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ON EXPONENTIAL STABILIZATION OF N -LEVEL QUANTUM ANGULAR MOMENTUM SYSTEMS*

WEICHAO LIANG[†], NINA H. AMINI[‡], AND PAOLO MASON[§]

Abstract. In this paper, we consider the feedback stabilization problem for N -level quantum angular momentum systems undergoing continuous-time measurements. By using stochastic and geometric control tools, we provide sufficient conditions on the feedback control law ensuring almost sure exponential convergence to a predetermined pure state corresponding to an eigenvector of the measurement operator. In order to achieve these results, we establish general features of quantum trajectories which are of interest by themselves. We illustrate the results by designing a class of feedback control laws satisfying the above-mentioned conditions and finally we demonstrate the effectiveness of our methodology through numerical simulations for three-level quantum angular momentum systems.

Key words. Stochastic stability, Exponential stability, Quantum control, Lyapunov techniques

AMS subject classifications. 81Q93, 93D15, 60H10

1. Introduction. The evolution of an open quantum system undergoing indirect continuous-time measurements is described by the so-called quantum stochastic master equation, which has been derived by Belavkin in quantum filtering theory [8]. The quantum filtering theory, relying on quantum stochastic calculus and quantum probability theory (developed by Hudson and Parthasarathy [20]) plays an important role in quantum optics and computation. The initial concepts of quantum filtering have been developed in the 1960s by Davies [16, 17] and extended by Belavkin in the 1980s [7, 8, 10, 9]. For a modern treatment of quantum filtering, we refer to [14].

A quantum stochastic master equation (or quantum filtering equation) is composed of a deterministic part and a stochastic part. The deterministic part, which corresponds to the average dynamics, is given by the well known Lindblad operator. The stochastic part represents the back-action effect of continuous-time measurements. The solutions of this equation are called quantum trajectories and their properties have been studied in [27, 28].

Quantum measurement-based feedback control, as a branch of stochastic control has been first developed by Belavkin in [7]. This field has attracted the interest of many theoretical and experimental researchers mainly starting from the early 2000s, yielding fundamental results [38, 6, 27, 37, 3, 40, 24]. In particular, theoretical studies carried out in [27, 18, 26, 4, 5] lead to the first experimental implementation of real-time quantum measurement-based feedback control in [34].

In [13], the authors established a quantum separation principle. Similarly to the classical separation principle, this result allows to interpret the control problem

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as a state-based feedback control problem for the filter (the best estimate, i.e., the conditional state), without caring of the actual quantum state. This motivates the state-based feedback design for the quantum filtering equation based on the knowledge of the initial state. In this context, stabilization of quantum filters towards pure states (i.e., the preparation of pure states) has major impact in developing new quantum technologies. According to [1], the stochastic part of the quantum filtering equation, unlike the deterministic one, contributes to increase the purity of the quantum state. Moreover, if we turn off the control acting on the quantum system, the measurement induces a collapse of the quantum state towards a pure state corresponding to either one of eigenvectors of the measurement operator, a phenomenon known as quantum state reduction [2, 38, 27, 33]. Thus, combining the continuous measurement with the feedback control may provide an effective strategy for preparing a selected target state in practice.

In [38], the authors design for the first time a quantum feedback controller that globally stabilizes a quantum spin- $\frac{1}{2}$ system (which is a special case of quantum angular momentum systems) towards a pure state corresponding to an eigenvector of σ_z in the presence of imperfect measurement. This feedback controller has been designed by looking numerically for an appropriate global Lyapunov function. Then, in [27], by analyzing the stochastic flow and by using the stochastic LaSalle theorem, the authors constructed a switching feedback controller which globally stabilizes the N -level quantum angular momentum system, in the presence of imperfect measurement, to the target eigenstate. A continuous version of this feedback controller has been proposed in [37]. The essential ideas in [38, 37] for constructing the continuous feedback controller remain the same: the controllers consist of two parts, the first one contributing to the local convergence to the target state, and the second one driving the system away from the antipodal states. Also, in [15], the authors have proven by simple Lyapunov arguments the exponential stabilizability for spin- $\frac{1}{2}$ systems by applying a proportional output feedback. For the same model, and as a preliminary step for this paper, we showed in [23] the exponential stabilizability by state feedback. In [12], the authors discussed the exponential stability of subspaces of the Hilbert space for general uncontrolled systems driven by Wiener and Poisson processes. They followed the approach established in [36] combining open-loop control and feedback design.

The main contribution of this paper is the derivation of some general conditions on the feedback law enforcing the exponential convergence towards the target state. These conditions are obtained mainly by studying the asymptotic behavior of quantum trajectories. Roughly speaking, under such conditions, and making use of the support theorem and other classical stochastic tools, we show that any neighborhood of the target state may be approached with probability one starting from any initial state (Lemma 6.1). This result allows to show the exponential convergence towards the target state by applying local Lyapunov-type arguments (Theorem 6.3). To show the convergence towards the target state, previous works applied stochastic LaSalle theorem (see e.g., [27, 38, 36, 37]) which, unlike Lyapunov-type arguments, does not guarantee exponential stability and does not provide any information on the convergence rate. As demonstration of the general result, explicit parametrized stabilizing feedback laws are exhibited (Theorem 6.4 and Theorem 6.5). In addition to the main result, in Theorem 5.1, we show the exponential convergence of the system with zero control towards the set of pure states corresponding to eigenvectors of the measurement operator (quantum state reduction with exponential rate). Note that to obtain our main results, some preliminary results on the asymptotic behavior of quantum

trajectories associated with the considered system were needed, see [Section 4](#). We believe that these results are significant by themselves.

Notations. The imaginary unit is denoted by i . We take $\mathbb{1}$ as the indicator function. We denote the conjugate transpose of a matrix A by A^* . The function $\text{Tr}(A)$ corresponds to the trace of a square matrix A . The commutator of two square matrices A and B is denoted by $[A, B] := AB - BA$.

2. System description. Consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. Let W_t be the one-dimensional standard Wiener process and assume that \mathcal{F}_t is the natural filtration of the process W_t . The dynamics of a N -level quantum angular momentum system is given by the following matrix-valued stochastic differential equation [\[8, 14, 38\]](#):

$$(2.1) \quad d\rho_t = F(\rho_t)dt + \sqrt{\eta}G(\rho_t)dW_t,$$

where

- the quantum state is described by the density operator ρ , which belongs to the compact space $\mathcal{S} := \{\rho \in \mathbb{C}^{N \times N} \mid \rho = \rho^*, \text{Tr}(\rho) = 1, \rho \geq 0\}$,
- the drift term is given by

$$F(\rho) := -i\omega[J_z, \rho] + M \left(J_z \rho J_z - \frac{1}{2} J_z^2 \rho - \frac{1}{2} \rho J_z^2 \right) - iu(\rho)[J_y, \rho],$$

and the diffusion term is given by $G(\rho) := \sqrt{M}(J_z \rho + \rho J_z - 2\text{Tr}(J_z \rho)\rho)$,

- the function $u : \mathcal{S} \rightarrow \mathbb{R}$ denotes the feedback law,
- J_z is the (self-adjoint) angular momentum along the axis z , and it is defined by

$$J_z e_n = (J - n)e_n, \quad n \in \{0, \dots, 2J\},$$

where $J = \frac{N-1}{2}$ represents the fixed angular momentum and $\{e_0, \dots, e_{2J}\}$ corresponds to an orthonormal basis of \mathbb{C}^N . With respect to this basis, the matrix form of J_z is given by

$$J_z = \begin{bmatrix} J & & & & \\ & J-1 & & & \\ & & \ddots & & \\ & & & -J+1 & \\ & & & & -J \end{bmatrix},$$

- J_y is the (self-adjoint) angular momentum along the axis y , and it is defined by

$$J_y e_n = -ic_n e_{n-1} + ic_{n+1} e_{n+1}, \quad n \in \{0, \dots, 2J\},$$

where $c_m = \frac{1}{2}\sqrt{(2J+1-m)m}$. The matrix form of J_y is given by

$$J_y = \begin{bmatrix} 0 & -ic_1 & & & \\ ic_1 & 0 & -ic_2 & & \\ & \ddots & \ddots & \ddots & \\ & & ic_{2J-1} & 0 & -ic_{2J} \\ & & & ic_{2J} & 0 \end{bmatrix},$$

- $\eta \in (0, 1]$ measures the efficiency of the detectors, $M > 0$ is the strength of the interaction between the light and the atoms, and $\omega \geq 0$ is a parameter characterizing the free Hamiltonian.

If the feedback u is in $C^1(\mathcal{S}, \mathbb{R})$, the existence and uniqueness of the solution of (2.1) as well as the strong Markov property of the solution are ensured by the results established in [27].

3. Basic stochastic tools. In this section, we will introduce some basic definitions and classical results which are fundamental for the rest of the paper.

Infinitesimal generator and Itô formula. Given a stochastic differential equation $dq_t = f(q_t)dt + g(q_t)dW_t$, where q_t takes values in $Q \subset \mathbb{R}^p$, the infinitesimal generator is the operator \mathcal{L} acting on twice continuously differentiable functions $V : Q \times \mathbb{R}_+ \rightarrow \mathbb{R}$ in the following way

$$\mathcal{L}V(q, t) := \frac{\partial V(q, t)}{\partial t} + \sum_{i=1}^p \frac{\partial V(q, t)}{\partial q_i} f_i(q) + \frac{1}{2} \sum_{i,j=1}^p \frac{\partial^2 V(q, t)}{\partial q_i \partial q_j} g_i(q) g_j(q).$$

Itô formula describes the variation of the function V along solutions of the stochastic differential equation and is given as follows

$$dV(q, t) = \mathcal{L}V(q, t)dt + \sum_{i=1}^p \frac{\partial V(q, t)}{\partial q_i} g_i(q) dW_t.$$

From now on, the operator \mathcal{L} is associated with the Equation (2.1).

Stochastic stability. We introduce some notions of stochastic stability needed throughout the paper by adapting classical notions (see e.g., [25, 22]) to our setting. In order to provide them, we first present the definition of Bures distance [11].

DEFINITION 3.1. *The Bures distance between two quantum states ρ_a and ρ_b in \mathcal{S} is defined as*

$$d_B(\rho_a, \rho_b) := \sqrt{2 - 2\text{Tr}\left(\sqrt{\sqrt{\rho_b}\rho_a\sqrt{\rho_b}}\right)}.$$

In particular, the Bures distance between a quantum state $\rho_a \in \mathcal{S}$ and a pure state $\rho_n := e_n e_n^$ with $n \in \{0, \dots, 2J\}$, is given by*

$$d_B(\rho_a, \rho_n) = \sqrt{2 - 2\sqrt{\text{Tr}(\rho_a \rho_n)}}.$$

Also, the Bures distance between a quantum state ρ_a and a set $E \subseteq \mathcal{S}$ is defined as

$$d_B(\rho_a, E) = \min_{\rho \in E} d_B(\rho_a, \rho).$$

Given $E \subseteq \mathcal{S}$ and $r > 0$, we define the neighborhood $B_r(E)$ of E as

$$B_r(E) = \{\rho \in \mathcal{S} \mid d_B(\rho, E) < r\}.$$

DEFINITION 3.2. *Let \bar{E} be an invariant set of system (2.1), then \bar{E} is said to be*

1. locally stable in probability, if for every $\varepsilon \in (0, 1)$ and for every $r > 0$, there exists $\delta = \delta(\varepsilon, r)$ such that,

$$\mathbb{P}(\rho_t \in B_r(\bar{E}) \text{ for } t \geq 0) \geq 1 - \varepsilon,$$

whenever $\rho_0 \in B_\delta(\bar{E})$.

