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# NON-BOUNDEDNESS OF THE NUMBER OF NODAL DOMAINS OF A SUM OF EIGENFUNCTIONS

PIERRE BÉRARD, PHILIPPE CHARRON, AND BERNARD HELFFER

ABSTRACT. Generalizing Courant’s nodal domain theorem, the “Extended Courant property” is the statement that a linear combination of the first  $n$  eigenfunctions has at most  $n$  nodal domains. In the first part of the paper, we prove that the Extended Courant property is false for the subequilateral triangle and for regular  $N$ -gons ( $N$  large), with the Neumann boundary condition. More precisely, we prove that there exists a Neumann eigenfunction  $u_k$  of the  $N$ -gon, with index  $4 \leq k \leq 6$ , such that the set  $\{u_k \neq 1\}$  has  $(N + 1)$  connected components. In the second part, we prove that there exist metrics  $g$  on  $\mathbb{T}^2$  (resp. on  $\mathbb{S}^2$ ) which are arbitrarily close to the flat metric (resp. round metric), and an eigenfunction  $f$  of the associated Laplace-Beltrami operator such that the set  $\{f \neq 1\}$  has infinitely many connected components. In particular the Extended Courant property is false for these closed surfaces. These results are strongly motivated by a recent paper by Buhovsky, Logunov and Sodin (arXiv:1811.03835). As for the positive direction, in Appendix B, we prove that the Extended Courant property is true for the isotropic quantum harmonic oscillator in  $\mathbb{R}^2$ .

## 1. INTRODUCTION

Let  $\Omega \subset \mathbb{R}^2$  be a bounded domain (open connected set) with piecewise smooth boundary, or a compact Riemannian surface, with or without boundary, and let  $\Delta$  be the Laplace-Beltrami operator. Consider the (real) eigenvalue problem

$$(1.1) \quad \begin{cases} -\Delta u = \lambda u \text{ in } \Omega, \\ B(u) = 0 \text{ on } \partial\Omega, \end{cases}$$

where the boundary condition  $B(u) = 0$  is either the Dirichlet or the Neumann boundary condition,  $u = 0$  or  $\partial_\nu u = 0$  on  $\partial\Omega$ , or the empty condition if  $\partial\Omega$  is empty.

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We arrange the eigenvalues of (1.1) in nondecreasing order, multiplicities taken into account,

$$(1.2) \quad \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$$

The *nodal set*  $\mathcal{Z}(u)$  of a (real) function  $u$  is defined to be

$$(1.3) \quad \mathcal{Z}(u) = \overline{\{x \in \Omega \mid u(x) = 0\}}.$$

The *nodal domains* of a function  $u$  are the connected components of  $\Omega \setminus \mathcal{Z}(u)$ . Call  $\beta_0(u)$  the number of nodal domains of the function  $u$ .

The following theorem can be found in [7, Chap. VI.6].

**Theorem 1.1** (Courant, 1923). *An eigenfunction  $u$ , associated with the  $n$ -th eigenvalue  $\lambda_n$  of the eigenvalue problem (1.1), has at most  $n$  nodal domains,  $\beta_0(u) \leq n$ .*

For  $n \geq 1$ , denote by  $\mathcal{L}_n(\Omega)$  the vector space of linear combinations of eigenfunctions of problem (1.1), associated with the first  $n$  eigenvalues,  $\lambda_1, \dots, \lambda_n$ .

**Conjecture 1.2** (Extended Courant Property). *Let  $w \in \mathcal{L}_n(\Omega)$  be a nontrivial linear combination of eigenfunctions associated with the  $n$  first eigenvalues of problem (1.1). Then,  $\beta_0(w) \leq n$ .*

This conjecture is motivated by a statement made in a footnote<sup>1</sup> of Courant-Hilbert's book.

Conjecture 1.2 is known to be true in dimension 1 (Sturm, 1833). In higher dimensions, it was pointed out by V. Arnold (1973), in relation with Hilbert's 16th problem. Arnold noted that the conjecture is true for  $\mathbb{RP}^2$ , the real projective space with the standard metric. It follows from [12] that Conjecture 1.2 is true when restricted to linear combinations of even (resp. odd) spherical harmonics on  $\mathbb{S}^2$  equipped with the standard metric  $g_0$ . Counterexamples to the conjecture were constructed by O. Viro (1979) for  $\mathbb{RP}^3$ . As far as we know,  $\mathbb{RP}^2$  is the only higher dimensional compact example for which Conjecture 1.2 is proven to be true. We refer to [1, 2] for references, and for several counterexamples, including convex domains in  $\mathbb{R}^2$ , with the Dirichlet or Neumann boundary condition, and to Appendix B for a proof of the Extended Courant property for the isotropic quantum harmonic oscillator in  $\mathbb{R}^2$ .

**Questions 1.3.** *Natural questions.*

- (1) *Fix  $\Omega$  as above, and  $N \geq 2$ . Can one bound  $\beta_0(w)$ , for  $w \in \mathcal{L}_N(\Omega)$ , in terms of  $N$  and geometric invariants of  $\Omega$ ?*
- (2) *Assume that  $\Omega \subset \mathbb{R}^2$  is a convex domain. Can one bound  $\beta_0(w)$ , for  $w \in \mathcal{L}_N(\Omega)$ , in terms of  $N$ , independently of  $\Omega$ ?*

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<sup>1</sup>p. 454 in [7].

- (3) Assume that  $\Omega$  is a simply-connected closed surface. Can one bound  $\beta_0(w)$ , for  $w \in \mathcal{L}_N(\Omega)$ , in terms of  $N$ , independently of  $\Omega$ ?

