified space $H_{\mathbf{C}}$, but not when a confusion seems possible. We denote in the same way the analogous operators defined

on H^2 or on $H_{\mathbf{C}}$. With the aim of underlining the role of the semiclassical parameter $h > 0$, we also set for all V belonging to H^2:

$$\Phi_{Sh}(V) = h^{\frac{1}{2}}\Phi_S(V). \tag{1.3}$$

Let M_ω be the operator with domain $D(M_\omega) \subset H$ such that $M_\omega q(k) = |k|q(k)$ almost everywhere in $k \in \mathbb{R}^3$. In the Fock space framework, the photon free energy Hamiltonian operator H_{ph} is usually defined as $h\mathrm{d}\Gamma(M_\omega)$. The photon number operator denoted by N is $N = \mathrm{d}\Gamma(I)$.

The three components of the magnetic field at each point x in $\mathbb{R}^3$ are defined using the elements B_{jx} belonging to H^2 and written as follows, when one identifies H^2 with the complexified space $H_{\mathbf{C}}$:

$$B_{jx}(k) = \frac{i\chi(|k|)|k|^{\frac{1}{2}}}{(2\pi)^{\frac{3}{2}}}e^{-i(k\cdot x)}\frac{k \times e_j}{|k|}, \quad k \in \mathbb{R}^3\backslash\{0\} \tag{1.4}$$

where the function χ (ultraviolet cutoff) belongs to $\mathcal{S}(\mathbb{R})$ and (e_1, e_2, e_3) is the canonical basis of $\mathbb{R}^3$. Also set,

$$E_{mx} = JB_{mx}, \quad 1 \leq m \leq 3 \tag{1.5}$$

where $J : H^2 \to H^2$ stands for the helicity operator defined by,

$$JX(k) = \frac{k \times X(k)}{|k|}, \quad k \in \mathbb{R}^3 \setminus \{0\}, \tag{1.6}$$

for each X in H^2 or $H_{\mathbf{C}}$. One then defines the electric and magnetic fields components operators at each point x of $\mathbb{R}^3$ by:

$$B_m(x) = \Phi_{Sh}(B_{mx}) = h^{\frac{1}{2}}\Phi_S(B_{mx}), \quad E_m(x) = \Phi_{Sh}(E_{mx}) = h^{\frac{1}{2}}\Phi_S(E_{mx}),$$

for $m = 1, 2, 3$.

The Hamiltonian of the QED model is a selfadjoint extension of the following operator, initially defined in a dense subspace of $\mathcal{H}_{ph} \otimes \mathcal{H}_{sp}$,

$$H(h) = H_{ph} \otimes I + hH_{int}, \tag{1.7}$$

where $H_{ph} = h\mathrm{d}\Gamma(M_\omega)$ is the photon free energy operator, acting in a domain $D(H_{ph}) \subset \mathcal{H}_{ph}$ and

$$H_{int} = \sum_{\lambda=1}^{P}\sum_{m=1}^{3}(\beta_m + B_m(x_\lambda)) \otimes \sigma_m^{[\lambda]}, \tag{1.8}$$

where $\beta = (\beta_1, \beta_2, \beta_3)$ is the external constant magnetic field and the x_λ $(1 \leq \lambda \leq P)$ are the points of $\mathbb{R}^3$ where the fixed particles are located.

Let us recall that $H(h)$ has a selfadjoint extension with the same domain as the free operator $H_0 = H_{ph} \otimes I$. If an element U of H^2 lies in the domain $D(M_\omega^{-1/2})$ then the Segal field $\Phi_S(U)$ is bounded from $D(H_{ph})$ into $\mathcal{H}_{ph}$, see point ii) of Proposition 3.4 in [7] or see [19]. This is therefore the case for the operators $B_j(x)$ and $E_j(x)$ according to the assumptions on the ultraviolet cutoff function χ in (1.4). The statement above follows thus from the Kato-Rellich Theorem.

Let A be a selfadjoint operator in $\mathcal{H}_{ph} \otimes \mathcal{H}_{sp}$, bounded or not. We are led to investigate the asymptotic properties of

$$A(t, h) = e^{i\frac{t}{h}H(h)}Ae^{-i\frac{t}{h}H(h)}. \tag{1.9}$$

If A is bounded, so is $A(t, h)$. If A is bounded from $D(H(h))$ in $\mathcal{H}_{ph} \otimes \mathcal{H}_{sp}$, so is $A(t, h)$.

In a finite dimensional setting one way of studying the asymptotic properties of an operator analogous to $A(t,h)$, in the case when A is a pseudodifferential operator, defined, for example, by the Weyl calculus, would be the following. One proves that, under suitable hypotheses, the analogue of $A(t,h)$ is also a pseudodifferential operator, and gives an asymptotic expansion of its symbol. This is the classical finite dimensional Egorov Theorem. We do not follow this way here but use the Wick symbol instead. The Wick calculus is another very standard way to establish a link between the quantum dynamics and the corresponding classical dynamics.

For any bounded operator A, one may associate with A and also with $A(t,h)$ their Wick symbols. This is a function defined on the phase space, with values in $\mathcal{L}(\mathcal{H}_{sp})$, which reflects the semiclassical properties of these operators when h tends to 0. In our situation, the phase space is H^2 where H is the one photon configuration Hilbert space defined above. The definition of the Wick symbol is recalled below.

Definition 1.1. *Let H be any arbitrary infinite dimensional separable Hilbert space. The coherent states $\Psi_{X,h}$ are the elements of the Fock space $\mathcal{F}_s(H_{\mathbf{C}})$ defined, for any $X=(a,b)$ in H^2 and every $h>0$, by*

$$\Psi_{Xh} = e^{-\frac{|X|^2}{4h}} \sum_{n\geq 0} \frac{(a+ib)\otimes\cdots\otimes(a+ib)}{(2h)^{n/2}\sqrt{n!}} \tag{1.10}$$

(with n tensor products in the above sum). When $X=0$, $\Psi_{0,h}$ denotes the (quantum) vacuum state.

If A denotes any bounded operator in $\mathcal{F}_s(H_{\mathbf{C}})$ or any unbounded operator $(A, D(A))$ with a domain $D(A)$ containing the coherent states, then the Wick symbol of A is the function $\sigma_h^{wick}(A)$, defined on H^2 by

$$\sigma_h^{wick}(A)(X) =< A\Psi_{X,h}, \Psi_{X,h} >, \tag{1.11}$$

where, for all $X=(a,b)$ in H^2 and every $h>0$, $\Psi_{X,h}$ is the corresponding coherent state recalled above. It is also called Husimi function.

If A is a similar operator in $\mathcal{F}_s(H_{\mathbf{C}})\otimes\mathcal{H}_{sp}$ where $\mathcal{H}_{sp}$ is a finite dimensional Hilbert space, then the Wick symbol of A is the mapping $\sigma_h^{wick}(A)$ defined from H^2 into $\mathcal{L}(\mathcal{H}_{sp})$ verifying

$$< \sigma_h^{wick}(A)(X)a, b >=< A(\Psi_{X,h}\otimes a), \Psi_{X,h}\otimes b >, \tag{1.12}$$

for all a and b in $\mathcal{H}_{sp}$.

Coherent states are a very general tool in quantum mechanics (see for example [26], [16], [47]). For Wick symbols, see *e.g.*, [21], [17] and also [34] (for finitely or infinitely many degrees of freedom), [3], [5], ...

The problem is now that of deducing the properties of the Wick symbol of $A(t,h)$ from the Wick symbol of A.

To this aim, one could use a description of the operator $e^{-i\frac{t}{h}H(h)}$, similar to the one given in [7]. But in [7], we supposed that the function χ (the ultraviolet cutoff) appearing in (1.4) is equal to 0 in a neighbourhood of the origin. This assumption is, physically, not very realistic. It was necessary because we did not use, in [7], the symbol class $S(H^2,Q)$ of Definition 1.2 below. Maybe it would be possible to rewrite the study of [7] using this symbol classes and so to suppress the unnecessary hypothesis. A method using commutators (Heisenberg type equations) is more straightforward.

For our physical applications, we do not need general operators A. It suffices to be able to take, for A, a spin observable (of the form $I\otimes\sigma_m^{[\lambda]}$), a field observable (of the form $\Phi_{Sh}(B_{mx})\otimes I$ or its analogue for the electric field) or the number operator. For all these observables A, we give an asymptotic expansion of the Wick symbol of $A(t,h)$ in powers of h. Note that the computation of the successive terms of the expansion needs only the

formal aspects of quantization (Mizrahi series), but the estimation of the error term needs the results of Section 2.

If A is a Segal field, its Wick symbol is well defined since coherent states belong to the number operator domain, which is itself included in the domain of any Segal field (see Lemma 2.1 in [19]). This also holds true for the operators $\mathrm{d}\Gamma(T)$ where $T \in \mathcal{L}(\mathcal{H})$. The Wick symbols of these operators are considered in Theorem 2.3. One denotes by X or (q,p) the variable of H^2 and by x,k variables of $\mathbb{R}^3$. The Wick symbol of an operator A will be here often denoted by $A(X,h)$ or $A(q,p,h)$.

For our purpose, the initial observables A are assumed to have the particular following form:

$$A = \Phi_S(F_A) \otimes I + I \otimes S_A \tag{1.13}$$

where S_A belongs to $\mathcal{L}(\mathcal{H}_{sp})$ and $\Phi_S(F_A)$ is the Segal field associated with an element F_A in the domain $D(M_\omega^{-1/2}) \subset H^2$. In this case, according to the above remarks, the product of (1.9) is well defined as an operator from $D(H(h))$ into $\mathcal{H}_{ph} \otimes \mathcal{H}_{sp}$. According to Proposition 5.2, if A has the form (1.13) then the operator $A(t,h)$ is the sum of a Segal field with a bounded operator. Since both of them have a Wick symbol then the Wick symbol $A(t,X,h)$ taking values in $\mathcal{L}(\mathcal{H}_{sp})$ is well defined.

For observables being as in (1.13) type, we shall for the function $A(t,X,h)$, on the one hand study bounds on the derivatives and on the other hand give an asymptotic expansion as h goes to 0 with the aim of obtaining quantum corrections for the Bloch model. In order to obtain bounds on the derivatives, we define a class of functions $F \in C^\infty$ on the phase space H^2 associated with a nonnegative quadratic form Q on H^2 (which can be degenerate) in the following way.

Definition 1.2. *For any real separable Hilbert space H and for each nonnegative quadratic form Q on H^2, let $S(H^2,Q)$ be the class of functions $f \in C^\infty(H^2)$ such that there exists $C(f) > 0$ satisfying, for any integer $m \geq 0$, for all vectors V_1 ... V_m in H^2:*

$$|(d^m f)(x)(V_1,\ldots,V_m)| \leq C(f) Q(V_1)^{1/2} \ldots Q(V_m)^{1/2}. \tag{1.14}$$

The smallest constant $C(f)$ satisfying (1.14) is denoted by $\|f\|_Q$.

In what follows, the quadratic form Q will be

$$Q(X) = (A_Q X) \cdot X, \tag{1.15}$$

with $A_Q \in \mathcal{L}(H^2)$, selfadjoint, nonnegative, trace class in H^2. Note that the idea of defining a class of symbols this way for quantization purposes, thanks to a quadratic form on the phase space, goes back to [35] and [56]. One however notes that, concerning these works, the constant C denoted by C_m involved there depends on m whereas it is independent of m here. Also observe that a function in $S(H^2,Q)$ extends as a holomorphic function on the complexified $(H_{\mathbf{C}})^2$, satisfying

$$|F(Z)| \leq \|F\|_Q \, e^{Q(\mathrm{Im}Z)^{1/2}}, \quad Z \in (H_{\mathbf{C}})^2.$$

For each time t, we shall show that the function $X \to A(t,X,h)$ belongs to the class of Definition 1.2 associated with the time dependent quadratic form Q_t on H^2 that we now define.

One knows that,

$$e^{-i\frac{t}{h}H_{ph}} = \Gamma(\chi_t) \tag{1.16}$$

where χ_t is the symplectic map defined, setting $\omega(k) = |k|$, by

$$\chi_t(q,p) = (q_t, p_t), \tag{1.17}$$

$$q_t(k) = \cos(t\omega(k))q(k) + \sin(t\omega(k))p(k), \qquad p_t(k) = -\sin(t\omega(k))q(k) + \cos(t\omega(k))p(k).$$

For each operator A, bounded from $D(H_{ph} \otimes I)$ to $\mathcal{H}_{ph} \otimes \mathcal{H}_{sp}$, we set:

$$A^{free}(t,h) = e^{i\frac{t}{h}(H_{ph}\otimes I)} A e^{-i\frac{t}{h}(H_{ph}\otimes I)}. \tag{1.18}$$

In particular, we define in this way $H_{int}^{free}(t)$. The proof of Theorem 4.3 shows that the Wick symbol $H_{int}^{free}(t,X)$ of the operator $H_{int}^{free}(t)$ is well defined and that:

$$H_{int}^{free}(t,X) = H_{int}(\chi_t(X)). \tag{1.19}$$

For all $t \in \mathbb{R}$, one defines a quadratic form Q_t on H^2 by:

$$Q_t(q,p) = |t| \int_0^t |dH_{int}^{free}(s,q,p)|^2 ds = |t| \int_0^t |dH_{int}(\chi_s(q,p))|^2 ds \tag{1.20}$$

where the norm in the integral is that of $\mathcal{L}(\mathcal{H}_{sp})$. For this equality to define a quadratic form, one chooses on $\mathcal{L}(\mathcal{H}_{sp})$ the Hilbert-Schmidt norm. One denotes by $dH_{int}^{free}(t,q,p)$ the differential with respect to (q,p) of H_{int}^{free} which is an affine function, that is to say the function obtained by replacing the constant field β by 0. Notice that the operator A_t satisfying $Q_t(X) = (A_t X) \cdot X$ for all X in H^2 (as in (1.15)) is trace class.

