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# Quadratic variation and quadratic roughness

Rama CONT and Purba DAS

July 7, 2019

## Abstract

We study the concept of quadratic variation of a continuous path along a sequence of partitions and its dependence with respect to the choice of the partition sequence. We define the quadratic roughness of a path along a partition sequence and show that, for Hölder-continuous paths satisfying this roughness condition, the quadratic variation along balanced partitions is invariant with respect to the choice of the partition sequence. Paths of Brownian motion are shown to satisfy this quadratic roughness property almost-surely. Using these results we derive a formulation of Föllmer's pathwise integration along paths with finite quadratic variation which is invariant with respect to the partition sequence.

**Keywords:** Quadratic variation, Pathwise integral, Itô calculus, Local time.

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The concept of *quadratic variation* plays a central role in stochastic analysis and the modern theory of stochastic integration [12, 29]. The quadratic variation of a (real-valued) random process  $(X(t), t \in [0, T])$  with continuous sample paths is defined as the limit in the sense of (uniform) convergence in probability, of the sum of squared increments

$$\sum_{\pi_n} (X(t_{k+1}^n \wedge t) - X(t_k^n \wedge t))^2 \quad (1)$$

along a sequence of partitions  $\pi^n = (0 = t_0^n < t_1^n < \dots < t_{N(\pi^n)}^n = T)$  with vanishing step size  $|\pi^n| = \sup_{i=1..N(\pi^n)} |t_i^n - t_{i-1}^n| \rightarrow 0$ . The relevance of this notion is underlined by the fact that large

classes of random processes –such as Brownian motion and diffusion processes– have finite (and non-zero) quadratic variation, while at the same time possessing infinite  $p$ -variation for  $p = 2$ .

Although quadratic variation for a stochastic process  $X$  is usually defined as a limit in probability of (1), it is essentially a pathwise property. In his seminal paper *Calcul d'Ito sans probabilités* [15], Hans Föllmer introduced the class of càdlàg paths  $X \in D([0, T], \mathbb{R})$  with finite quadratic variation along a sequence of partitions  $(\pi^n)$ , for which (1) has a limit with Lebesgue decomposition  $[X]_\pi(t) = [X]^c(t) + \sum_{0 \leq s \leq t} (\Delta X_s)^2$  and showed that one can define, for  $f \in C^2(\mathbb{R})$ ,

an integral

$$\int_0^T (\nabla f \circ X) d^\pi X = \lim_{n \rightarrow \infty} \sum_{\pi^n} \nabla f(X(t_i)) \cdot (X(t_{i+1}^n) - X(t_i^n)) \quad (2)$$

as a pointwise limit of (left) Riemann sums along the sequence  $(\pi^n)$  and this integral satisfies a change of variable formula:

$$\begin{aligned} f(X(t)) &= f(X(0)) + \int_0^t (\nabla f \circ X) d^\pi X + \frac{1}{2} \int_0^t \nabla^2 f(X(s)) \cdot d[X]_\pi^c \\ &+ \sum_{[0, t]} (f(X(s)) - f(X(s-)) - \nabla f(X(s)) \Delta X(s)). \end{aligned} \quad (3)$$

This ‘pathwise Itô formula’ could be potentially used as a starting point ingredient for a purely pathwise construction of the Itô calculus but, unlike the analogous theory for Riemann-Stieltjes or Young integrals, the construction in [15] seems to heavily depend on the choice of the sequence of partitions  $(\pi^n)$ : both the quadratic variation  $[X]_\pi$  and the pathwise integral (2) are defined as limits along this sequence of partitions: in fact, as shown by Freedman [16, p. 47], for any continuous function  $x$  one can construct a sequence of partitions  $\pi$  such that  $[x]_\pi = 0$ . This result was extended in [10] who show that given any continuous function  $x$  and any increasing function  $A$ , one can construct a partition  $\pi_{A,x}$  such that  $[x]_{\pi_{A,x}} = A$ . These negative results seem to suggest that  $[x]_\pi$  depends so strongly on  $\pi$  that there is no hope for uniqueness of the quantities in (3).

On the other hand, as shown by Lévy [26] and Dudley [13], for typical paths of Brownian motion the sums (1) converge to a unique limit along *any* sequence of partitions whose mesh decreases to zero fast enough. Therefore there exists a large set of paths –containing all typical Brownian paths– for which one should be able to define the quantities in (3) independently of the choice of the partition sequence  $(\pi^n)_{n \geq 1}$  for a large class of such sequences.

We clarify these issues by investigating in detail the dependence of quadratic variation with respect to the sequence of partitions, and deriving sufficient conditions for the stability of quadratic variation with respect to this choice. These conditions are related to an irregularity property of the path, which we call *quadratic roughness* (Def. 3.1): this property requires cross-products

of increments along the partition to average to zero at certain scales and is different from other notions of roughness such as Hölder roughness [17] or  $\rho$ -irregularity [4]. Importantly, we show that the quadratic roughness property is satisfied almostsurely by Brownian paths (Theorem 3.4).

Our main result is that, for Hölder-continuous paths satisfying this irregularity condition, the quadratic variation along balanced partitions (Def. 2.1) is invariant with respect to the choice of the partition sequence. This leads to an *intrinsic* notion of quadratic variation, a robust formulation of the pathwise Itô calculus (Theorem 5.1) and uniqueness of pathwise local time (Theorem 5.2) for such irregular paths.

Our results thus complement previous results on the pathwise approach to Itô calculus [2, 5, 7, 9, 9, 10, 15, 20, 24, 28] by identifying a set of paths for which these results are robust to the choice of the sequence of partitions involved in the construction. In contrast to the constructions in [21, 22, 23], our construction does not involve path-dependent partitions and does not rely on any probabilistic tools.

**Outline** Section 1 recalls the definition of quadratic variation along a sequence of partitions, following [5, 15]. Section 2 defines the class of balanced sequences of partitions and discusses asymptotic comparability of such partitions.

Section 3 introduces the concept of *quadratic roughness* and explores some of its properties. In particular we show that Brownian paths satisfy this property almost-surely (Theorem 3.4). Section 4 shows that quadratic roughness of a path is a sufficient condition for the invariance of quadratic variation with respect to the choice of partitions (Theorem 4.2). This result allows to give an intrinsic definition of quadratic variation without reference to a specific partition sequence (Proposition 4.5).

Section 5.1 builds on these result to arrive at a robust formulation of the Föllmer integral and the pathwise Itô formula. Section 5.2 extends these results to pathwise local time.

## 1 Quadratic variation along a sequence of partitions

Let  $T > 0$ . We denote  $D([0, T], \mathbb{R}^d)$  the space of  $\mathbb{R}^d$ -valued right-continuous functions with left limits (càdlàg functions),  $C^0([0, T], \mathbb{R}^d)$  the subspace of continuous functions and, for  $0 < \nu < 1$ ,  $C^\nu([0, T], \mathbb{R}^d)$  the space of Hölder continuous functions with exponent  $\nu$ :

$$C^\nu([0, T], \mathbb{R}^d) = \{x \in C^0([0, T], \mathbb{R}^d), \sup_{(t,s) \in [0,T]^2, t \neq s} \frac{\|x(t) - x(s)\|}{|t - s|^\nu} < +\infty\},$$

$$\text{and } C^{\nu-}([0, T], \mathbb{R}^d) = \bigcap_{0 \leq \alpha < \nu} C^\alpha([0, T], \mathbb{R}^d).$$

We denote by  $\Pi([0, T])$  the set of all finite partitions of  $[0, T]$ . A sequence of partitions of  $[0, T]$  is a sequence  $(\pi^n)_{n \geq 1}$  of elements of  $\Pi([0, T])$ :

$$\pi^n = (0 = t_0^n < t_1^n < \dots < t_{N(\pi^n)}^n = T).$$

We denote  $N(\pi^n)$  the number of intervals in the partition  $\pi^n$  and

$$|\pi^n| = \sup\{|t_i^n - t_{i-1}^n|, i = 1..N(\pi^n)\}, \quad \underline{\pi}^n = \inf\{|t_i^n - t_{i-1}^n|, i = 1..N(\pi^n)\} \quad (4)$$

the size of the largest (resp. the smallest) interval of  $\pi^n$ .

**Definition 1.1** (Quadratic variation of a path along a sequence of partitions).

Let  $\pi^n = (0 = t_0^n < t_1^n \dots < t_{N(\pi^n)}^n = T)$  be a sequence of partitions of  $[0, T]$  with vanishing mesh  $|\pi^n| = \sup_{i=0 \dots N(\pi^n)-1} |t_{i+1}^n - t_i^n| \rightarrow 0$ . A càdlàg function  $x \in D([0, T], \mathbb{R})$  is said to have finite quadratic variation along the sequence of partitions  $(\pi^n)_{n \geq 1}$  if

(i) the sequence of discrete measures

$$\sum_{t_j^n \in \pi^n} (x(t_{j+1}^n) - x(t_j^n))^2 \delta_{t_j^n} \xrightarrow{n \rightarrow \infty} \mu$$

converges weakly to a limit measure  $\mu$ .

(ii)  $[x]_\pi^c(t) = \mu([0, t]) - \sum_{0 < s \leq t} |\Delta x(s)|^2$  is continuous and increasing.

The increasing function  $[x]_\pi : [0, T] \rightarrow \mathbb{R}_+$  defined by

$$[x]_\pi(t) = \mu([0, t]) = \lim_{n \rightarrow \infty} \sum_{\pi_n} (x(t_{k+1}^n \wedge t) - x(t_k^n \wedge t))^2 \quad (5)$$

is called the *quadratic variation of  $x$  along the sequence of partitions  $\pi$* . We denote  $Q_\pi([0, T], \mathbb{R})$  the set of càdlàg paths with these properties.

Condition (ii) is equivalent to requiring that the increasing function  $[x]$  has the following *Lebesgue decomposition*:

$$[x]_\pi^c(t) = [x]_\pi(t) - \sum_{0 < s \leq t} |\Delta x(s)|^2.$$

$Q_\pi([0, T], \mathbb{R})$  is not a vector space (see e.g. [30]) so the corresponding definition for vector-valued paths requires some care [15]:

**Definition 1.2.** A càdlàg path  $x = (x^1, \dots, x^d) \in D([0, T], \mathbb{R}^d)$  is said to have finite quadratic variation along  $\pi = (\pi^n)_{n \geq 1}$  if for all  $i, j = 1 \dots d$  we have  $x^i \in Q_\pi([0, T], \mathbb{R})$  and  $x^i + x^j \in Q_\pi([0, T], \mathbb{R})$ . We then denote  $[x]_\pi \in D([0, T], S_d^+)$  the matrix-valued function defined by

$$[x]_\pi^{i,j}(t) = \frac{[x^i + x^j]_\pi(t) - [x^i]_\pi(t) - [x^j]_\pi(t)}{2} \in S_d^+,$$

where  $S_d^+$  is the set of symmetric semidefinite positive matrices. We denote by  $Q_\pi([0, T], \mathbb{R}^d)$  the set of functions satisfying these properties.

For  $x \in Q_\pi([0, T], \mathbb{R}^d)$ ,  $[x]_\pi \in D([0, T], S_d^+)$ .

As shown in [5], the above definitions may be more simply expressed in terms of convergence of discrete approximations. In the case of continuous paths, which is our focus in this paper, we have the following characterization [6, 5]:

**Proposition 1.3.**  $x \in C^0([0, T], \mathbb{R}^d)$  has finite quadratic variation along  $\pi = (\pi^n, n \geq 1)$  if and only if the sequence of functions  $[x]_{\pi^n}$  defined by

$$[x]_{\pi^n}(t) = \sum_{t_j^n \in \pi^n} (x(t_{j+1}^n \wedge t) - x(t_j^n \wedge t))^t (x(t_{j+1}^n \wedge t) - x(t_j^n \wedge t))$$

converges uniformly on  $[0, T]$  to a continuous (increasing) function  $[x]_\pi \in C^0([0, T], S_d^+)$ .