A negative answer to Question 1.3(1) for the 2-torus is given in a revised version of [4]. In this paper, the authors construct a smooth metric  $g$  on  $\mathbb{T}^2$ , and a family of eigenfunctions  $\phi_j$  with infinitely many isolated critical points. As a by-product of their construction, they prove that there exist a smooth metric  $g$ , a family of eigenfunctions  $\omega_j$ , and a family of real numbers  $c_j$  such that  $\beta_0(\omega_j - c_j) = +\infty$ , see Proposition 4.1.

**Remark 1.4.** When the metric is real analytic, an eigenfunction can only have finitely many isolated critical points. In [4, Introduction], the authors ask whether there exists an asymptotic upper bound for the number of critical points of an eigenfunction, in terms of the corresponding eigenvalue.

The main results of this paper are as follows. In Section 2, we prove that Conjecture 1.2 is false for a subequilateral (to be defined later on) triangle with Neumann boundary condition, see Proposition 2.3. In Section 3, we prove that the regular  $N$ -gons, with Neumann boundary condition, provide negative answers to both Conjecture 1.2, and Questions 1.3(2), at least for  $N$  large enough, see Proposition 3.1.

The second part of the paper, Sections 4 and 5, is strongly motivated by [9, 4] and [6]. We give a new proof that Conjecture 1.2 is false for the torus  $\mathbb{T}^2$ , and we prove that it is false for the sphere  $\mathbb{S}^2$  as well. More precisely, we prove the existence of smooth metrics  $g$  on  $\mathbb{T}^2$  (resp.  $\mathbb{S}^2$ ), which can be chosen arbitrarily close to the flat metric (resp. round metric), and an eigenfunction  $f$  of the associated Laplace-Beltrami operator, such that the set  $\{f \neq 1\}$  has infinitely many connected components. We refer to Proposition 4.3 for the torus, and to Propositions 5.1 and 5.2 for the sphere. In the case of  $\mathbb{T}^2$ , we also consider *analytic* metrics, see Proposition 4.5. As a by-product, we give a negative answer to the question raised in Remark 1.4.

In Appendix A, we prove the weaker result  $\beta_0(w) \leq 8d^2$  when  $w$  is the restriction to  $\mathbb{S}^2$  of a polynomial of degree  $d$  in  $\mathbb{R}^3$ . In Appendix B, we prove that Conjecture 1.2 is true for the isotropic quantum harmonic oscillator in  $\mathbb{R}^2$ . Both appendices rely on [5].

## 2. SUBEQUILATERAL TRIANGLE, NEUMANN BOUNDARY CONDITION

Let  $\mathcal{T}(b)$  denote the interior of the triangle with vertices  $A = (\sqrt{3}, 0)$ ,  $B = (0, b)$ , and  $c = (0, -b)$ . When  $b = 1$ ,  $\mathcal{T}(1)$  is an equilateral triangle with sides of length 2. From now on, we assume that  $0 < b < 1$ . The angle at the vertex  $A$  is less than  $\frac{\pi}{3}$ , and we say that  $\mathcal{T}(b)$  is a

subequilateral triangle, see Figure 2.1. Let  $\mathcal{T}(b)_+ = \mathcal{T}(b) \cap \{y > 0\}$ , and  $\mathcal{T}(b)_- = \mathcal{T}(b) \cap \{y < 0\}$ .

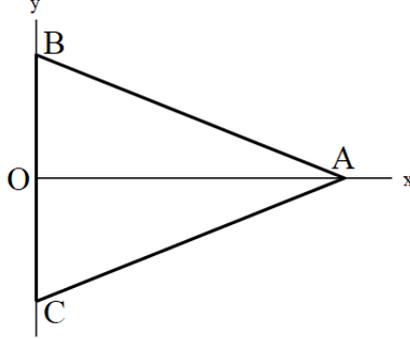


FIGURE 2.1. Subequilateral triangle,  $BC < AB = AC$

Call  $\nu_i(\mathcal{T}(b))$  the Neumann eigenvalues of  $\mathcal{T}(b)$ , and write them in non-decreasing order, with multiplicities, starting from the index 1,

$$(2.1) \quad 0 = \nu_1(\mathcal{T}(b)) < \nu_2(\mathcal{T}(b)) \leq \nu_3(\mathcal{T}(b)) \leq \dots$$

We recall the following theorems.

**Theorem 2.1** ([10], Theorem 3.1). *Every second Neumann eigenfunction of a subequilateral triangle  $\mathcal{T}(b)$  is even in  $y$ ,  $u(x, -y) = u(x, y)$ .*

**Theorem 2.2** ([13], Theorem B). *Let  $\mathcal{T}(b)$  be a subequilateral triangle. Then, the eigenvalue  $\nu_2(\mathcal{T}(b))$  is simple, and an associated eigenfunction  $u$  satisfies  $u(O) \neq 0$ , where  $O$  is the point  $O = (0, 0)$ . Normalize  $u$  by assuming that  $u(O) = 1$ . Then, the following properties hold.*

- (1) *The partial derivative  $u_x$  is negative in  $\overline{\mathcal{T}(b)} \setminus (\overline{BC} \cup \{A\})$ .*
- (2) *The partial derivative  $u_y$  is positive in  $\overline{\mathcal{T}(b)}_+ \setminus (\overline{OA} \cup \{B\})$ , and negative in  $\overline{\mathcal{T}(b)}_- \setminus (\overline{OA} \cup \{C\})$ .*
- (3) *The function  $u$  has exactly four critical points  $O, A, B$  and  $C$  in  $\overline{\mathcal{T}}$ .*
- (4) *The points  $B$  and  $C$  are the global maxima of  $u$ , and  $u(B) = u(C) > u(O) > 0$ .*
- (5) *The point  $A$  is the global minimum of  $u$ , and  $u(A) < 0$ .*
- (6) *The point  $O$  is the saddle point of  $u$ .*