2. exponentially stable in mean, if for some positive constants α and β ,

$$\mathbb{E}(d_B(\rho_t, \bar{E})) \leq \alpha d_B(\rho_0, \bar{E})e^{-\beta t},$$

whenever $\rho_0 \in \mathcal{S}$. The smallest value $-\beta$ for which the above inequality is satisfied is called the average Lyapunov exponent.

3. almost surely exponentially stable, if

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log d_B(\rho_t, \bar{E}) < 0, \quad \text{a.s.}$$

whenever $\rho_0 \in \mathcal{S}$. The left-hand side of the above inequality is called the sample Lyapunov exponent of the solution.

Note that any equilibrium $\bar{\rho}$ of (2.1), that is any quantum state satisfying $F(\bar{\rho}) = G(\bar{\rho}) = 0$, is a special case of invariant set.

Stratonovich equation and Support theorem. Any stochastic differential equation in Itô form in \mathbb{R}^K

$$dx_t = \widehat{X}_0(x_t)dt + \sum_{k=1}^n \widehat{X}_k(x_t)dW_t^k, \quad x_0 = x,$$

can be written in the following Stratonovich form [32]

$$dx_t = X_0(x_t)dt + \sum_{k=1}^n X_k(x_t) \circ dW_t^k, \quad x_0 = x,$$

where $X_0(x) = \widehat{X}_0(x) - \frac{1}{2} \sum_{l=1}^K \sum_{k=1}^n \frac{\partial \widehat{X}_k}{\partial x_l}(x)(\widehat{X}_k)_l(x)$, $(\widehat{X}_k)_l$ denoting the component l of the vector \widehat{X}_k , and $X_k(x) = \widehat{X}_k(x)$ for $k \neq 0$.

The following classical theorem relates the solutions of a stochastic differential equation with those of an associated deterministic one.

THEOREM 3.3 (Support theorem [35]). *Let $X_0(t, x)$ be a bounded measurable function, uniformly Lipschitz continuous in x and $X_k(t, x)$ be continuously differentiable in t and twice continuously differentiable in x , with bounded derivatives, for $k \neq 0$. Consider the Stratonovich equation*

$$dx_t = X_0(t, x_t)dt + \sum_{k=1}^n X_k(t, x_t) \circ dW_t^k, \quad x_0 = x.$$

Let \mathbb{P}_x be the probability law of the solution x_t starting at x . Consider in addition the associated deterministic control system

$$\frac{d}{dt}x_v(t) = X_0(t, x_v(t)) + \sum_{k=1}^n X_k(t, x_v(t))v^k(t), \quad x_v(0) = x.$$

with $v^k \in \mathcal{V}$, where \mathcal{V} is the set of all piecewise constant functions from \mathbb{R}_+ to \mathbb{R} . Now we define \mathcal{W}_x as the set of all continuous paths from \mathbb{R}_+ to \mathbb{R}^K starting at x , equipped with the topology of uniform convergence on compact sets, and \mathcal{I}_x as the smallest closed subset of \mathcal{W}_x such that $\mathbb{P}_x(x \in \mathcal{I}_x) = 1$. Then,

$$\mathcal{I}_x = \overline{\{x_v(\cdot) \in \mathcal{W}_x \mid v \in \mathcal{V}^n\}} \subset \mathcal{W}_x.$$

4. Preliminary results. Our aim here is to establish some basic properties of the quantum trajectories corresponding to Equation (2.1). This section is instrumental in order to prove our main results. In particular, the following lemmas are useful to identify invariant subsets of the state space \mathcal{S} . This allows to apply Itô formula to the Lyapunov functions considered in Section 5 and Section 6 which are \mathcal{C}^2 only on such invariant subsets. Moreover, Proposition 4.5 provides sufficient conditions on the feedback law which suppress the invariant subsets of the boundary (in the case $u \equiv 0$) for the case $\eta < 1$. In addition, Corollary 4.6 ensures that, under the assumptions of Proposition 4.5, the trajectories lying on the boundary converge to the target state.

Denote the projection of ρ onto the eigenstate ρ_k as $\rho_{k,k} := \text{Tr}(\rho \rho_k)$. In the following we state two lemmas inspired by analogous results established in [22, 25].

LEMMA 4.1. *Assume $u \equiv 0$. If $\rho_{k,k}(0) = 0$ for some $k \in \{0, \dots, 2J\}$, then $\mathbb{P}(\rho_{k,k}(t) = 0, \forall t \geq 0) = 1$, i.e., the set $\{\rho \in \mathcal{S} \mid \rho_{k,k} = 0\}$ is a.s. invariant for Equation (2.1). Otherwise, if the initial state satisfies $\rho_{k,k}(0) \neq 0$, then $\mathbb{P}(\rho_{k,k}(t) \neq 0, \forall t \geq 0) = 1$.*

Proof. For $u \equiv 0$, the dynamics of $\rho_{k,k}$ is given by

$$d\rho_{k,k}(t) = \sqrt{\eta}(G(\rho_t))_{k,k} dW_t = 2\sqrt{\eta M}(J - k - \text{Tr}(J_z \rho_t))\rho_{k,k}(t) dW_t.$$

In particular $|\sqrt{\eta}(G(\rho_t))_{k,k}| \leq R\rho_{k,k}(t)$, for some $R > 0$, yielding the first part of the lemma.

Let us now prove the second part of the lemma. Assume that $\rho_{k,k}(0) > 0$ and $\mathbb{P}(\rho_{k,k}(t) \neq 0, \forall t \geq 0) < 1$. In particular $\mathbb{P}(\tau < \infty) > 0$, where $\tau := \inf\{t \geq 0 \mid \rho_{k,k}(t) = 0\}$. Let T be sufficiently large so that $\mathbb{P}(\tau \leq T) > 0$. Now, let $\varepsilon \in (0, \rho_{k,k}(0))$, and consider any \mathcal{C}^2 function V defined on \mathcal{S} such that

$$V(\rho) = \frac{1}{\rho_{k,k}}, \quad \text{if } \rho_{k,k} > \varepsilon.$$

Then we have $\mathcal{L}V(\rho) = \rho_{k,k}^{-3}(\sqrt{\eta}G(\rho))_{k,k}^2 \leq R^2V(\rho)$ if $\rho_{k,k} > \varepsilon$. We further define the time-dependent function $f(\rho, t) = e^{-R^2t}V(\rho)$, whose infinitesimal generator is given by $\mathcal{L}f(\rho, t) = e^{-R^2t}(-R^2V(\rho) + \mathcal{L}V(\rho)) \leq 0$ if $\rho_{k,k} > \varepsilon$. Now, define the stopping time $\tau_\varepsilon := \inf\{t \geq 0 \mid \rho_{k,k}(t) \notin (\varepsilon, 1)\}$. By Itô formula, we have

$$\mathbb{E}(f(\rho_{\tau_\varepsilon \wedge T}, \tau_\varepsilon \wedge T)) = V_0 + \mathbb{E}\left(\int_0^{\tau_\varepsilon \wedge T} \mathcal{L}f(\rho_s, s) ds\right) \leq V_0 = \frac{1}{\rho_{k,k}(0)}.$$

Since $\tau \geq \tau_\varepsilon$ we deduce that, conditioning to the event $\{\tau \leq T\}$, $f(\rho_{\tau_\varepsilon \wedge T}, \tau_\varepsilon \wedge T) = f(\rho_{\tau_\varepsilon}, \tau_\varepsilon) = e^{-R^2T}\varepsilon^{-1}$, which implies

$$\mathbb{E}\left(e^{-R^2T}\varepsilon^{-1}\mathbb{1}_{\{\tau \leq T\}}\right) = \mathbb{E}\left(f(\rho_{\tau_\varepsilon}, \tau_\varepsilon)\mathbb{1}_{\{\tau \leq T\}}\right) \leq \mathbb{E}(f(\rho_{\tau_\varepsilon \wedge T}, \tau_\varepsilon \wedge T)) \leq \frac{1}{\rho_{k,k}(0)}.$$

Thus, $\mathbb{P}(\tau \leq T) = \mathbb{E}(\mathbb{1}_{\{\tau \leq T\}}) \leq \varepsilon e^{R^2 T / \rho_{k,k}(0)}$. Letting ε tend to 0, we get $\mathbb{P}(\tau \leq T) = 0$ which gives a contradiction. The proof is then complete. \square

LEMMA 4.2. *Let $n \in \{0, \dots, 2J\}$. Assume that $u \in \mathcal{C}^1(\mathcal{S}, \mathbb{R})$ and $u(\rho_n) = 0$. If the initial state satisfies $\rho_0 \neq \rho_n$, then $\mathbb{P}(\rho_t \neq \rho_n, \forall t \geq 0) = 1$.*

Proof. Given $\varepsilon > 0$, we consider any \mathcal{C}^2 function on \mathcal{S} such that

$$V(\rho) = \frac{1}{1 - \rho_{n,n}}, \quad \text{if } \rho_{n,n} < 1 - \varepsilon.$$

We find

$$\mathcal{L}V(\rho) = -\frac{u(\rho)\text{Tr}(i[J_y, \rho]\rho_n)}{(1 - \rho_{n,n})^2} + \frac{4\eta M[(J - n - \text{Tr}(J_z \rho))\rho_{n,n}]^2}{(1 - \rho_{n,n})^3},$$

whenever $\rho_{n,n} < 1 - \varepsilon$.

By applying the assumptions $u \in \mathcal{C}^1(\mathcal{S}, \mathbb{R})$ and $u(\rho_n) = 0$, we deduce that $|u(\rho)| = |u(\rho) - u(\rho_n)| \leq C\|\rho - \rho_n\| \leq \sqrt{2}C\sqrt{1 - \rho_{n,n}}$, where as matrix norm we have used the Hilbert-Schmidt norm, $\|A\| := \text{Tr}(AA^*)^{1/2}$. Then by

$$\begin{aligned} \text{Tr}(i[J_y, \rho]\rho_n) &= 2c_{n+1}\mathbf{Re}\{\rho_{n,n+1}\} - 2c_n\mathbf{Re}\{\rho_{n,n-1}\} \\ &\leq 2(c_{n+1} + c_n)\sqrt{\rho_{n,n}(1 - \rho_{n,n})}, \end{aligned}$$

we have $|u(\rho)\text{Tr}(i[J_y, \rho]\rho_n)| \leq 2C(c_{n+1} + c_n)(1 - \rho_{n,n})$. Also, as we have $|J - n - \text{Tr}(J_z \rho)| \leq 2J(1 - \rho_{n,n})$, we get $\mathcal{L}V(\rho) \leq KV(\rho)$, with $K = 2C(c_{n+1} + c_n) + 16J^2\eta M$. To conclude the proof, one just applies the same arguments as in the previous lemma. \square

Consider the observation process of the system y_t , whose dynamics satisfies $dy_t = dW_t + 2\sqrt{\eta M}\text{Tr}(J_z \rho_t)dt$. By Girsanov theorem [29], the process y_t is a standard Wiener process under a new probability measure \mathbb{Q} equivalent to \mathbb{P} . Denote by $\mathcal{F}_t^y := \sigma(y_s, 0 \leq s \leq t)$ the σ -field generated by the observation process up to time t . Then $u_t := u(\rho_t)$ is a bounded real càdlàg process adapted to \mathcal{F}_t^y . By applying the classical stochastic filtering theory [39], the Zakai equation associated with Equation (2.1) takes the following linear form

$$(4.1) \quad d\tilde{\rho}_t = F(\tilde{\rho}_t)dt + \sqrt{\eta}\tilde{G}(\tilde{\rho}_t)dy_t,$$

where $\tilde{\rho}_t = \tilde{\rho}_t^* \geq 0$, $F(\tilde{\rho})$ is defined as in (2.1), and $\tilde{G}(\tilde{\rho}) := \sqrt{M}(J_z \tilde{\rho}_t + \tilde{\rho}_t J_z)$. The equation (4.1) has a unique strong solution [39, 29], and the solutions of the equations (2.1) and (4.1) satisfy the relation

$$(4.2) \quad \rho_t = \tilde{\rho}_t / \text{Tr}(\tilde{\rho}_t),$$

which can be verified easily by applying Itô formula. In the following lemma, we adapt [27, Lemma 3.2] to the case of positive-definite matrices.