Now we can state our main result.

Theorem 1.3. *Let A be an observable of the form (1.13) with F_A in the domain $D(M_\omega^{-1/2})$. Let $A(t,h)$ be the operator defined in (1.9) and $A(t,X,h)$ be its Wick symbol. Then,*

i) The function $X \to A(t,X,h)$ is the sum of a linear function of the variable X with a function belonging to $S(H^2, 4Q_t)$ and taking values in $\mathcal{L}(\mathcal{H}_{sp})$.

More precisely, there exists a sequence of functions $A^{[j]}(t,\cdot)$, $j \geq 0$, taking values in $\mathcal{L}(\mathcal{H}_{sp})$ and satisfying the following properties:

ii) The function $A^{[0]}(t,\cdot)$ is the sum of a linear function with a function lying in $S(H^2, 4Q_t)$ and taking values in $\mathcal{L}(\mathcal{H}_{sp})$ with a norm bounded independently of t when t remains in a compact subset of $\mathbb{R}$.

iii) For $j \geq 1$, the function $A^{[j]}(t,\cdot)$ belongs to $S(H^2, 16^{j+1}Q_t)$ and takes values in $\mathcal{L}(\mathcal{H}_{sp})$ with a norm bounded independently of t when t remains in a compact subset of $\mathbb{R}$.

iv) The Wick symbol $X \to A(t,X,h)$ satisfies for any integer M:

$$A(t,X,h) = \sum_{j=0}^{M} h^j A^{[j]}(t,X) + h^{M+1} R^{[M]}(t,X,h) \tag{1.21}$$

where $R^{[M]}(t,\cdot,h)$ belongs to $S(H^2, 16^{M+5}Q_t)$ with a norm bounded independently of t and h when t remains in a compact subset of $\mathbb{R}$ and h varies in $(0,1]$.

The construction of the functions $A^{[j]}(t,\cdot)$ for $j \geq 0$ and $R^{[M]}(t,\cdot,h)$ for any integer M and $h > 0$ is given in Section 5.

We can be more precise about the expansion when the observable A of Theorem 1.3 is one of the Segal fields $B_m(x) = \Phi_{S,h}(B_{mx})$ or $E_m(x) = \Phi_{S,h}(E_{mx})$, or one of the operators $\sigma_m^{[\lambda]}$ defined in (1.2). These operators all have the form (1.13). We then denote by $B_m^{[j]}(x,t,X)$, $E_m^{[j]}(x,t,X)$ and $S_m^{[\lambda,j]}(t,X)$ $(1 \leq m \leq 3)$ the functions denoted by $A^{[j]}(X,t)$ in Theorem 1.3. The detailed construction of these functions is given in Theorem 6.2. Let us now give only the general idea.

The first terms $B_m^{[0]}(x,t,X)$ and $E_m^{[0]}(x,t,X)$ are functions of $X \in H^2$ and also of (x,t). As functions of (x,t), these functions satisfy the free Maxwell equations, the initial conditions being the fields corresponding to the initial coherent state. Then, the first terms $S_m^{[\lambda,0]}(t,X)$ satisfy the Bloch equations [15] (1946) but where the magnetic field is the sum of the constant external field with $\mathbf{B}^{[0]}$.

Next, the terms with $j \geq 1$ are determined by induction. Let us assume that $B_m^{[j-1]}(x,t,X)$ and $E_m^{[j-1]}(x,t,X)$ together with $S_m^{[\lambda,j-1]}(t,X)$ are already determined. Then, the functions $B_m^{[j]}(x,t,X)$ and $E_m^{[j]}(x,t,X)$ considered as functions of (x,t) satisfy Maxwell equations with entirely vanishing initial condition together with a zero charge density and a current density equal to

$$\mathbf{J}^{[j]}(x,t,X) = \sum_{\lambda=1}^{P} \mathbf{S}^{[\lambda,\mathbf{j-1}]}(t,X) \times \mathrm{grad}\rho(x - x_\lambda) \tag{1.22}$$

with

$$\rho(x) = (2\pi)^{-3} \int_{\mathbb{R}^3} |\chi(|k|)|^2 \cos(k \cdot x) dk, \tag{1.23}$$

where χ is the ultraviolet cutoff function appearing in Definition 1.4 of the magnetic field operators. This term expresses the radiation, between times 0 and t, of the spins in the $(j-1)$-th order of their movement. One finally determines $S_m^{[\lambda,j]}(t,X)$ solving the differential system (6.12). It depends on the one hand, on the mutual interactions of the spins and on the other hand, on quantum corrections coming from QED. This constitutes a quantum correction of the Bloch equations. The proof relies on some equations of Maxwell-Bloch type for operator valued functions (see [55] and also Theorem 6.1).

We now turn to the time evolution of the number operator N. The number operator is not under the form (1.13) and one cannot therefore directly apply Theorem 1.3. Moreover, the operator

$$N(t,h) = e^{i\frac{t}{h}H(h)}(N \otimes I)e^{-i\frac{t}{h}H(h)}.$$

is not precisely defined. We shall instead use its formal derivative:

$$N'(t,h) = (i/h)e^{i\frac{t}{h}H(h)}[H(h),(N \otimes I)]e^{-i\frac{t}{h}H(h)}. \tag{1.24}$$

This definition makes sense. We shall see that $[H(h), N \otimes I]$ is a bounded operator from $D(H(h))$ into $\mathcal{F}_s(H_{\mathbf{C}}) \otimes \mathcal{H}_{sp}$. In particular, the quantity $< N'(t,X,h)a, a >$ amounts to the photon number average value emitted by unit of time, at time t, when the initial state is taken as $\Psi_{X,h} \otimes a$ with a unit normalized element a in $\mathcal{H}_{sp}$.

Theorem 1.4. *i) The operator $N'(t,h)$ defined in (1.24) is well defined from $D(H(h))$ into $\mathcal{F}_s(H_{\mathbf{C}}) \otimes \mathcal{H}_{sp}$. Its Wick symbol $N'(t,X,h)$ defined for $X \in D(M_\omega) \subset H^2$ satisfies,*

$$N'(t,X,h) = \sum_{\lambda=1}^{P} \sum_{m=1}^{3} E_m^{free,pol}(x_\lambda, X, t) S_m^{[\lambda,0]}(t,X) + N^{res}(t,X,h), \tag{1.25}$$

where $X \to E_m^{free,pol}(x_\lambda, X, t)$ is a linear form on H^2 that is determined in Section 7 and $X \to N^{res}(t,X,h)$ is a function in $S(H^2, KQ_t)$, with $K > 0$ and its norm in this class is bounded independently of t belonging to a compact set of $\mathbb{R}$ and of h lying in $(0,1]$.

ii) There exists a sequence of functions $N^{[j]}$ on $\mathbb{R} \times H^2$ satisfying, for any integer $M \geq 1$,

$$N^{res}(t, X, h) = \sum_{j=1}^{M} h^j N^{[j]}(t, X) + h^{M+1} R^{[M]}(t, X, h). \tag{1.26}$$

For all $j \geq 1$, for every $M \geq 1$ and for any $t \in \mathbb{R}$, the function $X \to N^{[j]}(t, X)$ belongs to $S(H^2, K_j Q_t)$ and $R_M(t, \cdot, h)$ belongs to a class $S(H^2, L_M Q_t)$ where K_j and L_M are some constants and where the norms are bounded independently of t belonging to any compact set of $\mathbb{R}$ and of h lying in $(0, 1]$.

This theorem is proved in Section 7 where the functions $N^{[j]}(t, X)$ are determined. However, for a better understanding at this stage of equality (1.25), let us introduce here the element $E_m^{free,pol}(x_\lambda, X, t)$ involved here. We define $E_m^{free,pol}(x_\lambda, X, t)$ in a particular case, when the photon $X = (q, p)$ satisfies for almost all $k \in \mathbb{R}^3$,

$$(k \times q(k), k \times p(k)) = \varepsilon |k| (-p(k), q(k))$$

where $\varepsilon = \pm 1$. These two cases correspond to the circular right and left polarization notions in physics. In both cases we have,

$$E_m^{free,pol}(x_\lambda, X, t) = \varepsilon \mathbf{E}^{free}(x_\lambda, t, X).$$

Thus, in other words, (1.25) says that, the mean number of photons emitted by unit of time is related to the scalar product of the spin and of the electric field, corrected according to the polarization, up to corrections in $\mathcal{O}(h)$.

Note that we are not expecting to consider in these issues the limit as t goes to infinity in the semiclassical context recalling that even in finite dimension, semiclassical expansions should be valid up to Ehrenfest time. We also note that [16] gives first order quantum corrections for a related model, namely, the interaction of an electric dipole moment with the quantized electric field. Besides, it is observed in [50] that the polarization is involved in the photon emission.

The relevance of the semiclassical limit is also suggested by the following complementary remark. We point out that the semiclassical approximation of the model studied here not only concerns time evolutions of observables but other issues can naturally be studied. Indeed, it is known, see for example [11][24][36] (note that very often $h = 1$ in references for the third model), that the Hamiltonian $H(h)$ defined above has a ground state U_h satisfying:

$$H(h)U_h = E_h U_h, \quad E_h = \inf \sigma(H(h)), \quad \|U_h\| = 1$$

and it is proved in [9] under some conditions that the observable average values $B_m(x)$ and $E_m(x)$ related to the three components of the magnetic and electric fields at an arbitrary point $x \in \mathbb{R}^3$ satisfy

$$< B_m(x)U_h, U_h >= hB_m^{class}(x) + \mathcal{O}(h^{3/2}), \quad < E_m(x)U_h, U_h >= hE_m^{class}(x) + \mathcal{O}(h^{3/2})$$

where $\mathbf{B}^{class}(x)$ and $\mathbf{E}^{class}(x)$ are the magnetic and electric fields associated, according to elementary physics, with the P spins viewed as magnets all pointing in the direction of the non quantized constant external magnetic field. In particular, $\mathbf{E}^{class}(x) = 0$. We also derive in [9] a connection between the model studied here (the third model) with the Ising model.

Notations. The scalar product of two elements a and b of a real Hilbert space will be denoted by $a \cdot b$. It is the case for the configuration space H or the phase space H^2, when it is not identified with the complexified space $H_{\mathbf{C}}$. We denote by $< a, b >$ the hermitian product on a complex space, which always will be antilinear with respect to the second factor (such that $< a, ib >= -i < a, b >$). It is the case for $H_{\mathbf{C}}$, for the Fock space

$\mathcal{H}_{ph} = \mathcal{F}_s(H_{\mathbf{C}})$, and for the space $\mathcal{H}_{sp}$. The H or H^2 norm will be denoted by $|\cdot|$ and the norm of $\mathcal{F}_s(H_{\mathbf{C}})$ or $L^2(\mathcal{B}, \mu_{B,h/2})$, by $\|\cdot\|$.

Sketch of the proof. One first writes the equations that should be satisfied by the functions $A^{[j]}(t,X)$ in Theorem 1.3 in order to be, at least formally, a good approximation of the Wick symbol of the operator $A(t,h)$. These equations are explicitly written down in Sections 5.2 and 5.3, where we also prove existence and uniqueness properties of the solutions. It is in addition derived that these solutions belong to some classes of Definition 1.2. It remains to prove (1.21) and thus, to compare the true Wick symbol of $A(t,h)$ with its supposed approximation. Rather than comparing two functions, it seems easier to compare two operators. Therefore, with each function F belonging to a class $S(H^2,Q)$ of Definition 1.2, one associates an operator denoted by $Op_h^{wick}(F)$ whose Wick symbol is F.

This Wick quantization of a given function F is not always possible. It is possible if F is polynomial function. The corresponding operator is then defined using Wick (normal) ordering (see [3], [27], ...). It is also defined for some quadratic forms (see [42]).

We show in Section 2 that Wick quantization is also possible for functions F in a class $S(H^2,Q)$ of Definition 1.2. To this end, we first begin by giving a heat operator inverse to the function F. The fact that this is possible should probably be related to properties of the functions in $S(H^2,Q)$, which are stronger than analyticity. Next, we use the anti-Wick quantization commonly used in finite dimension (see [14] and [43]) which is however only an intermediate step in our case. Also in infinite dimension, this anti-Wick quantization has its own interest (see [3] for constructing semiclassical measures). This will enable a Wick quantization of the coefficients $X \to A^{[j]}(t,X)$ and also of the error terms appearing in computations. Thus, estimate (1.21) brings us back to a comparison between two operators which is easier to consider. Let us also mention that another technique of [40] concerning Wick quantization could be applied to our classes.

2 Quantization

The purpose of this section is to give an answer to the following issue when H is any arbitrary infinite dimensional separable Hilbert space and not necessarily the photon Hilbert space. For a given function F on H^2, can we find an operator A in the Fock space $\mathcal{F}_s(H_{\mathbf{C}})$, bounded or unbounded, with a Wick symbol equal to F ? If so, we can say that A is the Wick quantization of F. Before, we had to define an anti-Wick quantization which is here only a first step but can have is own interest. Analogous techniques may be found in [1, 2, 51].