The notion of quadratic variation along a sequence of partitions is different from the p-variation for  $p = 2$ . The p-variation involves taking a supremum over *all* partitions, whereas quadratic variation is a limit taken along a given sequence  $(\pi^n)_{n \geq 1}$ . In general  $[x]_\pi$  given by (5) is smaller than the p-variation for  $p = 2$ . In fact, for diffusion processes the typical situation is that the p-variation is (almost-surely) infinite for  $p = 2$  [14, 31] while the quadratic variation is finite for sequences satisfying some mesh size condition. For instance, typical paths of Brownian motion have finite quadratic variation along any sequence of partitions with mesh size  $o(1/\log n)$  [13, 11] while simultaneously having infinite p-variation almost surely for  $p \leq 2$  [27, p. 190]:

$$\inf_{\pi \in \Pi(0, T)} \sum_{\pi} |W(t_{k+1}) - W(t_k)|^2 = 0, \quad \text{while} \quad \sup_{\pi \in \Pi(0, T)} \sum_{\pi} |W(t_{k+1}) - W(t_k)|^2 = \infty$$

almost-surely.

The quadratic variation of a path along a sequence of partitions strongly depends on the chosen sequence. In fact, as shown by Freedman [16, p. 47], given any continuous functions, one can always construct a sequence of partitions along which the quadratic variation is zero. This result was extended by Davis et al. [10] who show that, given any continuous path  $x \in C^0([0, T], \mathbb{R})$  and any increasing function  $A : [0, T] \rightarrow \mathbb{R}_+$  one can construct a partition sequence  $\pi$  such that  $[x]_\pi = A$ . Notwithstanding these negative results, we will identify a property of the path  $x$  which guarantees the uniqueness of  $[x]_\pi$  across the class of *balanced* partition sequences, which we now define.

## 2 Balanced partition sequences

One of the difficulties in comparing quadratic variation along two different partition sequences is that lack of uniform bounds on the partition mesh size. We introduce in this section the class of *balanced* partition sequences which allow for such bounds.

We will say that two (real) sequences  $a = (a_n)_{n \geq 1}$  and  $b = (b_n)_{n \geq 1}$  are asymptotically comparable, denoted  $a_n \asymp b_n$ , if  $|a_n| = O(|b_n|)$  and  $|b_n| = O(|a_n|)$  as  $n \rightarrow \infty$ . If both sequences are strictly positive then

$$a_n \asymp b_n \iff \limsup_{n \rightarrow \infty} \frac{|b_n|}{|a_n|} < \infty \quad \text{and} \quad \limsup_{n \rightarrow \infty} \frac{|a_n|}{|b_n|} < \infty.$$

### 2.1 Definition and properties

**Definition 2.1** (Balanced partition sequence).

Let  $\pi^n = (0 = t_0^n < t_1^n \dots < t_{N(\pi^n)}^n = T)$  be a sequence of partitions of  $[0, T]$  and

$$\underline{\pi}^n = \inf_{i=0..N(\pi^n)-1} |t_{i+1}^n - t_i^n|, \quad |\pi^n| = \sup_{i=0..N(\pi^n)-1} |t_{i+1}^n - t_i^n|.$$

We say that  $(\pi^n)_{n \geq 1}$  is *balanced* if

$$\exists c > 0, \quad \forall n \geq 1, \quad \frac{|\pi^n|}{\underline{\pi}^n} \leq c. \quad (6)$$

This condition means that all of intervals in the partition  $\pi^n$  are asymptotically comparable. Note that since  $\underline{\pi}^n N(\pi^n) \leq T$ , any balanced sequence of partitions satisfies

$$|\pi^n| \leq c \underline{\pi}^n \leq \frac{cT}{N(\pi^n)}. \quad (7)$$

We will denote by  $\mathbb{B}([0, T])$  the set of all balanced partition sequences of  $[0, T]$ .

**Proposition 2.2.** *Let  $\pi = (\pi^n)_{n \geq 1}$  be a sequence of partitions of  $[0, T]$ . Then:*

$$(i) \quad \pi \in \mathbb{B}([0, T]) \iff \liminf_{n \rightarrow \infty} N(\pi^n) \underline{\pi}^n > 0 \text{ and } \limsup_{n \rightarrow \infty} N(\pi^n) |\pi^n| < \infty.$$

(ii) *Let  $N(\pi^n, t_1, t_2)$  be the number of partition points of  $\pi^n$  in the interval  $[t_1, t_2]$ . If  $\pi \in \mathbb{B}([0, T])$  then for any  $h > 0$ ,*

$$\limsup_{n \rightarrow \infty} \frac{\sup_{t \in [0, T-h]} N(\pi^n, t, t+h)}{\inf_{t \in [0, T-h]} N(\pi^n, t, t+h)} < \infty.$$

(iii) *If  $\pi = (\pi^n)_{n \geq 1} \in \mathbb{B}([0, T])$  then*

$$\limsup_n \frac{N(\pi^{n+1})}{N(\pi^n)} < \infty \iff \limsup_n \frac{|\pi^n|}{|\pi^{n+1}|} < \infty \iff \limsup_n \frac{\pi^n}{\underline{\pi}^{n+1}} < \infty. \quad (8)$$

(iv) *If  $g \in C^1([0, T], \mathbb{R})$  is strictly increasing with  $\inf g' > 0$  then  $g$  maps any balanced partition of  $[0, T]$  to a balanced partition sequence of  $g([0, T])$ .*

The proof of this proposition is given in Appendix A.1.

We will say that two balanced partition sequences  $\tau = (\tau^n)_{n \geq 1}$  and  $\sigma = (\sigma^n)_{n \geq 1}$  are (asymptotically) comparable if

$$0 < \liminf_{n \rightarrow \infty} \frac{|\sigma^n|}{|\tau^n|} \leq \limsup_{n \rightarrow \infty} \frac{|\sigma^n|}{|\tau^n|} < \infty \quad (9)$$

which then implies

$$0 < \liminf_{n \rightarrow \infty} \frac{N(\sigma^n)}{N(\tau^n)} \leq \limsup_{n \rightarrow \infty} \frac{N(\sigma^n)}{N(\tau^n)} < \infty \quad (10)$$

We denote  $\tau \asymp \sigma$  (or  $\tau^n \asymp \sigma^n$ ). Note that for general (not balanced) sequences of partitions (9) does not imply (10): this is a consequence of (7). If  $\tau \asymp \sigma$  then the number of points of  $\tau^n$  in any interval of  $\sigma^n$  remains bounded as  $n \rightarrow \infty$ .

The following lemma, whose proof is given in Appendix A.2, shows how one can adjust the rate at which the mesh of a balanced sequence decreases to zero.

**Lemma 2.3** (Adjusting the mesh of a balanced sequence). *Let  $\tau = (\tau^n)_{n \geq 1}$  and  $\sigma = (\sigma^n)_{n \geq 1}$  be two balanced partition sequences of  $[0, T]$  with  $\limsup_n \frac{|\sigma^n|}{|\tau^n|} < 1$  and  $|\tau^n| \xrightarrow{n \rightarrow \infty} 0$ .*

(i) *There exists a subsequence  $(\tau^{k(n)})_{n \geq 1}$  of  $\tau$  such that:*

$$\limsup_n \frac{|\sigma^n|}{|\tau^{k(n)}|} \geq 1.$$

(ii) *If  $\limsup_n \frac{|\tau^n|}{|\tau^{n+1}|} < \infty$ , there exists a subsequence  $(\tau^{k(n)})_{n \geq 1}$  of  $\tau$  which is asymptotically comparable to  $\sigma$ :  $\tau^{k(n)} \asymp \sigma^n$ .*

(iii) *There exists  $r : \mathbb{N} \mapsto \mathbb{N}$  such that*

$$\limsup_n \frac{|\sigma^{r(n)}|}{|\tau^n|} \geq 1.$$

(Note that  $r : \mathbb{N} \mapsto \mathbb{N}$  may not be injective i.e.  $(\sigma^{r(n)}, n \geq 1)$  is not a subsequence of  $\sigma$ .)

**Coarsening** A partition may be refined by adding points to it. The inverse operation, which we call *coarsening*, corresponds to removing points i.e. subsampling or grouping of partition points. We will be specifically interested in coarsenings which preserve the balance property but may modify the asymptotic rate of decrease of the mesh size:

**Definition 2.4** (Coarsening of a balanced partition sequence). Let  $\pi^n = (0 = t_0^n < t_1^n \cdots < t_{N(\pi^n)}^n = T)$  be a balanced sequence of partitions of  $[0, T]$  with vanishing mesh  $|\pi^n| \rightarrow 0$  and  $0 < \beta < 1$ . A  **$\beta$ -coarsening** of  $\pi$  is a sequence of subpartitions of  $\pi^n$ :

$$A^n = (0 = t_{p(n,0)}^n < t_{p(n,1)}^n < \cdots < t_{p(n,N(A^n))}^n = T)$$

such that  $(A^n)_{n \geq 1}$  is a balanced partition sequence of  $[0, T]$  and  $|A^n| \asymp |\pi^n|^\beta$ .

Note that since  $\beta < 1$ ,  $|A^n| \asymp |\pi^n|^\beta \gg |\pi^n|$  as  $n$  increases, so the number of points of  $\pi^n$  in each interval of  $A^n$  increases to infinity:

$$\inf_{j=1 \dots N(A^n)} p(n, j) - p(n, j-1) \xrightarrow{n \rightarrow \infty} \infty.$$

## 2.2 Quadratic variation along balanced partitions

If a path has quadratic variation along a sequence of partitions, then it also has (the same) quadratic variation along any sub-sequence. This simple remark has interesting implications when the partitions are balanced: comparing the sum of squared increments along the original sequence with the sum along a sub-sequence (with finer mesh), we obtain that under some scaling conditions on the mesh, cross-products increments along the finer partition average to zero across the coarser partition.

**Lemma 2.5** (Averaging property of cross-products of increments). *Let  $x \in C^\alpha([0, T], \mathbb{R}^d)$  for some  $\alpha > 0$  and  $\sigma^n = \{s_1^n, s_2^n, \dots, s_{N(\sigma^n)}^n\}$  be a balanced sequence of partitions of  $[0, T]$  such that  $x \in Q_\sigma([0, T], \mathbb{R}^d)$ . Let  $\kappa > \frac{1}{\alpha}$  and  $(\sigma^{l_n})_{n \geq 1}$  a subsequence of  $\sigma^n$  with  $|\sigma^{l_n}| \asymp |\sigma^n|^\kappa$ . Define  $p(k, n) = \inf\{m \geq 1 : s_m^{l_n} \in (s_k^n, s_{k+1}^n)\}$  for  $k = 1 \dots N(\sigma^n)$ . Then*

$$\sum_{k=1}^{N(\sigma^n)} \sum_{p(k,n) \leq i \neq j < p(k+1,n)-1} \left( x(s_{i+1}^{l_n}) - x(s_i^{l_n}) \right)^t \left( x(s_{j+1}^{l_n}) - x(s_j^{l_n}) \right) \xrightarrow{n \rightarrow \infty} 0.$$

We provide the proof for  $d = 1$ . The extension to  $d > 1$  is straightforward.