LEMMA 4.3. *Assume $u \in \mathcal{C}^1(\mathcal{S}, \mathbb{R})$. The set of positive-definite matrices is a.s. invariant for (2.1). More in general, the rank of ρ_t is a.s. non-decreasing.*

Proof. The initial state of (4.1) with respect to the basis of its eigenstates is given by $\tilde{\rho}_0 = \sum_i \tilde{\lambda}_i \tilde{\psi}_i \tilde{\psi}_i^*$, where $\tilde{\rho}_0 \tilde{\psi}_i = \tilde{\lambda}_i \tilde{\psi}_i$ for $i \in \{0, \dots, 2J\}$. If $\rho_0 > 0$, due to the relation (4.2), we have $\tilde{\rho}_0 > 0$, thus $\tilde{\lambda}_i > 0$ for all i . Extend the probability space by

defining $\mathcal{F}_t^{y, \widetilde{W}} := \sigma(y_s, \widetilde{W}_s, 0 \leq s \leq t)$, where \widetilde{W}_t is a Brownian motion independent of y_t . Set $B_t := \sqrt{\eta}y_t + \sqrt{1-\eta}\widetilde{W}_t$, whose quadratic variation satisfies $\langle B_t, B_t \rangle = t$. Following [27, Lemma 3.2], we consider the equations

$$\begin{aligned} d\rho_t^i &= F(\rho_t^i)dt + \widetilde{G}(\rho_t^i)\sqrt{\eta}dy_t + \widetilde{G}(\rho_t^i)\sqrt{1-\eta}d\widetilde{W}_t, \quad \rho_0^i = \tilde{\psi}_i\tilde{\psi}_i^*, \\ d\tilde{\psi}_i(t) &= (i\omega J_z - iu_t J_y - M/2J_z^2)\tilde{\psi}_i(t)dt + \sqrt{M}J_z\tilde{\psi}_i(t)dB_t, \quad \tilde{\psi}_i(0) = \tilde{\psi}_i, \end{aligned}$$

where $\tilde{\psi}_i(t) \in \mathbb{C}^N$. The solutions of the equations above satisfy $\rho_t^i = \tilde{\psi}_i(t)\tilde{\psi}_i^*(t)$ by Itô formula. In virtue of [29, Theorem 5.48], for all $t \geq 0$, there exists an almost surely invertible random matrix U_t such that $\psi_i(t) = U_t\tilde{\psi}_i$.

Let $\rho_t' = \sum_i \tilde{\lambda}_i \rho_t^i$, so that in particular $\rho_0' = \tilde{\rho}_0$ and $\rho_t' = U_t\tilde{\rho}_0U_t^*$. Due to the linearity of $F(\cdot)$ and $\widetilde{G}(\cdot)$, the stochastic Fubini theorem [39, Lemma 5.4] and the Itô isometry,

$$\mathbb{E}(\rho_t' | \mathcal{F}_t^y) = \rho_0' + \int_0^t F(\mathbb{E}(\rho_s' | \mathcal{F}_t^y))ds + \int_0^t \widetilde{G}(\mathbb{E}(\rho_s' | \mathcal{F}_t^y))\sqrt{\eta}dy_s.$$

By the uniqueness in law [30, Proposition 9.1.4] of the solution of the equation (4.1), the laws of $\tilde{\rho}_t$ and $\mathbb{E}(\rho_t' | \mathcal{F}_t^y) = \mathbb{E}(U_t\tilde{\rho}_0U_t^* | \mathcal{F}_t^y)$ are equal for all $t \geq 0$.

By what precedes $\rho_0 > 0$ implies $\rho_t' > 0$ a.s. which in turn yields $\rho_t = \tilde{\rho}_t/\text{Tr}(\tilde{\rho}_t) > 0$ a.s. We have thus proved that the set of positive-definite matrices is a.s. invariant for (2.1).

Let us now consider the general case in which ρ_0 is not necessarily full rank. We have

$$(4.3) \quad \text{rank}(\rho_t') = \text{rank}(U_t\tilde{\rho}_0U_t^*) = \text{rank}(\tilde{\rho}_0) = \text{rank}(\rho_0), \quad a.s.$$

Note that the kernel of any positive semi-definite matrix $\hat{\rho} \in \mathbb{C}^{N \times N}$ coincides with the space $\{\psi \in \mathbb{C}^N | \psi^*\hat{\rho}\psi = 0\}$, and that for almost every path $\rho_t'(\omega)$

$$\{\psi \in \mathbb{C}^N | \mathbb{E}(\psi^*\rho_t'\psi | \mathcal{F}_t^y) = 0\} \subseteq \{\psi \in \mathbb{C}^N | \psi^*\rho_t'(\omega)\psi = 0\}.$$

This implies $\text{rank}(\tilde{\rho}_t) \geq \text{rank}(\rho_t') = \text{rank}(\rho_0)$ for any $t \geq 0$ almost surely, which concludes the proof. \square

LEMMA 4.4. *Assume $u \in \mathcal{C}^1(\mathcal{S}, \mathbb{R})$. If $\eta = 1$, then the boundary of the state space $\partial\mathcal{S} := \{\rho \in \mathcal{S} | \det(\rho) = 0\}$ is a.s. invariant for (2.1).*

Proof. Based on the proof of Lemma 4.3, if $\eta = 1$, we have $B_t = y_t$ which implies $\tilde{\rho}_t = \rho_t'$. Then by applying the relation (4.3), we get the conclusion. \square

The Stratonovich form of Equation (2.1) is given by

$$(4.4) \quad d\rho_t = \widehat{F}(\rho_t)dt + \sqrt{\eta}G(\rho_t) \circ dW_t,$$

where

$$\begin{aligned} \widehat{F}(\rho) &:= -i\omega[J_z, \rho] + M \left((1-\eta)J_z\rho J_z - \frac{1+\eta}{2}(J_z^2\rho + \rho J_z^2) + 2\eta\text{Tr}(J_z^2\rho)\rho \right) \\ &\quad + 2\eta M\text{Tr}(J_z\rho)(J_z\rho + \rho J_z - 2\text{Tr}[J_z\rho]\rho) - iu(\rho)[J_y, \rho], \end{aligned}$$

and G is defined as in (2.1). The corresponding deterministic control system is given by

$$(4.5) \quad \dot{\rho}_v(t) = \widehat{F}(\rho_v(t)) + \sqrt{\eta}G(\rho_v(t))v(t), \quad \rho_v(0) = \rho_0,$$

where $v(t) \in \mathcal{V}$. By the support theorem ([Theorem 3.3](#)), the set \mathcal{S} is positively invariant for Equation [\(4.5\)](#).

In the following, we state some preliminary results that will be applied to our stabilization problem in the following sections. For this purpose, we fix a target state $\rho_{\bar{n}}$ for some $\bar{n} \in \{0, \dots, 2J\}$.

PROPOSITION 4.5. *Suppose that $\eta \in (0, 1)$, $u \in \mathcal{C}^1(\mathcal{S}, \mathbb{R})$ and $u(\rho_{\bar{n}}) = 0$. Assume that $\nabla u \cdot G(\rho_0) \neq 0$ or $\nabla u \cdot \widehat{F}(\rho_0) \neq 0$ for any $\rho_0 \in \{\rho \in \mathcal{S} \setminus \rho_{\bar{n}} \mid \rho_{k,k} = 0 \text{ for some } k, \text{ and } u(\rho) = 0\}$. Then for any initial condition $\rho_0 \in \{\rho \in \mathcal{S} \setminus \rho_{\bar{n}} \mid \rho_{k,k} = 0 \text{ for some } k\}$ and $\varepsilon > 0$, there exists at most one trajectory $\rho_v(t)$ of [\(4.5\)](#) starting from ρ_0 which lies in $\partial\mathcal{S}$ for t in $[0, \varepsilon]$. For any other initial state $\rho_0 \in \partial\mathcal{S} \setminus \rho_{\bar{n}}$ and $v \in \mathcal{V}$, $\rho_v(t) > 0$ for $t > 0$.*

Proof. Define $Z_1(t) := \text{Span}\{e_k \mid (\rho_v(t))_{k,k} = 0\}$ and $Z_2(t)$ the eigenspace corresponding to the eigenvalue 0 of $\rho_v(t)$. By definition, $Z_1(t) \subseteq Z_2(t)$ for all $t \geq 0$. Since all the subspaces which are invariant by J_z take the form $\text{Span}\{e_{k_1}, \dots, e_{k_n}\}$ for $\{k_1, \dots, k_n\} \subseteq \{0, \dots, 2J\}$, we deduce that $Z_1(t)$ is the largest subspace of $Z_2(t)$ invariant by J_z .

Denote by $\lambda_k(t)$ and $\psi_k(t)$ for $k \in \{0, \dots, 2J\}$ the eigenvalues and eigenvectors of $\rho_v(t)$, where, without loss of generality, we assume $\lambda_k(t) \in \mathcal{C}^1$ since $\rho_v(t) \in \mathcal{C}^1$ ([\[21, Theorem 2.6.8\]](#)). In addition, we suppose that the eigenvectors $\psi_k(t)$ form an orthonormal basis of \mathbb{C}^N .

Let $\psi_k(t) \in Z_2(t)$ for $t \in [0, \varepsilon]$. In order to provide an expression of the derivative for the eigenvalue λ_k along the path, we observe that

$$(4.6) \quad \frac{1}{t}(\lambda_k(t + \delta) - \lambda_k(t)) = \frac{1}{\psi_k^*(t + \delta)\psi_k(t)} \left(\psi_k^*(t + \delta) \frac{\rho_v(t + \delta) - \rho_v(t)}{t} \psi_k(t) \right).$$

Since ψ_k is a unit vector, then by compactness, we can extract a sequence $\delta_n \searrow 0$ such that $\psi_k(t + \delta_n)$ converges to an eigenvector $\psi_k(t)$ of $\rho_v(t)$. By passing to the limit on the left-hand and right-hand sides of Equation [\(4.6\)](#), we get $\dot{\lambda}_k(t) = \psi_k^*(t)\dot{\rho}_v(t)\psi_k(t) = M(1 - \eta)\psi_k^*(t)J_z\rho_v(t)J_z\psi_k(t)$.

If $\psi_k(t) \notin Z_1(t)$ then $J_z\psi_k(t) \notin Z_2(t)$, since otherwise $Z_1(t)$ would not be the largest subspace invariant by J_z contained in $Z_2(t)$. Thus $\dot{\lambda}_k(t) > 0$, which implies $\lambda_k(s) > 0$ for any $s - t > 0$ sufficiently small. We deduce that $\dim Z_2(s) \leq \dim Z_1(t)$. Moreover, by continuity of $\rho_v(t)$, we have $Z_1(s) \subseteq Z_1(t)$, for any $s - t > 0$ sufficiently small. Now we consider the case where $Z_1(t) \neq 0$ for $t \geq 0$. In this case, we have two possibilities: either $u(\rho_v(\cdot)) \equiv 0$ on $[0, \varepsilon]$ for some $\varepsilon > 0$; or $u(\rho_v(t)) \neq 0$ for arbitrarily small $t > 0$. Note that under the assumptions of the proposition there exists at most one v such that $u(\rho_v(\cdot)) \equiv 0$. It is therefore enough to show that, for the second possibility, $\rho_v(t)$ belongs to the interior of \mathcal{S} for all $t > 0$. For this purpose, we first show that for all $t > 0$ such that $u(\rho_v(t)) \neq 0$ and $Z_1(t) \neq 0$, there exists $s - t > 0$ arbitrarily small such that $u(\rho_v(s)) \neq 0$ and $Z_1(s) \subsetneq Z_1(t)$.