2.1 Anti-Wick quantization.

We recall that in finite dimension n, the anti-Wick operator associated with a bounded measurable function F on $\mathbb{R}^n$ is defined, for all f and g in $L^2(\mathbb{R}^n)$, for any $h > 0$, by

$$< Op_h^{AW}(F) f, g >= (2\pi h)^{-n} \int_{\mathbb{R}^{2n}} F(X) < f, \Psi_{X,h} > < \Psi_{X,h}, g > dX, \tag{2.1}$$

where the $\Psi_{X,h}$ are the standard coherent states on $\mathbb{R}^n$ indexed by $X \in \mathbb{R}^{2n}$ (see [21]). One of the advantages of this quantization used for example in [43] is the possibility to consider less regular functions F.

With the aim of defining corresponding operators in the Fock space $\mathcal{H}_{ph}$, one could imagine to consider integrals on H^2. Naturally, the Lebesgue measure is not existing anymore and it can be replaced by a Gaussian measure.

For this purpose, H is replaced in the integral by another space B.

Wiener Space. If H is an infinite dimensional separable Hilbert space, and $t > 0$, there is no probability measure ν on on the Borel $\sigma-$algebra of H such that, for each $a \in H$:

$$\int_B e^{ia\cdot x} d\nu(x) = e^{-\frac{t}{2}|a|^2}.$$

However, there exists one if H is finite dimensional. In this case, the measure satisfying the above equality will be denoted by $\mu_{H,t}$ and called a Gaussian measure. In the infinite dimensional case, we may use a greater Banach space B, containing H, and a suitable probability measure on B.

For these points, see [31], [41] and also Theorem 2.1. The exact conditions to be fulfilled by B together with the properties involved in this paper are recalled in [4].

Theorem 2.1. *([30]-[33],[41]). Let H be a real separable Hilbert space. Then there exists a (non unique) Banach space B containing H, such that $B' \subset H' = H \subset B$, each space being dense in the subsequent, and for all $t > 0$, there exists a probability measure $\mu_{B,t}$ on the Borel σ-algebra of B satisfying,*

$$\int_B e^{ia(x)} d\mu_{B,t}(x) = e^{-\frac{t}{2}|a|^2}, \quad a \in B'. \tag{2.2}$$

Here, $|a|$ denotes the norm in H and the notation $a(x)$ stands for the duality between B' and B. Moreover, for each finite dimensional space $E \subset H$, and $p > 1$, and $t > 0$, there is an injection j_E of $L^p(E, \mu_{E,t})$ in $L^p(B, \mu_{B,t})$ such that, if $E \subset F \subset H$, we have $j_F = j_E \circ i_{E,F}$, where, for each $u \in L^p(E, \mu_{E,t})$, $i_{E,F}(u) = u \otimes 1$, where 1 is the constant function equal to 1 in the orthogonal subspace of E in F.

One says that (H, B) is a Wiener space. One also says that B is a Wiener extension of H.

Gaussian variables. We recall ([41]) that, for all a in $B' \subset H$, the mapping $B \ni x \mapsto a(x)$ belongs to $L^2(B, \mu_{B,h})$, with a norm equal to $h^{\frac{1}{2}}|a|$. Thus, the mapping associating, with every $a \in B'$, the function above considered as an element of $L^2(B, \mu_{B,h})$, can be extended by density to a mapping $a \mapsto \ell_a$ from H into $L^2(B, \mu_{B,h})$. The functions ℓ_a are gaussian variables in the sense of [38].

Stochastic extensions. If f is a bounded continuous function on an infinite dimensional Hilbert space H, we would like to define its integral for a Gaussian measure, for instance in order to define an operator. It does not make sense in general but, with some functions f, we may associate a function $\widetilde{f}$ on a Wiener extension B of H, and we shall use the integral of $\widetilde{f}$ instead. The function $\widetilde{f}$ will be called a stochastic extension of f, according to the terminology of L. Gross, who introduced the notion. This notion is recalled in Definition 2.2 of [8].

Definition 2.2. *Let H be a Hilbert space and B a Wiener extension of H. Let f be a bounded continuous function on H. We say that f has a stochastic extension in $L^p(B, \mu_{B,t})$ if, for each increasing sequence (E_n) of finite dimensional subspaces in H, whose union is dense in H, denoting by f_n the restriction of f to E_n, the sequence $j_{E_n} f_n$ has a limit in $L^p(B, \mu_{B,t})$. This limit, which is independent of the sequence (E_n), will be denoted by $\widetilde{f}$ and called a stochastic extension of f.*

One may find in [8] and in [4] examples of functions admitting such extensions. For a given function f, having or not a stochastic extension does not depend on the chosen Wiener extension B. For $p \geq 1$, the integral of $\widetilde{f}$ does not depend either on the choice of B.

Segal-Bargmann transform. For each function f in $\mathcal{H}_{ph}$, for every X in H^2 and for each $h > 0$, we set:

$$(T_h f)(X) = \frac{< f, \Psi_{X,h} >}{< \Psi_{0,h}, \Psi_{X,h} >} = e^{\frac{|X|^2}{4h}} < f, \Psi_{X,h} > \tag{2.3}$$

where $\Psi_{X,h}$ denotes the coherent state defined in (1.10). This function $T_h f$ is Gâteaux anti-holomorphic on H^2 when one identifies $(x,\xi) \in H^2$ with $x+i\xi$. Its restriction to any finite dimensional subspace E of H^2 belongs to $L^2(E,\mu_{E,h})$, its norm being bounded independently of E. According to Theorem 8.8 in [4], its stochastic extension $\widetilde{T}_h f$ exists in $L^2(B^2,\mu_{B^2,h})$ and $\widetilde{T}_h$ is a partial isometry from $\mathcal{F}_s(H_{\mathbf{C}})$ in $L^2(B^2,\mu_{B^2,h})$ whose range is the closure of anti-holomorphic functions. The mapping $\widetilde{T}_h$ is called the Segal Bargmann transform.

Anti-Wick Operator. For every function F on H^2 (called the symbol) admitting a stochastic extension $\widetilde{F}$ measurable and bounded on B^2, one denotes by $Op_h^{AW}(F)$ the operator defined by,

$$< Op_h^{AW}(F)f, g >= \int_{B^2} \widetilde{F}(X)\widetilde{T}_h f(X)\overline{\widetilde{T}_h g(X)} d\mu_{B^2,h}(X), \tag{2.4}$$

for all f and g in $\mathcal{H}_{ph}$. Since $\widetilde{T}_h$ is a partial isometry from $\mathcal{F}_s(H_{\mathbf{C}})$ into $L^2(B^2,\mu_{B^2,h})$, this operator is bounded on $\mathcal{F}_s(H_{\mathbf{C}})$ and its norm is smaller than the sup norm of $\widetilde{F}$.

Unlike in the finite dimensional case, we cannot define an anti-Wick operator for any bounded measurable function F in H^2 since this function needs a stochastic extension. Still, one can associate an anti-Wick operator with every function F in $S(H^2,Q)$ where Q is a nonnegative quadratic form on H^2 written under the form (1.15) with A_Q selfadjoint and trace class. Indeed, according to [37], Proposition 3.10, such a function admits a stochastic extension $\widetilde{F}$ measurable and bounded on B^2. In this case, we have $\|\widetilde{F}\|_{L^\infty} \leq \|F\|_Q$ and therefore, since $\widetilde{T}_h$ is a partial isometry from $\mathcal{F}_s(H_{\mathbf{C}})$ into $L^2(B^2,\mu_{B^2,h})$, we have:

$$\|Op_h^{AW}(F)\| \leq \|F\|_Q. \tag{2.5}$$

Again, we shall use anti-Wick quantization as an intermediate step towards the Wick quantization studied in the following subsection. The anti-Wick quantization in infinite dimension has its own interest: it is used in [3] for the construction of semiclassical measures.

2.2 Wick quantization.

Any bounded operator T has a Wick symbol but it is not always possible for a function F on H^2 to find an operator whose Wick symbol is F. The theorem below shows that it is true in some cases. We recall here that the coherent states belong to $D(N^m)$ for any integer $m \geq 1$. Polynomial functions are finite linear combinations of multiplications of $e_j \cdot q$ and $e_j \cdot p$ as maps of (q,p), where (e_j) is an orthonormal basis of H. For any multi-index α we set $(q \pm ip)^\alpha = \prod_j (e_j \cdot q \pm i e_j \cdot p)^{\alpha_j}$ and we use similar notations for products of creation or annihilation operators.

Theorem 2.3. *i) For any function F belonging to $S(H^2,Q)$ where Q is a nonnegative quadratic form on H^2 written under the form (1.15), where A_Q is selfadjoint and trace class, and for each $h>0$, there exists a unique bounded operator B_h in $\mathcal{F}_s(H_{\mathbf{C}})$ such that $\sigma_h^{wick}(B_h) = F$. We then set $B_h = Op_h^{wick}(F)$. One has the following estimate:*

$$\|Op_h^{wick}(F)\| \leq e^{(h/2)\mathrm{Tr}A_Q}\|F\|_Q, \tag{2.6}$$

where the norm in the left hand side is the $\mathcal{L}(\mathcal{F}_s(H_{\mathbf{C}}))$ norm and A_Q is the operator verifying $Q(X) = (A_Q X)\cdot X$ for all $X \in H^2$.

ii) For any polynomial function F on H^2 of degree m and for all $h>0$ there exists an unbounded operator B_h, closable, with initial domain $D(N^m)$, satisfying $\sigma_h^{wick}(B_h) = F$. We thus set $B_h = Op_h^{wick}(F)$. In particular, the Wick symbol of a Segal field $\Phi_S(A)$ associated with $A \in H^2$ is the function $H^2 \ni X \to h^{-1/2} A \cdot X$, where $A \cdot X$ is the real scalar product on H^2.

iii) For any quadratic form F on H^2 written as $F(q,p) = (Aq)\cdot q + (Ap)\cdot p$, where $A \in \mathcal{L}(H)$ is selfadjoint, and for every $h > 0$, the operator $B_h = 2h\mathrm{d}\Gamma(A)$ satisfies $\sigma_h^{wick}(B_h) = F$.

We now need to involve the heat operator in order to first prove point i) of the theorem above. The heat operator is defined for each measurable bounded function F in H^2 admitting a stochastic extension $\widetilde{F}$ in $L^1(B^2, \mu_{B,h})$ and for each $h > 0$ by:

$$(H_h F)(X) = \int_{B^2} \widetilde{F}(X+Y) d\mu_{B^2,h}(Y), \tag{2.7}$$

for $X \in H^2$ (see Definition 5.1 and formula (28) of [37]). We also have:

$$(H_h F)(X) = e^{-\frac{|X|^2}{2h}} \int_{B^2} \widetilde{F}(U)\ e^{\frac{1}{h}\ell_X(U)} d\mu_{B^2,h}(U). \tag{2.8}$$

Theorem 2.4. *Let F be in $S(H^2,Q)$. Then the function $h \to H_h F$ is C^∞ on $[0,\infty[$ with values in the Banach space $S(H^2,Q)$. Moreover, for all $h > 0$, the operator H_h is an isomorphism from $S(H^2,Q)$ onto itself. As operators in $S(H^2,Q)$, the norm of H_h and the norm of its inverse denoted by H_{-h} satisfy the following bounds:*

$$\|H_h\| \leq 1, \qquad \|H_{-h}\| \leq e^{(h/2)\mathrm{Tr}A_Q}, \tag{2.9}$$

where A_Q is the operator satisfying $Q(X) = (A_Q X)\cdot X$.

Proof. These results are taken from [37]. The first claim is Theorem 5.17. It is proved in Proposition 5.12 that the Laplace operator Δ is well defined on $S(H^2,Q)$ and that it is bounded on $S(H^2,Q)$ into itself, with norm smaller or equal than $\mathrm{Tr}A_Q$. On the other hand, one has $H_h = e^{(h/2)\Delta}$ (Proposition 5.13 or Theorem 5.17). Therefore, the inverse of the bounded operator H_h is defined by:

$$H_{-h}F = \sum_{m=0}^{\infty} (-1)^m \frac{h^m}{2^m m!} \Delta^m F. \tag{2.10}$$

Its norm satisfies (2.9), which proves the result.

Proof of point i) of Theorem 2.3. One knows that, for all X and Y in H^2:

$$< \Psi_{X,h}, \Psi_{Yh} >= e^{-\frac{1}{4h}(|X-Y|^2) + \frac{i}{2h}\sigma(X,Y)} \tag{2.11}$$

where σ is the symplectic form $\sigma((x,\xi),(q,p)) = q\cdot\xi - p\cdot x$. Hence, for each $U = (a,b)$ in H^2, setting $\check{U} = (-b,a)$:

$$\widetilde{T}_h \Psi_{U,h}(X) = e^{-\frac{1}{4h}(|U|^2 + \frac{1}{2h}\ell_{U-i\check{U}}(X))}. \tag{2.12}$$

Then, for each F in $S(H^2,Q)$ and $h > 0$, one has, according to (2.4):

$$< Op_h^{AW}(F)\Psi_{U,h}, \Psi_{U,h} >= e^{-\frac{1}{2h}|U|^2} \int_{B^2} \widetilde{F}(q,p) e^{\frac{1}{h}(\ell_a(q)+\ell_b(p))} d\mu_{B^2,h}(q,p) = H_h F(U).$$

The last equality follows from (2.8). Therefore:

$$\sigma_h^{wick}(Op_h^{AW}(F)) = H_h F. \tag{2.13}$$

To prove point i) of Theorem 2.3, it is then sufficient to set

$$Op_h^W(F) = Op_h^{AW}(H_{-h}F). \tag{2.14}$$

The norm estimate (2.6) is then a consequence of the above definition, with (2.9) and (2.5).