*Proof.* Let  $\sigma^{l_n}$  be a sub-sequence of  $\sigma^n$  satisfying  $|\sigma^{l_n}| \asymp |\sigma^n|^\kappa$ . Define

$$[x]_{\sigma^n}(t) = \sum_{k=1}^{N(\sigma^n)-1} \left( x(s_{k+1}^n \wedge t) - x(s_k^n \wedge t) \right)^2, \quad [x]_{\sigma^{l_n}}(t) = \sum_{s_k^{l_n} \in \sigma^{l_n}} \left( x(s_{k+1}^{l_n} \wedge t) - x(s_k^{l_n} \wedge t) \right)^2.$$

Then  $[x]_{\sigma^{l_n}}(t) - [x]_{\sigma^n}(t) \rightarrow 0$ . Grouping the points of  $\sigma^{l_n}$  along intervals of  $\sigma^n$ , we obtain:

$$\begin{aligned} \left| [x]_{\sigma^n} - [x]_{\sigma^{l_n}} \right| &= \left| \sum_{\sigma^n} (x(s_{i+1}^n) - x(s_i^n))^2 - \sum_{\sigma^{l_n}} (x(s_{i+1}^{l_n}) - x(s_i^{l_n}))^2 \right| \\ &= \left| \sum_{\sigma^n} \left( (x(s_{i+1}^n) - x(s_i^n))^2 - \sum_{j=p(i,n)}^{p(i+1,n)-1} (x(s_{j+1}^{l_n}) - x(s_j^{l_n}))^2 \right) + \sum_{k=1}^{N(\sigma^n)} (x(s_{p(k,n)}^{l_n}) - x(s_{p(k,n)-1}^{l_n}))^2 \right| \end{aligned}$$



$$\geq \left| \sum_{\sigma^n} \left( (x(s_{i+1}^n) - x(s_i^n))^2 - \sum_{j=p(i,n)}^{p(i+1,n)-1} (x(s_{j+1}^n) - x(s_j^n))^2 \right) \right| - \sum_{k=1}^{N(\sigma^n)} (x(s_{p(k,n)}^n) - x(s_{p(k,n)-1}^n))^2.$$

Using the Hölder continuity of  $f$ , the last term in the above equation is bounded above by  $\frac{1}{N(\sigma^n)}$ .  
 $\sum_{k=1}^{N(\sigma^n)} C|\sigma^{l_n}|^{2\alpha} \leq CN(\sigma^n)|\sigma^{l_n}|^{2\alpha}$ . Now using the balanced property of  $\sigma^{l_n}$ , the last term is bounded above by

$$\sum_{i=1}^{N(\sigma^n)} C_1|\sigma^{l_n}|^{2\alpha} \leq N(\sigma^n)C_1|\sigma^{l_n}|^{2\alpha} \leq \frac{C_2N(\sigma^n)}{N(\sigma^{l_n})^{2\alpha}} \leq C_2 \times N(\sigma^n)^{1-2\alpha\kappa} \xrightarrow{n \rightarrow \infty} 0$$

since  $1 - 2\alpha\kappa < 0$ . So finally we obtain

$$\lim_{n \rightarrow \infty} \left| \sum_{k=1}^{N(\sigma^n)} \sum_{p(k,n) \leq i \neq j < p(k+1,n)} (x(s_{i+1}^n) - x(s_i^n)) (x(s_{j+1}^n) - x(s_j^n)) \right| \leq \lim_{n \rightarrow \infty} |[x]_{\sigma^n} - [x]_{\sigma^{l_n}}| = 0.$$

■

### 3 Quadratic roughness

#### 3.1 Quadratic roughness along a sequence of partitions

Lemma 2.5 shows that if a function has finite quadratic variation along a balanced partition sequence, then the product of the increments along any subsequence with *sufficiently small mesh* average to zero if we do the averaging along the original (coarser) sequence. Intuitively, this means that there is enough cancellation across neighboring increments such that their cross-products average to zero over a coarser grid. We will now introduce a slightly extended version of this property, which we call *quadratic roughness*, and show that plays a key role in the stability of quadratic variation with respect to the partition.

**Definition 3.1** (Quadratic roughness). Let  $\pi^n = (0 = t_0^n < t_1^n \dots < t_{N(\pi^n)}^n = T)$  be a balanced sequence of partitions of  $[0, T]$  with  $|\pi^n| \rightarrow 0$  and  $0 < \beta < 1$ . We say that  $x \in Q_\pi([0, T], \mathbb{R}^d)$  has the quadratic roughness property along  $\pi$  with coarsening index  $\beta$  on  $[0, T]$  if

- (i)  $[x]_\pi$  is strictly increasing on  $[0, T]$ .
- (ii) For any  $\beta$ -coarsening  $A^n = (0 = t_{p(n,0)}^n < t_{p(n,1)}^n < \dots < t_{p(n,N(A^n))}^n = T)$  of  $(\pi^n)_{n \geq 1}$  we have

$$\sum_{j=1}^{N(A^n)} \sum_{p(n,j-1) \leq i \neq i' < p(n,j)} (x(t_{i+1}^n \wedge t) - x(t_i^n \wedge t))^t (x(t_{i'+1}^n \wedge t) - x(t_{i'}^n \wedge t)) \xrightarrow{n \rightarrow \infty} 0.$$

We denote by  $R_\pi^\beta([0, T], \mathbb{R}^d) \subset Q_\pi([0, T], \mathbb{R}^d)$  the set of paths satisfying this property.

In other words, the quadratic roughness property states that cross-products of increments along  $\pi^n$  average to zero over *any* ( $\beta$ -)coarsening of  $\pi^n$  as the mesh is refined.

*Remark 3.2.* Let  $\sigma = (\sigma^n)_{n \geq 1}$  be a balanced partition sequence of  $[0, T]$  with  $|\sigma^n| \rightarrow 0$ . Then the quadratic roughness property along  $\sigma$  implies existence of quadratic variation along any  $\beta$ -coarsening of  $\sigma$  i.e. any sequence of subpartitions of  $\sigma$  satisfying Definition 2.4: for any  $0 < \beta \leq 1$ ,

$$R_\sigma^\beta([0, T], \mathbb{R}^d) \subset \bigcap_{\pi \in \mathcal{C}(\beta, \sigma)} Q_\pi([0, T], \mathbb{R}^d)$$

where  $\mathcal{C}(\beta, \sigma)$  denotes the set of all possible  $\beta$ -coarsenings of  $\sigma$ .

**Proposition 3.3.** *Let  $\pi = (\pi^n)_{n \geq 1}$  be a balanced partition sequence of  $[0, T]$  with  $|\pi^n| \rightarrow 0$  and  $x \in R_\pi^\beta([0, T], \mathbb{R}^d)$  with  $0 \leq \beta < 1$ . Then:*

1. *For any interval  $I \subset [0, T]$ ,  $x$  has the quadratic roughness property on  $I$  along  $\pi_I = (\pi_I^n)_{n \geq 1} = (\pi^n \cap I)_{n \geq 1}$ :  $x \in R_{\pi_I}^\beta([0, T], \mathbb{R}^d)$ .*
2. *For any subsequence  $\tau^n = \pi^{k(n)}$  of  $\pi$ , we have  $x \in R_\tau^\beta([0, T], \mathbb{R}^d)$ .*
3. *For any  $\lambda \in \mathbb{R}$ ,  $\lambda x \in R_\pi^\beta([0, T], \mathbb{R}^d)$ .*
4. *If  $y$  is a function with bounded variation then  $x + y \in R_\pi^\beta([0, T], \mathbb{R}^d)$ .*

The proof of this lemma is given in Appendix A.3.

## 3.2 Quadratic roughness of Brownian paths

We now show that the quadratic roughness property is satisfied by typical sample paths of Brownian motion.

**Theorem 3.4** (Quadratic roughness of Brownian paths). *Let  $W$  be a Wiener process on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ ,  $T > 0$  and  $(\pi^n)_{n \geq 1}$  a balanced sequence of partitions of  $[0, T]$  with*

$$(\log n)^\gamma |\pi^n| \xrightarrow{n \rightarrow \infty} 0 \quad \text{for some } \gamma > 2. \quad (11)$$

*Then the sample paths of  $W$  almost-surely satisfy the quadratic roughness property on  $[0, T]$ :*

$$\forall \beta \in \left[\frac{2}{\gamma}, 1\right), \quad \mathbb{P}(W \in R_\pi^\beta([0, T], \mathbb{R})) = 1.$$

*Proof.* Let  $W$  be a Wiener process on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , which we take to be the canonical Wiener space without loss of generality i.e  $W(t, \omega) = \omega(t)$ . Let  $\pi^n = (0 = t_0^n < t_1^n \dots < t_{N(\pi^n)}^n = T)$  be a balanced sequence of partitions of  $[0, T]$  satisfying (11). Then in particular  $|\pi^n| \log n \rightarrow 0$  so the results of Dudley [13] imply that

$$\mathbb{P}\left(\sum_{\pi^n} |W(t_{i+1}^n \wedge t) - W(t_i^n \wedge t)|^2 \xrightarrow{n \rightarrow \infty} 0\right) = 1.$$

So if we set  $\Omega_0 = \Omega \cap Q_\pi([0, T])$  then  $\mathbb{P}(\Omega_0) = 1$  and for any  $\omega \in \Omega_0$  we have  $[\omega]_\pi(t) = t$ . To prove the quadratic roughness property we use an estimate from [19]. From the properties of the Wiener process,

$$X_i^n = \frac{W(t_{i+1}^n) - W(t_i^n)}{\sqrt{t_{i+1}^n - t_i^n}} \sim N(0, 1)$$

are independent for  $i = 0, \dots, N(\pi^n) - 1$ . Now let  $\beta \in [\frac{2}{\gamma}, 1)$  and  $\mathcal{C}(\beta, \pi)$  be the set of all possible  $\beta$ -coarsenings of  $\pi$  i.e. sequences of subpartitions of  $\pi$  satisfying Definition 2.4, and  $\mathcal{C}_n(\beta, \pi) = \{A^n, A \in \mathcal{C}(\beta, \pi)\}$ . Let  $A = (A^n) \in \mathcal{C}(\beta, \pi)$  with

$$A^n = (0 = t_{p(n,0)}^n < t_{p(n,1)}^n < \dots < t_{p(n,N(A^n))}^n = T). \quad (12)$$

Define  $a_{ii'} = \sqrt{(t_{i+1}^n - t_i^n)(t_{i'+1}^n - t_{i'}^n)}$  if  $p(n, j-1) \leq i \neq i' < p(n, j)$  for some  $j \in \{1, 2, \dots, N(A^n)\}$  otherwise set  $a_{ii'} = 0$ . Let

$$S(A^n) = \sum_{j=1}^{N(A^n)} \sum_{p(n, j-1) \leq i \neq i' < p(n, j)} (W(t_{i+1}^n) - W(t_i^n))^T (W(t_{i'+1}^n) - W(t_{i'}^n)) = \sum_{i, i'=1}^{N(\pi^n)} a_{ii'} X_i^n X_{i'}^n.$$

Then

$$\begin{aligned} \Lambda^2 &= \sum_{1 \leq i, i' \leq N(\pi^n)} a_{ii'}^2 = \sum_{j=1}^{N(A^n)} \sum_{p(n, j-1) \leq i \neq i' < p(n, j)} a_{ii'}^2 \\ &\leq \sum_{j=1}^{N(A^n)} \sum_{p(n, j-1) \leq i \neq i' < p(n, j)} \Delta t_i^n \Delta t_{i'}^n \leq \sum_{j=1}^{N(A^n)} |A_j^n|^2 \leq |A^n| \sum_{j=1}^{N(A^n)} |A_j^n| = |A^n| T. \end{aligned}$$

The result of Hanson and Wright (1971)[19] then implies that for  $\delta > 0$  there exist constants  $C_1$  and  $C_2$  such that

$$\forall n \geq 1, \quad \mathbb{P}\left(|S(A^n)| > \delta\right) \leq 2 \exp\left(-\min\left\{C_1 \frac{\delta}{\sqrt{|A^n|}}, C_2 \frac{\delta^2}{|A^n|}\right\}\right) \stackrel{n \rightarrow \infty}{\leq} 2 \exp\left(-C_1 \frac{\delta}{\sqrt{|A^n|}}\right)$$

since  $|A^n| \asymp |\pi^n|^\beta \rightarrow 0$ . As  $|\pi^n|^\beta (\log n)^2 \rightarrow 0$ , if we denote  $\varepsilon_n = \sqrt{|A^n|} (\log n)$  then  $\varepsilon_n \rightarrow 0$  and we can rewrite this bound as

$$\mathbb{P}\left(|S(A^n)| > \delta\right) \leq 2 \exp\left(-\min\left\{\frac{C_1 \delta \log(n)}{\varepsilon_n}, \frac{C_2 \delta^2 \log(n)^2}{\varepsilon_n^2}\right\}\right) \leq \frac{2C}{n^{C_1 \delta / \varepsilon_n}}.$$

The above estimate is valid for any  $A^n \in \mathcal{C}_n(\beta, \pi)$ . We now note that  $\mathcal{C}_n(\beta)$  is a finite set, of which we estimate the number of elements. For  $k = 1, 2, \dots, N(A^n) - 1$  let  $B(n, k) = p(n, k+1) - p(n, k)$  designate the number of points of  $\pi^n$  in the  $k$ -th interval  $[t_{p(n,k)}^n, t_{p(n,k+1)}^n)$  of  $A^n$ . For any  $n \geq 1$ ,

$$N(A^n) \inf_k B(n, k) \leq N(\pi^n) \leq N(A^n) \sup_k B(n, k)$$

and, since both  $A^n$  and  $\pi^n$  are balanced partitions,

$$0 < 1 \leq \limsup_n \frac{\sup_k B(n, k)}{\inf_k B(n, k)} < \infty.$$

So there exists  $N_1 \geq 1$  such that for  $n \geq N_1$  we have

$$0 < 1 \leq \limsup_n \frac{\sup_k B(n, k)}{\inf_k B(n, k)} \leq c_1 < \infty,$$

where  $c_1$  is a constant. So for  $n \geq N_1$ ,

$$0 \leq \frac{\sup_k B(n, k) - \inf_k B(n, k)}{\inf_k B(n, k)} \leq c_1 - 1 < \infty, \quad \text{i.e.} \quad 0 \leq \sup_k B(n, k) - \inf_k B(n, k) \leq c_2 \inf_k B(n, k).$$