Let us pick k such that $e_k \in Z_1(t)$, and at least one between e_{k-1} and e_{k+1} is not contained in $Z_1(t)$ ¹. We now show by contradiction that $e_k \notin Z_1(s)$ for some $s - t > 0$ arbitrarily small. We assume that $e_k \in Z_1(\tau)$ for $\tau \in [t, t + \varepsilon]$, with $\varepsilon > 0$. By setting $q^n(\tau) := \rho_v(\tau)e_n$, for $n \in \{0, \dots, 2J\}$ and $\tau \geq 0$, the condition $(\rho_v(\tau))_{n,n} = 0$ is equivalent to $q^n(\tau) = 0$. In particular, by assumption, $q^k(\tau) = 0$ for $\tau \in [t, t + \varepsilon]$. On

¹If $k = 0$, the condition is replaced by $e_1 \notin Z_1(t)$ while if $k = 2J$, we assume $e_{2J-1} \notin Z_1(t)$.

this interval we have

$$\dot{q}^k(\tau) = iu(\rho_v(\tau))\rho_v(\tau)J_y e_k = u(\rho_v(\tau))\rho_v(\tau)\psi = 0,$$

where $\psi := c_k e_{k-1} - c_{k+1} e_{k+1}$. By taking ε small enough we may assume $u(\rho_v(\tau)) \neq 0$ and therefore the previous equality implies $\rho_v(\tau)\psi = 0$. This means that $\psi \in Z_2(\tau)$ and, since $\psi \notin Z_1(\tau)$, by the above argument we have $J_z \psi \notin Z_2(\tau)$ and

$$\psi^* \dot{\rho}_v(\tau)\psi = M(1 - \eta)\psi^* J_z \rho_v(\tau) J_z \psi > 0,$$

leading to a contradiction. Hence, there exists $s - t > 0$ arbitrarily small such that $Z_1(s) \subsetneq Z_1(t)$ and, by continuity of u , $u(\rho_v(s)) \neq 0$. Thus, by repeating the arguments for a finite number of steps, we can show that there exists $s - t > 0$ arbitrary small such that $Z_1(s) = 0$. As t may also be chosen arbitrarily small, this means that there exists an arbitrarily small $s > 0$ such that $\rho_v(s) > 0$.

To conclude the proof, we show that if $\rho_v(t_0) > 0$ for some $t_0 \geq 0$, then $\rho_v(t) > 0$ for all $t > t_0$. This can be done by considering the flow $\Phi_{t,v} : \mathcal{S} \rightarrow \mathcal{S}$ of Equation (4.5) which associates with each ρ_0 , the value $\rho_v(t)$. Since $\Phi_{t,v}$ is a diffeomorphism, if $\rho \in \mathcal{S} \setminus \partial\mathcal{S}$, there is an open neighborhood U of the state ρ such that $\Phi_{t,v}U \subset \mathcal{S}$ is also an open neighborhood of $\Phi_{t,v}\rho$. Thus, $\Phi_{t,v}\rho \in \mathcal{S} \setminus \partial\mathcal{S}$. The proof is then complete. \square

COROLLARY 4.6. *Suppose that the assumptions of Proposition 4.5 are satisfied. Then for all $\rho_0 \in \partial\mathcal{S} \setminus \rho_{\bar{n}}$, either ρ_t stays on the boundary of $\partial\mathcal{S}$ and converges to $\rho_{\bar{n}}$ as t goes to infinity or it exits the boundary in finite time and stays in the interior of \mathcal{S} afterwards, almost surely.*

Proof. By the support theorem, Theorem 3.3, and Proposition 4.5, we have $\mathbb{P}(\rho_\nu > 0) > 0$ for all $\nu > 0$ independently of the initial state $\rho_0 \in \mathcal{S} \setminus \rho_{\bar{n}}$. Define the set $\mathcal{S}_{\leq \zeta} := \{\rho \in \mathcal{S} \mid \det(\rho) \leq \zeta\} \setminus B_r(\rho_{\bar{n}})$ for any r arbitrary small and the stopping time $\tau_\zeta := \inf\{t \geq 0 \mid \rho_t \notin \mathcal{S}_{\leq \zeta}\}$. Now by compactness of $\mathcal{S}_{\leq \zeta}$ and the Feller continuity of ρ_t ([27, Lemma 4.5]), it is easy to see that for any $\nu > 0$ and $\zeta > 0$ small enough, there exists $\varepsilon > 0$ such that $\mathbb{P}_{\rho_0}(\tau_\zeta < \nu) > \varepsilon$,² independently of $\rho_0 \in \mathcal{S}_{\leq \zeta}$. Then we can conclude that $\sup_{\rho_0 \in \mathcal{S}_{\leq \zeta}} \mathbb{P}_{\rho_0}(\tau_\zeta \geq \nu) \leq 1 - \varepsilon$. By Dynkin inequality [19],

$$\sup_{\rho_0 \in \mathcal{S}_{\leq \zeta}} \mathbb{E}_{\rho_0}(\tau_\zeta) \leq \frac{\nu}{1 - \sup_{\rho_0 \in \mathcal{S}_{\leq \zeta}} \mathbb{P}_{\rho_0}(\tau_\zeta \geq \nu)} \leq \frac{\nu}{\varepsilon} < \infty.$$

By Markov inequality, for all $\rho_0 \in \mathcal{S}_{\leq \zeta}$, we have

$$\mathbb{P}_{\rho_0}(\tau_\zeta = \infty) = \lim_{n \rightarrow \infty} \mathbb{P}_{\rho_0}(\tau_\zeta \geq n) \leq \lim_{n \rightarrow \infty} \mathbb{E}_{\rho_0}(\tau_\zeta)/n = 0.$$

By arbitrariness of r we deduce that, either $\rho_t > 0$ for some positive time t or ρ_t converges to $\rho_{\bar{n}}$ as t tends to infinity while staying in $\partial\mathcal{S}$, almost surely. In addition, by the strong Markov property of ρ_t and Lemma 4.3, once ρ_t exits the boundary and enters the interior of \mathcal{S} , it stays in the interior afterwards. The proof is hence complete. \square

5. Quantum State Reduction. In this section, we study the dynamics of the N -level quantum angular momentum system (2.1) with the feedback $u \equiv 0$. First,

²Recall that \mathbb{P}_{ρ_0} corresponds to the probability law of ρ_t starting at ρ_0 ; the associated expectation is denoted by \mathbb{E}_{ρ_0} .

we can easily show, by Cauchy-Schwarz inequality, that in this case the equilibria of system (2.1) are exactly the eigenstates ρ_n , i.e., $F(\rho_n) = G(\rho_n) = 0$ with $n \in \{0, \dots, 2J\}$.

The following theorem shows that the quantum state reduction for the system (2.1) towards the invariant set $\bar{E} := \{\rho_0, \dots, \rho_{2J}\}$ occurs with exponential velocity. Note that in [27], the authors showed the almost sure asymptotic stability of the invariant set \bar{E} . Also, in the recent paper [15] the exponential stability in mean and the exact value of convergence probability have been shown in parallel to our results.

THEOREM 5.1 (*N-level quantum state reduction*). *For system (2.1), with $u \equiv 0$ and $\rho_0 \in \mathcal{S}$, the set \bar{E} is exponentially stable in mean and a.s. with average and sample Lyapunov exponent less or equal than $-\eta M/2$. Moreover, the probability of convergence to $\rho_n \in \bar{E}$ is $\text{Tr}(\rho_0 \rho_n)$ for $n \in \{0, \dots, 2J\}$.*

Proof. Let $I := \{k | \rho_{k,k}(0) = 0\}$ and $S_I := \{\rho \in \mathcal{S} | \rho_{k,k} = 0 \text{ if and only if } k \in I\}$. Then by Lemma 4.1, S_I is a.s. invariant for (2.1). Consider the function

$$(5.1) \quad V(\rho) = \frac{1}{2} \sum_{\substack{n,m=0 \\ n \neq m}}^{2J} \sqrt{\text{Tr}(\rho \rho_n) \text{Tr}(\rho \rho_m)} = \frac{1}{2} \sum_{\substack{n,m=0 \\ n \neq m}}^{2J} \sqrt{\rho_{n,n} \rho_{m,m}} \geq 0$$

as a candidate Lyapunov function. Note that $V(\rho) = 0$ if and only if $\rho \in \bar{E}$. As S_I is invariant for (2.1) with $u \equiv 0$ and V is twice continuously differentiable when restricted to S_I , we can compute $\mathcal{L}V(\rho) \leq -\frac{\eta M}{2} V(\rho)$. By Itô formula, for all $\rho_0 \in \mathcal{S}$, we have

$$\mathbb{E}(V(\rho_t)) = V(\rho_0) + \int_0^t \mathbb{E}(\mathcal{L}V(\rho_s)) ds \leq V(\rho_0) - \frac{\eta M}{2} \int_0^t \mathbb{E}(V(\rho_s)) ds.$$

In virtue of Grönwall inequality, we have $\mathbb{E}(V(\rho_t)) \leq V(\rho_0) e^{-\frac{\eta M}{2} t}$. Next, we show that the candidate Lyapunov function is bounded by the Bures distance from \bar{E} . Firstly, we have

$$V(\rho) = \frac{1}{2} \sum_{n=0}^{2J} \left(\sqrt{\rho_{n,n}} \sum_{m \neq n} \sqrt{\rho_{m,m}} \right) \geq \frac{1}{2} \sum_{n=0}^{2J} \sqrt{\rho_{n,n} (1 - \rho_{n,n})} \geq \frac{d_B(\rho, \bar{E})}{2} \sum_{n=0}^{2J} \sqrt{\rho_{n,n}}.$$

Combining with $\sum_{n=0}^{2J} \sqrt{\rho_{n,n}} \geq \sum_{n=0}^{2J} \rho_{n,n} = 1$, we have $\frac{1}{2} d_B(\rho, \bar{E}) \leq V(\rho)$. Let us now prove the converse inequality. Assume that $d_B(\rho, \bar{E}) = \sqrt{2 - 2\sqrt{\rho_{\bar{n},\bar{n}}}}$ for some index \bar{n} , then $\sqrt{\rho_{m,m}} \leq \sqrt{1 - \rho_{\bar{n},\bar{n}}} \leq d_B(\rho, \bar{E})$ for $m \neq \bar{n}$. In particular each addend in $V(\rho)$ is less or equal than $d_B(\rho, \bar{E})$, and $V(\rho) \leq J(2J+1) d_B(\rho, \bar{E})$. Thus, we have

$$(5.2) \quad C_1 d_B(\rho, \bar{E}) \leq V(\rho) \leq C_2 d_B(\rho, \bar{E}),$$

where $C_1 = 1/2$, $C_2 = J(2J+1)$. It implies,

$$\mathbb{E}(d_B(\rho_t, \bar{E})) \leq \frac{C_2}{C_1} d_B(\rho_0, \bar{E}) e^{-\frac{\eta M}{2} t}, \quad \forall \rho_0 \in \mathcal{S}.$$

which means that the set \bar{E} is exponentially stable in mean with average Lyapunov exponent less or equal than $-\eta M/2$.

