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BIVARIATE FCLT FOR THE SAMPLE QUANTILE AND MEASURES OF DISPERSION FOR AUGMENTED GARCH(p, q) PROCESSES

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Bivariate FCLT for the Sample Quantile and Measures of Dispersion for Augmented GARCH(p, q) processes

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Abstract

In this note, we build upon the asymptotic theory for GARCH processes, considering the general class of augmented GARCH(p, q) processes. Our contribution is to complement the well-known univariate asymptotics by providing a bivariate functional central limit theorem between the sample quantile and the r -th absolute centred sample moment. This extends existing results in the case of identically and independently distributed random variables.

We show that the conditions for the convergence of the estimators in the univariate case suffice even for the joint bivariate asymptotics. We illustrate the general results with various specific examples from the class of augmented GARCH(p, q) processes and show explicitly under which conditions on the moments and parameters of the process the joint asymptotics hold.

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JEL classification: C13; C14; C30

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1 Introduction and Notation

Since the introduction of the ARCH and GARCH processes in the seminal papers by Engle, [13], and Bollerslev, [6], respectively, various GARCH modifications and extensions have been proposed and their statistical properties analysed (see e.g. [7] for a non-exhaustive ARCH glossary). Conditions for the stationarity of such processes, as well as central limit theorems (CLT) or functional central limit theorems (FCLT) have been obtained in various ways by exploiting the different dependence concepts underlying these GARCH type processes (see the introduction in [19] for references on CLT's under different dependence conditions).

The limit theorems extend also to different estimators apart from the underlying process itself, as for example: Powers of the process (e.g. [18] for augmented GARCH(1,1); [4], [19] for augmented GARCH(p, q)), sample autocovariance and sample variance (e.g. [20] for the GARCH(1,1); [2] for augmented GARCH(1,1)), or the sample quantile.

Still, joint asymptotics of such estimators have not been considered yet. It is what we are developing in this paper, providing bivariate functional central limit theorems for the sample quantile with the r -th absolute centred sample moment. This includes the case of the sample variance and also the sample

mean absolute deviation around the sample mean (MAD), two well-known and widely used measures of dispersion, extending the results obtained in [9] for identically and independently distributed (iid) random variables.

Note that the theoretical question arised from previous studies in financial risk management, one (see [8]) where the correlation between a log-ratio of sample quantiles with the sample MAD is measured using log-returns from different stock indices, the other (see [29] and [30]) considering the correlation of ‘the realized volatilities with the centred volatility increment’ for different underlying processes. Thus, we think that those asymptotic results may be of great use for applications in statistics or other application fields. For instance, coming back to financial risk management and risk measure estimation, we could extend results obtained for the Value-at-Risk, when estimated by the sample quantile, to Expected Shortfall using once again the FCLT.

To cover a broad range of GARCH processes, we focus on so called augmented GARCH(p, q) processes, introduced by Duan in [12]. They contain many well-known GARCH processes as special cases. Previous works on univariate CLT’s and stationarity conditions for this class of GARCH processes are, inter alia, [2],[4],[18] and [19].

The structure of the paper is as follows. We present in Section 2 the main results about the bivariate FCLT for the sample quantile and the r -th absolute centred sample moment for augmented GARCH(p, q) processes. Then, we present specific examples of well-known GARCH models in Section 3 and show how the general conditions in the main result translate for these specific cases. The proofs are given in Section 4.

Notation

We introduce the same notation as in [9]: Let (X_1, \dots, X_n) be a sample of size n . Assuming the random variables X_i ’s have a common distribution, denote their parent random variable (rv) X with parent cumulative distribution function (cdf) F_X , (and, given they exist,) probability density function (pdf) f_X , mean μ , variance σ^2 , and quantile of order p defined as $q_X(p) := \inf\{x \in \mathbb{R} : F_X(x) \geq p\}$. We denote the ordered sample by $X_{(1)} \leq \dots \leq X_{(n)}$.

We consider the sample estimators of the two quantities of interest, i.e. first the sample quantile for any order $p \in [0, 1]$ defined as $q_n(p) = X_{(\lceil np \rceil)}$, where $\lceil x \rceil = \min\{m \in \mathbb{Z} : m \geq x\}$, $\lfloor x \rfloor = \max\{m \in \mathbb{Z} : m \leq x\}$ and $[x]$, are the rounded-up, rounded-off integer-parts and the nearest-integer of a real number $x \in \mathbb{R}$, respectively. Second, the r -th absolute centred sample moment defined, for $r \in \mathbb{N}$, by

$$\hat{m}(X, n, r) := \frac{1}{n} \sum_{i=1}^n |X_i - \bar{X}_n|^r, \quad (1)$$

\bar{X}_n denoting the empirical mean. Special cases of this latter estimator include the sample variance ($r = 2$) and the sample mean absolute deviation around the sample mean ($r = 1$).

Recall the standard notation u^T for the transpose of a vector u and, for the signum function, $\text{sgn}(x) := -\mathbb{I}_{(x < 0)} + \mathbb{I}_{(x > 0)}$. Moreover the notations \xrightarrow{d} , $\xrightarrow{a.s.}$, \xrightarrow{P} and $\xrightarrow{D_d[0,1]}$ correspond to the convergence in distribution, almost surely, in probability and in distribution of a random vector in the d -dimensional Skorohod space $D_d[0, 1]$. Further, for real-valued functions f, g , we write $f(x) = O(g(x))$ (as $x \rightarrow \infty$) if and only if there exists a positive constant M and a real number x_0 s.t. $|f(x)| \leq Mg(x)$ for all $x \geq x_0$, and $f(x) = o(g(x))$ (as $x \rightarrow \infty$) if for all $\epsilon > 0$ there exists a real number x_0 s.t. $|f(x)| \leq \epsilon g(x)$ for all $x \geq x_0$. Analogously, for a sequence of rv’s X_n and constants a_n , we denote by $X_n = o_P(a_n)$ the convergence in probability to 0 of X_n/a_n .

2 The Bivariate FCLT

Let us introduce the augmented GARCH(p, q) process $X = (X_t)_{t \in \mathbb{Z}}$, due to Duan in [12], namely, for integers $p \geq 1$ and $q \geq 0$, X_t satisfies

$$X_t = \sigma_t \epsilon_t \quad (2)$$

$$\text{with } \Lambda(\sigma_t^2) = \sum_{i=1}^p g_i(\epsilon_{t-i}) + \sum_{j=1}^q c_j(\epsilon_{t-j})\Lambda(\sigma_{t-j}^2), \quad (3)$$

where (ϵ_t) is a series of iid rv's with mean 0 and variance 1, $\sigma_t^2 = \text{Var}(X_t)$ and $g_i, c_j, i = 1, \dots, p, j = 1, \dots, q$, are real-valued measurable functions. Also, as in [19], we restrict the choice of Λ to the so-called group of either polynomial GARCH(p, q) or exponential GARCH(p, q) processes (see Figure 1 in the Appendix):

$$(Lee) \quad \Lambda(x) = x^\delta, \text{ for some } \delta > 0, \quad \text{or} \quad \Lambda(x) = \log(x).$$

Clearly, for a strictly stationary solution to (2) and (3) to exist, the functions Λ, g_i, c_j as well as the innovation process $(\epsilon_t)_{t \in \mathbb{Z}}$ have to fulfill some regularity conditions (see e.g. [19], Lemma 1). Alike, for the bivariate FCLT to hold, certain conditions need to be fulfilled; we list them in the following.