As a direct corollary of these theorems, we have

**Proposition 2.3.** *Conjecture 1.2 is false for the subequilateral triangle  $\mathcal{T}(b)$  ( $0 < b < 1$ ), with Neumann boundary condition.*

*Proof.* Fix some  $0 < b < 1$ , denote  $\mathcal{T}(b)$  by  $\mathcal{T}$ , and  $\nu_2(\mathcal{T}(b))$  by  $\nu_2$  for simplicity. Let  $u$  be an eigenfunction of  $\mathcal{T}$  associated with  $\nu_2$ , and

such that  $u(O) = 1$ . In the proof, we write (A1) for Assertion (1) of Theorem 2.2, etc..

For  $a \in \mathbb{R}$ , call  $v_a$  the function  $v_a := u - a$ . This is a linear combination of a second and first Neumann eigenfunctions of  $\mathcal{T}$ . We shall now describe the nodal set of  $v_a$  carefully.

According to Theorem 2.1, for all  $a$ , the function  $v_a$  is even in  $y$ , so that it is sufficient to determine its nodal set in the triangle  $\mathcal{T}_+ = OAB$ , see Figure 2.2.

Case  $u(A) < a < u(B)$ .

- By (A4) and (A5), the nodal set  $\mathcal{Z}(v_a)$  is nontrivial if and only if  $u(A) < a < u(B)$ .
- By (A1) and (A2), the directional derivative of  $v_a$  in the direction of  $\overrightarrow{BA}$  is negative in the open segment  $BA$ , so that  $v_a|_{BA}$  is strictly decreasing from  $v_a(B)$  to  $v_a(A)$ , and therefore vanishes at a unique point  $Z_a = (\xi_a, \eta_a) \in BA$ . We now consider three subcases.

Case  $u(A) < a < u(O) < u(B)$ .

- By (A1),  $v_a|_{OA}$  is strictly decreasing from  $v_a(O)$  to  $v_a(A)$ , and therefore vanishes at a unique point  $W_a = (\omega_a, 0) \in OA$ . By (A2),  $\omega_a < \xi_a$ .
- By (A1) and (A2), it follows that the nodal set  $\mathcal{Z}(v_a) \cap \mathcal{T}_+$  is contained in the rectangle  $[\omega_a, \xi_a] \times [0, \eta_a]$ , and that it is a smooth  $y$ -graph over  $[\omega_a, \xi_a]$ , and a smooth  $x$ -graph over  $[0, \eta_a]$ .

We have proved that  $v_a$  has exactly two nodal domains in  $\mathcal{T}$ .

Case  $u(A) < a_c = u(O) < u(B)$ .

The analysis is similar to the previous one, except that  $\omega_{a_c} = 0$ . As a consequence,  $v_{a_c}$  has exactly three nodal domains in  $\mathcal{T}$ .

Case  $u(A) < u(O) < a < u(B)$ .

- By (A2),  $v_a|_{OB}$  is strictly increasing from  $v_a(O)$  to  $v_a(B)$ , so that it vanishes at a unique point  $V_a = (0, \zeta_a) \in OB$ . From (A1), it follows that  $\zeta_a < \eta_a$ .
- From (A1) and (A2), it follows that the nodal set  $\mathcal{Z}(v_a) \cap \mathcal{T}_+$  is contained in the rectangle  $[0, \xi_a] \times [\zeta_a, \eta_a]$ , and that it is a smooth  $y$ -graph over  $[0, \xi_a]$ , and a smooth  $x$ -graph over  $[\zeta_a, \eta_a]$ .

It follows that  $v_a < 0$  in  $] -\zeta_a, \zeta_a[ \times [0, \sqrt{3}] \cap \mathcal{T}$ , and that  $v_a$  has precisely three nodal domains in  $\mathcal{T}$ . Proposition 2.3 is proved.  $\square$

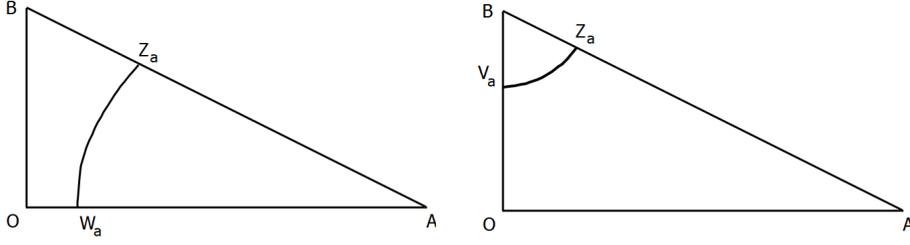


FIGURE 2.2. Nodal behaviour for the half subequilateral triangle

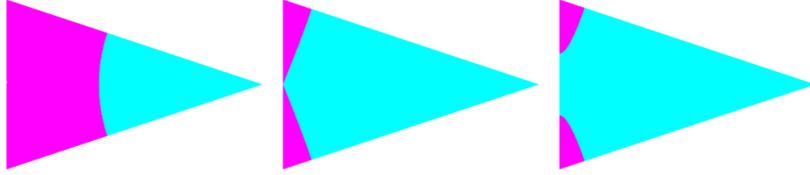


FIGURE 2.3. Nodal behaviour for the subequilateral triangle

### 3. REGULAR $N$ -GON, NEUMANN BOUNDARY CONDITION

**Proposition 3.1.** *Let  $\mathcal{P}_N$  denote the regular polygon with  $N$  sides, inscribed in the unit disk  $\mathcal{D}$ . Then, for  $N$  large enough, Conjecture 1.2 is false for  $\mathcal{P}_N$ , with the Neumann boundary condition. More precisely, there exist  $m \leq 6$ , an eigenfunction  $u_m$  associated with  $\nu_m(\mathcal{P}_N)$ , and a value  $a$  such that the function  $u_m + a$  has  $N + 1$  nodal domains.*