Now we consider the stochastic process $Q(\rho_t, t) = e^{\frac{\eta M}{2} t} V(\rho_t) \geq 0$ whose infinitesimal generator is given by $\mathcal{L}Q(\rho, t) = e^{\frac{\eta M}{2} t} (\eta M/2 V(\rho) + \mathcal{L}V(\rho)) \leq 0$. Hence, the

process $Q(\rho_t, t)$ is a positive supermartingale. Due to Doob martingale convergence theorem [30], the process $Q(\rho_t, t)$ converges almost surely to a finite limit as t tends to infinity. Consequently, $Q(\rho_t, t)$ is almost surely bounded, that is $\sup_{t \geq 0} Q(\rho_t, t) = A$, for some a.s. finite random variable A . This implies $\sup_{t \geq 0} V(\rho_t) = Ae^{-\frac{\eta M}{2}t}$ a.s. Letting t goes to infinity, we obtain $\limsup_{t \rightarrow \infty} \frac{1}{t} \log V(\rho_t) \leq -\frac{\eta M}{2}$ a.s. By the inequality (5.2),

$$(5.3) \quad \limsup_{t \rightarrow \infty} \frac{1}{t} \log d_B(\rho_t, \bar{E}) \leq -\frac{\eta M}{2}, \quad a.s.$$

which means that the set \bar{E} is a.s. exponentially stable with sample Lyapunov exponent less or equal than $-\eta M/2$.

In order to calculate the probability of convergence towards $\rho_n \in \bar{E}$, we follow an approach inspired by [5, 2]. According to the first part of the theorem, the process $\text{Tr}(\rho_t \rho_n)$ converges a.s. to $\mathbf{1}_{\{\rho_t \rightarrow \rho_n\}}$. Therefore, by applying the dominated convergence theorem, $\text{Tr}(\rho_t \rho_n)$ converges to $\mathbf{1}_{\{\rho_t \rightarrow \rho_n\}}$ in mean. As $\mathcal{L}\text{Tr}(\rho_t \rho_n) = 0$, then $\text{Tr}(\rho_t \rho_n)$ is a positive martingale. Hence,

$$\mathbb{P}(\rho_t \rightarrow \rho_n) = \lim_{t \rightarrow \infty} \mathbb{E}(\text{Tr}(\rho_t \rho_n)) = \text{Tr}(\rho_0 \rho_n),$$

and the proof is complete. \square

6. Exponential stabilization by continuous feedback. In this section, we study the exponential stabilization of system (2.1) towards a selected target state $\rho_{\bar{n}}$ with $\bar{n} \in \{0, \dots, 2J\}$. Firstly, we establish a general result ensuring the exponential convergence towards $\rho_{\bar{n}}$ under some assumptions on the feedback control law and an additional local Lyapunov type condition. Next, we design a parametrized family of feedback control laws satisfying such conditions.

6.1. Almost sure global exponential stabilization. Inspired by [37, Lemma 3.4] and [30, Proposition 3.1], in the following lemma we show that, wherever the initial state is, the trajectory ρ_t enters in $B_r(\rho_{\bar{n}})$ with $r > 0$ in finite time almost surely.

Before stating the result, we define $\mathbf{P}_{\bar{n}} := \{\rho \in \mathcal{S} \mid J - \bar{n} - \text{Tr}(J_z \rho) = 0\}$ and the ‘‘variance function’’ $\mathcal{V}(\rho) := \text{Tr}(J_z^2 \rho) - \text{Tr}(J_z \rho)^2$ of J_z .

LEMMA 6.1. *Assume that $u \in C^1(\mathcal{S}, \mathbb{R})$ and $u(\rho_{\bar{n}}) = 0$. Suppose that for any $\rho_0 \in \{\rho \in \mathcal{S} \mid \rho_{\bar{n}, \bar{n}} = 0\}$, there exists a control $v(t) \in \mathcal{V}$ such that for all $t \in (0, \varepsilon)$, with ε sufficiently small, $u(\rho_v(t)) \neq 0$, for some solution $\rho_v(t)$ of Equation (4.5). Assume moreover that*

$$(6.1) \quad \forall \rho \in \mathbf{P}_{\bar{n}} \setminus \rho_{\bar{n}}, \quad 2\eta M \mathcal{V}(\rho) \rho_{\bar{n}, \bar{n}} > u(\rho) \text{Tr}(i[J_y, \rho] \rho_{\bar{n}}).$$

Then for all $r > 0$ and any given initial state $\rho_0 \in \mathcal{S}$, $\mathbb{P}(\tau_r < \infty) = 1$, where $\tau_r := \inf\{t \geq 0 \mid \rho_t \in B_r(\rho_{\bar{n}})\}$ and ρ_t corresponds to the solution of system (2.1).

Proof. The lemma holds trivially for $\rho_0 \in B_r(\rho_{\bar{n}})$, as in that case $\tau_r = 0$. Let us thus suppose that $\rho_0 \in \mathcal{S} \setminus B_r(\rho_{\bar{n}})$. We show that there exists $T \in (0, \infty)$ and $\zeta \in (0, 1)$ such that $\mathbb{P}_{\rho_0}(\tau_r < T) > \zeta$. For this purpose, we make use of the support theorem. Therefore, we consider the differential equation

$$(6.2) \quad (\dot{\rho}_v(t))_{\bar{n}, \bar{n}} = \Delta_{\bar{n}}(\rho_v(t)) + 2\sqrt{\eta M P_{\bar{n}}}(\rho_v(t))(\rho_v(t))_{\bar{n}, \bar{n}} v(t),$$

where $v(t) \in \mathcal{V}$ is the control input, and

$$\begin{aligned}\Delta_{\bar{n}}(\rho) &:= 2\eta M [\text{Tr}(J_z^2 \rho) - (J - \bar{n})^2] \rho_{\bar{n}, \bar{n}} - u(\rho) \text{Tr}(i[J_y, \rho] \rho_{\bar{n}}) \\ &\quad + 4\eta M P_{\bar{n}}(\rho) \text{Tr}(J_z \rho) \rho_{\bar{n}, \bar{n}}, \\ P_{\bar{n}}(\rho) &:= J - \bar{n} - \text{Tr}(J_z \rho).\end{aligned}$$

Consider the special case in which $\rho_{\bar{n}, \bar{n}}(0) = 0$. By applying similar arguments as in the proof of [Proposition 4.5](#), there exists a control input $v \in \mathcal{V}$ such that $(\rho_v(t))_{\bar{n}, \bar{n}} > 0$ for all $t > 0$. Thus, without loss the generality, we suppose $\rho_{\bar{n}, \bar{n}}(0) > 0$. Then we show that there exist a control input v and a time $T \in (0, \infty)$ such that $\rho_v(t) \in B_r(\rho_{\bar{n}})$ for $t \leq T$ in the two following separate cases.

1. Let $\bar{n} \in \{0, 2J\}$. We have $\mathbf{P}_{\bar{n}} = \rho_{\bar{n}}$. Since $\mathcal{S} \setminus B_r(\rho_{\bar{n}})$ is compact, $\Delta_{\bar{n}}(\rho)$ is bounded from above in this domain and $|P_{\bar{n}}(\rho)|$ is bounded from below. Then by choosing the control input $v = KP_{\bar{n}}(\rho)/\rho_{\bar{n}, \bar{n}}$, with $K > 0$ sufficiently large, we can guarantee that $\rho_v(t) \in B_r(\rho_{\bar{n}})$ for $t \leq T$ with $T < \infty$ if $\rho_{\bar{n}, \bar{n}}(0) > 0$.
2. Now suppose $\bar{n} \in \{1, \dots, 2J-1\}$. Due to the compactness of $\mathbf{P}_{\bar{n}} \setminus B_r(\rho_{\bar{n}})$ and the condition [\(6.1\)](#), we have

$$\begin{aligned}m &:= \min_{\rho \in \mathbf{P}_{\bar{n}} \setminus B_r(\rho_{\bar{n}})} \Delta_{\bar{n}}(\rho) \\ &= \min_{\rho \in \mathbf{P}_{\bar{n}} \setminus B_r(\rho_{\bar{n}})} \left(2\eta M \mathcal{V}(\rho) \rho_{\bar{n}, \bar{n}} - u(\rho) \text{Tr}(i[J_y, \rho] \rho_{\bar{n}}) \right) > 0.\end{aligned}$$

Then we define an open set containing $\mathbf{P}_{\bar{n}} \setminus B_r(\rho_{\bar{n}})$,

$$\mathbf{P}_{\bar{n}} \setminus B_r(\rho_{\bar{n}}) \subseteq \mathbf{U} := \{\rho \in \mathcal{S} \mid \Delta_{\bar{n}}(\rho) > m/2\} \subseteq \mathcal{S}.$$

Thus, setting $v(t) = 0$ whenever $\rho_v(t) \in \mathbf{U}$, we have

$$(\dot{\rho}_v(t))_{\bar{n}, \bar{n}} = \Delta_{\bar{n}}(\rho_v(t)) > m/2 \quad \text{on } \mathbf{U}.$$

Moreover, $(\mathcal{S} \setminus B_r(\rho_{\bar{n}})) \setminus \mathbf{U}$ is compact, then $\Delta_{\bar{n}}(\rho)$ is bounded from above and $|P_{\bar{n}}(\rho)|$ is bounded from below in this domain. For all $\rho_v(t) \in \{\rho \in \mathcal{S} \mid \rho_{\bar{n}, \bar{n}} > 0\}$, we can take the feedback $v = KP_{\bar{n}}(\rho)/\rho_{\bar{n}, \bar{n}}$ with $K > 0$ sufficiently large, so that $(\dot{\rho}_v(t))_{\bar{n}, \bar{n}}$ is bounded from below on $(\mathcal{S} \setminus B_r(\rho_{\bar{n}})) \setminus \mathbf{U}$. The proposed input v guarantees that $\rho_v(t) \in B_r(\rho_{\bar{n}})$ for $t \leq T$ with $T < \infty$ if $\rho_{\bar{n}, \bar{n}}(0) > 0$. Therefore, there exists $T \in (0, \infty)$ such that, for all $\rho_0 \in \mathcal{S} \setminus B_r(\rho_{\bar{n}})$, there exists $v(t)$ steering the system from ρ_0 to $B_r(\rho_{\bar{n}})$ by time T . By compactness of $\mathcal{S} \setminus B_r(\rho_{\bar{n}})$ and the Feller continuity of ρ_t , we have $\sup_{\rho_0 \in \mathcal{S} \setminus B_r(\rho_{\bar{n}})} \mathbb{P}_{\rho_0}(\tau_r \geq T) \leq 1 - \zeta < 1$, for some $\zeta > 0$. By Dynkin inequality [\[19\]](#),

$$\sup_{\rho_0 \in \mathcal{S} \setminus B_r(\rho_{\bar{n}})} \mathbb{E}_{\rho_0}(\tau_r) \leq \frac{T}{1 - \sup_{\rho_0 \in \mathcal{S} \setminus B_r(\rho_{\bar{n}})} \mathbb{P}_{\rho_0}(\tau_r \geq T)} \leq \frac{T}{\zeta} < \infty.$$

Then by Markov inequality, for all $\rho_0 \in \mathcal{S} \setminus B_r(\rho_{\bar{n}})$, we have

$$\mathbb{P}_{\rho_0}(\tau_r = \infty) = \lim_{n \rightarrow \infty} \mathbb{P}_{\rho_0}(\tau_r \geq n) \leq \lim_{n \rightarrow \infty} \mathbb{E}_{\rho_0}(\tau_r)/n = 0,$$

which implies $\mathbb{P}_{\rho_0}(\tau_r < \infty) = 1$. The proof is complete. \square

Remark 6.2. In Step 1 and Step 2 of the proof of [\[27, Theorem 4.2\]](#), the authors showed, by using similar techniques, that if $u(\rho) = 1$, then there exists a $\gamma \in (0, 1)$ such

that for all $\rho_0 \in \{\rho \in \mathcal{S} \mid \rho_{\bar{n}, \bar{n}} \leq \gamma\}$, $\mathbb{P}(\hat{\tau}_\gamma < \infty) = 1$ with $\hat{\tau}_\gamma := \inf\{t > 0 \mid \rho_{\bar{n}, \bar{n}} > \gamma\}$. On the other hand, our result is stronger than that of [27] as ρ_t can enter in an arbitrarily small neighborhood of $\rho_{\bar{n}}$ in finite time almost surely. This will allow to localize the Lyapunov stability analysis around the target state. Note that the method developed to show Step 1 of the proof of [27, Theorem 4.2] strongly relies on the fact that the feedback control is constant, while our method works for \mathcal{C}^1 controls.