First, conditions concerning the dependence structure of the process X . We use the concept of L_p -near-epoch dependence (L_p -NED), using a definition due to Andrews in [1] but restricted to stationary processes. Let $(Z_n)_{n \in \mathbb{Z}}$, be a sequence of rv's and $\mathcal{F}_s^t = \sigma(Z_s, \dots, Z_t)$, for $s \leq t$, the corresponding sigma-algebra. The usual L_p -norm is denoted by $\|\cdot\|_p := \mathbb{E}^{1/p}[|\cdot|^p]$. Let us recall the L_p -NED definition.

Definition 1 (L_p -NED, [1]) For $p > 0$, a stationary sequence $(X_n)_{n \in \mathbb{Z}}$ is called L_p -NED on $(Z_n)_{n \in \mathbb{Z}}$ if it holds, for $k \geq 0$, that

$$\|X_1 - \mathbb{E}[X_1 | \mathcal{F}_{n-k}^{n+k}]\|_p \leq \nu(k),$$

for non-negative constants $\nu(k)$ such that $\nu(k) \rightarrow 0$ as $k \rightarrow \infty$.

If $\nu(k) = O(k^{-\tau-\epsilon})$ for some $\epsilon > 0$, we say that X_n is L_p -NED of size $(-\tau)$.

If $\nu(k) = O(e^{-\delta k})$ for some $\delta > 0$, we say that X_n is geometrically L_p -NED.

The second set of conditions concerns the distribution of the augmented GARCH(p, q) process. We impose three different types of conditions as in the iid case (see [9]): First, the existence of a finite $2k$ -th moment for any integer $k > 0$ for the innovation process (ϵ_t) . Then, given that the process X is stationary, the continuity or l -fold differentiability of its distribution function F_X (at a given point or neighbourhood) for any integer $l > 0$, and the positivity of its density f_X (at a given point or neighbourhood). Those conditions are named as:

- (M_k) $\mathbb{E}[|\epsilon_0|^{2k}] < \infty$,
- (C_0) F_X is continuous,
- (C_l') F_X is l -times differentiable,
- (P) f_X is positive.

The third type of conditions is set on the parameters and functions of the augmented GARCH(p, q) process of the (*Lee*) family: Positivity of the functions used and boundedness in L_r -norm for either the polynomial GARCH, (P_r), or exponential/logarithmic GARCH, (L_r), respectively, for a given integer $r > 0$,

$$\begin{aligned}
(A) \quad & g_i \geq 0, c_j \geq 0, i = 1, \dots, p, j = 1, \dots, q, \\
(P_r) \quad & \sum_{i=1}^p \|g_i(\epsilon_0)\|_r < \infty, \quad \sum_{j=1}^q \|c_j(\epsilon_0)\|_r < 1, \\
(L_r) \quad & \mathbb{E}[\exp(4r \sum_{i=1}^p |g_i(\epsilon_0)|^2)] < \infty, \quad \sum_{j=1}^q |c_j(\epsilon_0)| < 1.
\end{aligned}$$

Note that condition (L_r) requires the c_j to be constant functions. Now, let us state the main result. To ease its presentation we introduce a trivariate normal random vector (functionals of X), $(U, V, W)^T$, with mean zero and the following covariance matrix:

$$(D) \left\{ \begin{aligned}
\text{Var}(U) &= \text{Var}(X_0) + 2 \sum_{i=1}^{\infty} \text{Cov}(X_i, X_0) \\
\text{Var}(V) &= \text{Var}(|X_0|^r) + 2 \sum_{i=1}^{\infty} \text{Cov}(|X_i|^r, |X_0|^r) \\
\text{Var}(W) &= \text{Var}\left(\frac{p - \mathbf{1}_{(X_0 \leq q_X(p))}}{f_X(q_X(p))}\right) + 2 \sum_{i=1}^{\infty} \text{Cov}\left(\frac{p - \mathbf{1}_{(X_i \leq q_X(p))}}{f_X(q_X(p))}, \frac{p - \mathbf{1}_{(X_0 \leq q_X(p))}}{f_X(q_X(p))}\right) \\
&= \frac{p(1-p)}{f_X^2(q_X(p))} + \frac{2}{f_X^2(q_X(p))} \sum_{i=1}^{\infty} (\mathbb{E}[\mathbf{1}_{(X_0 \leq q_X(p))} \mathbf{1}_{(X_i \leq q_X(p))}] - p^2) \\
\text{Cov}(U, V) &= \sum_{i \in \mathbb{Z}} \text{Cov}(|X_i|^r, X_0) = \sum_{i \in \mathbb{Z}} \text{Cov}(|X_0|^r, X_i) \\
\text{Cov}(U, W) &= \frac{-1}{f_X(q_X(p))} \sum_{i \in \mathbb{Z}} \text{Cov}(\mathbf{1}_{(X_i \leq q_X(p))}, X_0) = \frac{-1}{f_X(q_X(p))} \sum_{i \in \mathbb{Z}} \text{Cov}(\mathbf{1}_{(X_0 \leq q_X(p))}, X_i) \\
\text{Cov}(V, W) &= \frac{-1}{f_X(q_X(p))} \sum_{i \in \mathbb{Z}} \text{Cov}(|X_0|^r, \mathbf{1}_{(X_i \leq q_X(p))}) = \frac{-1}{f_X(q_X(p))} \sum_{i \in \mathbb{Z}} \text{Cov}(|X_0|^r, \mathbf{1}_{(X_0 \leq q_X(p))}).
\end{aligned} \right.$$

Theorem 2 (bivariate FCLT) Consider an augmented GARCH(p, q) process X as defined in (2) and (3) satisfying condition (Lee) , (C_0) at the mean μ for $r = 1$, and both conditions (C_2') , (P) at $q_X(p)$. Assume also conditions (M_r) , (A_0) , and either $(P_{max(1,r/\delta)})$ for polynomial GARCH, or (L_r) for exponential GARCH. Introducing the random vector $T_{n,r}(X) = \begin{pmatrix} q_n(p) - q_X(p) \\ \hat{m}(X, n, r) - m(X, r) \end{pmatrix}$, for $r \in \mathbb{Z}$, we have the following FCLT: For $t \in [0, 1]$, as $n \rightarrow \infty$,

$$\sqrt{n} t T_{[nt],r}(X) \xrightarrow{D_2[0,1]} \mathbf{W}_{\Gamma^{(r)}}(t)$$

where $(\mathbf{W}_{\Gamma^{(r)}}(t))_{t \in [0,1]}$ is the 2-dimensional Brownian motion with covariance matrix $\Gamma^{(r)} \in \mathbb{R}^{2 \times 2}$ defined for any $(s, t) \in [0, 1]^2$ by $\text{Cov}(\mathbf{W}_{\Gamma^{(r)}}(t), \mathbf{W}_{\Gamma^{(r)}}(s)) = \min(s, t)\Gamma^{(r)}$, where

$$\begin{aligned}
\Gamma_{11}^{(r)} &= \text{Var}(W), \\
\Gamma_{22}^{(r)} &= r^2 \mathbb{E}[X_0^{r-1} \text{sgn}(X_0)^r]^2 \text{Var}(U) + \text{Var}(V) - 2r \mathbb{E}[X_0^{r-1} \text{sgn}(X_0)^r] \text{Cov}(U, V), \\
\Gamma_{12}^{(r)} &= \Gamma_{21}^{(r)} = -r \mathbb{E}[X_0^{r-1} \text{sgn}(X_0)^r] \text{Cov}(U, W) + \text{Cov}(V, W),
\end{aligned}$$

$(U, V, W)^T$ being the trivariate normal vector (functionals of X) with mean zero and covariance given in (D) , all series being absolute convergent.