*Proof.* When  $N$  tends to infinity, the polygon  $\mathcal{P}_N$  tends to the disk in the Hausdorff distance. According to [11, Remark 2, p. 206], it follows that, for all  $j \geq 1$ , the Neumann eigenvalue  $\nu_j(\mathcal{P}_N)$  tends to the Neumann eigenvalue  $\nu_j(\mathcal{D})$  of the unit disk. The Neumann eigenvalues of the unit disk satisfy

$$(3.1) \quad \nu_1(\mathcal{D}) < \nu_2(\mathcal{D}) = \nu_3(\mathcal{D}) < \nu_4(\mathcal{D}) = \nu_5(\mathcal{D}) < \nu_6(\mathcal{D}) < \nu_7(\mathcal{D}) \cdots$$

and are given respectively by the squares of the zeros of the derivatives of Bessel functions:  $0 = j'_{0,1}, j'_{1,1}, j'_{2,1}, j'_{0,2}$ , and  $j'_{3,1}$ . It follows that, for  $N$  large enough, the eigenvalue  $\nu_6(\mathcal{P}_N)$  is simple.

From now on, we assume that  $N$  is sufficiently large to ensure that  $\nu_6(\mathcal{P}_N)$  is a simple eigenvalue. Let  $u_6$  be an associated eigenfunction.

Call  $A_i, 1 \leq i \leq N$ , the vertices of  $\mathcal{P}_N$ , so that the triangles  $OA_iA_{i+1}$  are subequilateral triangles with apex angle  $\frac{\pi}{N}$ .

Call  $D_j$  the  $2N$  lines of symmetry of  $\mathcal{P}_N$ . When  $N = 2m$  is even, the lines of symmetry are the  $m$  diagonals joining opposite vertices, and the  $m$  lines joining the mid-points of opposite sides. When  $N = 2m + 1$  is odd, the lines of symmetries are the  $N$  lines joining the vertex  $A_i$  to the mid-point of the opposite side. Call  $D_1$  the line of symmetry passing through the first vertex. Call  $D_2$  the line of symmetry such

that the angle  $(D_1, D_2)$  is equal to  $\pi/N$ . Denote the corresponding mirror symmetries by  $D_1$  and  $D_2$  as well. The symmetry group of the regular  $N$ -gon is the dihedral group  $\mathbb{D}_N$  with presentation,

$$(3.2) \quad \mathbb{D}_N = \left\{ D_1, D_2 \mid D_1^2 = D_2^2 = 1, (D_2 D_1)^N = 1 \right\}.$$

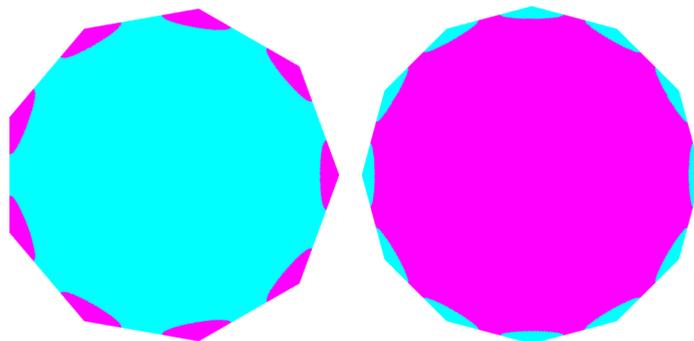


FIGURE 3.1.  $\mathcal{P}_9$  and  $\mathcal{P}_{12}$ , Neumann boundary condition

The group  $\mathbb{D}_N$  acts on functions, and commutes with the Laplacian. It leaves the eigenspaces invariant, and we therefore have a representation of degree 1 in the eigenspace  $\mathcal{E}(\nu_6)$ . This representation must be equivalent to one of the irreducible representations of  $\mathbb{D}_N$  of degree 1. When  $N$  is even, there are 4 such representations,  $\rho_{\sigma_1, \sigma_2}$  with  $\sigma_1, \sigma_2 \in \{-1, 1\}$ , and such that  $\rho_{\sigma_1, \sigma_2}(D_1) = \sigma_1$  and  $\rho_{\sigma_1, \sigma_2}(D_2) = \sigma_2$ . When  $N$  is odd, there are only 2 irreducible representations of degree 1,  $\rho_{\sigma, \sigma}$ , with  $\sigma \in \{-1, 1\}$ . Eigenfunctions corresponding to simple eigenvalues must be invariant or anti-invariant under  $D_1$  and  $D_2$  depending on the signs of  $\sigma_1$  and  $\sigma_2$ . Anti-invariant eigenfunctions must vanish on the corresponding line of symmetry. If  $(\sigma_1, \sigma_2) \neq (1, 1)$ , the functions must have at least  $N$  nodal domains. For  $N \geq 7$ , this is not possible for  $\mathcal{E}(\nu_6)$ . An eigenfunction in  $\mathcal{E}(\nu_6)$  must be  $D_1$  and  $D_2$  invariant, and must therefore correspond to an eigenfunction of  $\mathcal{T}$  with Neumann boundary condition, and with eigenvalue  $\nu_6 \geq \nu_2(\mathcal{T})$ . We can now apply Proposition 3.1.  $\square$

**Remark 3.2.** The above proposition also shows that the regular  $N$ -gons, with the Neumann boundary condition, provides a counterexample to Questions 1.3(2), when  $N$  is large enough. This is illustrated by Figure 3.1.