Proof of point ii) of Theorem 2.3. If F is a polynomial function of degree m on H^2, we can again define $Op_h^W(F)$ by (2.14). Indeed, $G = H_{-h}F$ is also a polynomial function G on H^2 with the same degree. One proves in [37] (Proposition 3.17) that each polynomial function G of degree m has a stochastic extension $\widetilde{G}$. Moreover,

$$|(GT_hf)(q,p)| \leq C|(T_hf)(q-ip)| \sum |(q-ip)^\alpha|,$$

where the sum above is finite. A Hilbert basis (e_j) of H is fixed. Note that $(q-ip)^\alpha (Tf)(q-ip)$ is, up to a multiplicative factor, the Segal Bargmann transform of $(a^\star(e))^\alpha f$, which is well defined as an element of $\mathcal{F}_s(H_{\mathbf{C}})$ if f belongs to the domain of N^m. Thus, the function $\widetilde{G}\widetilde{T}_hf$ belongs to $L^2(B^2, \mu_{B^2,H})$. Consequently, the integral (2.4) makes sense. It indeed defines an unbounded operator $Op_h^{AW}(H_{-h}F)$ with (initial) domain $D(N^m)$ where m is the degree of F. We recall that the coherent states are in the domain of $D(N^m)$. Therefore, the Wick symbol of $Op_h^{AW}(H_{-h}F)$ is well defined. Reasoning as in the above point i) shows that this Wick symbol is equal to F.

Let us now see the usual relation between the Wick quantization of a polynomial function with the Wick (normal) ordering.

Degree one polynomial function. First, we prove the last claim of point ii). In order to derive it, we remark, from the Definition (1.10) of coherent states, that, for each $X \in H^2$:

$$\Psi_{X,h} = e^{-\frac{i}{\sqrt{h}}\Phi_S(\widehat{X})}\Psi_0 \tag{2.15}$$

where Ψ_0 is the vacuum (independent of h) and $\widehat{X} = (-b, a)$ for $X = (a, b)$. Therefore

$$e^{it\Phi_S(A)}\Psi_{X,h} = e^{it\Phi_S(A)}e^{-\frac{i}{\sqrt{h}}\Phi_S(\widehat{X})}\Psi_0.$$

We know that, for all U and V in H^2:

$$e^{i\Phi_S(U)}e^{i\Phi_S(V)} = e^{\frac{i}{2}\sigma(U,V)}e^{i\Phi_S(U+V)} \tag{2.16}$$

where σ is the symplectic form $\sigma((a,b),(q,p)) = b\cdot q - a\cdot p$. Therefore:

$$e^{it\Phi_S(A)}\Psi_{X,h} = e^{-\frac{it}{2\sqrt{h}}\sigma(A,\widehat{X})}e^{i\Phi_S(tA-(1/\sqrt{h})\widehat{X})}\Psi_0 = e^{-\frac{it}{2\sqrt{h}}\sigma(A,\widehat{X})}\ \Psi_{X+t\sqrt{h}\widehat{A},h}.$$

According to (2.11),

$$< e^{it\Phi_S(A)}\Psi_{X,h}, \Psi_{X,h} >= e^{-\frac{it}{2\sqrt{h}}\sigma(A,\widehat{X})}\ e^{-\frac{t^2}{4}|\widehat{A}|^2}\ e^{\frac{i}{2h}\mathrm{Im}<X+t\sqrt{h}\widehat{A},X>}.$$

Differentiating with respect to t, at $t=0$, and using $\sigma(A,\widehat{X}) = -A\cdot X$ where $A\cdot X$ is the real scalar product on H^2, we indeed obtain that:

$$< \Phi_S(A)\Psi_{X,h}, \Psi_{X,h} >= h^{-1/2}A\cdot X.$$

If F is a degree one polynomial function on H^2 and if (e_j) is a Hilbert basis of H, we can write:

$$F(q,p) = \sum_j a_j(e_j\cdot q + ie_j\cdot p) + b_j(e_j\cdot q - ie_j\cdot p).$$

Here the sums are finite. From the above computations:

$$Op_h^{wick}(F) = \sqrt{2h}\sum_j a_j a(e_j) + b_j a^\star(e_j). \tag{2.17}$$

Wick normal ordering. Degree m polynomial functions. As we have stated at the beginning of Section 2.2, any polynomial function F of degree m can be written as:

$$F(q,p) = \sum_{|\alpha|+|\beta|\leq m} a_{\alpha\beta}(q+ip)^\alpha(q-ip)^\beta$$

with $(q \pm ip)^\alpha = \prod_j (e_j \cdot q \pm i e_j \cdot p)^{\alpha_j}$. Let us check that

$$Op_h^{wick}(F) = (2h)^{m/2} \sum_{|\alpha|+|\beta|\leq m} a_{\alpha\beta} a^\star(e)^\beta a(e)^\alpha \tag{2.18}$$

where we set:

$$a(e)^\alpha = \prod_j a(e_j)^{\alpha_j}.$$

This is proved by iteration on m. For $m = 1$, it is (2.17). Assume that the equality is proved for all polynomial function of degree $\leq m-1$. Each degree m polynomial function can be expressed as:

$$F(q,p) = (e_j.q + ie_j p)A(q,p) + (e_k.q - ie_k p)B(q,p)$$

where A and B are of degree $\leq m-1$. Then, the operator

$$T_h = \sqrt{2h}\Big(a^\star(e_k) Op_h^{wick}(B) + Op_h^{wick}(A) a(e_j)\Big)$$

has F as Wick symbol. This follows from (2.17) and from an analog of Theorem 4.1 valid for finite sums for the Wick symbol of the composition of two operators. Using the induction hypothesis for A and B, we indeed deduce that (2.18) holds true for all m.

Proof of point iii) of Theorem 2.3. By Definition (1.11) of the Wick symbol and according to Definition (1.10) of the coherent states together with the usual definition of $d\Gamma(A)$, it follows that the Wick symbol of an operator $d\Gamma(A)$ where $A \in \mathcal{L}(H)$ and is selfadjoint is given by

$$\sigma_h^{wick}(d\Gamma(A))(q,p) = \frac{1}{2h}\Big[(Aq)\cdot q + (Ap)\cdot p\Big]. \tag{2.19}$$

For instance, the Wick symbol of the number operator $N = d\Gamma(I)$ is:

$$N(q,p) = \frac{1}{2h}(|q|^2 + |p|^2). \tag{2.20}$$

The Wick symbol of $H_{ph} = h d\Gamma(M_\omega)$ is defined only for $X = (q,p)$ in $D(M_\omega)$, by:

$$H_{ph}(q,p) = \frac{1}{2}\Big[(M_\omega q)\cdot q + (M_\omega p)\cdot p\Big]. \tag{2.21}$$

3 Beals characterization.

Wick quantization implements a one to one correspondance between the space $S(H^2, Q)$ in Definition 1.2 and a space of operators acting in $\mathcal{F}_s(H_{\mathbf{C}})$ that we now specify.

This set of operators is inspired by the Beals characterization [13], see also [57]. In the finite dimensional case, the multiplication operators by a coordinate function and the partial differentiation operators play an essential part. In the infinite dimensional case, this role is played by Segal fields. The usual Beals hypothesis is here modified in order to avoid domain definition issues.

Definition 3.1. *([8]). For each $V = (a,b)$ in H^2, let $\widehat{V} = (-b,a)$. Set $h > 0$. Let Q be a quadratic form on H^2 of the form (1.15) with $A_Q \in \mathcal{L}(H^2)$ selfadjoint, nonnegative and trace class in H^2. We denote by $\mathcal{L}(Q)$ the space of all bounded operators A in $\mathcal{F}_s(H_{\mathbf{C}})$ satisfying:*

i) For each integer $m \geq 1$, for all $V_1, \ldots, V_m$ in H^2 and for each $h > 0$, the function

$$\mathbb{R}^m \ni t \to A_h(t) = e^{\frac{i}{\sqrt{h}}\Phi_S(t_1\widehat{V}_1+\ldots+t_m\widehat{V}_m)}\ A\ e^{-\frac{i}{\sqrt{h}}\Phi_S(t_1\widehat{V}_1+\ldots+t_m\widehat{V}_m)} \tag{3.1}$$

is of class C^m from $\mathbb{R}^m$ into $\mathcal{L}(\mathcal{F}_s(H_{\mathbf{C}}))$, that is to say, $t \to < A_h(t)f, g >$ is C^m for any f and g.

ii) There exists a positive real number $C(A,Q)$ satisfying, for each integer m and for all $V_1, \ldots, V_m$ in H^2, for any h in $(0,1]$:

$$\|\partial_{t_1}....\partial_{t_m}A_h(0)\| \leq C(A,Q)\prod_{j=1}^{m} Q(V_j)^{1/2}, \quad h \in (0,1]. \tag{3.2}$$

The norm in the above right hand side is the $\mathcal{L}(\mathcal{F}_s(H_{\mathbf{C}}))$ norm. The smallest constant $C(A,Q)$ such that (3.2) is valid is denoted by $\|A\|_{\mathcal{L}(Q)}$.

In the usual Beals hypothesis, the estimate (3.2) would be written as:

$$\|\mathrm{ad}\Phi_S(\widehat{V}_1)\cdots\mathrm{ad}\Phi_S(\widehat{V}_m)A\| \leq C(A,Q)\ h^{m/2}\prod_{j=1}^{m} Q(V_j)^{1/2}, \quad h \in (0,1].$$

If A and B are in $\mathcal{L}(Q)$ then the composition $A \circ B$ belongs to $\mathcal{L}(4Q)$ and

$$\|A \circ B\|_{\mathcal{L}(4Q)} \leq \|A\|_{\mathcal{L}(Q)}\ \|B\|_{\mathcal{L}(Q)}. \tag{3.3}$$

Theorem 3.2. *Let Q be the quadratic form written as in (1.15), with $A_Q \in \mathcal{L}(H^2)$, selfadjoint, nonnegative, trace class in H^2. Then:*

i) For each operator A in $\mathcal{L}(Q)$ and for each h in $(0,1]$, the Wick symbol $\sigma_h^{wick}(A)$ belongs to $S(H^2,Q)$ and we also have:

$$\|\sigma_h^{wick}(A)\|_Q \leq \|A\|_{\mathcal{L}(Q)}.$$

ii) For each function F being in $S(H^2,Q)$ and for every $h > 0$, the operator $Op_h^{wick}(F)$ belongs to $\mathcal{L}(Q)$ and the following estimates holds true:

$$\|Op_h^{wick}(F)\|_{\mathcal{L}(Q)} \leq \|F\|_Q e^{(h/2)\mathrm{Tr}(A_Q)}. \tag{3.4}$$

Theorem 3.2 is an infinite dimensional type of Beals characterization Theorem (see [13], and also [6, 8]).

Proof of Theorem 3.2.

i) By (2.15) and (2.16), we have, for all X in H^2 and for each finite system $(V_1, ...V_m)$ in H^2:

$$\sigma_h^{wick}(A_h(t))(X) = \sigma_h^{wick}(A)(X + t \cdot V).$$

Therefore:

$$(d^m\sigma_h^{wick}(A))(X)(V_1, ..., V_m) = \sigma_h^{wick}\Big(\partial_{t_1}....\partial_{t_m}A_h(0)\Big)(X).$$

Point i) then follows since $|\sigma_h^{wick}(B)(X)| \leq \|B\|$ for any bounded operator B and for all $X \in H^2$.

ii) Let F be in $S(H^2,Q)$, $h > 0$ and $A = Op_h^{wick}(F)$. The computations above show that $A_h(t) = Op_h^{wick}(F(\cdot + t \cdot V))$ for any finite sequence $(V_1, ..., V_m)$ in H^2. The function $\mathbb{R}^m \ni t \to F(\cdot + t \cdot V)$ belongs to the class C^m from $\mathbb{R}^m$ into $S(H^2,Q)$. The mapping Op_h^{wick} is a continuous linear map from $S(H^2,Q)$ into $\mathcal{L}(\mathcal{F}_s(H_{\mathbf{C}}))$ with a

norm bound given in (2.6). Consequently, the function $t \to A_t(t)$ is in the class C^m from $\mathbb{R}^m$ into $\mathcal{L}(\mathcal{F}_s(H_{\mathbf{C}}))$. We have:

$$\partial_{t_1}....\partial_{t_m} F(X + t \cdot V)\Big|_{t=0} = (d^m F)(X)(V_1, ...V_m).$$

Point ii) then follows.

4 Compositions, commutators and covariance.

4.1 Mizrahi series.

The operator composition will be used in two cases. In Section 5, the commutator of two operators is studied, one being a Segal field and the other one being a bounded operator with a Wick symbol belonging to the class $S(H^2, Q)$. In Section 6, the composition of two bounded operators is considered, when the Wick symbols are in a class $S(H^2, Q)$.