Remark 3 Note that for the bivariate FCLT in Theorem 2, apart from $(P_{\max(1,r/\delta)})$ or (L_r) respectively, which are the conditions for the univariate CLT of the r -th centred sample moment, also the conditions (C_0) at μ for $r = 1$, and (C_2') , (P) at $q_X(p)$, are needed. Requiring (C_2') , (P) at $q_X(p)$ exactly correspond to the conditions for the CLT of the sample quantile. Hence, the bivariate FCLT between the sample quantile and the r -th absolute centred sample moment does not require any additional condition in comparison to the respective univariate convergence.

Choosing $t = 1$ in Theorem 2 provides the usual CLT that we state for completeness:

Corollary 4 Consider an augmented GARCH(p, q) process as defined in (2) and (3). Under the same conditions as in Theorem 2, the joint behaviour of the sample quantile $q_n(p)$ (for $p \in (0, 1)$) and the r -th absolute centred sample moment $\hat{m}(X, n, r)$, is asymptotically bivariate normal:

$$\sqrt{n} \begin{pmatrix} q_n(p) - q_X(p) \\ \hat{m}(X, n, r) - m(X, r) \end{pmatrix} \xrightarrow[n \rightarrow \infty]{d} \mathcal{N}(0, \Gamma^{(r)}), \quad (4)$$

where the asymptotic covariance matrix $\Gamma^{(r)} = (\Gamma_{ij}^{(r)}, 1 \leq i, j \leq 2)$ is as in Theorem 2.

As special case we can also recover the CLT between the sample quantile and the r -th absolute centred sample moment in the iid case, given by Theorem 7 in [9]:

Corollary 5 Consider an augmented GARCH(p, q) process as defined in (2) and (3), choosing g_i, c_j, Λ such that σ_i^2 is a constant. Under the same conditions as in Theorem 2, the joint behaviour of the sample quantile $q_n(p)$ (for $p \in (0, 1)$) and the r -th absolute centred sample moment $\hat{m}(X, n, r)$, is asymptotically bivariate normal:

$$\sqrt{n} \begin{pmatrix} q_n(p) - q_X(p) \\ \hat{m}(X, n, r) - m(X, r) \end{pmatrix} \xrightarrow[n \rightarrow \infty]{d} \mathcal{N}(0, \Gamma^{(r)}), \quad (5)$$

where the asymptotic covariance matrix $\Gamma^{(r)} = (\Gamma_{ij}^{(r)}, 1 \leq i, j \leq 2)$ simplifies to

$$\begin{aligned} \Gamma_{11}^{(r)} &= \frac{p(1-p)}{f_X^2(q_X(p))}; & \Gamma_{22}^{(r)} &= r^2 \mathbb{E}[X_0^{r-1} \operatorname{sgn}(X_0)^r]^2 \sigma^2 + \operatorname{Var}(|X_0|^r) - 2r \mathbb{E}[X_0^{r-1} \operatorname{sgn}(X_0)^r] \operatorname{Cov}(X_0, |X_0|); \\ \Gamma_{12}^{(r)} &= \Gamma_{21}^{(r)} = \frac{1}{f_X(q_X(p))} (r \mathbb{E}[X_0^{r-1} \operatorname{sgn}(X_0)^r] \operatorname{Cov}(\mathbf{1}_{(X_0 \leq q_X(p))}, X_0) - \operatorname{Cov}(\mathbf{1}_{(X_0 \leq q_X(p))}, |X_0|^r)). \end{aligned}$$

Idea of the proof - Let us briefly describe the idea of the proof of Theorem 2, developed in Section 4. To prove the FCLT, we need to show that two conditions are fulfilled for the vector $T_{n,r}(X)$. These specific conditions arise from the application of the multivariate FCLT (Theorem A.1 in [3], which extends the univariate counterpart from, e.g., Billingsley in [5]) and are the following:

(H_1) A representation t_j of $T_{n,r}$ with $\mathbb{E}[t_j] = 0$ and $\mathbb{E}[t_j^2] < \infty$, such that we have, for $j \in \mathbb{Z}$

$$\frac{1}{n} \sum_{j=1}^n t_j = T_{n,r}(X) \text{ and } t_j = f(\epsilon_j, \epsilon_{j-1}, \dots),$$

where $f : \mathbb{R}^{2 \times \infty} \rightarrow \mathbb{R}^2$ is a measurable function and $(\epsilon_j, j \in \mathbb{Z})$ is a sequence of \mathbb{R}^2 -valued iid rv's with mean 0 and variance 1.

(H_2) A Δ -dependent approximation of t_j , i.e. the existence of a sequence of random vectors $(t_j^{(\Delta)}, j \in \mathbb{Z})$ such that, for any $\Delta \geq 1$, we have

$$t_j^{(\Delta)} = f^{(\Delta)}(\epsilon_{j-\Delta}, \dots, \epsilon_j, \dots, \epsilon_{j+\Delta}), \text{ for measurable functions } f^{(\Delta)} : \mathbb{R}^{2 \times (\Delta+1)} \rightarrow \mathbb{R}^2,$$

and $\sum_{\Delta \geq 1} \|t_0 - t_0^{(\Delta)}\|_2 < \infty.$

Checking the first condition, (H_1), is done in two steps. First, we show why we can use the Bahadur representation of the sample quantile given in [27] (Theorem 1). Second, we prove a corresponding representation for the r -th absolute centred sample moment (extending results from [9], [25]). These representations of the sample quantile and r -th absolute centred sample moment then naturally fulfil (H_1).

To proof the second condition (H_2), we need to find a Δ -dependent approximation $t_j^{(\Delta)}$. For this, we show that the existence of our chosen $t_j^{(\Delta)}$ can be reduced to the existence of a Δ -dependent approximation for the process X_j or powers of the process $|X_j|$, for which results in [19] can be used.

3 Examples

In this section we review some well-known examples of augmented GARCH(p, q) processes and discuss which conditions these models need to fulfill in order for the bivariate asymptotics of Theorem 2 to be valid.

Note that the moment condition on the innovations, (M_r), as well as the continuity and differentiability conditions, (C_2'), (P), each at $q_X(p)$, and (C_0) at μ for $r = 1$, remain the same for the whole class of augmented GARCH processes. But, depending on the specifications of the process, (2) and (3), the conditions, ($P_{\max(1, r/\delta)}$) for polynomial GARCH or (L_r) for exponential GARCH respectively, translate differently in the various examples.

For this, we introduce in Table 1 different augmented GARCH(p, q) models by providing for each the corresponding volatility equation, (3), and the specifications of the functions g_i and c_j . We consider 10 models which belong to the group of polynomial GARCH ($\Lambda(x) = x^\delta$) and two examples of exponential GARCH ($\Lambda(x) = \log(x)$). As the nesting of the different models presented is not obvious, we give a schematic overview in Figure 1 in the Appendix. An explanation of the abbreviations for, and authors of, the different models can be found there too. Note that the presented selection of augmented GARCH (p, q) processes is not exhaustive.

Note that in Table 1 the specification of g_i is the same for the whole APGARCH family (only the c_j change), whereas for the two exponential GARCH models, it is the reverse. The general restrictions on the parameters are as follows: $\omega > 0, \alpha_i \geq 0, -1 \leq \gamma_i \leq 1, \beta_j \geq 0$ for $i = 1, \dots, p, j = 1, \dots, q$. Further, the parameters in the GJR-GARCH (TGARCH) are denoted with an asterisk (with a plus or minus) as they are not the same as in the other models (see the Appendix for details).