**Remark 3.3.** As shown in [2], Conjecture 1.2 is false for the regular hexagon  $\mathcal{P}_6$  with Neumann boundary condition. In this case,  $\nu_6(\mathcal{P}_6) = \nu_7(\mathcal{P}_6)$ , and has multiplicity 2, with two eigenfunctions associated with different irreducible representations of  $\mathbb{D}_6$ .

**Remark 3.4.** Numerical computations indicate that the first eight Neumann eigenvalues of  $\mathcal{P}_7$  to  $\mathcal{P}_{12}$  have the same multiplicities as the

first eight eigenvalues of the disk and, in particular, that  $\nu_6$  is simple. Proposition 3.1 is probably true for all  $N \geq 6$ . Numerical computations also indicate that this proposition should be true for  $\mathcal{P}_N$  with the Dirichlet boundary condition as well. The argument in the proof of Proposition 3.1 fails in the cases  $N = 4$  and  $N = 5$  which remain open.

#### 4. COUNTEREXAMPLES ON $\mathbb{T}^2$

**4.1. Previous results.** The following result appears in Section 3 in an updated version<sup>2</sup> of [4].

**Proposition 4.1.** *There exist a smooth metric  $g$  on the torus  $\mathbb{T}^2$ , in the form  $g = Q(x)(dx^2 + dy^2)$ , an infinite sequence  $\phi_j$  of eigenfunctions of the Laplace-Beltrami operator  $\Delta_g$ , and an infinite sequence  $c_j$  of real numbers, such that the level sets  $\{(x, y) \mid \phi_j(x, y) = c_j\}$  have infinitely many connected components.*

**Remarks 4.2.** (i) The proof of Proposition 4.1 shows that the functions  $\phi_j - c_j$  have an infinite number of nodal domains, contradicting any extension of Conjecture 1.2.

(ii) A weaker result was independently proved by the second author [6]. Namely, for any integer  $n$ , there exist a conformal metric  $g_n$ , a set  $\{\phi_{n,j}, 1 \leq j \leq n\}$  of eigenfunctions of  $\Delta_{g_n}$ , and a set  $\{c_{n,j}, 1 \leq j \leq n\}$  of real numbers, such that the level sets  $\{\phi_{n,j} = c_{n,j}\}$  have infinitely many connected components. Furthermore, it is possible to choose the metrics  $g_n$  arbitrarily close to the flat metric.

In this section, we give an easy proof of Proposition 4.1, in the particular case of one eigenfunction only, avoiding the subtleties of [4]. We note that this particular case is sufficient to prove that Conjecture 1.2 is false on  $(\mathbb{T}^2, g)$  for some Liouville metrics which can be chosen arbitrarily close to the flat metric.

**4.2. Metrics on  $\mathbb{T}^2$  with a prescribed eigenfunction.** In this Section, we apply an approach due to Jakobson and Nadirashvili [9], see also [4].

Consider the torus  $\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$  with the flat metric  $g_0 = dx^2 + dy^2$ , and associated Laplace-Beltrami operator  $\Delta_0 = \partial_x^2 + \partial_y^2$ . Consider Liouville metrics on  $\mathbb{T}^2$ , of the form  $Q(x)(dx^2 + dy^2)$ , where  $Q$  is a positive  $C^\infty$  function on  $\mathbb{T}^1 = \mathbb{R}/2\pi\mathbb{Z}$ .

Generally speaking, identify functions on  $\mathbb{T}^1$  (resp.  $\mathbb{T}^2$ ) with periodic functions on  $\mathbb{R}$  (resp.  $\mathbb{R}^2$ ).

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<sup>2</sup>The authors would like to thank I. Polterovich for making the new version of [4] available to them.

For a given function  $Q > 0$ , and for  $k \in \mathbb{N}$ , consider the family of eigenvalue problems

$$(4.1) \quad -y''(x) + k^2 y(x) = \sigma Q(x) y(x) \text{ on } \mathbb{T}^1,$$

with associated complete sets of spectral pairs  $\{(\sigma_{k,j}, F_{k,j}) \mid j \in \mathbb{N}^\bullet\}$  (here  $\mathbb{N}^\bullet = \mathbb{N} \setminus \{0\}$ ).

A complete set of spectral pairs  $(\lambda, \phi)$  for the eigenvalue problem

$$(4.2) \quad -\Delta_0 \phi(x, y) = \lambda Q(x) \phi(x, y) \text{ on } \mathbb{T}^2,$$

is given by

$$(4.3) \quad \left\{ (\sigma_{k,j}, F_{k,j}(x) \cos(ky)), (\sigma_{k,j}, F_{k,j}(x) \sin(ky)) \mid k \in \mathbb{N}, j \in \mathbb{N}^\bullet \right\}.$$

Given a positive  $C^\infty$  function  $F$  on  $\mathbb{T}^2$ , and an integer  $m \in \mathbb{N}^\bullet$ , define the function  $\Phi(x, y) = F(x) \cos(my)$ . The function  $\Phi$  is an eigenfunction of the eigenvalue problem (4.2), for some positive function  $Q$ , if and only if there exists some  $\lambda > 0$  such that

$$Q(x) = \frac{m^2}{\lambda} \left( 1 - \frac{1}{m^2} \frac{F''(x)}{F(x)} \right),$$

with  $m$  chosen such that

$$(4.4) \quad F''(x) < m^2 F(x), \quad \text{for all } x \in \mathbb{R}.$$

Since the flat metric  $g_0$  corresponds to  $Q \equiv 1$ , we choose  $\lambda$  to be  $m^2$ , and

$$(4.5) \quad Q(x) = 1 - \frac{1}{m^2} \frac{F''(x)}{F(x)}.$$

so that the associated metric on  $\mathbb{T}^2$  appears as a perturbation of  $g_0$ .