In the following, we state our general result concerning the exponential stabilization of N -level quantum angular momentum systems.

THEOREM 6.3. *Assume that the feedback control law satisfies the assumptions of Lemma 6.1. Additionally, suppose that there exists a positive-definite function $V(\rho)$ such that $V(\rho) = 0$ if and only if $\rho = \rho_{\bar{n}}$, and V is continuous on \mathcal{S} and twice continuously differentiable on the set $\mathcal{S} \setminus \rho_{\bar{n}}$. Moreover, suppose that there exist positive constants C , C_1 and C_2 such that*

- (i) $C_1 d_B(\rho, \rho_{\bar{n}}) \leq V(\rho) \leq C_2 d_B(\rho, \rho_{\bar{n}})$, for all $\rho \in \mathcal{S}$, and
- (ii) $\limsup_{\rho \rightarrow \rho_{\bar{n}}} \frac{\mathcal{L}V(\rho)}{V(\rho)} = -C$.

Then, $\rho_{\bar{n}}$ is a.s. exponentially stable for the system (2.1) with sample Lyapunov exponent less or equal than $-C - \frac{K}{2}$, where $K := \liminf_{\rho \rightarrow \rho_{\bar{n}}} g^2(\rho)$ and $g(\rho) := \sqrt{\eta} \frac{\partial V(\rho)}{\partial \rho} \frac{G(\rho)}{V(\rho)}$.

Proof. The proof proceeds in three steps:

1. First we show that $\rho_{\bar{n}}$ is locally stable in probability;
2. Next we show that for any fixed $r > 0$ and almost all sample paths, there exists $T < \infty$ such that for all $t \geq T$, $\rho_t \in B_r(\rho_{\bar{n}})$;
3. Finally, we prove that $\rho_{\bar{n}}$ is a.s. exponentially stable with sample Lyapunov exponent less or equal than $-C - \frac{K}{2}$.

Step 1: By the condition (ii), we can choose $r > 0$ sufficiently small such that $\mathcal{L}V(\rho) \leq -C(r)V(\rho)$ for $\rho \in B_r(\rho_{\bar{n}}) \setminus \rho_{\bar{n}}$, for some $C(r) > 0$. Let $\varepsilon \in (0, 1)$ be arbitrary. By the continuity of $V(\rho)$ and the fact that $V(\rho) = 0$ if and only if $d_B(\rho, \rho_{\bar{n}}) = 0$, we can find $\delta = \delta(\varepsilon, r) > 0$ such that

$$(6.3) \quad 1/\varepsilon \sup_{\rho_0 \in B_\delta(\rho_{\bar{n}})} V(\rho_0) \leq C_1 r.$$

Assume that $\rho_0 \in B_\delta(\rho_{\bar{n}})$ and let τ be the first exit time of ρ_t from $B_r(\rho_{\bar{n}})$. By Itô formula, we have

$$\mathbb{E}(V(\rho_{t \wedge \tau})) \leq V(\rho_0) - C(r) \mathbb{E} \left(\int_0^{t \wedge \tau} V(\rho_s) ds \right) \leq V(\rho_0).$$

For all $t \geq \tau$, $d_B(\rho_{t \wedge \tau}, \rho_{\bar{n}}) = d_B(\rho_\tau, \rho_{\bar{n}}) = r$. Hence, by the condition (i),

$$\mathbb{E}(V(\rho_{t \wedge \tau})) \geq \mathbb{E}(\mathbb{1}_{\{\tau \leq t\}} V(\rho_\tau)) \geq \mathbb{E}(\mathbb{1}_{\{\tau \leq t\}} C_1 d_B(\rho_\tau, \rho_{\bar{n}})) = C_1 r \mathbb{P}(\tau \leq t).$$

Combining with the inequality (6.3), we have

$$\mathbb{P}(\tau \leq t) \leq \frac{\mathbb{E}(V(\rho_{t \wedge \tau}))}{C_1 r} \leq \frac{V(\rho_0)}{C_1 r} \leq \varepsilon.$$

Letting t tend to infinity, we get $\mathbb{P}(\tau < \infty) \leq \varepsilon$ which implies

$$\mathbb{P}(d_B(\rho_t, \rho_{\bar{n}}) < r \text{ for } t \geq 0) \geq 1 - \varepsilon.$$

Step 2: Since $u(\rho) = 0$ in \bar{E} if and only if $\rho = \rho_{\bar{n}}$ by Lemma 6.1 we obtain, for all $\rho_0 \in \mathcal{S}$, $\mathbb{P}(\tau_\delta < \infty) = 1$, where $\tau_\delta := \inf\{t \geq 0 \mid \rho_t \in B_\delta(\rho_{\bar{n}})\}$. It implies that ρ_t enters $B_\delta(\rho_{\bar{n}})$ in a finite time almost surely. Due to Step 1, for all $\rho_0 \in B_\delta(\rho_{\bar{n}})$, $\mathbb{P}(\sigma_r < \infty) \leq \varepsilon$, where $\sigma_r := \inf\{t \geq 0 \mid \rho_t \notin B_r(\rho_{\bar{n}})\}$.

We define two sequences of stopping times $\{\sigma_r^k\}_{k \geq 0}$ and $\{\tau_\delta^k\}_{k \geq 1}$ such that $\sigma_r^0 = 0$, $\tau_\delta^{k+1} = \inf\{t \geq \sigma_r^k \mid \rho_t \in B_\delta(\rho_{\bar{n}})\}$ and $\sigma_r^{k+1} = \inf\{t \geq \tau_\delta^{k+1} \mid \rho_t \notin B_r(\rho_{\bar{n}})\}$. By the strong Markov property, we find

$$\begin{aligned} \mathbb{P}_{\rho_0}(\sigma_r^m < \infty) &= \mathbb{P}_{\rho_0}(\tau_\delta^1 < \infty, \sigma_r^1 < \infty, \dots, \sigma_r^m < \infty) \\ &= \mathbb{P}_{\rho_{\tau_\delta^1}}(\sigma_r < \infty) \cdots \mathbb{P}_{\rho_{\tau_\delta^m}}(\sigma_r < \infty) \leq \varepsilon^m. \end{aligned}$$

Thus, for all $\rho_0 \in \mathcal{S}$, we have $\mathbb{P}(\sigma_r^m < \infty, \forall m > 0) = 0$. We deduce that, for almost all sample paths, there exists $T < \infty$ such that, for all $t \geq T$, $\rho_t \in B_r(\rho_{\bar{n}})$, which concludes Step 2.

Step 3: In this step, we obtain an upper bound of the sample Lyapunov exponent by employing an argument inspired by [25, Theorem 4.3.3]. For $\rho \neq \rho_{\bar{n}}$, $\mathcal{L} \log V(\rho) = \frac{\mathcal{L}V(\rho)}{V(\rho)} - \frac{g^2(\rho)}{2}$. Due to Lemma 4.2, $\rho_{\bar{n}}$ cannot be attained in finite time almost surely, then by Itô formula, we have

$$\log V(\rho_t) = \log V(\rho_0) + \int_0^t \frac{\mathcal{L}V(\rho_s)}{V(\rho_s)} ds + \int_0^t g(\rho_s) dW_s - \frac{1}{2} \int_0^t g^2(\rho_s) ds.$$

Let $m \in \mathbb{Z}_{>0}$ and take arbitrarily $\varepsilon \in (0, 1)$. By the exponential martingale inequality (see e.g. [25, Theorem 1.7.4]), we have

$$\mathbb{P}\left(\sup_{0 \leq t \leq m} \left[\int_0^t g(\rho_s) dW_s - \frac{\varepsilon}{2} \int_0^t g^2(\rho_s) ds \right] > \frac{2}{\varepsilon} \log m\right) \leq \frac{1}{m^2}.$$

Since $\sum_{m=1}^{\infty} \frac{1}{m^2} < \infty$, by Borel-Cantelli lemma we have that for almost all sample paths there exists m_0 such that, if $m > m_0$, then

$$\sup_{0 \leq t \leq m} \left(\int_0^t g(\rho_s) dW_s - \frac{\varepsilon}{2} \int_0^t g^2(\rho_s) ds \right) \leq \frac{2}{\varepsilon} \log m.$$

Thus, for $0 \leq t \leq m$ and $m > m_0$,

$$\int_0^t g(\rho_s) dW_s \leq \frac{2}{\varepsilon} \log m + \frac{\varepsilon}{2} \int_0^t g^2(\rho_s) ds, \quad a.s.$$

We have

$$\log V(\rho_t) \leq \log V(\rho_0) + \int_0^t \frac{\mathcal{L}V(\rho_s)}{V(\rho_s)} ds + \frac{2}{\varepsilon} \log m - \frac{1-\varepsilon}{2} \int_0^t g^2(\rho_s) ds, \quad a.s.$$

It gives

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log V(\rho_t) \leq \limsup_{t \rightarrow \infty} \frac{1}{t} \left(\int_0^t \frac{\mathcal{L}V(\rho_s)}{V(\rho_s)} ds - \frac{1-\varepsilon}{2} \int_0^t g^2(\rho_s) ds \right) \quad a.s.$$

Letting ε tend to zero, we have

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log V(\rho_t) \leq \limsup_{t \rightarrow \infty} \frac{1}{t} \left(\int_0^t \frac{\mathcal{L}V(\rho_s)}{V(\rho_s)} ds - \frac{1}{2} \int_0^t g^2(\rho_s) ds \right) \quad a.s.$$

For every fixed $T > 0$ consider the event

$$\Omega_T = \{\rho_t \in B_r(\rho_{\bar{n}}) \text{ for all } t \geq T\}.$$

Due to the condition (ii), for almost all $\omega \in \Omega_T$,

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{1}{t} \left(\int_0^t \frac{\mathcal{L}V(\rho_s)}{V(\rho_s)} ds - \frac{1}{2} \int_0^t g^2(\rho_s) ds \right) \\ \leq \limsup_{t \rightarrow \infty} \frac{1}{t} \left(\int_T^t \frac{\mathcal{L}V(\rho_s)}{V(\rho_s)} ds - \frac{1}{2} \int_T^t g^2(\rho_s) ds \right) \\ \leq -C(r) - \inf_{\rho \in B_r(\rho_{\bar{n}}) \setminus \rho_{\bar{n}}} \frac{g^2(\rho)}{2}. \end{aligned}$$

Since T can be taken arbitrarily large and Step 2 implies that $\lim_{T \rightarrow \infty} \mathbb{P}(\Omega_T) = 1$, we can conclude that

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log V(\rho_t) \leq -C(r) - \inf_{\rho \in B_r(\rho_{\bar{n}}) \setminus \rho_{\bar{n}}} \frac{g^2(\rho)}{2}, \quad a.s.$$

Finally, due to the condition (i) and since r can be taken arbitrarily small, we have

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log d_B(\rho_t, \rho_{\bar{n}}) \leq -C - \frac{K}{2}, \quad a.s.$$

which yields the result. \square

6.2. Feedback controller design. The purpose of this subsection is to design parametrized feedback laws which stabilize exponentially the system (2.1) almost surely towards a target state $\rho_{\bar{n}}$. For the choice of target state, we consider first the particular case $\bar{n} \in \{0, 2J\}$ and then the general case $\bar{n} \in \{0, \dots, 2J\}$.