Table 1: Presentation of the volatility equation (3) and the corresponding specifications of functions g_i, c_j for selected augmented GARCH models.

	standard formula for $\Lambda(\sigma_t^2)$	corresponding specifications of g_i, c_j in (3)
Polynomial GARCH		
APGARCH family	$\sigma_t^{2\delta} = \omega + \sum_{i=1}^p \alpha_i (y_{t-i} - \gamma_i y_{t-i})^{2\delta} + \sum_{j=1}^q \beta_j \sigma_{t-j}^{2\delta}$	$g_i = \omega/p \text{ and } c_j = \alpha_j (\epsilon_{t-j} - \gamma_j \epsilon_{t-j})^{2\delta} + \beta_j$
AGARCH	$\sigma_t^2 = \omega + \sum_{i=1}^p \alpha_i (y_{t-i} - \gamma_i y_{t-i})^2 + \sum_{j=1}^q \beta_j \sigma_{t-j}^2$	$c_j = \alpha_j (\epsilon_{t-j} - \gamma_j \epsilon_{t-j})^2 + \beta_j$
GJR-GARCH	$\sigma_t^2 = \omega + \sum_{i=1}^p (\alpha_i^* + \gamma_i^* \mathbb{I}_{(y_{t-i} < 0)}) y_{t-i}^2 + \sum_{j=1}^q \beta_j \sigma_{t-j}^2$	$c_j = \beta_j + \alpha_j^* \epsilon_{t-j}^2 + \gamma_j^* \max(0, -\epsilon_{t-j})^2$
GARCH	$\sigma_t^2 = \omega + \sum_{i=1}^p \alpha_i y_{t-i}^2 + \sum_{j=1}^q \beta_j \sigma_{t-j}^2$	$c_j = \alpha_j \epsilon_{t-j}^2 + \beta_j$
ARCH	$\sigma_t^2 = \omega + \sum_{i=1}^p \alpha_i y_{t-i}^2$	$c_j = \alpha_j \epsilon_{t-j}^2$
TGARCH	$\sigma_t = \omega + \sum_{i=1}^p (\alpha_i^+ \max(y_{t-i}, 0) - \alpha_i^- \min(y_{t-i}, 0)) + \sum_{j=1}^q \beta_j \sigma_{t-j}$	$c_j = \alpha_j \epsilon_{t-j} - \alpha_j \gamma_j \epsilon_{t-j} + \beta_j$
TSGARCH	$\sigma_t = \omega + \sum_{i=1}^p \alpha_i y_{t-i} + \sum_{j=1}^q \beta_j \sigma_{t-j}$	$c_j = \alpha_j \epsilon_{t-j} + \beta_j$
PGARCH	$\sigma_t^\delta = \omega + \sum_{i=1}^p \alpha_i y_{t-i} ^\delta + \sum_{j=1}^q \beta_j \sigma_{t-j}^\delta$	$c_j = \alpha_j \epsilon_{t-j} ^\delta + \beta_j$
VGARCH	$\sigma_t^2 = \omega + \sum_{i=1}^p \alpha_i (\epsilon_{t-i} + \gamma_i)^2 + \sum_{j=1}^q \beta_j \sigma_{t-j}^2$	$g_i = \omega/p + \alpha_i (\epsilon_{t-i} + \gamma_i)^2 \text{ and } c_j = \beta_j$
NGARCH	$\sigma_t^2 = \omega + \sum_{i=1}^p \alpha_i (y_{t-i} + \gamma_i \sigma_{t-i})^2 + \sum_{j=1}^q \beta_j \sigma_{t-j}^2$	$g_i = \omega/p \text{ and } c_j = \alpha_j (\epsilon_{t-j} + \gamma_j)^2 + \beta_j$
Exponential GARCH		$c_j = \beta_j \text{ and}$
MGARCH	$\log(\sigma_t^2) = \omega + \sum_{i=1}^p \alpha_i \log(\epsilon_{t-i}^2) + \sum_{j=1}^q \beta_j \log(\sigma_{t-j}^2)$	$g_i = \omega/p + \alpha_i \log(\epsilon_{t-i}^2)$
EGARCH	$\log(\sigma_t^2) = \omega + \sum_{i=1}^p \alpha_i (\epsilon_{t-i} - \mathbb{E} \epsilon_{t-i}) + \gamma_i \epsilon_{t-i} + \sum_{j=1}^q \beta_j \log(\sigma_{t-j}^2)$	$g_i = \omega/p + \alpha_i (\epsilon_{t-i} - \mathbb{E} \epsilon_{t-i}) + \gamma_i \epsilon_{t-i}$

In Tables 2 and 3 we present how the conditions ($P_{\max(1,r/\delta)}$) or (L_r) translate for each model. Table 2 treats the specific case of an augmented GARCH(p, q) process with $p = q = 1$ and is presented here whereas Table 3 treats the general case for arbitrary $p \geq 1, q \geq 0$ and is deferred to the Appendix. In the first column we consider the conditions for the general r -th absolute centred sample moment, $r \in \mathbb{N}$. Of biggest interest to us are the specific cases of the sample MAD ($r = 1$) and the sample variance ($r = 2$) as measure of dispersion estimators respectively, presented in the second and third column.

For the selected polynomial GARCH models the requirement $\sum_{i=1}^p \|g_i(\epsilon_0)\|_{\max(1,r/\delta)} < \infty$ in condition ($P_{\max(1,r/\delta)}$) will always be fulfilled. Thus, we only need to analyse the condition $\sum_{j=1}^q \|c_j(\epsilon_0)\|_{\max(1,r/\delta)} < 1$.

Note that in Table 2 (and also Table 3) the restrictions on the parameter space, given by ($P_{\max(1,r/\delta)}$) or (L_r) respectively, are the same as the conditions for univariate FCLT's of the process X_t^r itself (see [4], [18]). For $r = 1$, they coincide with the conditions for e.g. β -mixing with exponential decay (see [10]).

Table 2: Conditions ($P_{\max(1,r/\delta)}$) or (L_r) respectively translated for different augmented GARCH(1,1) models. Left column for the general r -th absolute centred sample moment, middle for the MAD ($r = 1$) and right for the variance ($r = 2$).