Theorem 4.1. *Let G be a function in $S(H^2, Q)$ with a quadratic form Q given by a nonnegative and trace class operator A_Q as in (1.15). Let V be an element of H. Then we have:*

$$[a(V), Op_h^{wick}(G)] = \sqrt{h/2}\ Op_h^{wick}(K) \qquad \text{with} K(q,p) = (V \cdot \partial_q + iV \cdot \partial_p)G(q,p). \tag{4.1}$$

One has:

$$\sigma_h^{wick}\Big(\Phi_{S,h}(V) \circ Op_h^{wick}(G)\Big)(X) = \varphi_V(X)G(X) + hC^{1,wick}(\varphi_V, G)(X)$$

where $\varphi_V(X) = V \cdot X$. Fixing a Hilbert basis (e_j) of H, one sets $\partial_{q_j} = e_j \cdot \partial_q$ and for all differentiable functions F and G,

$$C^{1,wick}(F,G) = \frac{1}{2}\sum_j (\partial_{q_j} - i\partial_{p_j})F\ (\partial_{q_j} + i\partial_{p_j})G.$$

The above sum is independent of the chosen basis. We also have:

$$\sigma_h^{wick}\Big(Op_h^{wick}(G) \circ \Phi_{S,h}(V)\Big)(X) = \varphi_V(X)G(X) + hC^{1,wick}(G, \varphi_V)(X).$$

See [3]. Fix a Hilbert basis (e_j) of H. We define, for all multi-index $\alpha = (\alpha_j)$ (which means $\alpha_j = 0$ except for a finite number of values of j), two differential operators on H^2, denoting by (q,p) the variable of H^2:

$$(\partial_q \pm i\partial_p)^\alpha = \prod_j (e_j \cdot \partial_q \pm ie_j \cdot \partial_p)^{\alpha_j}\ .$$

Theorem 4.2. *Let Q be a quadratic form on H^2 as in (1.15), with A_Q nonnegative and trace class. Then, for each F and G in $S(H^2, Q)$, we can write:*

$$Op_h^{wick}(F) \circ Op_h^{wick}(G) = Op_h^{wick}(C_h^{wick}(F,G)) \tag{4.2}$$

$$C_h^{wick}(F,G) = \sum_\alpha \left(\frac{h^{|\alpha|}}{2^{|\alpha|}\alpha!}\right)(\partial_q - i\partial_p)^\alpha F\ (\partial_q + i\partial_p)^\alpha G. \tag{4.3}$$

The series above is absolutely converging. For all integers m, we have:

$$\left| C_h^{wick}(F,G)(X) - \sum_{|\alpha| \le m} \left(\frac{h^{|\alpha|}}{2^{|\alpha|}\alpha!}\right)(\partial_q - i\partial_p)^\alpha F(X)\ (\partial_q + i\partial_p)^\alpha G(X) \right| \le \tag{4.4}$$

$$\|F\|_Q\ \|G\|_Q\ h^{m+1}[\mathrm{Tr}A_Q]^{m+1} e^{h\mathrm{Tr}A_Q}.$$

See [10] or [48] ~~in the case of finite dimension. See also [3] formula (17)(i).~~

Proof. For all multi-indices α, we define an element u_α of $\mathcal{F}_s(H_{\mathbf{C}})$ by:

$$u_\alpha = (\alpha!)^{-1/2} \left[\prod_j \Big(a^\star(e_j) \Big)^{\alpha_j} \right] \Psi_0 \tag{4.5}$$

where Ψ_0 is the vacuum state. Thus, (u_α) is a Hilbert basis of $\mathcal{F}_s(H_{\mathbf{C}})$. We see that:

$$< Op_h^{wick}(G)\Psi_0, u_\alpha >= \left(\frac{h^{|\alpha|}}{2^{|\alpha|}\alpha!} \right)^{1/2} (\partial_q + i\partial_p)^\alpha G(0). \tag{4.6}$$

Indeed,

$$< Op_h^{wick}(G)\Psi_0, u_\alpha >= (\alpha!)^{-1/2} < \prod_j \Big(\text{ad } a(e_j) \Big)^{\alpha_j} Op_h^{wick}(G)\Psi_0, \Psi_0 > .$$

Applying several times (4.1) with $X = e_j$, we indeed obtain (4.6). For every X in H^2, we have:

$$\Big| (\partial_q - i\partial_p)^\alpha F(X) \Big| \, \Big| (\partial_q + i\partial_p)^\alpha G(X) \Big| \leq \|F\|_Q \, \|G\|_Q \prod_j (Q(e_j,0)^{1/2} + Q(0,e_j)^{1/2})^{2\alpha_j}.$$

$$\leq 2^{|\alpha|} \|F\|_Q \, \|G\|_Q \prod_j (Q(e_j,0) + Q(0,e_j))^{\alpha_j}.$$

As a consequence,

$$\sum_{|\alpha|=m} \left(\frac{h^{|\alpha|}}{2^{|\alpha|}\alpha!} \right) \Big| (\partial_q - i\partial_p)^\alpha F(X) \Big| \, \Big| (\partial_q + i\partial_p)^\alpha G(X) \Big| \leq \|F\|_Q \, \|G\|_Q \, \frac{h^m}{m!} \left[\sum_j Q(e_j,0) + Q(0,e_j) \right]^m$$

$$\leq \|F\|_Q \, \|G\|_Q \, \frac{h^m}{m!} [\mathrm{Tr} A_Q]^m.$$

Therefore, the series (4.3) converges absolutely for each X. According to these points,

$$C_h^{wick}(F,G)(0) = \sum_\alpha < Op_h^{wick}(G)\Psi_0, u_\alpha > < u_\alpha, Op_h^{wick}(\overline{F})\Psi_0 >$$

$$=< Op_h^{wick}(G)\Psi_0, Op_h^{wick}(\overline{F})\Psi_0 >=< Op_h^{wick}(F) \circ Op_h^{wick}(G)\Psi_0, \Psi_0 >= \sigma_h^{wick}\Big(Op_h^{wick}(F) \circ Op_h^{wick}(G) \Big)(0).$$

Recall that $Op_h^{wick}(\overline{F})$ is the adjoint of $Op_h^{wick}(F)$. Consequently, equalities (4.2) and (4.3) are proved for $X = 0$. For any arbitrary other X one notes that, according to (2.15) and (2.16),

$$\sigma_h^{wick}\Big(Op_h^{wick}(F) \circ Op_h^{wick}(G) \Big)(X) = \sigma_h^{wick}\Big(Op_h^{wick}(F_X) \circ Op_h^{wick}(G_X) \Big)(0)$$

where $F_X(U) = F(X+U)$. Equalities (4.2) and (4.3) then follow for any X. The last inequality in the theorem comes from the fact that:

$$\sum_{m+1}^{\infty} \frac{h^k}{k!} [\mathrm{Tr} A_Q]^k \leq h^{m+1} [\mathrm{Tr} A_Q]^{m+1} e^{h \mathrm{Tr} A_Q}.$$

4.2 Covariance by some Bogoliubov transformations.

Theorem 4.3. *Let U be a symplectic unitary linear map in H^2. Then:*

i) For each bounded operator A in $\mathcal{F}_s(H_{\mathbf{C}})$, we have:

$$\sigma_h^{wick}(\Gamma(U)^\star A\Gamma(U))(X) = \sigma_h^{wick}(A)(UX).$$

ii) Let $\widetilde{Q}(X) = Q(UX)$. Then, for each $F \in S(H^2, Q)$, the function $F \circ U$ belongs to $S(H^2, \widetilde{Q})$, with the same norm.

iii) For each bounded operator A in $\mathcal{L}(Q)$, the operator $\Gamma(U)^\star A\Gamma(U)$ is in $\mathcal{L}(\widetilde{Q})$, with the same norm.

Let us recall that, if a symplectic map U in H^2 is not unitary, but verifying $U^\star U - I$ is trace class, then there is still a covariance for the Bogoliubov transformation associated with U, but only for the Weyl calculus, not for the Wick quantization. See [8].

Proof. If U is both symplectic and unitary then it is also a $\mathbf{C}-$linear map when H^2 is identified with $H_{\mathbf{C}}$. In other words, U commutes with the map $X \to \widehat{X}$, where $\widehat{X} = (-b, a)$ if $X = (a, b)$. Then, the coherent states $\Psi_{X,h}$ (with $X \in H^2$) defined in (1.10) satisfy:

$$\Gamma(U)\Psi_{X,h} = \Psi_{UX,h}.$$

Point i) follows easily. Point ii) follows from the chain rule. According to the above point i) with point ii) of Theorem 2.3 together with the facts that U is a unitary operator commuting with the mapping $X \to \widehat{X}$, we have

$$\Gamma(U)\Phi_S(\widehat{V})\Gamma(U)^\star = \Phi_S(\widehat{UV}).$$

As a consequence:

$$[\Phi_S(\widehat{V}), \Gamma(U)^\star A\Gamma(U)] = \Gamma(U)^\star[\Phi_S(\widehat{UV}), A]\Gamma(U).$$

Iterating that process we obtain point iii).

5 Proof of Theorem 1.3.

5.1 Proof of point i).

In the previous Sections 2, 3 and 4, the space H is an arbitrary infinite dimensional separable Hilbert space. We now turn back to the photon Hilbert space of Section 1. We also consider the space $\mathcal{H}_{sp}$ of Section 1. Let Q be a quadratic form on H^2. The space $S(H^2, Q)$ of Definition 1.2 now refers to a space of functions taking values in $\mathcal{L}(\mathcal{H}_{sp})$ and the norm in the left hand side of (1.14) is the $\mathcal{L}(\mathcal{H}_{sp})$ norm. With a function F in $S(H^2, Q)$, we define an operator $Op_h^{wick}(F)$ being bounded in $\mathcal{H}_{ph} \otimes \mathcal{H}_{sp}$ where $\mathcal{H}_{ph} = \mathcal{F}_s(H_{\mathbf{C}})$.

Theorem 4.1 is then adapted. Fixing a Hilbert basis (e_j) of H, we set using the notations in Section 4, for all differentiable functions F and G, with values in $\mathcal{L}(\mathcal{H}_{sp})$,

$$C^{1,wick}(F, G) = \frac{1}{2}\sum_j (\partial_{q_j} - i\partial_{p_j})F \circ (\partial_{q_j} + i\partial_{p_j})G,$$

where $\circ$ stands for the composition of operators in $\mathcal{L}(\mathcal{H}_{sp})$. Let A be any operator written as $A = \Phi_{S,h}(V) \otimes S$ where $V \in H^2$ and $S \in \mathcal{L}(\mathcal{H}_{sp})$. Let $B = Op_h^{wick}(G)$ where $G \in S(H^2, Q)$. The Wick symbol of A is the function $H^2 \ni X \to \varphi_V(X) = (V \cdot X)S$. The following equality comes from Theorem 4.1:

$$\sigma_h^{wick}([A,B])(X) = [\varphi_V(X), G(X)] + hC^{1,wick}(\varphi_V, G)(X) - hC^{1,wick}(G, \varphi_V)(X) \tag{5.1}$$

where the bracket in the first term denotes the commutator of two operators in $\mathcal{L}(\mathcal{H}_{sp})$.

We denote by $\mathcal{L}(Q)$ the space of bounded operators in $\mathcal{H}_{ph} \otimes \mathcal{H}_{sp}$ as in Definition 3.1 with $e^{\frac{i}{\sqrt{h}}\Phi_S(t_1\widehat{V}_1+\ldots+t_m\widehat{V}_m)}$ replaced by $e^{\frac{i}{\sqrt{h}}\Phi_S(t_1\widehat{V}_1+\ldots+t_m\widehat{V}_m)} \otimes I$.

For each $t \in \mathbb{R}$, we shall use the quadratic form Q_t defined in (1.20) and also use the quadratic form:

$$\widetilde{Q}_t(q,p) = Q_t(\chi_{-t}(q,p)) = |t| \int_0^t |dH_{int}^{free}(-s,q,p)|^2 ds. \tag{5.2}$$

The following theorem is proved in [8].

Theorem 5.1. *For any $t \in \mathbb{R}$, the family $U_h^{red}(t)$ for any $h \in (0,1]$ defined by:*

$$U_h^{red}(t) = e^{-i\frac{t}{h}(H_{ph}\otimes I)} e^{i\frac{t}{h}H(h)} \tag{5.3}$$

belongs to the class $\mathcal{L}(\widetilde{Q}_t)$ with the quadratic form defined in (5.2). Moreover,

$$\|U_h^{red}(t)\|_{\mathcal{L}(\widetilde{Q}_t, \mathcal{L}(\mathcal{H}_{sp}))} = 1. \tag{5.4}$$

This function satisfies:

$$\frac{d}{dt}U_h^{red}(t) = iH_{int}^{free}(-t)U_h^{red}(t). \tag{5.5}$$

The point i) of Theorem 1.3 will be a consequence of the following proposition.

Proposition 5.2. *Let A be a selfadjoint operator, bounded or unbounded in $\mathcal{H}_{ph} \otimes \mathcal{H}_{sp}$. We suppose that A is written under the form (1.13) where F_A is an element of H^2 and with S_A lying in $\mathcal{L}(\mathcal{H}_{sp})$. Then:*

i) One has:

$$A^{free}(t,h) = \Phi_S(\chi_{-t}(F_A)) \otimes I + I \otimes S_A.$$

ii) If $A(t,h)$ is defined in (1.9) then the operator $A(t,h) - A^{free}(t,h)$ is bounded in $\mathcal{L}(4Q_t)$ where Q_t is the quadratic form defined in (1.20).