**4.3. Example 1.** In this subsection, we prove the following result by describing an explicit construction.

**Proposition 4.3.** *There exists a metric  $g_Q = Q(x) (dx^2 + dy^2)$  on the torus  $\mathbb{T}^2$ , and an eigenfunction  $\Phi$  of the associated Laplace-Beltrami operator,  $-\Delta_Q \Phi := Q^{-1} \Delta_0 \Phi = \Phi$ , such that the super-level set  $\{\Phi > 1\}$  has infinitely many connected components. As a consequence, Conjecture 1.2 is false for  $(\mathbb{T}^2, g_Q)$ .*

**Remarks 4.4.** (i) The metric  $g_Q$  in Proposition 4.3 is smooth, but not real analytic. In Subsection 4.4, we give a different kind of result for real analytic metrics. (ii) This proposition also implies that  $\Phi$  has infinitely many isolated critical points, a particular case of [4, Theorem 1].

*Proof.*

Step 1. Let  $\phi : [-\pi, \pi] \rightarrow \mathbb{R}$  be a function such that

$$(4.6) \quad \begin{cases} 0 \leq \phi(x) \leq 1, \\ \text{Supp}(\phi) \subset [-\frac{\pi}{2}, \frac{\pi}{2}], \\ \phi \equiv 1 \text{ on } [-\frac{\pi}{3}, \frac{\pi}{3}]. \end{cases}$$

Define the function  $F_1 : [-\pi, \pi] \rightarrow \mathbb{R}$  by

$$(4.7) \quad F_1(x) = \phi(x) \exp\left(-\frac{1}{x^2}\right) \cos\left(\frac{1}{x^2}\right) + 1 - \phi(x).$$

It is clear that  $F_1$  satisfies

$$(4.8) \quad \begin{cases} |F_1(x)| \leq 1, \\ |x| > \frac{\pi}{2} \Rightarrow F_1(x) = 1, \\ |x| > \frac{\pi}{3} \Rightarrow F_1(x) > 0. \end{cases}$$

It follows that  $F_1$  vanishes only in  $[-\frac{\pi}{3}, \frac{\pi}{3}]$ , with zero set  $\mathcal{Z}(F_1) = \{x \mid F_1(x) = 0\}$  given by

$$(4.9) \quad \mathcal{Z}(F_1) = \{0\} \cup \left\{ \pm \frac{1}{\sqrt{\frac{\pi}{2} + k\pi}} \mid k \in \mathbb{N} \right\}.$$

The zero set  $\mathcal{Z}(F_1)$  is an infinite sequence of distinct points with 0 as only accumulation point, and the function  $F_1$  changes sign at each zero.

Step 2. Define  $F_0$  to be the function  $F_1$  extended as a  $2\pi$ -periodic function on  $\mathbb{R}$ , and  $F$  to be  $F := 1 + \frac{1}{2}F_0$ . Given  $m \in \mathbb{N}$ , define the function  $\Phi_m : \mathbb{T}^2 \rightarrow \mathbb{R}$  to be  $\Phi_m(x, y) = F(x) \cos(my)$ .

The functions  $F$  and  $\Phi_m$  satisfy,

$$(4.10) \quad \begin{cases} F \in C^\infty(\mathbb{T}^2), \\ F \geq \frac{1}{2}, \\ \{F_0 < 0\} \times \mathbb{T}^1 \subset \{\Phi_m < 1\}, \\ \{\Phi_m \geq 1\} \subset \{F_0 \geq 0\} \times \mathbb{T}^1, \\ \{F_0 \geq 0\} \times \{0\} \subset \{\Phi_m \geq 1\}. \end{cases}$$

It follows from (4.9) that  $\{F_0 \geq 0\} \subset \mathbb{T}^1$  is the union of infinitely many pairwise disjoint closed intervals,  $I_\ell, \ell \in \mathbb{Z}$ . It follows from (4.10) that there is at least one connected component of the super-level set  $\{\Phi_m > 1\}$  in each  $I_\ell \times \mathbb{T}^1$ .

We have constructed a family of functions,  $\Phi_m, m \in \mathbb{N}$ , whose super-level set  $\{\Phi_m > 1\}$  has infinitely many connected components in  $\mathbb{T}^2$ . This is illustrated in Figure 4.1 (in the figure,  $m = 1$ , the red curve is the graph of a function which is rapidly oscillating, like the function  $F_1$

defined in (4.7), and the closed blue curves are part of the corresponding level set  $\{\Phi = 1\}$ .)