augmented GARCH (1, 1)	$r \in \mathbb{N}$	$r = 1$	$r = 2$
APGARCH	$\mathbb{E}[\alpha_1 (\epsilon_0 - \gamma_1 \epsilon_{t-1})^{2\delta} + \beta_1 ^r] < 1$	$\alpha_1 \mathbb{E}[(\epsilon_0 - \gamma_1 \epsilon_{t-1})^{2\delta}] + \beta_1 < 1$	$\mathbb{E}[\alpha_1 (\epsilon_0 - \gamma_1 \epsilon_{t-1})^{2\delta} + \beta_1 ^2] < 1$
AGARCH	$\mathbb{E}[\alpha_1 (\epsilon_0 - \gamma_1 \epsilon_{t-1})^2 + \beta_1 ^r] < 1$	$\alpha_1 \mathbb{E}[(\epsilon_0 - \gamma_1 \epsilon_{t-1})^2] + \beta_1 < 1$	$\mathbb{E}[\alpha_1 (\epsilon_0 - \gamma_1 \epsilon_{t-1})^2 + \beta_1 ^2] < 1$
GJR-GARCH	$\mathbb{E}[\alpha_1^* \epsilon_0^2 + \beta_1 + \gamma_1^* \max(0, -\epsilon_0^2) ^r] < 1$	$\alpha_1^* + \beta_1 + \gamma_1^* \mathbb{E}[\max(0, -\epsilon_0^2)] < 1$	$\mathbb{E}[\alpha_1^* \epsilon_0^2 + \beta_1 + \gamma_1^* \max(0, -\epsilon_0^2) ^2] < 1$
GARCH	$\mathbb{E}[(\alpha_1 \epsilon_0^2 + \beta_1)^r] < 1$	$\alpha_1 + \beta_1 < 1$	$\alpha_1^2 \mathbb{E}[\epsilon_0^4] + \alpha_1 \beta_1 + \beta_1^2 < 1$
ARCH	$\alpha_1^r \mathbb{E}[\epsilon_0^{2r}] < 1$	$\alpha_1 < 1$	$\alpha_1^2 \mathbb{E}[\epsilon_0^4] < 1$
TGARCH	$\mathbb{E}[\alpha_1 \epsilon_{t-1} - \alpha_1 \gamma_1 \epsilon_{t-1} + \beta_1 ^r] < 1$	$\alpha_1 \mathbb{E} \epsilon_{t-1} + \beta_1 < 1$	$\mathbb{E}[\alpha_1 \epsilon_{t-1} - \alpha_1 \gamma_1 \epsilon_{t-1} + \beta_1 ^2] < 1$
TSGARCH	$\mathbb{E}[\alpha_1 \epsilon_{t-1} + \beta_1 ^r] < 1$	$\alpha_1 \mathbb{E} \epsilon_{t-1} + \beta_1 < 1$	$\mathbb{E}[\alpha_1 \epsilon_{t-1} + \beta_1 ^2] < 1$
PGARCH	$\mathbb{E}[\alpha_1 \epsilon_0 + \beta_1 ^{2r}] < 1$	$\alpha_1 + 2\alpha_1 \beta_1 \mathbb{E} \epsilon_0 + \beta_1^2 < 1$	$\mathbb{E}[\alpha_1 \epsilon_0 + \beta_1 ^4] < 1$
VGARCH		for any $r \in \mathbb{N}$: $\beta_1 < 1$	
NGARCH	$\mathbb{E}[\alpha_1 (\epsilon_0 + \gamma_1)^2 + \beta_1 ^r] < 1$	$\alpha_1 (1 + \gamma_1^2) + \beta_1 < 1$	$\mathbb{E}[\alpha_1 (\epsilon_0 + \gamma_1)^2 + \beta_1 ^2] < 1$
MGARCH		for any $r \in \mathbb{N}$: $\mathbb{E}[\exp(4r \omega/p + \alpha_1 \log(\epsilon_0^2) ^2)] < \infty$ and $ \beta_1 < 1$	
EGARCH		for any $r \in \mathbb{N}$: $\mathbb{E}[\exp(4r \omega/p + \alpha_1 (\epsilon_0 - \mathbb{E} \epsilon_0) + \gamma_1 \epsilon_0 ^2)] < \infty$ and $ \beta_1 < 1$	

4 Proofs

Before stating the proof of the main theorem, let us start with two auxiliary results. As it requires some work to find the asymptotics of $\hat{m}(X, n, r) = \frac{1}{n} \sum_{i=1}^n |X_i - \bar{X}_n|^r$ for any integer $r \geq 1$, and such a result is of interest in its own right, we give it separately in Proposition 7. To prove it, we need the following Lemma, which extends Lemma 2.1 in [25] (case $v = 1$) to any moment $v \in \mathbb{N}$, and the iid case presented in Lemma 8 in [9].

Lemma 6 *Consider a stationary and ergodic time-series $(X_n, n \geq 1)$ with parent rv X . Then, for $v = 1$ or 2, given that the 2nd moment of X exists, or, for any integer $v > 2$, given that the v -th moment of X exists, it holds, as $n \rightarrow \infty$, almost surely that*

$$\frac{1}{n} \sum_{i=1}^n (X_i - \mu)^v (|X_i - \bar{X}_n| - |X_i - \mu|) = (\bar{X}_n - \mu) \left(\mathbb{E}[(X - \mu)^v \operatorname{sgn}(\mu - X)] + o_P(1) \right) + o_P(1/\sqrt{n}). \quad (6)$$

Proof of Lemma 6. The proof starts exactly as its equivalent in the iid case; see proof of Lemma 8 in [9]. The argumentation needs to be adapted at the end in two points, using the stationarity and ergodicity of the process. Here, it follows by ergodicity and stationarity that $\sqrt{n}|\bar{X}_n - \mu|^{v+1} \xrightarrow[n \rightarrow \infty]{P} 0$ for any integer $v \geq 1$. Further, as a last step, we use the ergodicity of the process, instead of the law of large numbers, to conclude that

$$\frac{1}{n} \sum_{i=1}^n (X_i - \mu)^v \operatorname{sgn}(\mu - X_i) \xrightarrow[n \rightarrow \infty]{a.s.} \mathbb{E}[(X - \mu)^v \operatorname{sgn}(\mu - X)]. \quad \square$$

Now we are ready to state the asymptotic relation between the r -th absolute centred sample moment with known and unknown mean, respectively. This enables us to compute the asymptotics of $\hat{m}(X, n, r)$. As for Lemma 6, it is an extension to the stationary and ergodic case of Proposition 9 in the iid case [9].

Proposition 7 *Consider a stationary and ergodic time series $(X_n, n \geq 1)$ with parent rv X . Then, given that the r -th moment of X exists, with $r \in \mathbb{N}$, it holds, as $n \rightarrow \infty$, that*

$$\sqrt{n} \left(\frac{1}{n} \sum_{i=1}^n |X_i - \bar{X}_n|^r \right) = \sqrt{n} \left(\frac{1}{n} \sum_{i=1}^n |X_i - \mu|^r \right) - r \sqrt{n} (\bar{X}_n - \mu) \mathbb{E}[(X - \mu)^{r-1} \text{sgn}(X - \mu)^r] + o_P(1).$$

Proof of Proposition 7. The proof can be extended from the proof of Proposition 9 in the iid case in [9]. We distinguish again three different cases for r : Even integers r , $r = 1$, and odd integers $r > 1$. Whereas the first two cases (r even and $r = 1$) hold true without modifications from the iid case, for odd integers $r > 1$, we point out the three differences to the proof in the iid case. First, $\sqrt{n}(\bar{X}_n - \mu)^v \xrightarrow{P} 0$, for $v \geq 2$, follows from the stationarity and ergodicity of the process. Second, we use the ergodicity instead of the law of large numbers. Third, we use Lemma 6 instead of its counterpart in the iid case, Lemma 8 in [9]. \square

Proof of Theorem 2. The proof consists of four steps. We first show that the process (X_t) fulfils the conditions required for having a Bahadur representation of the sample quantile, second, that a similar representation holds for the r -th absolute centred sample moment, third, the construction of a so called Δ -dependent approximation, which we then use in the fourth step for a multivariate FCLT.

Step 1: Bahadur representation of the sample quantile - conditions.

The Bahadur representation of the sample quantile is well known, see e.g. Theorem 1 of [27]. It holds under some conditions that we need to verify. For the ease of comparison, we adapt some of the notation of Theorem 1, [27]. We have the following:

- Choosing the bivariate function $g(x, t) := \mathbb{I}_{(x \leq t)}$, the non-negativity, boundedness, measurability, and non-decreasingness in the second variable, are straightforward. The function g also satisfies the variation condition uniformly in some neighbourhood of $q_X(p)$ if it is Lipschitz-continuous (see Example 1.5 in [27]). But the latter follows from condition (C_2') .
- The differentiability of $\mathbb{E}[g(X, t)] = F_X(t)$ and positivity of its derivative at $t = q_X(p)$ are given by condition (P) at $q_X(p)$.
- Equation (12) in [27] is fulfilled as, by our assumption (C_2') , F_X is twice differentiable in $q_X(p)$ (see Remark 2, [27]).
- The stationarity of the process follows from assumption $(P_{\max(1, r/\delta)})$ or (L_r) , respectively, and Lemma 1 of [19].
- Lastly, let us verify that the process (X_t) is L_1 -NED with polynomial rate. Denoting, for $s \leq t$, the sigma-algebra $\mathcal{F}_s^t = \sigma(\epsilon_s, \dots, \epsilon_t)$, we can write for any integer $\Delta \geq 1$

$$\|X_t - \mathbb{E}[X_t | \mathcal{F}_{t-\Delta}^{t+\Delta}]\|_2^2 = \|\sigma_t - \mathbb{E}[\sigma_t | \mathcal{F}_{t-\Delta}^{t+\Delta}]\|_2^2 \sqrt{\mathbb{E}[\epsilon_t^2]}.$$