In the following theorem, we consider the case $\bar{n} \in \{0, 2J\}$. Before stating the result, we note that we can describe the set $B_{r(\lambda)}(\rho_{\bar{n}}) \setminus \rho_{\bar{n}}$ as follows

$$D_\lambda(\rho_{\bar{n}}) := \{\rho \in \mathcal{S} \mid 0 < \lambda < \rho_{\bar{n}, \bar{n}} < 1\} = B_{r(\lambda)}(\rho_{\bar{n}}) \setminus \rho_{\bar{n}},$$

where $r(\lambda) = \sqrt{2 - 2\sqrt{\lambda}}$.

THEOREM 6.4. *Consider system (2.1) with $\rho_0 \in \mathcal{S}$ and assume $\eta \in (0, 1)$. Let $\rho_{\bar{n}} \in \{\rho_0, \rho_{2J}\}$ be the target eigenstate and define the feedback controller*

$$(6.4) \quad u_{\bar{n}}(\rho) = \alpha(1 - \text{Tr}(\rho\rho_{\bar{n}}))^\beta - \gamma \text{Tr}(i[J_y, \rho]\rho_{\bar{n}}),$$

where $\gamma \geq 0$, $\beta \geq 1$ and $\alpha > 0$. Then the feedback controller (6.4) exponentially stabilizes system (2.1) almost surely to the equilibrium $\rho_{\bar{n}}$ with sample Lyapunov exponent less or equal than $-\eta M$.

Proof. To prove the theorem, we show that we can apply Theorem 6.3 with the Lyapunov function $V_{\bar{n}}(\rho) = \sqrt{1 - \text{Tr}(\rho\rho_{\bar{n}})}$ for $\bar{n} = 0$ and $\bar{n} = 2J$. First, it is easy to see that $u_{\bar{n}}$ satisfies the assumptions of Lemma 6.1 and Lemma 4.2. Then, we need to show that the conditions (i) and (ii) of Theorem 6.3 hold true. Note that $\frac{\sqrt{2}}{2} d_B(\rho, \rho_{\bar{n}}) \leq V_{\bar{n}}(\rho) \leq d_B(\rho, \rho_{\bar{n}})$, so that the condition (i) is shown. We are left to check the condition (ii). The infinitesimal generator $\mathcal{L}V_{\bar{n}}$ takes the following form

$$(6.5) \quad \mathcal{L}V_{\bar{n}}(\rho) = \frac{u_{\bar{n}}}{2} \frac{\text{Tr}(i[J_y, \rho]\rho_{\bar{n}})}{V_{\bar{n}}(\rho)} - \frac{\eta M}{2} \frac{(J - \bar{n} - \text{Tr}(J_z \rho))^2 \text{Tr}(\rho\rho_{\bar{n}})^2}{V_{\bar{n}}^3(\rho)}.$$

If $\bar{\mathbf{n}} = \mathbf{0}$, and $\rho \in D_\lambda(\rho_0)$, we find

$$\frac{u_0}{2} \frac{\text{Tr}(i[J_y, \rho] \rho_0)}{V_0(\rho)} \leq \alpha c_1 (V_0(\rho))^\beta \leq \alpha c_1 (1 - \lambda)^{\frac{\beta-1}{2}} V_0(\rho),$$

since $|\text{Tr}(i[J_y, \rho] \rho_0)| = 2c_1 |\mathbf{Re}\{\rho_{0,1}\}| \leq 2c_1 |\rho_{0,1}| \leq 2c_1 V_0(\rho)$. Moreover, we have

$$J - \text{Tr}(J_z \rho) = \sum_{k=1}^{2J} k \rho_{k,k} \geq \sum_{k=1}^{2J} \rho_{k,k} = 1 - \rho_{0,0} = (V_0(\rho))^2.$$

Thus, for all $\rho \in D_\lambda(\rho_0)$, $\mathcal{L}V_0(\rho) \leq -C_{0,\lambda} V_0(\rho)$, where $C_{0,\lambda} = \frac{\eta M \lambda^2}{2} - \alpha c_1 (1 - \lambda)^{\frac{\beta-1}{2}}$. The case $\bar{\mathbf{n}} = \mathbf{2J}$ may be treated similarly. In particular, for all $\rho \in D_\lambda(\rho_{2J})$, one gets $\mathcal{L}V_{2J}(\rho) \leq -C_{2J,\lambda} V_{2J}(\rho)$, where $C_{2J,\lambda} = \frac{\eta M \lambda^2}{2} - \alpha c_{2J} (1 - \lambda)^{\frac{\beta-1}{2}} = C_{0,\lambda}$.

Furthermore, for $\bar{n} \in \{0, 2J\}$, we have $g^2(\rho) \geq \eta M \lambda^2$, for all $\rho \in D_\lambda(\rho_{\bar{n}})$. Hence, we can apply [Theorem 6.3](#) for $\bar{n} \in \{0, 2J\}$, with $C = \frac{\eta M}{2}$ and $K = \eta M$. The proof is complete. \square

In the following theorem, we consider the general case $\bar{n} \in \{0, \dots, 2J\}$.

THEOREM 6.5. *Consider system (2.1) with $\rho_0 \in \mathcal{S} \setminus \partial\mathcal{S}$. Let $\rho_{\bar{n}} \in \bar{E}$ be the target state and define the feedback*

$$(6.6) \quad u_{\bar{n}}(\rho) = \alpha (P_{\bar{n}}(\rho))^\beta = \alpha (J - \bar{n} - \text{Tr}(J_z \rho))^\beta,$$

where $\beta \geq 1$ and $\alpha > 0$. Then the feedback (6.6) exponentially stabilizes system (2.1) almost surely to the equilibrium $\rho_{\bar{n}}$ with sample Lyapunov exponent less or equal than $-\eta M$ if $\bar{n} \in \{0, 2J\}$ and $-\eta M/2$ if $\bar{n} \in \{1, \dots, 2J-1\}$.

Proof. Consider the following candidate Lyapunov function

$$(6.7) \quad V_{\bar{n}}(\rho) = \sum_{k \neq \bar{n}} \sqrt{\text{Tr}(\rho \rho_k)}.$$

Due to [Lemma 4.3](#), all diagonal elements of ρ_t remain strictly positive for all $t \geq 0$ almost surely. Since $V_{\bar{n}}(\rho)$ is C^2 in $\mathcal{S} \setminus \partial\mathcal{S}$, we can make use of similar arguments as those in [Theorem 6.3](#). First, we show that the following conditions are satisfied.

C.1. $2\eta M \mathcal{V}(\rho) \rho_{\bar{n}, \bar{n}} > u_{\bar{n}} \text{Tr}(i[J_y, \rho] \rho_{\bar{n}})$, $\forall \rho \in \mathbf{P}_{\bar{n}} \setminus \rho_{\bar{n}}$,

C.2. $u_{\bar{n}}(\rho) \leq \bar{C} V_{\bar{n}}(\rho)$ with $\bar{C} > 0$, $\forall \rho \in D_\lambda(\rho_{\bar{n}})$.

Roughly speaking, by [Lemma 6.1](#), C.1 provides a sufficient condition guaranteeing the accessibility of any arbitrary small neighborhood of $\rho_{\bar{n}}$. C.2 is helpful to obtain a bound of the type $\mathcal{L}V_{\bar{n}} \leq -C V_{\bar{n}}$ on $D_\lambda(\rho_{\bar{n}})$.

We now show that these conditions are satisfied. The property C.1 follows from the fact that, for all $\rho \in \mathbf{P}_{\bar{n}} \setminus \rho_{\bar{n}}$, we have $u_{\bar{n}}(\rho) = 0$ and $\mathcal{V}(\rho) > 0$.

Next, we can show that the property C.2 holds true, because

$$|P_{\bar{n}}(\rho)| = \left| \sum_{k \neq \bar{n}} k \rho_{k,k} - \bar{n} (1 - \rho_{\bar{n}, \bar{n}}) \right| \leq \Upsilon (1 - \rho_{\bar{n}, \bar{n}}),$$

where $\Upsilon := \max\{\bar{n}, 2J - \bar{n}\}$. Then, for all $\rho \in D_\lambda(\rho_{\bar{n}})$,

$$u_{\bar{n}}(\rho) \leq \alpha \Upsilon^\beta (1 - \rho_{\bar{n}, \bar{n}})^{\beta-1/2} \sqrt{1 - \rho_{\bar{n}, \bar{n}}} \leq \alpha \Upsilon^\beta (1 - \lambda)^{\beta-1/2} V_{\bar{n}}(\rho).$$

Consider the Lyapunov function (6.7). In the following, we verify the conditions (i) and (ii) of [Theorem 6.3](#). First note that by Jensen inequality, we have

$V_{\bar{n}}(\rho) \leq \sqrt{2J} \sqrt{1 - \rho_{\bar{n}, \bar{n}}}$. Then we get $\frac{\sqrt{2}}{2} d_B(\rho, \rho_{\bar{n}}) \leq V_{\bar{n}}(\rho) \leq \sqrt{2J} d_B(\rho, \rho_{\bar{n}})$, hence the condition (i) is shown. In order to verify the condition (ii), we write the infinitesimal generator of the Lyapunov function which has the following form

$$(6.8) \quad \mathcal{L}V_{\bar{n}}(\rho) = -\frac{u_{\bar{n}}}{2} \sum_{k \neq \bar{n}} \frac{\text{Tr}(i[J_y, \rho] \rho_k)}{\sqrt{\rho_{k,k}}} - \frac{\eta M}{2} \sum_{k \neq \bar{n}} (P_k(\rho))^2 \sqrt{\rho_{k,k}}.$$

We find

$$\begin{aligned} \frac{|\text{Tr}(i[J_y, \rho] \rho_k)|}{\sqrt{\rho_{k,k}}} &= \frac{|c_k \mathbf{Re}\{\rho_{k,k-1}\} - c_{k+1} \mathbf{Re}\{\rho_{k,k+1}\}|}{\sqrt{\rho_{k,k}}} \leq \frac{c_k |\rho_{k,k-1}| + c_{k+1} |\rho_{k,k+1}|}{\sqrt{\rho_{k,k}}} \\ &\leq c_k \sqrt{\rho_{k-1,k-1}} + c_{k+1} \sqrt{\rho_{k+1,k+1}} \leq c_k + c_{k+1}. \end{aligned}$$

For $k \neq \bar{n}$ and for all $\rho \in D_\lambda(\rho_{\bar{n}})$ with $\lambda > 1 - 1/\Upsilon$, we have

$$|J - k - \text{Tr}(J_z \rho)| \geq |\bar{n} - k| - |P_{\bar{n}}(\rho)| \geq 1 - \Upsilon(1 - \rho_{\bar{n}, \bar{n}}) \geq 1 - \Upsilon(1 - \lambda) > 0.$$

Thus, for all $\rho \in D_\lambda(\rho_{\bar{n}})$,

$$\mathcal{L}V_{\bar{n}}(\rho) \leq -\left(\frac{\eta M(1 - \Upsilon(1 - \lambda))^2}{2} - \alpha \Gamma \Upsilon^\beta (1 - \lambda)^{\beta-1/2} \right) V_{\bar{n}}(\rho) \leq -C_{\bar{n}, \lambda} V_{\bar{n}}(\rho),$$

where $\Gamma := \sum_{k \neq \bar{n}} (c_k + c_{k+1})$ and $C_{\bar{n}, \lambda} := \frac{\eta M(1 - \Upsilon(1 - \lambda))^2}{2} - \alpha \Gamma \Upsilon^\beta (1 - \lambda)^{\beta-1/2}$.