So for  $n \geq N_1$  an upper-bound for the number of distinct partitions in  $\mathcal{C}_n(\beta, \pi)$  is given by

$$|\mathcal{C}_n(\beta, \pi)| \leq \left( \sup_k B(n, k) - \inf_k B(n, k) \right) \times N(A^n) \leq c_2 \inf_k B(n, k) \times N(A^n) \leq c_2 N(\pi^n).$$

Combining this with our previous estimate yields

$$\mathbb{P}\left(\exists A^n \in \mathcal{C}_n(\beta, \pi), |S(A^n)| > \delta\right) \leq \sum_{A^n \in \mathcal{C}_n(\beta, \pi)} \mathbb{P}\left(|S(A^n)| > \delta\right) \leq \frac{2c_2 N(\pi^n)}{n^{C_1 \delta / \varepsilon_n}}.$$

Since  $\pi$  is balanced,  $N(\pi^n) \asymp T/|\pi^n| \asymp (\log n)^{2/\beta} \varepsilon_n^{-2/\beta}$  so for  $n$  large we have

$$\forall \delta > 0, \quad \mathbb{P}\left(\exists A^n \in \mathcal{C}_n(\beta, \pi), |S(A^n)| > \delta\right) \leq \frac{2c(\log n)^{2/\beta} \varepsilon_n^{-2/\beta}}{n^{C_1 \delta / \varepsilon_n}} = O\left(\frac{(\log n)^{2/\beta} \varepsilon_n^{-2/\beta}}{n^{C_1 \delta / \beta \varepsilon_n}}\right).$$

Since  $\varepsilon_n \rightarrow 0$ , the series  $\sum_n \frac{(\log n)^{2/\beta} \varepsilon_n^{-2/\beta}}{n^{C_1 \delta / \beta \varepsilon_n}} < \infty$  is absolutely convergent, so we can apply the Borel-Cantelli lemma to obtain for each  $\delta > 0$  a set  $\Omega_\delta$  with  $\mathbb{P}(\Omega_\delta) = 1$  and  $N_\delta \in \mathbb{N}$  such that

$$\forall \omega \in \Omega_\delta, \quad \forall A \in \mathcal{C}(\beta, \pi), \quad \forall n \geq N_\delta, \quad |S(A^n)(\omega)| \leq \delta.$$

Now if we set

$$\Omega_\pi = \Omega_0 \cap \left( \bigcap_{k \geq 1} \Omega_{1/k} \right) \quad \text{then} \quad \mathbb{P}(\Omega_\pi) = 1$$

and for paths in  $\Omega_\pi$  we have  $S(A^n) \rightarrow 0$  simultaneously for *all*  $\beta$ -coarsenings  $A \in \mathcal{C}(\beta, \pi)$ :

$$\forall \omega \in \Omega_\pi, \quad \forall A \in \mathcal{C}(\beta, \pi), \quad S(A^n)(\omega) \xrightarrow{n \rightarrow \infty} 0,$$

which allows to conclude that  $\mathbb{P}(W \in R_\pi^\beta([0, T], \mathbb{R})) = 1$ . ■

Combining Theorem 3.4 with Remark 3.2 we obtain:

**Corollary 3.5.** *Under the assumptions of Theorem 3.4,*

$$\mathbb{P}\left(W \in \bigcap_{\pi \in \mathcal{C}(\beta, \sigma)} Q_\pi([0, T], \mathbb{R}^d)\right) = 1.$$

## 4 Uniqueness of quadratic variation along balanced partitions

### 4.1 Main result

The following lemma shows that the quadratic roughness property is a necessary condition for the stability of quadratic variation with respect to the choice of the partition sequence:

**Lemma 4.1.** *Let  $0 < \beta < 1$  and  $\pi = (\pi^n)_{n \geq 1}$  be a balanced partition sequence such that  $x \in Q_\pi([0, T], \mathbb{R}^d)$  with  $[x]_\pi$  strictly increasing. If for all  $\tau \in \mathcal{C}(\beta, \sigma)$  we have  $[x]_\tau = [x]_\pi$  then  $x$  has the quadratic roughness property:  $x \in R_\pi^\beta([0, T], \mathbb{R}^d)$ .*

*Proof.* Let  $\tau \in \mathcal{C}(\beta, \sigma)$  be a  $\beta$ -coarsening of  $\pi$ . Using the same notation as in Def. 3.1,

$$\tau^n = (0 = t_{p(n,0)}^n < t_{p(n,1)}^n < \dots < t_{p(n,N(\pi^n))}^n = T).$$

From the assumption we have  $x \in Q_\tau([0, T], \mathbb{R}^d)$  and  $[x]_\tau = [x]_\pi$ . So for all  $t \in [0, T]$ :

$$\begin{aligned} & \lim_{n \rightarrow \infty} [x]_{\pi^n}(t) - [x]_{\tau^n}(t) = 0 \\ &= \lim_{n \rightarrow \infty} \sum_{j=1}^{N(\tau^n)} \sum_{p(n,j-1) \leq i \neq i' < p(n,j)} (x(t_{i+1}^n \wedge t) - x(t_i^n \wedge t))^t (x(t_{i'+1}^n \wedge t) - x(t_{i'}^n \wedge t)) \end{aligned}$$

which is the quadratic roughness property, so the result follows.  $\blacksquare$

We will now show that quadratic roughness is also a *sufficient* condition for the uniqueness of quadratic variation along (balanced) partition sequences.

We consider a sequence of partitions satisfying the following assumption:

**Assumption 1.**  $\sigma = (\sigma^n)_{n \geq 1}$  is a balanced sequence of partitions of  $[0, T]$  with  $|\sigma^n| \rightarrow 0$  and

$$\limsup_n \frac{|\sigma^n|}{|\sigma^{n+1}|} < \infty, \quad \text{or equivalently} \quad \limsup_n \frac{|N(\sigma^{n+1})|}{|N(\sigma^n)|} < \infty \quad (13)$$

The dyadic partition (and any geometric partition) obviously satisfies this assumption.

Our main result is that quadratic roughness along such a sequence implies uniqueness of pathwise quadratic variation along *all* balanced partition sequences:

**Theorem 4.2.** *Let  $\sigma$  be a sequence of partitions satisfying Assumption 1 and  $x \in C^\alpha([0, T], \mathbb{R}^d) \cap R_\sigma^\beta([0, T], \mathbb{R}^d)$  for some  $0 < \beta \leq \alpha$ . Then for any balanced sequence of partitions  $\tau = (\tau^n)_{n \geq 1}$ , if  $x \in Q_\tau([0, T], \mathbb{R}^d)$  then  $[x]_\tau = [x]_\sigma$ .*

*Proof.* We first prove the result in the case where  $\limsup_n \frac{|\tau^n|}{|\sigma^n|} < 1$ . We will then show that the

case  $\limsup_n \frac{|\tau^n|}{|\sigma^n|} \geq 1$  may be reduced to this one.

If  $\limsup_n \frac{|\tau^n|}{|\sigma^n|} < 1$  then from Lemma 2.3(i) if we define

$$k(n) = \inf\{k \geq n, \quad |\tau^k| \leq |\sigma^n|\} < \infty \quad \text{since} \quad |\tau^k| \xrightarrow{k \rightarrow \infty} 0. \quad (14)$$

then the subsequence  $(\sigma^{k(n)})_{n \geq 1}$  of  $\sigma$  satisfies

$$\limsup_n \frac{|\tau^n|}{|\sigma^{k(n)}|} \geq 1 \quad \text{and} \quad \limsup_n \frac{|\tau^n|}{|\sigma^{k(n)-1}|} < 1.$$

For  $i = 1..N(\tau^n)$ ,

$$j(i, n) = \inf\{j \geq 1, \quad s_j^{k(n)} \in (t_i^n, t_{i+1}^n]\}.$$

Then we have

$$s_{j(i,n)-1}^{k(n)} \leq t_k^n < s_{j(i,n)}^{k(n)} < \cdots < s_{j(i+1,n)-1}^{k(n)} \leq t_{i+1}^n < s_{j(i+1,n)}^{k(n)}.$$

We now show that the size  $j(i+1, n) - j(i, n)$  of these clusters is uniformly bounded. Assume that  $\sup_{i=1..N(\tau^n)} j(i+1, n) - j(i, n) \rightarrow \infty$  as  $n \rightarrow \infty$ . Then, from the above definition of  $k(n)$  and using

the balanced property of  $\tau^n$  and  $\sigma^{k(n)}$  we have  $\limsup_n \frac{|\tau^n|}{|\sigma^{k(n)}|} \rightarrow \infty$ . Since  $\limsup_n \frac{|\tau^n|}{|\sigma^{k(n)-1}|}$  is bounded from (14) we thus have

$$\limsup_n \left( \frac{|\tau^n|}{|\sigma^{k(n)}|} - \frac{|\tau^n|}{|\sigma^{k(n)-1}|} \right) = \limsup_n \frac{|\tau^n|}{|\sigma^{k(n)-1}|} \left( \frac{|\sigma^{k(n)-1}|}{|\sigma^{k(n)}|} - 1 \right) = \infty$$

which contradicts the assumption that  $\limsup_j \frac{|\sigma^{j-1}|}{|\sigma^j|} < \infty$ . Hence the sequence

$\sup_{i=1..N(\tau^n)} j(i+1, n) - j(i, n)$  is bounded as  $n \rightarrow \infty$ :

$$\exists M > 0, \forall i, n \geq 1, \quad j(i+1, n) - j(i, n) \leq M < \infty.$$

Therefore  $(\sigma^{k(n)})_{n \geq 1}$  and  $(\tau^n)_{n \geq 1}$  are (asymptotically) comparable i.e. the sequences

$$\frac{N(\sigma^{k(n)})}{N(\tau^n)} \quad \text{and} \quad \frac{|\sigma^{k(n)}|}{|\tau^n|} \quad (15)$$

are bounded. Define now

$$l_n = \inf\{l \geq n : N(\sigma^l) \geq N(\sigma^{k(n)})^{1/\beta}\}$$

Then  $N(\sigma^{l_n-1}) < N(\sigma^{k(n)})^{1/\beta} \leq N(\sigma^{l_n})$ . Since the subsequence  $(\sigma^{l_n})$  is also balanced, there exists constants  $c_1$  and  $c_2$  such that  $c_1 |\sigma^{l_n-1}| > |\sigma^{k(n)}|^{1/\beta} \geq c_2 |\sigma^{l_n}|$ . The points of the partition  $\tau^n$  are interspersed among those of  $\sigma^{l_n}$ . Define for  $k = 1 \cdots N(\tau^n)$ :

$$p(n, k) = \inf\{m \geq 1 : s_m^{l_n} \in (t_k^n, t_{k+1}^n]\}.$$

Then we have

$$s_{p(n,k)-1}^{l_n} \leq t_k^n < s_{p(n,k)}^{l_n} < \cdots < s_{p(n,k+1)-1}^{l_n} \leq t_{k+1}^n < s_{p(n,k+1)}^{l_n} \quad (16)$$

where  $p(n, N(\tau^n)) - 1 = N(\sigma^{l_n})$ . We will show that by grouping the points of  $\sigma^{l_n}$  according to the intervals defined by  $\tau^n$  we obtain a  $\beta$ -coarsening of  $\sigma^{l_n}$  and use the quadratic roughness property of  $x$  to conclude that for all  $t \in [0, T]$ :

$$[x]_{\tau^n}(t) = \sum_{k=0}^{N(\tau^n)-1} (x(t_{k+1}^n \wedge t) - x(t_k^n \wedge t))^t (x(t_{k+1}^n \wedge t) - x(t_k^n \wedge t)) \quad \text{and}$$

$$[x]_{\sigma^{l_n}}(t) = \sum_{k=0}^{N(\sigma^{l_n})-1} (x(s_{k+1}^{l_n} \wedge t) - x(s_k^{l_n} \wedge t))^t (x(s_{k+1}^{l_n} \wedge t) - x(s_k^{l_n} \wedge t))$$

have the same limits. We shall give the proof for  $t = T$ ; for  $t < T$  we have an additional boundary term which goes to zero.