But $\mathbb{E}[\epsilon_t^2] < \infty$ since (M_r) holds. Notice that the property of being geometrically L_2 -NED, $\mathbb{E}[\sigma_t | \mathcal{F}_{t-\Delta}^{t+\Delta}]\|_2 = O(e^{-\kappa\Delta})$ for some $\kappa > 0$, implies L_1 -NED with polynomial rate, as

$$\|\sigma_t - \mathbb{E}[\sigma_t | \mathcal{F}_{t-\Delta}^{t+\Delta}]\|_1 \leq \|\sigma_t - \mathbb{E}[\sigma_t | \mathcal{F}_{t-\Delta}^{t+\Delta}]\|_2 = O(e^{-\kappa\Delta}) = O(\Delta^{-(\beta+3)}), \quad (7)$$

for some $\beta > 3$. So it suffices showing that σ_t is geometrically L_2 -NED. For the polynomial GARCH, it follows from Corollary 1 in [19], which can be applied as (A_0) and (P_1) hold. For the logarithmic GARCH case, it follows from Corollary 3 in [19] as (A_0) and (L_r) hold.

Thus, we can use Theorem 1 of [27] and write, a.s., as $n \rightarrow \infty$,

$$q_n(p) - q_X(p) + \frac{F(q_X(p)) - F_n(q_X(p))}{f_X(q_X(p))} = o\left(n^{-\frac{5}{8} - \frac{1}{8}\gamma} (\log n)^{\frac{3}{4}} (\log \log n)^{\frac{1}{2}}\right), \quad (8)$$

for some $\gamma \geq 1/5$.

Step 2: Representation of the r-th absolute centred sample moment -conditions.

The representation being given in Proposition 7 under some conditions, we only need to check that we fulfil them.

- The stationarity of the process is satisfied under $(P_{\max(1,r/\delta)})$ or (L_r) as observed in Step 1.
- For the moment condition and ergodicity, we simply verify that the conditions for a CLT of X_t^r (or $|X_t|^r$) are fulfilled, distinguishing between the polynomial and logarithmic case. Conditions (M_r) , (A) , $(P_{\max(1,r/\delta)})$ in the polynomial case, and (M_r) , (A) , (L_r) in the logarithmic case respectively, imply the CLT, using Corollary 2 and 3 in [19], respectively.

Step 3: Conditions for applying the FCLT

To use a multivariate FCLT, we need to verify the two conditions (H_1) and (H_2) given after Theorem 2. In fact we adapt them to a three-dimensional version to simplify the computation, then apply the continuous mapping theorem to get back a two-dimensional representation. This will be made explicit in Step 4.

Condition (H_1) translates to first finding a representation u_j such that

$$\mathbb{E}[u_j] = 0, \quad \mathbb{E}[u_j^2] < \infty, \quad u_j = f(\epsilon_j, \epsilon_{j-1}, \dots), j \in \mathbb{Z}, \quad (9)$$

where $f : \mathbb{R}^{3 \times \infty} \rightarrow \mathbb{R}^3$ is a measurable function and $(\epsilon_j, j \in \mathbb{Z})$ a sequence of \mathbb{R}^3 -valued iid rv's with mean 0 and variance 1. Second, condition (H_2) comes back to finding a Δ -dependent approximation $u_j^{(\Delta)}$ such that, for any integer $\Delta \geq 1$, it holds, for $j \in \mathbb{Z}$,

$$u_j^{(\Delta)} = f^{(\Delta)}(\epsilon_{j-\Delta}, \dots, \epsilon_j, \dots, \epsilon_{j+\Delta}) \quad (10)$$

$$\text{and } \sum_{\Delta \geq 1} \|u_0 - u_0^{(\Delta)}\|_2 < \infty, \quad (11)$$

where $f^{(\Delta)} : \mathbb{R}^{3 \times (2\Delta+1)} \rightarrow \mathbb{R}^3$ is a measurable function.

Therefore, let us define, anticipating its use in Step 4 for the FCLT,

$$u_j = \begin{pmatrix} X_j \\ |X_j|^r - m(X, r) \\ \frac{p - \mathbf{1}_{(X_j \leq q_X(p))}}{f_X(q_X(p))} \end{pmatrix}.$$

We need to verify that u_j fulfils (9): $\mathbb{E}[u_j] = 0$ holds by construction. $\mathbb{E}[|X_j|^{2r}] < \infty$ is guaranteed since $|X_t|^r$ satisfies a CLT (see Step 2), thus also $\mathbb{E}[u_j^2] < \infty$. Finally $X_j = f(\epsilon_j, \epsilon_{j-1}, \dots)$ follows from Lemma 1 in [19], as we assume (A) . Hence, this latter relation also holds for functionals of X_j , so for u_j .

Then, we define a Δ -dependent approximation $u_0^{(\Delta)}$ satisfying (10) and (11). Denote, for the ease of notation, $X_{0\Delta} := \mathbb{E}[X_0 | \mathcal{F}_{-\Delta}^{+\Delta}]$, and set $u_0^{(\Delta)} = \begin{pmatrix} X_{0\Delta} \\ \mathbb{E}[|X_0|^r | \mathcal{F}_{-\Delta}^{+\Delta}] - m(X, r) \\ \frac{p - \mathbf{1}_{(X_{0\Delta} \leq q_X(p))}}{f_X(q_X(p))} \end{pmatrix}$ with $\mathcal{F}_s^t = \sigma(\epsilon_s, \dots, \epsilon_t)$

for $s \leq t$. Thus, (10) is fulfilled by construction. Let us verify (11). We can write

$$\begin{aligned}
& \sum_{\Delta \geq 1} \|u_0 - u_0^{(\Delta)}\|_2 \\
&= \sum_{\Delta \geq 1} \mathbb{E} \left[(X_0 - X_{0\Delta})^2 + \left(|X_0|^r - \mathbb{E}[|X_0|^r | \mathcal{F}_{-\Delta}^{+\Delta}] \right)^2 + \frac{1}{f_X^2(q_X(p))} \left(-\mathbf{1}_{(X_0 \leq q_X(p))} + \mathbf{1}_{(X_{0\Delta} \leq q_X(p))} \right)^2 \right]^{1/2} \\
&\leq \sum_{\Delta \geq 1} \left(\|X_0 - X_{0\Delta}\|_2 + \left\| |X_0|^r - \mathbb{E}[|X_0|^r | \mathcal{F}_{-\Delta}^{+\Delta}] \right\|_2 + \frac{1}{f_X(q_X(p))} \left\| -\mathbf{1}_{(X_0 \leq q_X(p))} + \mathbf{1}_{(X_{0\Delta} \leq q_X(p))} \right\|_2 \right).
\end{aligned} \tag{12}$$