Proof. Point i) is a consequence of Theorem 4.3. Let A be an operator as in (1.13). Set $A(t,h)$ the operator defined in (1.9). Using the operator $U_h^{red}(t)$, $h \in (0,1]$, defined in (5.3), one has:

$$A(t,h) = e^{i\frac{t}{h}(H_{ph}\otimes I)} U_h^{red}(t) A U_h^{red}(t)^\star e^{-i\frac{t}{h}(H_{ph}\otimes I)}.$$

Since $U_h^{red}(t)$ is unitary:

$$A(t,h) = e^{i\frac{t}{h}(H_{ph}\otimes I)} \Big(A + [U_h^{red}(t), A] U_h^{red}(t)^\star \Big) e^{-i\frac{t}{h}(H_{ph}\otimes I)}.$$

From Theorem 4.1 and the expression (1.13) for A, the commutator $[U_h^{red}(t), A]$ belongs to $\mathcal{L}(\widetilde{Q}_t)$. Consequently, by (3.3), the composition $[U_h^{red}(t), A]U_h^{red}(t)^\star$ lies in $\mathcal{L}(4\widetilde{Q}_t)$. According to Theorem 4.3:

$$e^{i\frac{t}{h}(H_{ph}\otimes I)} [U_h^{red}(t), A] U_h^{red}(t)^\star e^{-i\frac{t}{h}(H_{ph}\otimes I)} \in \mathcal{L}(4\widetilde{Q}_t \circ \chi_t).$$

Since $\widehat{Q}_t \circ \chi_t = Q_t$, this proves Proposition 5.2. □

Set

$$A^{[red]}(t,h) = e^{-i\frac{t}{h}(H_{ph}\otimes I)} A(t,h) e^{i\frac{t}{h}(H_{ph}\otimes I)} = e^{-i\frac{t}{h}(H_{ph}\otimes I)} e^{i\frac{t}{h}H(h)}\ A\ e^{-i\frac{t}{h}H(h)} e^{i\frac{t}{h}(H_{ph}\otimes I)}.$$

This operator satisfies:

$$\frac{d}{dt}A^{[red]}(t,h) = i[H_{int}^{free}(-t), A^{[red]}(t,h)], \qquad A^{[red]}(0,h) = A. \tag{5.6}$$

According to (5.1):

$$\sigma_h^{wick}([H_{int}^{free}(-t), A(t,h)])(X) = [H_{int}^{free}(-t,X,h), A(t,X,h)] + hC_1^{wick}(H_{int}^{free}(-t,\cdot,h), A(t,\cdot,h))(X) \tag{5.7}$$
$$-hC_1^{wick}(A(t,\cdot,h)\ ,\ H_{int}^{free}(-t,\cdot,h))(X)$$

Consequently:

$$\frac{d}{dt}A^{[red]}(t,X,h) = i[H_{int}^{free}(-t,X,h), A^{[red]}(t,X,h)] + \tag{5.8}$$
$$+iC_1^{wick}\Big(H_{int}^{free}(-t,\cdot)\ ,\ A^{[red]}(t,\cdot,h)\Big)(X) - iC_1^{wick}\Big(A^{[red]}(t,\cdot,h)\ ,\ H_{int}^{free}(-t,\cdot)\Big)(X).$$

5.2 Zeroth order term.

The operator $A^{[red]}(t,h)$ satisfying (5.6) is now approximated by a sum written under the following form:

$$S^{[M,red]}(t,h) = \sum_{j=0}^{M} h^j A^{[j,red]}(t)$$

where the operators $A^{[j,red]}(t)$ will be determined by their Wick symbols denoted $A^{[j,red]}(t,X)$. Since the Wick symbol $A^{[red]}(t,X,h)$ has to satisfy (5.8) then it makes sense to have in the case $j=0$:

$$\frac{d}{dt}A^{[0,red]}(t,X) = i[H_{int}^{free}(-t,X), A^{[0,red]}(t,X)] \tag{5.9}$$

and $A^{[0,red]}(0,X) = A(X)$, where $A(X)$ is the Wick symbol of the observable A. For $j \geq 1$, we must have:

$$\frac{d}{dt}A^{[j,red]}(t,X) = i[H_{int}^{free}(-t,X), A^{[j,red]}(t,X)] + \Phi^{[j-1,red]}(t,X) \tag{5.10}$$

where

$$\Phi^{[j-1,red]}(t,X) = iC_1^{wick}\Big(H_{int}^{free}(-t,\cdot)\ ,\ A^{[j-1,red]}(t,\cdot,h)\Big)(X) - iC_1^{wick}\Big(A^{[j-1,red]}(t,\cdot,h)\ ,\ H_{int}^{free}(-t,\cdot)\Big)(X) \tag{5.11}$$

and $A^{[j,red]}(0,X) = 0$.

In order to solve the equation (5.9), we use the following proposition, which is close to Propositions 6.1 and 6.2 in [7].

Proposition 5.3. *There exists a unique function G on $\mathbb{R}^2 \times H^2$ taking values in $\mathcal{L}(\mathcal{H}_{sp})$ satisfying, for all X in H^2:*

$$\frac{d}{dt}G(t,s,X) = iG(t,s,X)H_{int}^{free}(t,X), \qquad H_{int}^{free}(t,X) = H_{int}(\chi_t(X)), \tag{5.12}$$
$$G(s,s,X) = I.$$

Moreover, $G(t,s,q,p)$ is unitary. If $0 \leq s < t$ then the function $X \to G(t,s,X)$ belongs to $S(H^2, Q_t)$ where Q_t is defined in (1.20) with a norm equal to 1. This function also satisfies, for each s and t in $\mathbb{R}$, for each X in H^2:

$$G(s,0,X)^\star G(t,0,X) = G(t-s,0,\chi_s(X)). \tag{5.13}$$

Proof. The existence property and the fact that $G(t,s,X)$ is unitary can be considered as standard facts. Next, we see that, for all V in H^2:

$$\frac{d}{dt}dG(t,s,X)(V) = iG(t,s,X)dH_{int}^{free}(t,V) + idG(t,s,X)(V)H_{int}^{free}(t,X).$$

From Duhamel's principle and since $dG(s,s,X)(V)=0$ we then deduce:

$$dG(t,s,X)(V) = i\int_s^t G(\sigma,s,X)dH_{int}^{free}(\sigma,V)G(t,\sigma,X)d\sigma.$$

Iterating, for all sequences $(V_1,...,V_m)$ in H^2:

$$d^mG(t,s,\cdot)(V_1,...,V_m) = i^m\sum_{\varphi\in S_m}\int_{\Delta_m(t,s)} G(\sigma_1,s,\cdot)dH_{int}^{free}(\sigma_1,V_{\varphi(1)})G(\sigma_2,\sigma_1,\cdot)...$$

$$...dH_{int}^{free}(\sigma_2,V_{\varphi(2)})...dH_{int}^{free}(\sigma_m,V_{\varphi(m)})G(t,\sigma_m,\cdot)d\sigma_1...d\sigma_m$$

where S_m is the permutation group of m elements and:

$$\Delta_m(t,s) = \{(\sigma_1,...\sigma_m),\quad s<\sigma_1<...<\sigma_m<t\}.$$

Since $G(t,s,q,p)$ is unitary, one obtains:

$$|d^mG(t,s,\cdot)(V_1,...,V_m)| \leq \sum_{\varphi\in S_m}\int_{\Delta_m(t,s)} |dH_{int}^{free}(\sigma_1,V_{\varphi(1)})|...|dH_{int}^{free}(\sigma_m,V_{\varphi(m)})|d\sigma_1...d\sigma_m.$$

Consequently:

$$|d^mG(t,s,\cdot)(V_1,...,V_m)| \leq \prod_{j=1}^m\int_s^t |dH_{int}^{free}(\sigma,V_j)|d\sigma$$

where the above norm is the $\mathcal{L}(\mathcal{H}_{sp})$ norm. One notices that:

$$\int_0^t |dH_{int}^{free}(\sigma,V)|d\sigma \leq Q_t(V)^{1/2}.$$

The first point then follows. Concerning equality (5.13), we remark that, as functions of the variable t, the two hand sides satisfy the same differential equation:

$$\frac{d}{dt}F(t,X) = iF(t,X)H_{int}^{free}(t,X).$$

For the right hand side of (5.13), we see that according to (1.19), $H_{int}^{free}(t-s,\chi_s(X)) = H_{int}^{free}(t,X)$. Besides, the two hand sides of (5.13) are equal for $t=s$. Therefore, they are equal everywhere.

If G is the function of Proposition 5.3 and if A is written as in (1.13) then the function $A^{[0,red]}$ defined by:

$$A^{[0,red]}(t,X) = F_A(X)\otimes I + G(-t,0,X)^\star S_A G(-t,0,X) \tag{5.14}$$

satisfies (5.9). The first term, as a function of X, is a continuous linear function on H^2 and by Proposition 5.3, the second term belongs to $S(H^2,4\widetilde{Q}_t)$. The function defined by:

$$A^{[0]}(t,X) = F_A(\chi_t(X))\otimes I + G(t,0,X)S_A G(t,0,X)^\star \tag{5.15}$$

satisfies $A^{[0,red]}(t,X) = A^{[0]}(t,\chi_{-t}(X))$ and this function belongs to $S(H^2,4Q_t)$.

5.3 j-th order term, $j \geq 1$.

Fix $j \geq 1$. Being given a function $A^{[j-1,red]}(t,X)$ in $S(H^2, 16^j \widetilde{Q}_t)$ we want to find a function $A^{[j,red]}(t,X)$ satisfying (5.10) with $\Phi^{[j-1,red]}$ defined in (5.11). If G is the function of Proposition 5.3 then the function defined, for each X in H^2 by:

$$A^{[j,red]}(t,X) = \int_0^t G(-t,0,X)^\star G(-s,0,X)\Phi^{[j-1,red]}(s,X)G(-s,0,X)^\star G(-t,0,X)ds \tag{5.16}$$

satisfies (5.10) and $A^{[j,red]}(0,X) = 0$. By (5.13), we also have:

$$A^{[j,red]}(t,X) = \int_0^t G(t-s,0,\chi_{-t}(X))\Phi^{[j-1,red]}(s,X)G(t-s,0,\chi_{-t}(X))^\star ds.$$

According to the induction hypothesis, if $0 \leq s \leq t$ then the function $\Phi^{[j-1,red]}(s,\cdot)$ belongs to $S(H^2, 16^j \widetilde{Q}_s)$ also contained in $S(H^2, 16^j \widetilde{Q}_t)$. The function $X \to G(t-s,0,\chi_{-t}(X))$ is in $S(H^2, \widetilde{Q}_t)$. By the result (3.3) about composition, the function $A^{[j,red]}(t,\cdot)$ belongs to $S(H^2, 16^{j+1}\widetilde{Q}_t)$. The function $A^{[j]}(t,X)$ defined by:

$$A^{[j]}(t,X) = \int_0^t G(t-s,0,X)\Phi^{[j-1,red]}(s,\chi_t(X))G(t-s,0,X)^\star ds \tag{5.17}$$

where

$$\Phi^{[j-1]}(t,\cdot) = iC_1^{wick}(H_{int}(\cdot), A^{[j-1]}(t,\cdot)) - iC_1^{wick}(A^{[j-1]}(t,\cdot), H_{int}(\cdot)) \tag{5.18}$$

satisfies $A^{[j,red]}(t,X) = A^{[j]}(t,\chi_{-t}(X))$. This function is in $S(H^2, 16^{j+1}Q_t)$.

5.4 End of the proof of Theorem 1.3.

The function $A^{[0]}(t,\cdot)$ of Theorem 1.3 is defined in (5.15) and the functions $A^{[j]}(t,\cdot)$ $(j \geq 1)$ are constructed by iteration and given in (5.17) and (5.18). We saw that $A^{[0]}(t,\cdot)$ is the sum of a linear function on H^2 and of a function in $S(H^2, 4Q_t)$. For $j \geq 1$, $A^{[j]}(t,\cdot)$ belongs to $S(H^2, 16^{j+1}Q_t)$. Therefore, the claims ii) and iii) of Theorem 1.3 are already proved for these functions.