Decomposing  $\Delta_k^n = x(t_{k+1}^n) - x(t_k^n)$  along the partition points of  $\sigma^{l_n}$ , we obtain,

$$\underbrace{x(t_{k+1}^n) - x(t_k^n)}_{\Delta_k^n} = \underbrace{(x(t_{k+1}^n) - x(s_{p(n,k+1)-1}^{l_n}))}_{D_k} - \underbrace{(x(t_k^n) - x(s_{p(n,k)-1}^{l_n}))}_{B_k} + \underbrace{\sum_{i=p(n,k)}^{p(n,k+1)-1} (x(s_i^{l_n}) - x(s_{i-1}^{l_n}))}_{C_k}$$

Grouping together the terms in  $[x]_{\sigma^{l_n}}$  according to (16) yields

$$\begin{aligned} [x]_{\tau^n}(T) - [x]_{\sigma^{l_n}}(T) &= \sum_{k=1}^{N(\tau^n)} \left[ \Delta_k^{n,t} \Delta_k^n - \sum_{i=p(n,k)}^{p(n,k+1)-1} (x(s_i^{l_n}) - x(s_{i-1}^{l_n}))^t (x(s_i^{l_n}) - x(s_{i-1}^{l_n})) \right] \\ &= \sum_{k=1}^{N(\tau^n)} [\Delta_k^{n,t} \Delta_k^n - C_k^t C_k] + \sum_{k=1}^{N(\tau^n)} \left[ C_k^t C_k - \sum_{i=p(n,k)}^{p(n,k+1)-1} (x(s_i^{l_n}) - x(s_{i-1}^{l_n}))^t (x(s_i^{l_n}) - x(s_{i-1}^{l_n})) \right]. \end{aligned}$$

Since

$$\begin{aligned} \sum_{k=1}^{N(\tau^n)} C_k^t C_k &= \sum_{k=1}^{N(\tau^n)} (\Delta_k^n - D_k + B_k)^t (\Delta_k^n - D_k + B_k) \\ &= \sum_{k=1}^{N(\tau^n)} \Delta_k^{n,t} \Delta_k^n + \sum_{k=1}^{N(\tau^n)} (D_k - B_k)^t (D_k - B_k) + \sum_{k=1}^{N(\tau^n)} \Delta_k^{n,t} B_k - \sum_{k=1}^{N(\tau^n)} \Delta_k^{n,t} D_k \end{aligned}$$

we finally obtain

$$\left| \sum_{k=1}^{N(\tau^n)} [\Delta_k^{n,t} \Delta_k^n - C_k^t C_k] \right| \leq \left| \sum_{k=1}^{N(\tau^n)} (D_k - B_k)^t (D_k - B_k) \right| + \left| \sum_{k=1}^{N(\tau^n)} \Delta_k^{n,t} B_k \right| + \left| \sum_{k=1}^{N(\tau^n)} \Delta_k^{n,t} D_k \right|.$$

Now we will show that as  $n \rightarrow \infty$ ,  $\sum_{k=1}^{N(\tau^n)} (\Delta_k^{n,t} \Delta_k^n - C_k^t C_k) \rightarrow 0$ . Since  $x \in C^\alpha([0, T], \mathbb{R}^d)$  we have

$$\forall t \in [0, T - h], \quad \forall h > 0, \quad \|x(t+h) - x(t)\| \leq \|x\|_\alpha h^\alpha.$$

Now

$$\left\| \sum_{k=1}^{N(\tau^n)} D_k^t D_k \right\| \leq \sum_{i=1}^{N(\tau^n)} \|D_k\|^2 \leq \sum_{k=1}^{N(\tau^n)} \|x\|_\alpha^2 |\sigma^{l_n}|^{2\alpha} \leq \|x\|_\alpha^2 N(\tau^n) |\sigma^{l_n}|^{2\alpha} \leq cN(\sigma^{k(n)}) |\sigma^{k(n)}|^{2\alpha/\beta} \xrightarrow{n \rightarrow \infty} 0$$

since  $\alpha \geq \beta$ . Similarly we have  $\sum_{k=1}^{N(\tau^n)} B_k^t B_k \rightarrow 0$ . Therefore

$$\sum_{k=1}^{N(\tau^n)} |(D_k - B_k)^t (D_k - B_k)| \leq 2 \left| \sum_{k=1}^{N(\tau^n)} D_k^t D_k \right| + 2 \left| \sum_{k=1}^{N(\tau^n)} B_k^t B_k \right| \rightarrow 0.$$

Using the Cauchy-Schwarz inequality,

$$\left\| \sum_{k=1}^{N(\tau^n)} D_k^t \Delta_k^n \right\| \leq \left( \sum_{k=1}^{N(\tau^n)} \|D_k\|^2 \right)^{\frac{1}{2}} \left( \sum_{k=1}^{N(\tau^n)} \|\Delta_k^n\|^2 \right)^{\frac{1}{2}}.$$

Since the quadratic variation of  $x$  along the sequence of partitions  $\tau$  is finite the sequence

$\sum_{k=1}^{N(\tau^n)} \|\Delta_k^n\|^2$  is bounded. Combining this with the estimate above we obtain

$$\sum_{k=1}^{N(\tau^n)} D_k^t \Delta_k^n \rightarrow 0.$$

Similarly, we have,  $\sum_{k=1}^{N(\tau^n)} B_k^t \Delta_k^n \rightarrow 0$ . Therefore,  $\sum_{k=1}^{N(\tau^n)} [\Delta_k^{nt} \Delta_k^n - C_k^t C_k] \rightarrow 0$ . Hence

$$\lim_{n \rightarrow \infty} [x]_{\tau^n}(T) - [x]_{\sigma^n}(T) = \lim_{n \rightarrow \infty} \sum_{k=1}^{N(\tau^n)} \left[ C_k^t C_k - \sum_{i=p(n,k)}^{p(n,k+1)-1} \left( x(s_i^{ln}) - x(s_{i-1}^{ln}) \right)^t \left( x(s_i^{ln}) - x(s_{i-1}^{ln}) \right) \right].$$

The partition points of  $\sigma^{ln}$  and  $\tau^n$  are ordered according to

$$s_{p(n,k)-1}^{ln} \leq t_k^n < s_{p(n,k)}^{ln} < s_{p(n,k)+1}^{ln} < \cdots < s_{p(n,k+1)-1}^{ln} \leq t_{k+1}^n < s_{p(n,k+1)}^{ln}. \quad (17)$$

Expanding the term  $C_k^t C_k$  we obtain:

$$\begin{aligned} & \sum_{k=1}^{N(\tau^n)} \left[ C_k^t C_k - \sum_{i=p(n,k)}^{p(n,k+1)-1} \left( x(s_i^{ln}) - x(s_{i-1}^{ln}) \right)^t \left( x(s_i^{ln}) - x(s_{i-1}^{ln}) \right) \right] \\ &= \sum_{k=1}^{N(\tau^n)} \left[ \left( x(s_{p(n,k+1)-1}^{ln}) - x(s_{p(n,k)-1}^{ln}) \right)^t \left( x(s_{p(n,k+1)-1}^{ln}) - x(s_{p(n,k)-1}^{ln}) \right) \right. \\ & \quad \left. - \sum_{i=p(n,k)}^{p(n,k+1)-1} \left( x(s_i^{ln}) - x(s_{i-1}^{ln}) \right)^t \left( x(s_i^{ln}) - x(s_{i-1}^{ln}) \right) \right] \\ &= \sum_{k=1}^{N(\tau^n)} \sum_{\substack{i,j=p(n,k) \\ i \neq j}}^{p(n,k+1)-1} \left( x(s_i^{ln}) - x(s_{i-1}^{ln}) \right)^t \left( x(s_j^{ln}) - x(s_{j-1}^{ln}) \right). \end{aligned}$$

Since  $x \in R_\sigma^\beta([0, T], \mathbb{R}^d)$ , by Proposition 3.3(iii)  $x$  also has the quadratic roughness property along the subsequence  $\sigma^l = (\sigma^{ln})_{n \geq 1}$  with the same coarsening index  $\beta$ . We will now show that

$$A^n = (0 = s_0^{ln} < s_{p(n,1)-1}^{ln} < s_{p(n,2)-2}^{ln} \cdots < s_{p(n,N(\tau^n))-1}^{ln} = T)$$

is a  $\beta$ -coarsening of  $(\sigma^{ln})_{n \geq 1}$ . For large enough  $n$ ,  $\frac{1}{2} \tau^n \leq \underline{A}^n \leq |A^n| \leq 2|\tau^n|$  so  $(A^n)$  is a balanced sequence of partitions. Also,

$$\limsup_n \frac{|\tau^n|}{|\sigma^{ln}|^\beta} \leq c_3 \limsup_n \frac{|\tau^n|}{|\sigma^{k(n)+1}|} \leq c_4 \limsup_n \frac{|\tau^n|}{|\sigma^{k(n)}|} \times \frac{|\sigma^{k(n)}|}{|\sigma^{k(n)+1}|} < \infty.$$

On the other hand

$$\liminf_n \frac{|\tau^n|}{|\sigma^{ln}|^\beta} \geq c_5 \liminf_n \frac{|\tau^n|}{|\sigma^{k(n)}|} > 0.$$



Therefore  $(A^n, n \geq 1)$  is a  $\beta$ -coarsening of  $(\sigma^{l_n})_{n \geq 1}$ . The quadratic roughness property of  $x$  along  $(\sigma^{l_n})_{n \geq 1}$  then implies

$$\sum_{k=1}^{N(\tau^n)} \sum_{\substack{i,j=p(n,k) \\ i \neq j}}^{p(n,k+1)-1} \left( x(s_i^{l_n}) - x(s_{i-1}^{l_n}) \right)^t \left( x(s_j^{l_n}) - x(s_{j-1}^{l_n}) \right) \xrightarrow{n \rightarrow \infty} 0.$$

Therefore  $[x]_\sigma - [x]_\tau = \lim_{n \rightarrow \infty} [x]_{\tau^n}(T) - [x]_{\sigma^{t_n}}(T) = 0$ .

Let us now examine the case where  $\limsup_n \frac{|\tau^n|}{|\sigma^n|} \geq 1$ . Using Lemma 2.3 (i) there exists a subsequence  $(\pi^n)_{n \geq 1} = (\tau^{k(n)})_{n \geq 1}$  of  $\tau$  such that:

$$\limsup_n \frac{|\sigma^n|}{|\pi^n|} \geq 1.$$

So using the above result  $[x]_\pi = [x]_\sigma$ . Since  $\pi$  is a subsequence of  $\tau$  we have  $[x]_\pi = [x]_\sigma = [x]_\tau$ . ■

As an application, we show that the roughness property and the quadratic variation of a path are invariant under a reparameterization of the path:

**Proposition 4.3** (Stability of quadratic variation under reparametrization). *Let  $\alpha > 0$  and  $g \in C^1([0, T], \mathbb{R}_+)$  be an increasing function with  $\inf g' > 0$  and  $\pi$  a sequence of partitions satisfying Assumption 1. Then for  $x \in C^\alpha([0, T], \mathbb{R}) \cap R_\pi^\beta([0, T], \mathbb{R})$ , we have  $x \circ g \in Q_\pi([0, T], \mathbb{R})$  and*

$$\forall t \in [0, T], \quad [x]_\pi(g(t)) = [x \circ g]_\pi(t).$$

*Proof.* Let  $\pi^n = (0 = t_1^n < t_2^n < \dots < t_{N(\pi^n)}^n = T)$  be a balanced partition. Then  $g(\pi^n) = (g(t_k^n), k = 0..N(\pi^n))$  defines a partition of  $[g(0), g(T)]$ . From Proposition 2.2(iii) the sequence of partitions  $g(\pi) = (g(\pi^n))_{n \geq 1}$  is balanced. From the mean value theorem there exists  $U_k^n \in [t_k^n, t_{k+1}^n]$  such that  $g(t_{k+1}^n) - g(t_k^n) = g'(U_k^n)(t_{k+1}^n - t_k^n)$ . Therefore