Noticing that

$$\begin{aligned}
|-\mathbf{1}_{(X_0 \leq q_X(p))} + \mathbf{1}_{(X_{0\Delta} \leq q_X(p))}| &= \mathbf{1}_{(X_{0\Delta} > q_X(p) \geq X_0)} + \mathbf{1}_{(X_0 > q_X(p) \geq X_{0\Delta})} \\
&\leq \mathbf{1}_{(X_{0\Delta} - X_0 > 0)} \mathbf{1}_{(q_X(p) - X_0 \geq 0)} + \mathbf{1}_{(0 > q_X(p) - X_0)} \mathbf{1}_{(0 > X_{0\Delta} - X_0)} \\
&= \mathbf{1}_{(X_0 \leq q_X(p))} (\mathbf{1}_{(X_{0\Delta} - X_0 > 0)} - \mathbf{1}_{(0 > X_{0\Delta} - X_0)}) + \mathbf{1}_{(0 > X_{0\Delta} - X_0)} \\
&\leq \mathbf{1}_{(X_0 \leq q_X(p))} \mathbf{1}_{(|X_{0\Delta} - X_0| > 0)} + \mathbf{1}_{(|X_{0\Delta} - X_0| > 0)} \leq 2 \mathbf{1}_{(|X_{0\Delta} - X_0| > 0)},
\end{aligned} \tag{13}$$

we deduce that

$$\left\| -\mathbf{1}_{(X_0 \leq q_X(p))} + \mathbf{1}_{(X_{0\Delta} \leq q_X(p))} \right\|_2 \leq \sqrt{\mathbb{E}[2 \mathbf{1}_{(|X_{0\Delta} - X_0| > 0)}]} = \sqrt{2} \sqrt{\mathbb{P}(|X_{0\Delta} - X_0| > 0)}.$$

But, by the Markov inequality and Jensen's inequality, we have, for any $\epsilon > 0$,

$$\mathbb{P}(|X_{0\Delta} - X_0| \geq \epsilon) \leq \frac{\|X_{0\Delta} - X_0\|_1}{\epsilon} \leq \frac{\|X_{0\Delta} - X_0\|_2}{\epsilon}. \tag{14}$$

Thus, we can reduce (12) to

$$\sum_{\Delta \geq 1} \left(\|u_0 - u_0^{(\Delta)}\|_2 \leq \sum_{\Delta \geq 1} \|X_0 - X_{0\Delta}\|_2 + \left\| |X_0|^r - \mathbb{E}[|X_0|^r | \mathcal{F}_{-\Delta}^{+\Delta}] \right\|_2 + \frac{\sqrt{2}}{f_X(q_X(p))} \sqrt{\mathbb{P}(|X_{0\Delta} - X_0| > 0)} \right). \tag{15}$$

Assuming that X_0 is geometrically L_2 -NED, i.e. $\|X_{0\Delta} - X_0\|_2 = O(e^{-\kappa\Delta})$ for some $\kappa > 0$, implies by (14) that $\sum_{\Delta \geq 1} \sqrt{\mathbb{P}(|X_{0\Delta} - X_0| \geq \epsilon)} < \infty$ holds for any $\epsilon > 0$. Hence, we will have shown that

$\sum_{\Delta \geq 1} \sqrt{\mathbb{P}(|X_{0\Delta} - X_0| > 0)} < \infty$, reducing the finiteness of $\sum_{\Delta \geq 1} \left\| -\mathbf{1}_{(X_0 \leq q_X(p))} + \mathbf{1}_{(X_{0\Delta} \leq q_X(p))} \right\|_2$ to X_0 being geometrically L_2 -NED.

Therefore, a sufficient condition for the finiteness of (15) is the geometric L_2 -NED of X_0 and $|X_0^r|$ respectively. This condition is satisfied, on the one hand in the polynomial case under $(M_r), (A_0)$ and $(P_{\max(1, r/\delta)})$ via Corollary 2 in [19], on the other hand under $(M_r), (A_0)$ and (L_r) via Corollary 3 in [19].

Step 4: Multivariate FCLT

Finally, we can apply a trivariate FCLT for u_j using Theorem A.1 from [3], as equations (9), (10), (11) are exactly the conditions needed in this theorem. Note that we adapted (10) from originally being $u_j^{(\Delta)} = f^{(\Delta)}(\epsilon_j, \dots, \epsilon_{j-\Delta})$. Indeed, it is straightforward to show that the proof of [3] still holds with this modification.

Using the Bahadur representation (8) of the sample quantile (ignoring the rest term for the moment), we can state:

$$\sqrt{n} \frac{1}{n} \sum_{j=1}^{[nt]} u_j = \sqrt{n} t \left(\frac{\bar{X}_{[nt]}}{\frac{p-F_{[nt]}(q_X(p))}{f_X(q_X(p))}} - m(X, r) \right) \xrightarrow{D_3[0,1]} \mathbf{W}_{\tilde{\Gamma}^{(r)}}(t) \quad \text{as } n \rightarrow \infty, \quad (16)$$

where $\mathbf{W}_{\tilde{\Gamma}^{(r)}}(t), t \in [0, 1]$ is the 3-dimensional Brownian motion with covariance matrix $\tilde{\Gamma}^{(r)} \in \mathbb{R}^{3 \times 3}$, i.e. the components $\tilde{\Gamma}_{ij}^{(r)}, 1 \leq i, j \leq 3$, satisfy the dependence structure (D), with all series being

absolutely convergent. By the multivariate Slutsky theorem, we can add $\begin{pmatrix} 0 \\ 0 \\ R_{[nt],p} \end{pmatrix}$ to the asymptotics

in (16) without changing the resulting distribution (as $\sqrt{n}R_{[nt],p} \xrightarrow{P} 0$). Hence, as $n \rightarrow \infty$,

$$\sqrt{n} t \left(\frac{\bar{X}_{[nt]} - \mu}{\frac{p-F_{[nt]}(q_X(p))}{f_X(q_X(p))}} - m(X, r) \right) + \sqrt{n} t \begin{pmatrix} 0 \\ 0 \\ R_{[nt],p} \end{pmatrix} = \sqrt{n} t \left(\frac{\bar{X}_{[nt]} - \mu}{q_{[nt]}(p) - q_X(p)} - m(X, r) \right) \xrightarrow{D_3[0,1]} \mathbf{W}_{\tilde{\Gamma}^{(r)}}(t). \quad (17)$$

Recalling the representation of $\hat{m}(X, n, r)$ in Proposition 7, we apply to (17) the multivariate continuous mapping theorem using the function $f(x, y, z) \mapsto (ax + y + b, z)$ with $a = -r \mathbb{E}[(X - \mu)^{r-1} \text{sgn}(X - \mu)^r]$ and $b = o_P(1/\sqrt{n})$, and obtain

$$\sqrt{n} t \left(\frac{a(\bar{X}_{[nt]} - \mu) + \frac{1}{[nt]} \sum_{j=1}^{[nt]} |X_j|^r - m(X, r) + b}{q_{[nt]}(p) - q_X(p)} \right) = \sqrt{n} t \left(\frac{\hat{m}(X, [nt], r) - m(X, r)}{q_{[nt]}(p) - q_X(p)} \right) \xrightarrow{D_2[0,1]} \mathbf{W}_{\Gamma^{(r)}}(t). \quad (18)$$

where $\Gamma^{(r)}$ follows from the specifications of $\tilde{\Gamma}^{(r)}$ above and the continuous mapping theorem. \square

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Appendix A Different Augmented GARCH models

As mentioned in the paper, we give an overview over the acronyms, authors and relation to each other of the augmented GARCH processes used.

The restrictions on the parameters, if not specified differently, are $\omega \geq 0, \alpha_i \geq 0, -1 \leq \gamma_i \leq 1, \beta_j \geq 0$ for $i = 1, \dots, p, j = 1, \dots, q$.