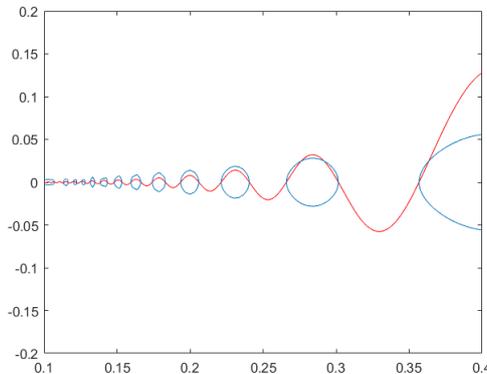


FIGURE 4.1. Level set  $\{\Phi = 1\}$

**Step 3.** Since  $F \in C^\infty(\mathbb{T}^1)$ , and  $F \geq \frac{1}{2}$ , the function  $\frac{F''}{F}$  is bounded from above. We choose  $m$  such that

$$(4.11) \quad m^2 > \sup_{x \in \mathbb{T}^1} \frac{F''(x)}{F(x)},$$

and we define the function  $Q_m : \mathbb{T}^1 \rightarrow \mathbb{R}$ ,

$$(4.12) \quad Q_m(x) = 1 - \frac{1}{m^2} \frac{F''(x)}{F(x)}.$$

Under the condition (4.11), the function  $Q_m$  defines a Liouville metric  $g_m$  on  $\mathbb{T}^2$ ,

$$(4.13) \quad g_m = Q_m(x) (dx^2 + dy^2)$$

and this metric can be chosen arbitrarily close to the flat metric  $dx^2 + dy^2$  as  $m$  goes to infinity. For the associated Laplace-Beltrami operator  $\Delta_{g_m}$ , we have

$$(4.14) \quad -\Delta_{g_m} \Phi_m(x, y) = m^2 \Phi_m(x, y),$$

so that the function  $\Phi_m$  is an eigenfunction of  $\Delta_{g_m}$ , with eigenvalue  $m^2$ . The super-level set  $\{\Phi_m > 1\}$  has infinitely many connected components in  $\mathbb{T}^2$ . In particular, the function  $\Phi_m - 1$  has infinitely many nodal domains.  $\square$

**4.4. Example 2.** The metric constructed in Proposition 4.3 is smooth, not real analytic. In this subsection, we prove the following result in which we have a real analytic metric.

**Proposition 4.5.** *Let  $n$  be any given integer. Then there exists a real analytic Liouville metric  $g = Q(x) (dx^2 + dy^2)$  on  $\mathbb{T}^2$ , and an eigenfunction  $\Phi$  of the associated Laplace-Beltrami operator,  $-\Delta_g \Phi = \Phi$ , with eigenvalue 1, such that the super-level set  $\{\Phi > 1\}$  has at least  $n$*

connected components. One can choose the metric  $g$  arbitrarily close to the flat metric  $g_0$ . Taking  $n \geq 4$ , and  $g$  close enough to  $g_0$ , the eigenvalue 1 is either the second, the third or the fourth eigenvalue of  $\Delta_g$ .

**Remarks 4.6.** (i) It follows from the proposition that the function  $\Phi - 1$  provides a counterexample to Conjecture 1.2 for  $(\mathbb{T}^2, g)$ .

(ii) The proposition also gives a negative answer to the question raised in Remark 1.4. Indeed, given any  $n \geq 4$ , the function  $\Phi$  given by the proposition is associated with the eigenvalue 1, whose labelling is at most 4, and  $\Phi$  has at least  $n$  isolated critical points.

*Proof.* Fix the integer  $n$ . For  $0 < a < 1$ , define the functions

$$(4.15) \quad \begin{cases} F_a(x) = 1 + a \cos(nx), \\ \Phi_a(x, y) = F_a(x) \cos(y). \end{cases}$$

Following the construction of Subsection 4.2, equation (4.5) yields the function

$$(4.16) \quad Q_a(x) = 1 + an^2 \frac{\cos(nx)}{1 + a \cos(nx)},$$

which is positive provided that  $an^2$  is small enough.

Call  $\lambda_{a,j}$  (written in non-decreasing order, with multiplicities) the eigenvalues of the Laplace-Beltrami operator  $-\Delta_a = -(Q_a(x))^{-1} \Delta_0$ , associated with the metric  $Q_a(x)(dx^2 + dy^2)$  on  $\mathbb{T}^2$ . The eigenvalues of  $-\Delta_0$  are given by

$$(4.17) \quad \begin{cases} \lambda_{0,1} = 0, \\ \lambda_{0,2} = 1 \text{ for } 2 \leq j \leq 5, \\ \lambda_{0,j} \geq 2 \text{ for } j \geq 6. \end{cases}$$

For  $n$  fixed, and  $a$  small enough (depending on  $n$ ), the eigenvalues  $\lambda_{a,j}$  satisfy

$$(4.18) \quad \begin{cases} \lambda_{a,1} = 0, \\ \lambda_{a,j} \in ]0.8, 1.2[ \text{ for } 2 \leq j \leq 5, \\ \lambda_{a,j} \geq 1.8 \text{ for } j \geq 6. \end{cases}$$

We note that the operator  $\Delta_a$  is invariant under the symmetries  $\Sigma_1 : (x, y) \rightarrow (-x, y)$  and  $\Sigma_2 : (x, y) \rightarrow (x, -y)$ , which commute. Consequently, the space  $L^2(\mathbb{T}^2, g_a)$  decomposes into four orthogonal subspaces

$$(4.19) \quad \mathcal{S}_{\varepsilon_1, \varepsilon_2} = \left\{ \phi \in L^2(\mathbb{T}^2) \mid \Sigma_1^* \phi = \varepsilon_1 \phi, \Sigma_2^* \phi = \varepsilon_2 \phi \right\},$$

and the eigenvalue problem for  $\Delta_a$  on  $L^2(\mathbb{T}^2)$  splits into independent problems by restriction to the subspaces  $\mathcal{S}_{\varepsilon_1, \varepsilon_2}$ , with  $\varepsilon_1, \varepsilon_2 \in \{-, +\}$ . The eigenvalue 0 corresponds to the first eigenvalue of  $-\Delta_a|_{\mathcal{S}_{+, +}}$ .