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{\sup_{\pi^n} (g(t_{k+1}^n) - g(t_k^n))}{\sup_{\pi^{n+1}} (g(t_{k+1}^{n+1}) - g(t_k^{n+1}))} &= \limsup_{n \rightarrow \infty} \frac{\sup_{\pi^n} g'(U_k^n)(t_{k+1}^n - t_k^n)}{\sup_{\pi^{n+1}} g'(U_k^{n+1})(t_{k+1}^{n+1} - t_k^{n+1})} \\ &\leq \frac{\max g'}{\min g'} \limsup_{n \rightarrow \infty} \frac{|\pi^n|}{|\pi^{n+1}|} < \infty. \end{aligned}$$

The last inequality follows from Assumption 1 on  $\pi$ . The assumptions on  $g$  then imply that  $g(\pi) \asymp \pi$ . The sequence of partitions  $g(\pi^n)$  then satisfies Assumption 1. Therefore we can apply theorem 4.2 to conclude that

$$\forall t \in [0, T], \quad [x]_\pi(g(t)) = [x \circ g]_\pi(t). \quad \blacksquare$$

## 4.2 Intrinsic definition of quadratic variation

For  $\gamma \geq 0$ , define  $\mathcal{P}_\gamma([0, T])$  as the set of balanced partition sequences  $(\sigma^n)_{n \geq 1}$  satisfying Assumption 1 such that  $|\sigma^n| = o(|\log n|^{-(\gamma+\epsilon)})$  for some  $\epsilon > 0$ :

$$\mathcal{P}_\gamma([0, T]) = \left\{ \sigma \in \mathbb{B}([0, T]), \quad \limsup_{n \rightarrow \infty} \frac{|\sigma^n|}{|\sigma^{n+1}|} < \infty, \quad \exists \epsilon > 0, (\log n)^{\gamma+\epsilon} |\sigma^n| \xrightarrow{n \rightarrow \infty} 0 \right\}. \quad (18)$$

Let  $\mathcal{Q}([0, T], \mathbb{R}^d)$  be the set of  $(\frac{1}{2} - \varepsilon)$  Hölder-continuous paths which satisfy the quadratic roughness property along *some* partition sequence  $\sigma \in \mathcal{P}_4([0, T])$ :

$$\mathcal{Q}([0, T], \mathbb{R}^d) = C^{\frac{1}{2}-}([0, T], \mathbb{R}^d) \cap \left( \bigcup_{\sigma \in \mathcal{P}_4([0, T])} R_{\sigma}^{\frac{1}{2}}([0, T], \mathbb{R}^d) \right). \quad (19)$$

**Lemma 4.4.** *The class  $\mathcal{Q}([0, T], \mathbb{R}^d)$  is non-empty and contains all 'typical' Brownian paths.*

*Proof.* Let  $W$  be a Wiener process on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , which we take to be the canonical Wiener space without loss of generality. For any  $\sigma \in \mathcal{P}_4([0, T])$ , Theorem 3.4 implies that

$$\mathbb{P} \left( W \in R_{\sigma}^{\frac{1}{2}}([0, T], \mathbb{R}^d) \right) = 1.$$

Brownian paths are almost-surely  $\alpha$ -Hölder for  $\alpha < \frac{1}{2}$  [25], so  $\mathbb{P} \left( W \in C^{\frac{1}{2}-}([0, T], \mathbb{R}^d) \right) = 1$  so finally

$$\mathbb{P} \left( W \in R_{\sigma}^{\frac{1}{2}}([0, T], \mathbb{R}^d) \cap C^{\frac{1}{2}-}([0, T], \mathbb{R}^d) \right) = 1.$$

and the result follows. ■

Based on the results above we can now give an 'intrinsic' definition of pathwise quadratic variation for paths in  $\mathcal{Q}([0, T], \mathbb{R}^d)$  which does not rely on a particular partition sequence:

**Proposition 4.5** (Quadratic variation map). *There exists a unique map*

$$[\cdot] : \mathcal{Q}([0, T], \mathbb{R}^d) \rightarrow C^0([0, T], S_d^+)$$

*such that*

$$\forall \pi \in \mathbb{B}([0, T]), \quad \forall x \in Q_{\pi}([0, T], \mathbb{R}^d) \cap \mathcal{Q}([0, T], \mathbb{R}^d), \quad \forall t \in [0, T], \quad [x]_{\pi}(t) = [x](t).$$

*We call  $[x]$  the quadratic variation of  $x$ .*

*Proof.* Let  $\pi \in \mathbb{B}([0, T])$ . For any  $x \in Q_{\pi}([0, T], \mathbb{R}^d) \cap \mathcal{Q}([0, T], \mathbb{R}^d)$  there exists  $\sigma \in \mathcal{P}_4([0, T])$  such that  $x \in R_{\sigma}^{\frac{1}{2}}([0, T], \mathbb{R}^d) \cap C^{\frac{1}{2}-}([0, T], \mathbb{R}^d)$ . Then Theorem 4.2 implies that for any balanced partition sequence  $\pi \in \mathbb{B}([0, T])$  we have:

$$\forall t \in [0, T], \quad [x]_{\pi}(t) = [x]_{\sigma}(t).$$

By the same argument the quadratic variation does not depend on the choice of  $\sigma \in \mathcal{P}_4([0, T])$  such that  $x \in R_{\sigma}^{\frac{1}{2}}([0, T], \mathbb{R}^d)$ , so the result follows. ■

*Remark 4.6.* If  $X$  is a continuous  $\mathbb{P}$ -semimartingale then its image  $[X]$  under the map defined in Proposition 4.5 coincides almost-surely with the usual definition of quadratic variation. Building on [21], Karandikar and Rao [23] construct a (different) quadratic variation map which shares this property. However, in contrast to [23], our construction does not involve any path-dependent partitions, identifies explicitly the domain of definition of the map (rather than implicitly in terms of the support of a probability measure) and does not use any probabilistic tools.

## 5 Pathwise Itô calculus

### 5.1 Pathwise integration and the pathwise Itô formula

Theorem 4.2 and Proposition 4.5 allow to give an 'intrinsic' formulation of Föllmer's pathwise integration and pathwise Itô calculus without relying on a specific sequence of partitions.

**Theorem 5.1** (Uniqueness of the Föllmer integral). *There exists a unique map*

$$\begin{aligned} I & : C^2(\mathbb{R}^d) \times \mathcal{Q}([0, T], \mathbb{R}^d) \rightarrow \mathcal{Q}([0, T], \mathbb{R}) \\ (f, x) & \rightarrow I(f, x) = \int_0^\cdot (\nabla f \circ x).dx \end{aligned}$$

such that:  $\forall \pi \in \mathbb{B}([0, T]), \quad \forall x \in Q_\pi([0, T], \mathbb{R}^d) \cap \mathcal{Q}([0, T], \mathbb{R}^d), \quad \forall t \in [0, T],$

$$I(f, x)(t) = \int_0^t (\nabla f \circ x).d^\pi x = \lim_{n \rightarrow \infty} \sum_{\pi^n} \nabla f(x(t_i^n)) \cdot (x(t_{i+1}^n \wedge t) - x(t_i^n \wedge t)).$$

We denote  $I(f, x) = \int_0^\cdot (\nabla f \circ x).dx$ . Furthermore

$$\begin{aligned} \forall f \in C^2(\mathbb{R}^d), \quad \forall \pi \in \mathbb{B}([0, T]), \quad \forall x \in Q_\pi([0, T], \mathbb{R}^d) \cap \mathcal{Q}([0, T], \mathbb{R}^d), \\ f(x(t)) - f(x(0)) = \int_0^t (\nabla f \circ x).dx + \frac{1}{2} \int_0^t \langle \nabla^2 f(x), d[x] \rangle > \end{aligned} \quad (20)$$

$$\text{and} \quad \left[ \int_0^\cdot (\nabla f \circ x).dx \right]_\pi (t) = \int_0^t \langle (\nabla f \circ x)^t(\nabla f \circ x), d[x] \rangle. \quad (21)$$

*Proof.* For any balanced partition sequence  $\pi \in \mathbb{B}([0, T])$  if  $x \in Q_\pi([0, T], \mathbb{R}^d) \cap \mathcal{Q}([0, T], \mathbb{R}^d)$  then there exists  $\sigma \in \mathcal{P}_4([0, T])$  such that  $x \in R_\sigma^{\frac{1}{2}}([0, T], \mathbb{R}^d) \cap C^{\frac{1}{2}-}([0, T], \mathbb{R}^d)$ . Then the pathwise Ito formula [15] implies

$$\begin{aligned} \int_0^t (\nabla f \circ x).d^\pi x = f(x(t)) - f(x(0)) - \frac{1}{2} \int_0^t \langle \nabla^2 f(x), d[x]_\pi \rangle > \\ \text{and} \quad \int_0^t (\nabla f \circ x).d^\sigma x = f(x(t)) - f(x(0)) - \frac{1}{2} \int_0^t \langle \nabla^2 f(x), d[x]_\sigma \rangle. \end{aligned}$$

From Theorem 4.2 we have  $[x]_\sigma = [x]_\pi$  so

$$\forall t \in [0, T], \quad \int_0^t (\nabla f \circ x).d^\pi x = \int_0^t (\nabla f \circ x).d^\sigma x$$

i.e. the pathwise integral  $\int_0^t (\nabla f \circ x).d^\pi x$  along a balanced sequence of partitions  $\pi$  does not depend on choice of  $\pi$ . By the same argument, it does not depend on the choice of  $\sigma \in \mathcal{P}_4([0, T])$ .

To show  $I(f, x) \in \mathcal{Q}([0, T], \mathbb{R})$  we first note that by [1, Lemma 4.11] we have  $I(f, x) \in C^{\frac{1}{2}-}([0, T], \mathbb{R})$ .

Applying the pathwise isometry formula [2, Theorem 2.1], to the integral  $\int_0^\cdot (\nabla f \circ x).d^\pi x$  we obtain that  $\int_0^\cdot (\nabla f \circ x).d^\pi x = \int_0^\cdot (\nabla f \circ x).d^\sigma x \in Q_\pi([0, T], \mathbb{R}) \cap Q_\sigma([0, T], \mathbb{R})$  and

$$\left[ \int_0^\cdot \nabla f \circ x dx \right]_\pi (t) = \int_0^t \langle (\nabla f \circ x)^t (\nabla f \circ x), d[x]_\pi \rangle .$$

From Theorem 4.2 we have  $[x]_\sigma = [x]_\pi$ , so  $\left[ \int_0^\cdot \nabla f \circ x dx \right]_\pi (t)$  does not depend on choice of balanced partition  $\pi$ . As a consequence:

$$\left[ \int_0^\cdot \nabla f \circ x dx \right]_\sigma (t) = \left[ \int_0^\cdot \nabla f \circ x dx \right]_\pi (t) = \int_0^t \langle (\nabla f \circ x)^t (\nabla f \circ x), d[x] \rangle .$$

Since  $[x]$  is strictly increasing by assumption, the right hand side is a strictly increasing function as soon as  $\nabla f \circ x \neq 0$  (otherwise the result trivially holds). Since we can choose any  $\pi \in \mathcal{C}(\frac{1}{2}, \sigma)$  in the above, we can apply Lemma 4.1 to conclude that  $I(f, x)$  has the quadratic roughness property along  $\sigma$ :  $I(f, x) \in R_\sigma^{1/2}([0, T], \mathbb{R})$ . So finally  $I(f, x) \in Q_\pi([0, T], \mathbb{R})$ .  $\blacksquare$

## 5.2 Local time

Pathwise analogues of (semimartingale) local time have been considered in [3, 8, 9, 24, 28, 32] in the context of extension of Föllmer's pathwise Ito formula to convex functions or functions with Sobolev regularity. In the aforementioned studies, local time of a path is constructed as a limit of a sequence of discrete approximations along a sequence of time partitions.

Given a partition sequence  $\sigma = (\sigma^n)_{n \geq 1}$  and a path  $x \in C^0([0, T], \mathbb{R}) \cap Q_\sigma([0, T], \mathbb{R})$ , one defines the function  $L_t^{\sigma^n} : \mathbb{R} \rightarrow \mathbb{R}$  by

$$L_t^{\sigma^n}(u) := 2 \sum_{t_j^n \in \sigma^n \cap [0, t]} \mathbb{1}_{[[x(t_j^n), x(t_{j+1}^n))]}(u) |x(t_{j+1}^n \wedge t) - u|.$$

where  $[[u, v]] := [u, v]$  if  $u \leq v$  and  $[[u, v]] := [v, u]$  if  $u > v$ .  $L_t^{\sigma^n}$  is bounded and zero outside  $[\min x, \max x]$ .