- APGARCH: Asymmetric power GARCH, introduced by Ding et al. in [11]. One of the most general polynomial GARCH models.
- AGARCH: Asymmetric GARCH, defined also by Ding et al. in [11], choosing $\delta = 1$ in APGARCH.
- GJR-GARCH: This process is named after its three authors Glosten, Jaganathan and Runkle and was defined by them in [16]. For the parameters α_i^*, γ_i^* it holds that $\alpha_i^* = \alpha_i(1 - \gamma_i)^2$ and $\gamma_i^* = 4\alpha_i\gamma_i$.
- GARCH: Choosing all $\gamma_i = 0$ in the AGARCH model (or $\gamma_i^* = 0$ in the GJR-GARCH), gives back the well-known GARCH(p, q) process by Bollerslev in [6].
- ARCH: Introduced by Engle in [13]. We recover it by setting all $\gamma_i = \beta_j = 0, \forall i, j$.
- TGARCH: Choosing $\delta = 1/2$ in the APGARCH model leads us to the so called threshold GARCH (TGARCH) by Zakoian in [28]. For the parameters α_i^+, α_i^- it holds that $\alpha_i^+ = \alpha_i(1 - \gamma_i), \alpha_i^- = \alpha_i(1 + \gamma_i)$.
- TSGARCH: Choosing $\gamma_i = 0$ in the TGARCH model we get, as a subcase, the TSGARCH model, named after its authors, i.e. Taylor, [26], and Schwert, [24].
- PGARCH: Another subfamily of the APGARCH processes is the Power-GARCH (PGARCH), also called sometimes NGARCH (i.e. non-linear GARCH) due to Higgins and Bera in [17].
- VGARCH: The volatility GARCH (VGARCH) model by Engle and Ng in [14] is also a polynomial GARCH model but is not part of the APGARCH family.
- NGARCH: This non-linear asymmetric model is due to Engle and Ng in [14], and sometimes also called NAGARCH.
- MGARCH: This model is called multiplicative or logarithmic GARCH and goes back to independent suggestions, in slightly different formulations, of Geweke in [15], Pantula in [23] and Milhøj in [21].
- EGARCH: This model is called exponential GARCH, introduced by Nelson in [22].

Then we give a schematic overview of the nesting of the different models in Figure 1.

Lastly, we present in Table 3 how the conditions ($P_{\max(1, r/\delta)}$) or (L_r) respectively translate for those augmented GARCH(p, q) processes - this is the generalization of Table 2. As, in contrast to Table 2, we do not gain any insight by considering the choices of $r = 1$ or $r = 2$, we only present the general case, $r \in \mathbb{N}$.

When $p \neq q$ we need to consider coefficients $\alpha_j, \beta_j, \gamma_j$ for $j = 1, \dots, \max(p, q)$. In case they are not defined, we set them equal to 0.

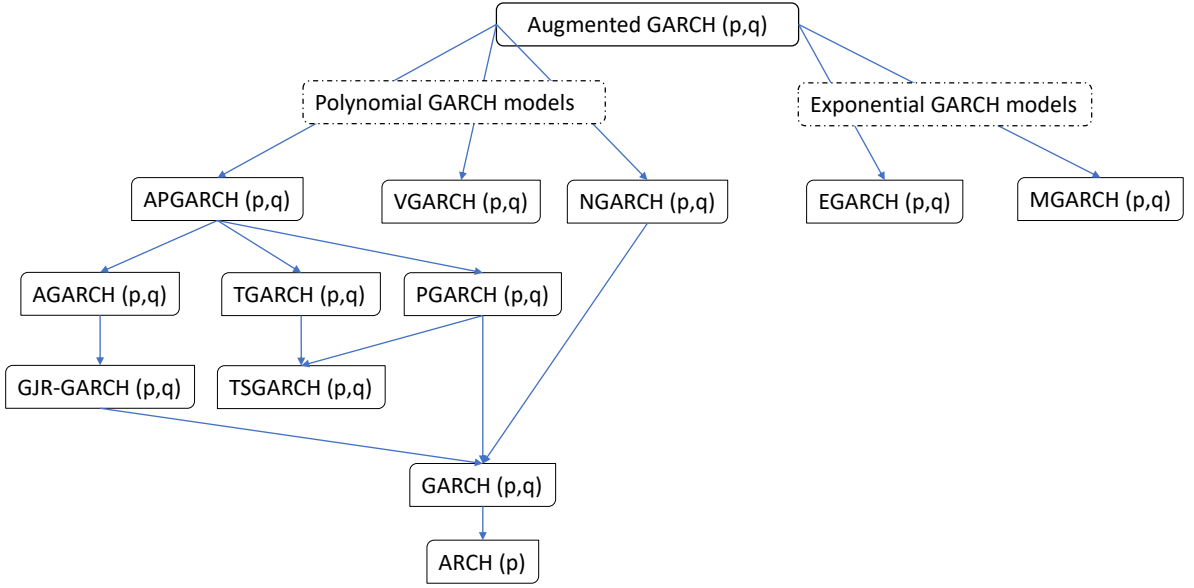


Figure 1: Schematic overview of the nesting of some augmented $GARCH(p, q)$ models.

Table 3: Conditions $(P_{\max(1, r/\delta)})$ or (L_r) respectively translated for different augmented $GARCH(p, q)$ models for the general r -th absolute centred sample moment, $r \in \mathbb{N}$.

augmented $GARCH(p, q)$	$r \in \mathbb{N}$
APGARCH	$\sum_{j=1}^{\max(p, q)} \mathbb{E}[\alpha_j (\epsilon_0 - \gamma_j \epsilon_{t-j})^{2\delta} + \beta_j ^r]^{1/r} < 1$
AGARCH	$\sum_{j=1}^{\max(p, q)} \mathbb{E}[\alpha_j \epsilon_0 - \gamma_j \epsilon_{t-j} ^2 + \beta_j ^r]^{1/r} < 1$
GJR-GARCH	$\sum_{j=1}^{\max(p, q)} \mathbb{E}[\alpha_j^* \epsilon_0^2 + \beta_j + \gamma_j^* \max(0, -\epsilon_0^2) ^r]^{1/r} < 1$
GARCH	$\sum_{j=1}^{\max(p, q)} \mathbb{E}[(\alpha_j \epsilon_0^2 + \beta_j)^r]^{1/r} < 1$
ARCH	$\sum_{j=1}^{\max(p, q)} \alpha_j \mathbb{E}[\epsilon_0^{2r}]^{1/r} < 1$
TGARCH	$\sum_{j=1}^{\max(p, q)} \mathbb{E}[\alpha_j \epsilon_{t-j} - \alpha_j \gamma_j \epsilon_{t-j} + \beta_j ^r]^{1/r} < 1$
TSGARCH	$\sum_{j=1}^{\max(p, q)} \mathbb{E}[\alpha_j \epsilon_{t-j} + \beta_j ^r]^{1/r} < 1$
PGARCH	$\sum_{j=1}^{\max(p, q)} \mathbb{E}[\alpha_j \epsilon_0 + \beta_j ^{2r}]^{1/(2r)} < 1$
VGARCH	$\sum_{j=1}^q \beta_j < 1$
NGARCH	$\sum_{j=1}^{\max(p, q)} \mathbb{E}[\alpha_j (\epsilon_0 + \gamma_j)^2 + \beta_j ^r]^{1/r} < 1$
MGARCH	$\mathbb{E}[\exp(4r \sum_{i=1}^p \omega/p + \alpha_i \log(\epsilon_0^2) ^2)] < \infty$ and $\sum_{j=1}^q \beta_j < 1$
EGARCH	$\mathbb{E}[\exp(4r \sum_{i=1}^p \omega/p + \alpha_i (\epsilon_0 - \mathbb{E} \epsilon_0) + \gamma_i \epsilon_0 ^2)] < \infty$ and $\sum_{j=1}^q \beta_j < 1$

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