In order to prove (1.21), we first derive a similar result for the functions $A^{[j,red]}(t,\cdot)$ defined in (5.14) for $j=0$, and in (5.16) and (5.11) for $j \geq 1$. For each integer M, let us set:

$$S^{[M,red]}(t,X,h) = \sum_{j=0}^{M} h^j A^{[j,red]}(t,X).$$

By (5.14), (5.16) and (5.11), this function satisfies:

$$\frac{d}{dt}S^{[M,red]}(t,X,h) = i[H_{int}^{free}(-t,X), S^{[M,red]}(t,X,h)] +$$
$$+ihC_1^{wick}\Big(H_{int}^{free}(-t,\cdot)\ ,\ S^{[M,red]}(t,\cdot,h)\Big)(X) - ihC_1^{wick}\Big(S^{[M,red]}(t,\cdot,h)\ ,\ H_{int}^{free}(-t,\cdot)\Big)(X)$$
$$-h^{M+1}\Phi^{[M,red]}(t,X).$$

We saw that $A^{[0,red]}(t,\cdot)$ is the sum of a linear function on H^2 with a function in $S(H^2, 4\widetilde{Q}_t)$. For $j \geq 1$, $A^{[j,red]}(t,\cdot)$ belongs to $S(H^2, 16^{j+1}\widetilde{Q}_t)$. By theorem 2.3, there is an operator $S^{[M,red]}(t,h)$ whose Wick symbol is $S^{[M,red]}(t,X,h)$. By Theorem 3.2, this operator is the sum of a Segal field with an operator in $\mathcal{L}(16^{M+1}\widetilde{Q}_t)$. Consequently, according to (5.1):

$$\frac{d}{dt}S^{[M,red]}(t,h) = i[H_{int}^{free}(-t), S^{[M,red]}(t,h)] - h^{M+1}Op_h^{wick}\Big(\Phi^{[M,red]}(t,\cdot)\Big).$$

Comparing with (5.6), we obtain:

$$\frac{d}{dt}\Big(A^{[red]}(t,h)-S^{[M,red]}(t,h)\Big)=i[H_{int}^{free}(-t),A^{[red]}(t,h)-S^{[M,red]}(t,h)]+$$

$$+h^{M+1}Op_h^{wick}\Big(\Phi^{[M,red]}(t,\cdot)\Big).$$

We also have $A^{[red]}(0,h)-S^{[M,red]}(0,h)=0$ and therefore, by Duhamel principle:

$$A^{[red]}(t,h)-S^{[M,red]}(t,h)=h^{M+1}R^{[M,red]}(t,h)$$

where

$$R^{[M,red]}(t,h)=-\int_0^t U_h^{red}(t)U_h^{red}(s)^\star Op_h^{wick}\Big(\Phi^{[M,red]}(s,\cdot)\Big)U_h^{red}(s)U_h^{red}(t)^\star ds,$$

and $U_h^{red}(t)=e^{-i\frac{t}{h}H_{ph}}e^{i\frac{t}{h}H(h)}$. By Theorem 5.1 and since the quadratic form $\widetilde{Q}_t$ is an increasing function of $t>0$, then the operators $U_h^{red}(t)$, $U_h^{red}(s)$ $(0\leq s\leq t)$ and their adjoints are in $\mathcal{L}(\widetilde{Q}_t)$. We have seen that the function $\Phi^{[M,red]}(s,\cdot)$ belongs to $S(H^2,16^{M+1}\widetilde{Q}_s)$. As a consequence, $Op_h^{wick}\Big(\Phi^{[M,red]}(s,\cdot)\Big)$ belongs to $\mathcal{L}(16^{M+1}\widetilde{Q}_s)$, thus in $\mathcal{L}(16^{M+1}\widetilde{Q}_t)$ if $0\leq s\leq t$. According to the property (3.3) concerning the composition of operators in these classes, one observes that the operator $R^{[M,red]}(t,h)$ is in $\mathcal{L}(16^{M+3}\widetilde{Q}_t)$. According the covariance property, one therefore obtains that,

$$A(t,h)-S^{[M]}(t,h)=h^{M+1}R^{[M]}(t,h)$$

where $R^{[M]}(t,h)$ is in $\mathcal{L}(16^{M+3}Q_t)$. Turning back to the Wick symbols, point (1.21) of Theorem 1.3 is then proved.

6 Relation with physics equations (Maxwell and Bloch).

We set, according to (1.9):

$$B_j(x,t,h)=e^{i\frac{t}{h}H(h)}(B_j(x)\otimes I)e^{-i\frac{t}{h}H(h)},\qquad E_j(x,t,h)=e^{i\frac{t}{h}H(h)}(E_j(x)\otimes I)e^{-i\frac{t}{h}H(h)},\tag{6.1}$$

$$S_j^{[\lambda]}(t,h)=e^{i\frac{t}{h}H(h)}(I\otimes\sigma_j^{[\lambda]})e^{-i\frac{t}{h}H(h)}.\tag{6.2}$$

Since the initial observables $B_j(x)\otimes I$, $E_j(x)\otimes I$ and $I\otimes\sigma_j^{[\lambda]}$ are all under the form (1.13) then we can apply Theorem 1.3. The above Wick symbols have an asymptotic expansion in powers of h with coefficients defined by iteration in (5.17). The aim of this section is to write this iteration under a form closer to the usual physics equations when the observables under consideration are the above observables. This is the content of Theorem 6.2 below.

To this end, we shall show that these operator valued functions satisfy the following equations which are similar to those in [55]. We use vectorial notations $\mathbf{B}(x,t,h)=(B_1(x,t,h),B_2(x,t,h),B_3(x,t,h))$ and similarly for $\mathbf{E}(x,t,h)$ and $\mathbf{S}^{[\lambda]}(t,h)$. Given two operators triplets $\mathbf{A}=(A_1,A_2,A_3)$ and $\mathbf{B}=(B_1,B_2,B_3)$ we denote by $\mathbf{A}\times\mathbf{B}$ and $\mathbf{A}\times^{sym}\mathbf{B}$ operators triplets defined by:

$$\Big(\mathbf{A}\times\mathbf{B}\Big)_1=A_2B_3-A_3B_2,\qquad\Big(\mathbf{A}\times^{sym}\mathbf{B}\Big)_1=\frac{1}{2}(A_2B_3+B_3A_2-A_3B_2-B_2A_3),$$

and the other components being similarly defined by circular permutations. Thus, one has $\mathbf{A}\times^{sym}\mathbf{B}=(1/2)(\mathbf{A}\times\mathbf{B}-\mathbf{B}\times\mathbf{A})$.

Theorem 6.1. *The operator valued functions defined in (6.1) and (6.2) satisfy, using the above vectorial notations:*

$$\mathrm{div}\mathbf{B}(x,t,h) = \mathrm{div}\mathbf{E}(x,t,h) = 0 \tag{6.3}$$

$$\frac{\partial}{\partial t}\mathbf{B}(x,t,h) = -\mathrm{curl}\mathbf{E}(x,t,h) \tag{6.4}$$

$$\frac{\partial}{\partial t}\mathbf{E}(x,t,h) = \mathrm{curl}\mathbf{B}(x,t,h) + h\sum_{\lambda=1}^{P}\mathbf{S}^{[\lambda]}(t,h)\times \mathrm{grad}\rho(x-x_\lambda) \tag{6.5}$$

where ρ is defined in (1.23). One also has:

$$\frac{d}{dt}\mathbf{S}^{[\lambda]}(t,h) = 2(\beta + \mathbf{B}(x_\lambda,t,h))\times^{sym}\mathbf{S}^{[\lambda]}(t,h). \tag{6.6}$$

Proof. Denoting, for instance, $B_j(x,q,p)$ as the Wick symbol of the operator $B_j(x)$, we see that, for all (q,p) in H^2,

$$\sum_{j=1}^{3}\frac{\partial B_j}{\partial x_j}(x,X) = \sum_{j=1}^{3}\frac{\partial E_j}{\partial x_j}(x,X) = 0.$$

One then deduces (6.3), first for $t=0$ and then for arbitrary t. We also have, for all X in $D(M)$

$$\{H_{ph}(\cdot),\mathbf{E}\}(x,X) = \mathrm{curl}\ \mathbf{B}(x,X), \qquad \{H_{ph}(\cdot),\mathbf{B}\}(x,X) = -\mathrm{curl}\ \mathbf{E}(x,X).$$

Consequently, from Theorem 2.4:

$$[H_{ph},\mathbf{E}(x)] = (h/i)\mathrm{curl}\ \mathbf{B}(x), \qquad [H_{ph},\mathbf{B}(x)] = -(h/i)\mathrm{curl}\ \mathbf{E}(x).$$

Using Definitions (1.4), (1.5) and (1.23), we see that, for the Poisson brackets:

$$\{B_m(x,\cdot),B_n(y,\cdot)\} = 0$$

$$\{E_m(x,\cdot),B_n(y,\cdot)\} = \mathrm{grad}\rho(x-y)\cdot(e_m\times e_n)$$

with ρ defined in (1.23) and where (e_j) denotes the canonical basis of $\mathbb{R}^3$. The Poisson bracket of two continuous linear forms on H^2 is independent of $(q,p)\in H^2$. It is here a function depending only on $x\in\mathbb{R}^3$. We then deduce, concerning the operators:

$$[E_m(x),B_n(y)] = (h/i)\mathrm{grad}\rho(x-y)\cdot(e_m\times e_n).$$

Let us prove, for instance (6.5). One has

$$\frac{d}{dt}\mathbf{E}(x,t,h) = (i/h)e^{i(t/h)H(h)}[H(h),\mathbf{E}(x)\otimes I]e^{-i(t/h)H(h)}$$

$$= (i/h)e^{i(t/h)H(h)}\Big([H_{ph},\mathbf{E}(x)]\otimes I\Big)e^{-i(t/h)H(h)} + ...$$

$$... + i\sum_{\mu=1}^{P}\sum_{m=1}^{3}e^{i(t/h)H(h)}\Big([B_m(x_\mu),\mathbf{E}(x)]\otimes\sigma_m^{[\mu]}\Big)e^{-i(t/h)H(h)}$$

$$= e^{i(t/h)H(h)}\mathrm{curl}\ \mathbf{B}(x)e^{-i(t/h)H(h)} + ...$$

$$... - h\sum_{\mu=1}^{P}e^{i(t/h)H(h)}\Big(I\otimes\mathrm{grad}\ \rho(x-x_\mu)\times\sigma^{[\mu]}\Big)e^{-i(t/h)H(h)}.$$

We then deduce (6.5). In order to prove (6.6), we see that, from (6.1), (6.2) and (1.7), (1.8):

$$\frac{d}{dt}S_j^{[\lambda]}(t,h) = (i/h)e^{i(t/h)H(h)}[H(h), I\otimes\sigma_j^{[\lambda]}]e^{i(t/h)H(h)}$$

$$= i\sum_{\mu=1}^{P}\sum_{m=1}^{3} e^{i(t/h)H(h)}(\beta_m + B_m(x_\mu))\otimes[\sigma_m^{[\mu]},\sigma_j^{[\lambda]}]e^{-i(t/h)H(h)}.$$

One notices that $[\sigma_m^{[\mu]},\sigma_j^{[\lambda]}] = 0$ if $\mu\neq\lambda$. We then deduce that, for instance

$$\frac{d}{dt}S_1^{[\lambda]}(t,h) = 2e^{i(t/h)H(h)}\Big((\beta_2+B_2(x_\lambda))\otimes\sigma_3^{[\lambda]} - (\beta_3+B_3(x_\lambda))\otimes\sigma_2^{[\lambda]}\Big)e^{-i(t/h)H(h)}$$

$$= 2\Big((\beta_2+B_2(x_\lambda,t,h))S_3^{[\lambda]}(t,h) - (\beta_3+B_3(x_\lambda,t,h))S_2^{[\lambda]}(t,h)\Big).$$

According to (6.1), (6.2), the operators $B_j(x,t,h)$ and $S_k^{[\lambda]}(t,h)$ are commuting. Thus one obtains (6.6). □

Given two functions $\mathbf{F}$ and $\mathbf{G}$, C^∞ on H^2 and taking values in $(\mathcal{L}(\mathcal{H}_{sp}))$, we set, for each integer $j\geq 0$:

$$C^{j,wick}(F,G) = \sum_{|\alpha|=j}\left(\frac{1}{2^{|\alpha|}\alpha!}\right)(\partial_q - i\partial_p)^\alpha F\circ(\partial_q+i\partial_p)^\alpha G$$

where $\circ$ denotes the composition of two operators in $\mathcal{L}(\mathcal{H}_{sp})$. Being given two triplets $\mathbf{F}$ and $\mathbf{G}$ of functions on H^2 taking values in $(\mathcal{L}(\mathcal{H}_{sp}))$, we denote by $C^{j,wick,\times}(\mathbf{F},\mathbf{G})$ the function taking values in $(\mathcal{L}(\mathcal{H}_{sp}))^3$ defined by:

$$\Big(C^{j,wick,\times}(\mathbf{F},\mathbf{G})\Big)_1 = C^{j,wick}(F_2,G_3) - C^{j,wick}(F_3,G_2),$$

the other component being similarly defined by circular permutations and set:

$$C^{j,wick,\times,sym}(\mathbf{F},\mathbf{G}) = \frac{1}{2}\Big(C^{j,wick,\times}(\mathbf{F},\mathbf{G}) - C^{j,wick,\times}(\mathbf{G},\mathbf{F})\Big).$$

We denote by, for instance, $B_m(x,t,X,h)$ the Wick symbol of the operator $B_m(x,t,h)$ defined in (6.1). Then, $B_m^{[j]}(x,t,X)$ stands for the coefficient of h^j in the asymptotic expansion of this symbol, given by Theorem 1.3 (the operators considered here are all under the form (1.13)).