Following [32, 3, 9, 28] we say that  $x$  has  $(L^2)$ -local time on  $[0, T]$  along  $\sigma$  if the sequence  $(L_t^{\sigma^n}, n \geq 1)$  converges weakly in  $L^2(\mathbb{R})$  to a limit  $L_t^\sigma$  for all  $t \in [0, T]$ :

$$\forall t \in [0, T], \quad \forall h \in L^2(\mathbb{R}), \quad \int L_t^{\sigma^n}(u) h(u) du \xrightarrow{n \rightarrow \infty} \int L_t^\sigma(u) h(u) du.$$

The local time along  $\pi$  satisfies the occupation time formula [32, 3, 28]: for every Borel set  $A \in \mathcal{B}(\mathbb{R})$ ,

$$\int_A L_t^\pi(u) du = \frac{1}{2} \int_0^t \mathbb{1}_A(x) d[x]_\pi$$

and the following extension of the pathwise Ito formula (20) to functions in the Sobolev space  $W^{2,2}(\mathbb{R})$  (see e.g.[10, Thm 3.1]):

$$\forall f \in W^{2,2}(\mathbb{R}), \quad f(x(t)) - f(x(0)) = \int_0^t (f' \circ x).d^\pi x + \frac{1}{2} \int_{\mathbb{R}} L_t^\pi(u) f''(u) du, \quad (22)$$

where the first integral is a limit of left Riemann sums along  $\pi$ :

$$\int_0^t (f' \circ x) \cdot d^\pi x = \lim_{n \rightarrow \infty} \sum_{\pi^n} f'(x(t_i^n)) \cdot (x(t_{i+1}^n) - x(t_i^n)).$$

The above construction depends on the choice of the partition sequence  $\pi$  and a natural question is therefore to clarify the dependence of this local time on choice of the partition sequence. Note that, differently from [18],  $L_t^\pi$  is the density of a *weighted* occupation measure, weighted by quadratic variation  $[x]_\pi$  so a necessary condition for the uniqueness of  $L_t^\pi$  is the uniqueness of  $[x]_\pi$ .

We now show that the quadratic roughness property implies an invariance property of the local time with respect to the sequence of partitions:

**Theorem 5.2** (Uniqueness of local time for rough functions). *Let  $\sigma$  be a sequence of partitions satisfying Assumption 1 and  $x \in C^\alpha([0, T], \mathbb{R}) \cap R_\sigma^\beta([0, T], \mathbb{R})$  with  $0 < \beta \leq \alpha \leq \frac{1}{2}$ . Assume  $x$  has local time  $L_t^\sigma$  on  $[0, t]$  along  $\sigma$ . Then if  $x$  has local time  $L_t^\pi$  on  $[0, t]$  along some balanced partition sequence  $\pi \in \mathbb{B}(\cdot, T)$  then*

$$L_t^\pi(u) = L_t^\sigma(u) \quad du - a.e.$$

This defines a unique element  $L_t \in L^2(\mathbb{R})$  which we call the local time of  $x$  on  $[0, t]$ .

This result shows that for paths satisfying the quadratic roughness property, the ( $L^2$ -)local time is an intrinsic object associated with the path  $x$ , independent of the (balanced) sequence of partitions used in the construction.

*Proof.* From [32, Satz 9] for any Borel set  $A \in \mathcal{B}(R)$  we have the occupation density formula:

$$\int_A L_t^\sigma(u) du = \frac{1}{2} \int_0^t \mathbb{1}_A(x) d[x]_\sigma.$$

If  $\pi$  is a balanced sequence of partitions and the local time along  $\pi$  exists, we also have

$$\forall A \in \mathcal{B}(R), \quad \int_A L_t^\pi(u) du = \frac{1}{2} \int_0^t \mathbb{1}_A(x) d[x]_\pi.$$

Since  $\pi$  is balanced and  $\limsup_n \frac{|\sigma^n|}{|\sigma^{n+1}|} < \infty$ , Theorem 4.2 implies that  $[x]_\pi = [x]_\sigma$ . Hence

$$\forall A \in \mathcal{B}(R), \quad \int_A L_t^\pi(u) du = \int_A L_t^\sigma(u) du,$$

which implies  $L_t^\pi = L_t^\sigma$  almost everywhere. ■

An important consequence of this result is the uniqueness of limits of left Riemann sums for integrands in the Sobolev space  $W^{1,2}(\mathbb{R})$  and a robust version of the pathwise Tanaka formula:

**Corollary 5.3** (Uniqueness of Föllmer integral on  $W^{1,2}(\mathbb{R})$  and pathwise Tanaka formula). *Under the assumptions of theorem 5.2 we have:*

$$\forall h \in W^{1,2}(\mathbb{R}), \forall t \in [0, T], \quad \int_0^t (h \circ x) d^\pi x = \int_0^t (h \circ x) d^\sigma x.$$

Designating this common value by  $\int_0^t (h \circ x)dx$ , we obtain

$$\forall f \in W^{2,2}(\mathbb{R}), \forall t \in [0, T], \quad f(x(t)) - f(x(0)) = \int_0^t (f' \circ x).dx + \frac{1}{2} \int_{\mathbb{R}} L_t(u) f''(u) du, \quad (23)$$

where the pathwise integral and the local time may be computed with respect to any balanced partition sequence along which  $x$  has local time.

## References

- [1] A. ANANOVA, *Pathwise integration and functional calculus for paths with finite quadratic variation*, PhD thesis, Imperial College London, 2018.
- [2] A. ANANOVA AND R. CONT, *Pathwise integration with respect to paths of finite quadratic variation*, Journal de Mathématiques Pures et Appliquées, 107 (2017), pp. 737–757.
- [3] J. BERTOIN, *Temps locaux et intégration stochastique pour les processus de Dirichlet*, in Séminaire de Probabilités, XXI, vol. 1247 of Lecture Notes in Math., Springer, Berlin, 1987, pp. 191–205.
- [4] R. CATELLIER AND M. GUBINELLI, *Averaging along irregular curves and regularisation of odes*, Stochastic Processes and their Applications, 126 (2016), pp. 2323 – 2366.
- [5] H. CHIU AND R. CONT, *On pathwise quadratic variation for cadlag functions*, Electronic Communications in Probability, 23 (2018).
- [6] R. CONT, *Functional Ito Calculus and functional Kolmogorov equations*, in Stochastic Integration by Parts and Functional Ito Calculus (Lecture Notes of the Barcelona Summer School in Stochastic Analysis, July 2012), Advanced Courses in Mathematics, Birkhauser Basel, 2016, pp. 115–208.
- [7] R. CONT AND D.-A. FOURNIÉ, *Change of variable formulas for non-anticipative functionals on path space*, J. Funct. Anal., 259 (2010), pp. 1043–1072.
- [8] R. CONT AND N. PERKOWSKI, *Pathwise integration and change of variable formulas for continuous paths with arbitrary regularity*, Transactions of the American Mathematical Society, 6 (2019), pp. 134–138.
- [9] M. DAVIS, J. OBLOJ, AND V. RAVAL, *Arbitrage bounds for prices of weighted variance swaps*, Mathematical Finance, 24 (2014), pp. 821–854.
- [10] M. DAVIS, J. OBLÓJ, AND P. SIORPAES, *Pathwise stochastic calculus with local times*, Ann. Inst. H. Poincaré Probab. Statist., 54 (2018), pp. 1–21.
- [11] W. F. DE LA VEGA, *On almost sure convergence of quadratic Brownian variation*, Ann. Probab., 2 (1974), pp. 551–552.
- [12] C. DELLACHERIE AND P.-A. MEYER, *Probabilities and potential*, vol. 29 of North-Holland Mathematics Studies, North-Holland Publishing Co., Amsterdam, 1978.
- [13] R. M. DUDLEY, *Sample functions of the gaussian process*, Ann. Probab., 1 (1973), pp. 66–103.

- [14] R. M. DUDLEY AND R. NORVAIŠA, *Concrete functional calculus*, Springer Monographs in Mathematics, Springer, New York, 2011.
- [15] H. FÖLLMER, *Calcul d'Itô sans probabilités*, in Seminar on Probability, XV (Univ. Strasbourg, Strasbourg, 1979/1980) (French), vol. 850 of Lecture Notes in Math., Springer, Berlin, 1981, pp. 143–150.
- [16] D. FREEDMAN, *Brownian Motion and Diffusion*, Springer, 1983.
- [17] P. K. FRIZ AND M. HAIRER, *A course on rough paths*, Universitext, Springer, Cham, 2014. With an introduction to regularity structures.
- [18] D. GEMAN AND J. HOROWITZ, *Occupation densities*, Ann. Probab., 8 (1980), pp. 1–67.
- [19] D. L. HANSON AND F. T. WRIGHT, *A bound on tail probabilities for quadratic forms in independent random variables*, Ann. Math. Statist., 42 (1971), pp. 1079–1083.
- [20] P. IMKELLER AND D. J. PRÖMEL, *Existence of lévy's area and pathwise integration*, Communications on Stochastic Analysis.
- [21] R. L. KARANDIKAR, *On the quadratic variation process of a continuous martingale*, Illinois J. Math., 27 (1983), pp. 178–181.
- [22] ———, *On pathwise stochastic integration*, Stochastic Process. Appl., 57 (1995), pp. 11–18.
- [23] R. L. KARANDIKAR AND B. V. RAO, *On quadratic variation of martingales*, Proc. Indian Acad. Sci. Math. Sci., 124 (2014), pp. 457–469.
- [24] D. KIM, *Local time for continuous paths with arbitrary regularity*, arxiv, (2019).
- [25] P. LÉVY, *Théorie de l'addition des variables aléatoires*, Gauthier-Villars, 1937.
- [26] ———, *Le mouvement brownien plan*, American Journal of Mathematics, 62 (1940), pp. 487–550.
- [27] P. LÉVY, *Processus stochastiques et mouvement brownien*, Gauthier-Villars & Cie, Paris, 2nd ed., 1965.
- [28] N. PERKOWSKI AND D. J. PRÖMEL, *Local times for typical price paths and pathwise Tanaka formulas*, Electron. J. Probab., 20 (2015), pp. no. 46, 15.
- [29] P. E. PROTTER, *Stochastic integration and differential equations*, Springer-Verlag, Berlin, 2005. Second edition.
- [30] A. SCHIED, *On a class of generalized Takagi functions with linear pathwise quadratic variation*, J. Math. Anal. Appl., 433 (2016), pp. 974–990.
- [31] S. J. TAYLOR, *Exact asymptotic estimates of Brownian path variation*, Duke Math. J., 39 (1972), pp. 219–241.
- [32] M. WUERMLI, *Lokalzeiten für martingale*, diploma thesis, Universität Bonn, 1980.

## A Technical proofs

### A.1 Proof of Proposition 2.2

(i) For any sequence of partitions  $\pi$  of  $[0, T]$  and for any  $n \geq 1$ :

$$N(\pi^n) \underline{\pi}^n \leq T \leq N(\pi^n) |\pi^n|.$$

For proof of  $(\Rightarrow)$ : Using the balanced property,  $\liminf_{n \rightarrow \infty} N(\pi^n) \underline{\pi}^n = \liminf_{n \rightarrow \infty} N(\pi^n) |\pi^n| \frac{\underline{\pi}^n}{|\pi^n|} \geq \liminf_{n \rightarrow \infty} \frac{1}{c} N(\pi^n) |\pi^n| \geq \frac{T}{c} > 0$ .

Similarly,  $\liminf_{n \rightarrow \infty} N(\pi^n) |\pi^n| = \liminf_{n \rightarrow \infty} N(\pi^n) \underline{\pi}^n \frac{|\pi^n|}{\underline{\pi}^n} \leq \liminf_{n \rightarrow \infty} c N(\pi^n) \underline{\pi}^n \leq cT < \infty$ .