Furthermore, for $\bar{n} \in \{0, 2J\}$, we have $g^2(\rho) \geq \eta M \lambda^2$, for all $\rho \in D_\lambda(\rho_{\bar{n}})$. Since $C_{\bar{n}, \lambda}$ and $\eta M \lambda^2$ converge respectively to $\frac{\eta M}{2}$ and ηM as λ tends to one, by employing the same arguments used earlier in the proof of [Theorem 6.3](#), we find that the sample Lyapunov exponent is less or equal than $-C - K/2$ where $C = \frac{\eta M}{2}$ for $\bar{n} \in \{0, \dots, 2J\}$, $K = \eta M$ for $\bar{n} \in \{0, 2J\}$ and $K = 0$ for $\bar{n} \in \{1, \dots, 2J - 1\}$. \square

Remark 6.6. Locally around the target state $\rho_{\bar{n}}$, the asymptotic behavior of the Lyapunov function (6.7) is the same as the one of the Lyapunov function (5.1). This is related to the fact that, under the assumptions on $u_{\bar{n}}$, the behavior of the system around the target state is similar to the case $u \equiv 0$. In particular, without feedback and conditioning to the event $\{\exists t' \geq 0 | \rho_t \in B_r(\rho_{\bar{n}}), \forall t \geq t'\}$, one can show that the trajectories converge a.s. to $\rho_{\bar{n}}$ with sample Lyapunov exponent equal to the one in [Theorem 6.5](#).

Remark 6.7. Note that the feedback controller satisfies the assumptions of [Proposition 4.5](#), that is $\sum_k \frac{\partial P_{\bar{n}}(\rho)}{\partial \rho_{k,k}} (G(\rho))_{k,k} \neq 0$ when $u_{\bar{n}}(\rho) = 0$ and $\rho \neq \rho_{\bar{n}}$ (this can be easily shown by applying Cauchy-Schwarz inequality). If $\eta \in (0, 1)$, [Theorem 6.5](#) and [Corollary 4.6](#) guarantee the convergence of almost all trajectories to the target state even if the initial state ρ_0 lies in the boundary of \mathcal{S} (the argument is no more valid if $\eta = 1$ because of [Lemma 4.4](#)). Unfortunately, these results do not ensure the almost sure exponential convergence towards the target state whenever ρ_0 lies in $\partial \mathcal{S} \setminus \rho_{\bar{n}}$. However, we believe that under the assumptions imposed on the feedback, we can still guarantee such convergence property. This is suggested by the following arguments.

Set the event $\Omega_{>0} = \bigcap_{t>0} \{\rho_t > 0\}$ which is \mathcal{F}_{0+} -measurable. By the strong Markov property of ρ_t , and by applying Blumenthal zero-one law [31], we have that either $\mathbb{P}(\Omega_{>0}) = 0$ or $\mathbb{P}(\Omega_{>0}) = 1$. In order to conclude that $\mathbb{P}(\Omega_{>0}) = 1$, it would be enough to show that $\mathbb{P}(\Omega_{>0}) > 0$, i.e., ρ_t exits the boundary and enters the interior of \mathcal{S} immediately with non-zero probability. [Proposition 4.5](#) provides some intuitions

about the validity of this property, as it proves that the majority of the trajectories of the associated deterministic equation (4.5) enter the interior of \mathcal{S} immediately. It is then tempting to conjecture that under the assumption of Proposition 4.5, for all $\rho_0 \in \partial\mathcal{S} \setminus \rho_{\bar{n}}$, $\rho_t > 0$ for all $t > 0$ almost surely. If this conjecture is correct, we can generalize Theorem 6.5 to the case $\rho_0 \in \mathcal{S}$.

7. Simulations. In this section, we illustrate our results by numerical simulations in the case of a three-level quantum angular momentum system. First, we consider the case $u \equiv 0$ (Theorem 5.1). Then, we illustrate the convergence towards the target states ρ_0 and ρ_1 by applying feedback laws of the form (6.4) and (6.6), respectively.

The simulations in the case $u \equiv 0$ are shown in Figure 1. In particular, we observe that the expectation of the Lyapunov function $\mathbb{E}(V(\rho_t))$ is bounded by the exponential function $V(\rho_0)e^{-\frac{\eta M}{2}t}$, and the expectation of the Bures distance $\mathbb{E}(d_B(\rho_t, \bar{E}))$ is always below the exponential function $C_2/C_1 d_B(\rho_0, \bar{E})e^{-\frac{\eta M}{2}t}$, with $C_1 = 1/2$ and $C_2 = 3$ (see Equation (5.2)) in accordance with the results of Section 5. Next, we set ρ_0 as the

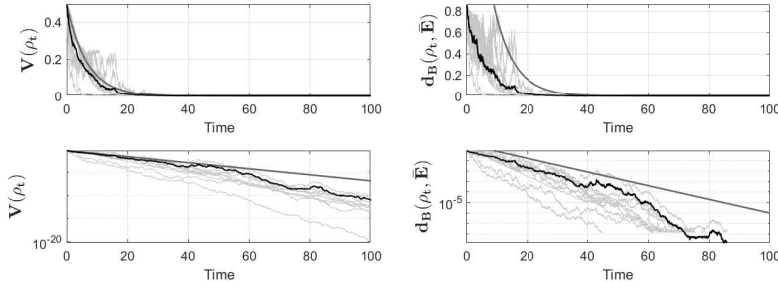


FIG. 1. Quantum state reduction of a three-level quantum angular momentum system with $u \equiv 0$ starting at $\text{diag}(0.3, 0.4, 0.3)$ when $\omega = 0$, $\eta = 0.3$ and $M = 1$: the black curve represents the mean value of 10 arbitrary sample trajectories, the gray curve represents the exponential reference with exponent $-\eta M/2$. The figures at the bottom are the semi-log versions of the ones at the top.

target eigenstate; the corresponding simulations with a feedback law of the form (6.4) and initial condition ρ_2 are shown in Figure 2. For this case, we note that a larger α can speed up the exit of the trajectories from a neighborhood of the antipodal state ρ_2 . Similarly, a larger γ may speed up the accessibility of a neighborhood of the target state ρ_0 . Finally, a larger β can weaken the role of the first term in the feedback law (6.4) on neighborhoods of the target state (a more detailed discussion for the two-level case may be found in [23]).

Then, we set ρ_1 as the target eigenstate; the simulations with a feedback law of the form (6.6) and initial condition $\text{diag}(0.3, 0.4, 0.3)$ (in the interior of \mathcal{S}) are shown in Figure 3. Finally, we repeat the last simulations for the case where the initial condition is ρ_2 . As simulations show, the trajectories enter immediately in the interior of \mathcal{S} and converge exponentially towards the target state (see Figure 4).

8. Conclusion and perspectives. In this paper, we have studied the asymptotic behavior of trajectories associated with quantum angular momentum systems for the cases with and without feedback law. Firstly, for the system with zero control, we have shown the exponential convergence towards the set of pure states corresponding to eigenvectors of the measurement operator J_z (quantum state reduction with exponential rate $\eta M/2$). We next proved the exponential convergence of N -level quantum

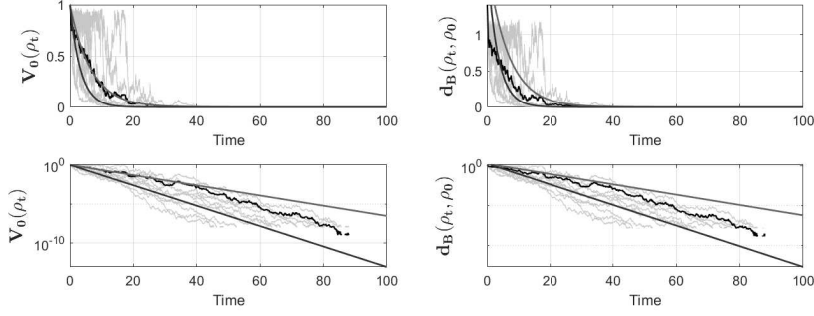


FIG. 2. Exponential stabilization of a three-level quantum angular momentum system towards ρ_0 with the feedback law (6.4) starting at ρ_2 with $\omega = 0$, $\eta = 0.3$, $M = 1$, $\alpha = 10$, $\beta = 5$ and $\gamma = 10$: the black curve represents the mean value of 10 arbitrary sample trajectories, the light gray and dark gray curves represent the exponential references with exponents $-\eta M/2$ and $-\eta M$ respectively. The figures at the bottom are the semi-log versions of the ones at the top.

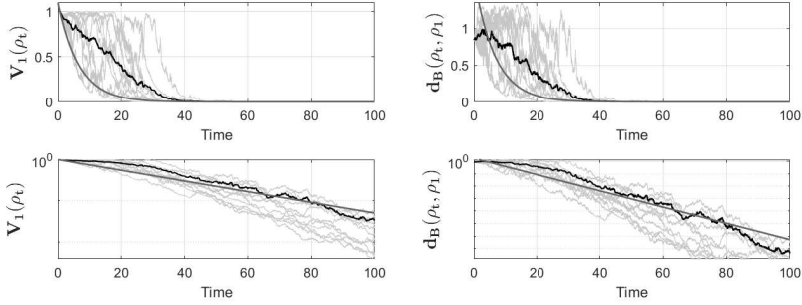


FIG. 3. Exponential stabilization of a three-level quantum angular momentum system towards ρ_1 with the feedback law (6.6) starting at $\text{diag}(0.3, 0.4, 0.3)$ with $\omega = 0$, $\eta = 0.3$, $M = 1$, $\alpha = 0.3$, $\beta = 10$: the black curve represents the mean value of 10 arbitrary sample trajectories, the gray curve represents the exponential reference with exponent $-\eta M/2$. The figures at the bottom are the semi-log versions of the ones at the top.

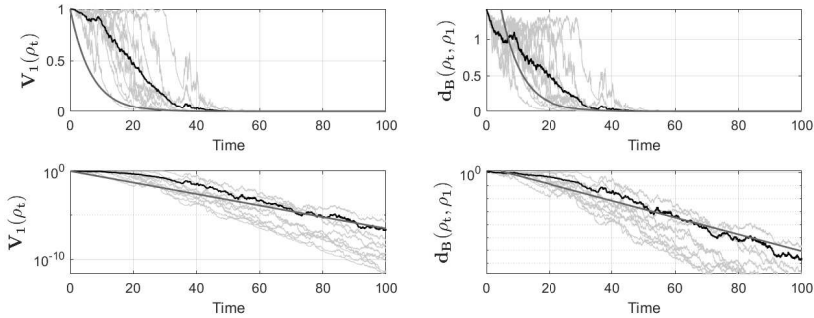


FIG. 4. Exponential stabilization of a three-level quantum angular momentum system towards ρ_1 with the feedback law (6.6) starting at ρ_2 with $\omega = 0$, $\eta = 0.3$, $M = 1$, $\alpha = 0.3$, $\beta = 10$: the black curve represents the mean value of 10 arbitrary sample trajectories, the gray curve represents the exponential reference with exponent $-\eta M/2$. The figures at the bottom are the semi-log versions of the ones at the top.

angular momentum systems towards an arbitrary predetermined target state under some general conditions on the feedback law. This was obtained by applying stochastic Lyapunov techniques and analyzing the asymptotic behavior of quantum trajectories. For illustration, we have provided a parametrized feedback law satisfying our general conditions to stabilize the system exponentially towards the target state.

Further research lines will address the possibility of extending our results in presence of delays, or for exponential stabilization of entangled states with applications in quantum computing. In particular, alternative choices of the measurement operator may be investigated to prepare predetermined entangled target states, such as Dicke or GHZ states.

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