Theorem 6.2. *With the above notations, one has, for all $j\geq 0$:*

$$\mathrm{div}\mathbf{B}^{[j]}(x,t,X) = \mathrm{div}\mathbf{E}^{[j]}(x,t,X) = 0 \tag{6.7}$$

$$\frac{\partial}{\partial t}\mathbf{B}^{[j]}(x,t,X) = -\mathrm{curl}\mathbf{E}^{[j]}(x,t,X). \tag{6.8}$$

One has, for $j=0$:

$$\frac{\partial}{\partial t}\mathbf{E}^{[0]}(x,t,X) = \mathrm{curl}\mathbf{B}^{[0]}(x,t,X)) \tag{6.9}$$

and for $j\geq 1$:

$$\frac{\partial}{\partial t}\mathbf{E}^{[j]}(x,t,X) = \mathrm{curl}\mathbf{B}^{[j]}(x,t,X) + \sum_{\lambda=1}^{P}\mathbf{S}^{[\lambda,\mathbf{j-1}]}(t,X)\times\mathrm{grad}\rho(x-x_\lambda). \tag{6.10}$$

One has for $j=0$:

$$\frac{d}{dt}\mathbf{S}^{[\lambda,0]}(t,X) = 2(\beta+\mathbf{B}^{[0]}(x_\lambda,t,X))\times\mathbf{S}^{[\lambda,0]}(t,X) \tag{6.11}$$

and for $j\geq 1$:

$$\frac{d}{dt}\mathbf{S}^{[\lambda,j]}(t,X) = 2(\beta+\mathbf{B}^{[0]}(x_\lambda,t,X))\times\mathbf{S}^{[\lambda,j]}(t,X) + ... \tag{6.12}$$

$$... + \sum_{p+k+n=j,n<j} C^{p,wick,\times,sym}\Big(\mathbf{B}^{[k]}(x_\lambda,t,\cdot)\ ,\ \mathbf{S}^{[\lambda,n]}(t,\cdot)\Big)(X).$$

One has for $t = 0$ and $j = 0$:

$$\mathbf{B}^{[0]}(x,0,X) = \mathbf{B}(x,X), \qquad \mathbf{E}^{[0]}(x,0,X) = \mathbf{E}(x,X), \qquad \mathbf{S}^{[\lambda,0]}(0,X) = \sigma^{[\lambda]}$$

and for $t = 0$, $j \geq 1$:

$$\mathbf{B}^{[j]}(x,0,X) = 0, \qquad \mathbf{E}^{[j]}(x,0,X) = 0, \qquad \mathbf{S}^{[\lambda,j]}(0,X) = 0.$$

Proof of equalities (6.7)-(6.10). Let us prove, for example, equality (6.10). Equality (6.5) implies

$$\frac{\partial}{\partial t}\mathbf{E}(x,t,X,h) = \mathrm{curl}\mathbf{B}(x,t,X,h) + h\sum_{\lambda=1}^{P}\mathbf{S}^{[\lambda]}(t,X,h)\times \mathrm{grad}\rho(x-x_\lambda). \tag{6.13}$$

Let us show that the terms $\mathbf{E}^{[j]}(x,t,X)$ are differentiable with respect to t and that we have, in the sense of Theorem 1.3:

$$\frac{\partial}{\partial t}\mathbf{E}(x,t,X,h) \sim \sum_{r\geq 0} h^j \frac{\partial}{\partial t}\mathbf{E}^{[j]}(x,t,X). \tag{6.14}$$

To do this, we apply Theorem 1.3 with, for instance, the observable $A = E_1(x)$ and also with the observable $B = (i/h)[H(h), A]$. The observable A is under the form (1.13). The operator B is also under this form since:

$$B = (i/h)[H_{ph}, E_1(x)]\otimes I + i\sum_{\mu=1}^{P}\sum_{m=1}^{3}[B_m(x_\mu), E_1(x)]\otimes \sigma_m^{[\mu]}$$

$$= \left(\frac{\partial B_3(x)}{\partial x_2} - \frac{\partial B_2(x)}{\partial x_3}\right)\otimes I - h\sum_{\mu=1}^{P}\sum_{m=1}^{3}(\mathrm{grad}\rho(x-x_\mu)\cdot(e_1\times e_m))(I\otimes\sigma_m^{[\mu]}).$$

According to Theorem 1.3, the Wick symbol of the operator $(i/h)e^{i(t/h)H(h)}[H(h), E_1(x)]e^{-i(t/h)H(h)}$ has an asymptotic expansion described in this theorem. The coefficient of h^j in this development is the derivative of $E_1^{[j]}(x,t,q,p)$ and we indeed have (6.14). We similarly prove that, in the sense of Theorem 1.3:

$$\mathrm{curl}\ \mathbf{B}(x,t,X,h) \sim \sum_{j\geq 0} h^j \mathbf{B}^{[j]}(x,t,X).$$

Consequently the two hand sides in (6.13) have asymptotic expansions in powers of h. Identifying the coefficients of h^j in the two hand sides, we then deduce (6.10).

Proof of equalities (6.11) and (6.12). Equality (6.6) implies:

$$\frac{d}{dt}\mathbf{S}^{[\lambda]}(t,X,h) = \sigma_h^{wick}\Big((\beta+\mathbf{B}(x_\lambda,t,X,h))\times\mathbf{S}^{[\lambda]}(t,X,h)\Big) + ...$$

$$... - \sigma_h^{wick}\Big(\mathbf{S}^{[\lambda]}(t,h)\times(\beta+\mathbf{B}(x_\lambda,t,h))\Big)(X).$$

From Theorem 1.3, we have, for all integers M:

$$\sigma_h^{wick}(\mathbf{B}(x_\lambda,t))(X) = \sum_{k=0}^{M} h^k\mathbf{B}^{[k]}(x_\lambda,t,q,p) + h^{M+1}R_M(t,q,p,h)$$

$$\sigma_h^{wick}(\mathbf{S}^{[\lambda]}(t,h)) = \sum_{n=0}^{M} h^n\mathbf{S}^{[\lambda,n]}(x_\lambda,t,X) + h^{M+1}S_M(t,X,h)$$

where $R_M(t,\cdot,h)$ and $S_M(t,\cdot,h)$ belong to $S(H^2, 16^{M+5}Q_t)$, with a norm bounded uniformly in t and h when t belongs to a compact set of $\mathbb{R}$ and when h belongs to $(0,1]$. In view of Theorem 4.2, we then deduce:

$$\sigma_h^{wick}\Big((\beta+\mathbf{B}(x_\lambda,t,h))\times\mathbf{S}^{[\lambda]}(t,h)\Big) = \sum_{j=0}^{M} h^j(\beta+\mathbf{B}^{[0]}(x_\lambda,t,\cdot))\times\mathbf{S}^{[\lambda,j]}(t,\cdot) + ... \tag{6.15}$$

$$\cdots + \sum_{p+k+n\leq M, p+k>0} h^{p+k+n} C^{p,wick,\times,sym}\Big(\mathbf{B}^{[k]}(x_\lambda,t,.),\mathbf{S}^{[\lambda,n]}(t,.)\Big) + h^{M+1}T_M(t,.,h)$$

where $T_M(t,\cdot,h)$ belongs to some space $S(H^2,KQ_t)$ with some $K>0$ and with a norm bounded uniformly in t and h when t belongs to a compact set of $\mathbb{R}$ and when h is in $(0,1]$. As above, we show that $\mathbf{S}^{[\lambda,j]}(t,X)$ is differentiable with respect to t and that we have, in the sense of Theorem 1.3,

$$\frac{d}{dt}\mathbf{S}^{[\lambda]}(t,X,h) \sim \sum_{j\geq 0} h^j \frac{d}{dt}\mathbf{S}^{[\lambda,j]}(t,X). \tag{6.16}$$

When identifying the coefficients of h^j in (6.15) and in (6.16), one then obtains (6.11) and (6.12). □

For example, the first term satisfies the Bloch equations:

$$\frac{d}{dt}\mathbf{S}^{[\lambda,0]}(t,X) = 2(\beta+\mathbf{B}^{[0]}(x_\lambda,t,X))\times\mathbf{S}^{[\lambda,0]}(t,X). \tag{6.17}$$

The second term satisfies, from (6.12) and Theorem 4.2:

$$\frac{d}{dt}\mathbf{S}^{[\lambda,1]}(t,X) = 2(\beta+\mathbf{B}^{[0]}(x_\lambda,t,X))\times\mathbf{S}^{[\lambda,1]}(t,X)+\cdots \tag{6.18}$$

$$\cdots + 2\mathbf{B}^{[1]}(x_\lambda,t,X)\times\mathbf{S}^{[\lambda,0]}(t,X)+\mathbf{K}^{[\lambda,1]}(t,X)$$

with, for instance:

$$K_1^{[\lambda,1]}(t,\cdot) = dB_2^{[0]}(x_\lambda,t,\cdot)\cdot dS_3^{[\lambda,0]}(t,\cdot) - dB_3^{[0]}(x_\lambda,t,\cdot)\cdot dS_2^{[\lambda,0]}(t,\cdot). \tag{6.19}$$

We have denoted by $(dF)(q,p)\cdot(dG)(q,p)$ the scalar product of the differentials of two functions on H^2. The second term in the right hand side of (6.18) only reflects the influence, according to the classical Bloch equations, of the radiated field, according to Maxwell equations, by all the spins between times 0 and t. Only the term $K_1^{[\lambda,1]}(t,\cdot)$ is genuinely a quantum correction.

7 Photon emission semiclassical study.

This section is concerned with the proof of Theorem 1.4. To this end, the operators $E_j^{pol}(x)$ involved in this result are now precisely defined.

We need to introduce, at each point x in $\mathbb{R}^3$, six operators $B_j^{pol}(x)$ and $E_j^{pol}(x)$ $(1\leq j\leq 3)$ having no counterpart in classical physics. We set $\mathcal{F}(q,p)=(-p,q)$. We denote by E_+ the subspace of all $X\in H^2$ satisfying $JX=\mathcal{F}X$, where J is defined in (1.6), and by E_- the subspace of all $X\in H^2$ verifying $JX=-\mathcal{F}X$. These two subspaces correspond to the circular right and left polarization notions in physics. Then, $\Pi_\pm : H^2\to E_\pm$ stands for the corresponding orthogonal projections. One has,

$$\Pi_+X=\frac{1}{2}\Big(X-J\mathcal{F}X\Big),\quad \Pi_-X=\frac{1}{2}\Big(X+J\mathcal{F}X\Big),$$

and thus, $\Pi_+-\Pi_-=-J\mathcal{F}$. Set,

$$B_j^{pol}(x,q,p) = -B_j(x,J\mathcal{F}(q,p)),\quad E_j^{pol}(x,q,p) = -E_j(x,J\mathcal{F}(q,p)). \tag{7.1}$$

We denote by $B_j^{pol}(x)$ and $E_j^{pol}(x)$ the operators whose Wick symbols are $B_j^{pol}(x,q,p)$ and $E_j^{pol}(x,q,p)$. We set:

$$B_j^{pol}(x,t,h) = e^{i\frac{t}{h}H(h)}(B_j^{pol}(x)\otimes I)e^{-i\frac{t}{h}H(h)},\qquad E_j^{pol}(x,t,h) = e^{i\frac{t}{h}H(h)}(E_j^{pol}(x)\otimes I)e^{-i\frac{t}{h}H(h)}. \tag{7.2}$$

Lemma 7.1. *The operator $N'(t,h)$ defined in (1.24) satisfies:*

$$N'(t,h) = \sum_{\lambda=1}^{P} \sum_{m=1}^{3} e^{i\frac{t}{h}H(h)} (E_m^{pol}(x_\lambda) \otimes \sigma_m^{[\lambda]}) e^{-i\frac{t}{h}H(h)}. \tag{7.3}$$

This operator is bounded from $D(H(h))$ to $\mathcal{F}_s(H_{\mathbf{C}}) \otimes \mathcal{H}_{ph}$. We have also:

$$N'(t,h) = -\sum_{\lambda=1}^{P} \sum_{m=1}^{3} E_m^{pol}(x_\lambda, t, h) \circ S_m^{[\lambda]}(t,h). \tag{7.4}$$

Proof. We begin with (1.24). Clearly, one has $[H_{ph} \otimes I, N \otimes I] = 0$ since $H_{ph} = h\mathrm{d}\Gamma(M_\omega)$, $N = \mathrm{d}\Gamma(I)$ and since M_ω commutes with I. Therefore:

$$[H_{int}, N \otimes I] = \sum_{\lambda=1}^{P} \sum_{m=1}^{3} [B_m(x_\lambda), N] \otimes \sigma_m^{[\lambda]}.$$

By [19] (Lemma 2.5 iii), third identity), we have:

$$i[B_m(x), N] = -E_m^{pol}(x),$$

for all $x \in \mathbb{R}^3$. This point comes from standard commutations properties (see *e.g.*, Lemma 2.5 (ii) in [19]). Therefore, (7.3) follows from (1.24) and these previous computations. We saw that the Segal fields $E_m^{pol}(x_\lambda)$ are bounded from $D(H_{ph})$ to $\mathcal{F}_s(H_{\mathbf{C}})$, because the corresponding elements of H^2 are in $D(M_\omega^{-1/2})$. Moreover the domain of $H(h)$ is $D(H_{ph}) \otimes \mathcal{H}_{sp}$. The equality (7.4) is another formulation of (7.3). □

End of the proof of Theorem 1.4. The operators $B_j^{pol}(x)$ and $E_j^{pol}(x)$ defined in (7.1) are under the form (1.13). Theorem 1.3 shows that each operator $B_j^{pol}(x,t.h)$ and $E_j^{pol}(x,t,h)$ defined in (7.2) is the sum of a Segal field with an operator belonging to $\mathcal{L}(4Q_t)$, in the sense of Section 5. Theorem 1.3 shows that the symbols have asymptotic expansions that can be written, in order to simplify notations, as:

$$\sigma_h^{wick}(\mathbf{E}^{pol}(x,t,h))(X) = \mathbf{E}^{pol}(x,t,X,h) \sim \sum_{j\geq 0} h^j \mathbf{E}^{pol,j}(x,t,X).$$

Then Theorem 1.4 follows from (7.4), from Theorems 4.1 and 4.2, and from the above asymptotic expansions. We also have:

$$N^{[j]}(t,\cdot) = \sum_{\lambda=1}^{P} \sum_{q=1}^{3} \sum_{k+m+n=j} C^{k,wick}\Big(E_q^{pol,m}(x_\lambda,t,\cdot)\ ,\ S_q^{[\lambda,n]}(t,q,p)\Big).$$

□

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