For proof of  $(\Leftarrow)$ :  $\limsup_{n \rightarrow \infty} \frac{|\pi^n|}{\underline{\pi}^n} = \limsup_{n \rightarrow \infty} \frac{N(\pi^n) |\pi^n|}{N(\pi^n) \underline{\pi}^n} = \frac{\limsup_{n \rightarrow \infty} N(\pi^n) |\pi^n|}{\limsup_{n \rightarrow \infty} N(\pi^n) \underline{\pi}^n} < \infty$ .

(ii) For any sequence of partitions  $\pi$ , and any fixed  $h > 0$  there exists a  $N_0$  such that for all  $n \geq N_0$ ,  $|\pi^n| < h$ . So for all  $n \geq N_0$  and for all  $t \in [0, T - h]$ ,  $N(\pi^n, t, t + h) \geq 1$ . Hence:

$$\underline{\pi}^n \leq \frac{h}{N(\pi^n, t, t + h)} \leq |\pi^n|.$$

So

$$\limsup_{n \rightarrow \infty} \frac{\sup_{t \in [0, T-h]} N(\pi^n, t, t + h)}{\inf_{t \in [0, T-h]} N(\pi^n, t, t + h)} \leq \limsup_{n \rightarrow \infty} \frac{|\pi^n|}{h} \times \frac{h}{\underline{\pi}^n} < \infty.$$

(iii) For any balanced sequence of partitions  $\pi$  of  $[0, T]$  and for any  $n \geq 1$ :

$$c_1 N(\pi^n) |\pi^n| \leq N(\pi^n) \underline{\pi}^n \leq T \leq N(\pi^n) |\pi^n| \leq c_2 N(\pi^n) \underline{\pi}^n.$$

$c_1$  and  $c_2$  are constants  $> 0$ . So the equivalence follows.

(iv) Let  $\pi = (\pi^n)_{n \geq 1}$  be any balanced sequence of partitions of  $[0, T]$ :

$$\pi^n = (0 = t_1^n < t_2^n < \dots < t_{N(\pi^n)}^n = T).$$

Now, define the new partition  $g(\pi) = (g(\pi^n))_{n \geq 1}$  as follows:

$$g(\pi^n) = \left( g(0) = g(t_1^n) < g(t_2^n) < \dots < g(t_{N(\pi^n)}^n) = g(T) \right).$$

Now, from mean value theorem there exists  $u_k^n, v_k^n \in [t_k^n, t_{k+1}^n]$  such that,

$$\begin{aligned} \left| \limsup_{n \rightarrow \infty} \frac{|g(\pi^n)|}{g(\pi^n)} \right| &= \left| \limsup_{n \rightarrow \infty} \frac{\sup_{\pi^n} (g(t_{k+1}^n) - g(t_k^n))}{\inf_{\pi^n} (g(t_{k+1}^n) - g(t_k^n))} \right| = \left| \limsup_{n \rightarrow \infty} \frac{\sup_{\pi^n} g'(u_k^n) (t_{k+1}^n - t_k^n)}{\inf_{\pi^n} g'(v_k^n) (t_{k+1}^n - t_k^n)} \right| \\ &\leq \left| \limsup_{n \rightarrow \infty} \frac{\sup_{\pi^n} g'(u_k^n)}{\inf_{\pi^n} g'(v_k^n)} \right| \times \left| \limsup_{n \rightarrow \infty} \frac{\sup_{\pi^n} (t_{k+1}^n - t_k^n)}{\inf_{\pi^n} (t_{k+1}^n - t_k^n)} \right| \leq \frac{\max g'}{\inf g'} c < \infty. \end{aligned}$$



## A.2 Proof of Lemma 2.3

Denote the partition points of  $\tau^n$  and  $\sigma^n$  respectively by  $(t_k^n, k = 0..N(\tau^n))$  and  $(s_l^n, l = 0..N(\sigma^n))$ .

*Proof of (i):* From the assumption we have,  $\limsup_n \frac{|\tau^n|}{|\sigma^n|} > 1$ . Then there exists  $N_0 \in \mathbb{N}$  such

that for  $n \geq N_0$ ,  $\frac{|\tau^n|}{|\sigma^n|} \geq 1$ . Since we are only concerned about the limiting behaviour when  $n \rightarrow \infty$  we will only consider  $n > N_0$  throughout the rest of the proof.

If  $\limsup_n \frac{|\tau^n|}{|\sigma^n|} < \infty$  we set  $k(n) = n$ ; otherwise if  $\limsup_n \frac{|\tau^n|}{|\sigma^n|} = +\infty$  we define:

$$k(n) = \inf\{k \geq n, |\tau^k| \leq |\sigma^n|\} < \infty \quad \text{since} \quad |\tau^k| \xrightarrow{k \rightarrow \infty} 0. \quad (24)$$

We now consider the subsequence  $(\tau^{k(n)})_{n \geq 1}$  of  $\tau$ . From the definition of  $k(n)$ :

$$\limsup_n \frac{|\sigma^n|}{|\tau^{k(n)}|} \geq 1.$$

*Proof of (ii):* Define  $k(n)$  as in (24) for  $i = 1..N(\sigma^n)$ ,

$$j(i, n) = \inf\{j \geq 1, t_j^{k(n)} \in (s_i^n, s_{i+1}^n]\}.$$

Then we have

$$t_{j(i,n)-1}^{k(n)} \leq s_i^n < t_{j(i,n)}^{k(n)} < \dots < t_{j(i+1,n)-1}^{k(n)} \leq s_{i+1}^n < t_{j(i+1,n)}^{k(n)}.$$

If for some  $i$ ,  $j(i+1, n) - j(i, n) \rightarrow \infty$  as  $n \rightarrow \infty$  then, from the above construction of  $k(n)$  and using the well balanced property of  $\sigma^n$  and  $\tau^{k(n)}$  we have:  $\limsup_n \frac{|\sigma^n|}{|\tau^{k(n)}|} \rightarrow \infty$  and

$\limsup_n \frac{|\sigma^n|}{|\tau^{k(n)-1}|} < 1$ . Hence,  $\limsup_n \frac{|\sigma^n|}{|\tau^{k(n)-1}|} \left[ \frac{|\tau^{k(n)-1}|}{|\tau^{k(n)}|} - 1 \right] \rightarrow \infty$  which is a contradiction because of our assumption. Hence the size  $j(i+1, n) - j(i, n)$  of clusters is uniformly bounded:

$$\forall k, n \geq 1, j(i+1, n) - j(i, n) \leq M < \infty.$$

So there exists a constant  $c_0$  such that

$$1 \leq \limsup_n \frac{|\sigma^n|}{|\tau^{k(n)}|} \leq c_0 < \infty. \quad (25)$$

Therefore  $(\tau^{k(n)})_{n \geq 1}$  and  $(\sigma^n)_{n \geq 1}$  are (asymptotically) comparable.

*Proof of (iii):* If  $\limsup_n \frac{|\sigma^n|}{|\tau^n|} < 1$  then the set  $\{n \geq 1, \frac{|\sigma^n|}{|\tau^n|} \geq 1\}$  is finite and the set,

$$A = \{n \geq 1, \frac{|\sigma^n|}{|\tau^n|} < 1\}$$

is infinite. Now define  $r : \mathbb{N} \mapsto \mathbb{N}$  as follows: we set  $r(n) = n$  for  $n \notin A$  and

$$r(n) = \inf\{k \geq 1, |\sigma^k| > |\tau^n|\} < \infty \quad \text{for} \quad n \in A.$$

Then

$$\limsup_{n \rightarrow \infty} \frac{|\sigma^{r(n)}|}{|\tau^n|} = \limsup_{n \in A} \frac{|\sigma^{r(n)}|}{|\tau^n|} \geq 1.$$

### A.3 Proof of Proposition 3.3

The proof of 1 – 3 are direct consequences of Definition 3.1. The proof of 4 is as follows. Let  $A^n = (0 = t_{p(n,0)}^n < t_{p(n,1)}^n < \dots < t_{p(n,N(A^n))}^n = T)$  be any  $\beta$ -coarsening of  $\pi$ . Then  $|A^n| \asymp |\pi^n|^\beta$ . Let  $I_j^n = (t_{p(n,j-1)}^n, t_{p(n,j)}^n] \cap [0, t]$ . Then for all  $t \in (0, T]$  :

$$\begin{aligned}
& \sum_{j=1}^{N(A^n)} \sum_{t_i^n \neq t_{i'}^n \in I_j^n} ((x+y)(t_{i+1}^n) - (x+y)(t_i^n))^t ((x+y)(t_{i'+1}^n) - (x+y)(t_{i'}^n)) \\
&= \sum_{j=1}^{N(A^n)} \sum_{t_i^n \neq t_{i'}^n \in I_j^n} (x(t_{i+1}^n) - x(t_i^n))^t (x(t_{i'+1}^n) - x(t_{i'}^n)) \\
&+ \sum_{j=1}^{N(A^n)} \sum_{t_i^n \neq t_{i'}^n \in I_j^n} (y(t_{i+1}^n) - y(t_i^n))^t (y(t_{i'+1}^n) - y(t_{i'}^n)) \\
&+ \sum_{j=1}^{N(A^n)} \sum_{t_i^n \neq t_{i'}^n \in I_j^n} (x(t_{i+1}^n) - x(t_i^n))^t (y(t_{i'+1}^n) - y(t_{i'}^n)).
\end{aligned}$$

From the quadratic roughness property of  $x$ , the first term in the sum goes to zero as  $n$  increases. For the last sum we have

$$\begin{aligned}
& \sum_{j=1}^{N(A^n)} \sum_{t_i^n \neq t_{i'}^n \in I_j^n} (x(t_{i+1}^n) - x(t_i^n))^t (y(t_{i'+1}^n) - y(t_{i'}^n)) \\
&= \sum_{j=1}^{N(A^n)} \sum_{t_{i'}^n \in I_j^n} (y(t_{i'+1}^n) - y(t_{i'}^n))^t \sum_{t_i^n \in I_j^n, t_i^n \neq t_{i'}^n} (x(t_{i+1}^n) - x(t_i^n)) \\
&\leq \sum_{j=1}^{N(A^n)} \sum_{t_{i'}^n \in I_j^n} |y(t_{i'+1}^n) - y(t_{i'}^n)|^t \left| \sum_{t_i^n \in I_j^n, t_i^n \neq t_{i'}^n} (x(t_{i+1}^n) - x(t_i^n)) \right| \\
&\leq \sum_{j=1}^{N(A^n)} \sum_{t_{i'}^n \in I_j^n} |y(t_{i'+1}^n) - y(t_{i'}^n)|^t \left[ \left| \sum_{t_i^n \in I_j^n} (x(t_{i+1}^n) - x(t_i^n)) \right| + |(x(t_{i'+1}^n) - x(t_{i'}^n))| \right] \\
&\leq \left[ \sup_j \left| \sum_{t_i^n \in I_j^n} (x(t_{i+1}^n) - x(t_i^n)) \right| + \sup_k |(x(t_{k+1}^n) - x(t_k^n))| \right] \sum_{j=1}^{N(A^n)} \sum_{t_{i'}^n \in I_j^n} |y(t_{i'+1}^n) - y(t_{i'}^n)| \\
&\leq \left[ \sup_j \left| \sum_{t_i^n \in I_j^n} (x(t_{i+1}^n) - x(t_i^n)) \right| + \sup_k |(x(t_{k+1}^n) - x(t_k^n))| \right] \sum_{\pi^n} |y(t_{i'+1}^n) - y(t_{i'}^n)|
\end{aligned}$$

Since  $A^n$  is a balanced sequence and  $N(A^n) \rightarrow \infty$ , we have  $|A^n| \rightarrow 0$  so by continuity of  $x$ , the first term goes to zero as  $n \rightarrow \infty$ . The second term is bounded as  $y$  has bounded variation. So we have

$$\sum_{j=1}^{N(A^n)} \sum_{t_i^n \neq t_{i'}^n \in I_j^n} (x(t_{i+1}^n) - x(t_i^n))^t (y(t_{i'+1}^n) - y(t_{i'}^n)) \xrightarrow{n \rightarrow \infty} 0.$$

Similarly,

$$\sum_{j=1}^{N(A^n)} \sum_{t_i \neq t_{i'} \in I_j^n} (y(t_{i+1}^n) - y(t_i^n))^t (y(t_{i'+1}^n) - y(t_{i'}^n)) \xrightarrow{n \rightarrow \infty} 0.$$

This completes the proof.