When  $a = 0$ , the eigenvalue 1 arises with multiplicity 2 from  $-\Delta_a|\mathcal{S}_{+,+}$  (the functions  $\cos x$  and  $\cos y$ ), with multiplicity 1 from  $-\Delta_a|\mathcal{S}_{-,+}$  (the function  $\sin x$ ), and multiplicity 1 from  $-\Delta_a|\mathcal{S}_{+,-}$  (the function  $\sin y$ ). For  $a$  small enough, the same spaces yield the eigenvalues  $\lambda_{a,j}$ ,  $2 \leq j \leq 5$ . From the above construction, the functions  $F_a(x) \cos(y) \in \mathcal{S}_{+,+}$  and  $F_a(x) \sin(y) \in \mathcal{S}_{+,-}$  correspond to the eigenvalue 1. There are also a second eigenvalues for  $-\Delta_a|\mathcal{S}_{+,+}$ ,  $\sigma_a$ , and  $\tau_a$  an eigenvalue for  $-\Delta_a|\mathcal{S}_{-,+}$ . It follows that

$$\{\lambda_{a,j}, 2 \leq j \leq 5\} = \{1, 1, \sigma_a, \tau_a\},$$

so that the eigenvalue 1 of  $-\Delta_a$  is either  $\lambda_{a,2}$ ,  $\lambda_{a,3}$ , or  $\lambda_{a,4}$ .

Arguments similar to those used in Subsection 4.3 show that the super-level set  $\{\Phi_a > 1\}$  has at least  $n$  connected components. It follows that the function  $\Phi_a - 1$  has at least  $n + 1$  nodal domains. When  $n \geq 4$ , this also tells us that  $\Phi_a - 1$  provides a counterexample to Conjecture 1.2. This is illustrated in Figure 4.2 (in the figure,  $n = 4$ ,  $a = 0.01$ , the red curve is the graph of  $a \cos(nx)$ , and the blue curves are components of the corresponding level set  $\{\Phi_a = 1\}$ ).  $\square$

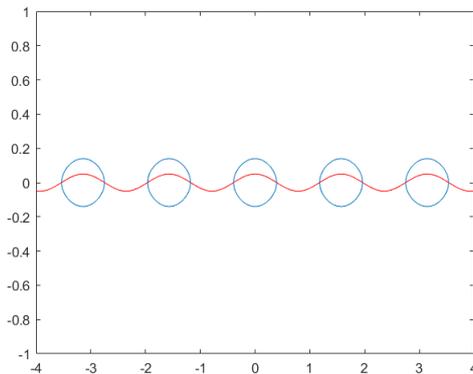


FIGURE 4.2. Level set  $\{\Phi = 1\}$

**4.5. Perturbation theory.** We use the same notation as in Subsection 4.4. Using perturbation theory, we now analyze the location of the eigenvalue 1 in the spectrum of the operator  $\Delta_a$ , and refine Proposition 4.5. More precisely, we prove

**Proposition 4.7.** *For  $n \geq 4$ , and  $a$  small enough, the eigenvalue 1 is the fourth eigenvalue of the operator  $-\Delta_a$  associated with the metric  $g_a = Q_a(x) (dx^2 + dy^2)$ , with  $Q_a(x)$  defined in (4.16)*

*Proof.* We have chosen the metrics  $g_a$  such that 1 is always an eigenvalue of multiplicity at least 2. Assuming (4.18), we analyze the behaviour of the two other eigenvalues. It will actually be sufficient to

compute the first three terms in their expansions in powers of  $a$ . We note that the operator  $\Delta_a$  is invariant under  $(x, y) \rightarrow (-x, y)$  and  $(x, y) \rightarrow (x, -y)$ . Consequently, we get four orthogonal subspaces depending on the symmetries. We first concentrate on the eigenfunction which is even with respect to the two symmetries, and then consider the eigenfunction which is even in  $y$  and odd in  $x$ .

We first concentrate on the eigenfunction which is even with respect to the two symmetries, and then consider the eigenfunction which is even in  $y$  and odd in  $x$ .

**Even case.** For  $a = 0$ , the eigenvalue  $\sigma_a$  is equal to 1, and simple, with corresponding eigenvector  $\cos x$ . Hence the perturbative analysis is easy (perturbation of a simple eigenvalue). Using Fourier decomposition in the  $y$ -variable, we are reduced to a 1D-problem,

$$(4.20) \quad -u''(x) = \sigma Q_a(x) u(x) \text{ on } \mathbb{T}^1,$$

and we only consider even functions. We look for the expansions of the second eigenvalue  $\sigma_a$  of (4.20) restricted to even functions, and of a corresponding eigenfunction  $u_a$ , in the form,

$$\sigma_a \sim 1 + \sum_{j>0} \sigma_j a^j, \quad u_a(x) \sim \cos x + \sum_{j>0} u_j(x) a^j,$$

with the additional condition that the functions  $u_j$  are orthogonal to  $\cos x$  in  $L^2(-\pi, \pi)$ . We observe that

$$Q_a(x) \sim 1 + n^2 \sum_{j>0} (-1)^{j-1} \cos(nx)^j a^j.$$

Recalling that  $u_a$  is even, we plug the above expansions into (4.20). The coefficients of the term  $a^0$  satisfy the equation. The coefficients of the term  $a$  in (4.20) give:

$$(4.21) \quad -u_1''(x) = u_1(x) + n^2 \cos(nx) \cos x + \sigma_1 \cos x,$$

together with the orthogonality condition

$$(4.22) \quad \int_{-\pi}^{\pi} u_1(x) \cos x dx = 0.$$

Taking the  $L^2$  scalar product with  $\cos x$  in (4.21), we get (for  $n > 2$ )

$$(4.23) \quad \sigma_1 = 0.$$

We now have to find an even function  $u_1$  satisfying (4.22), and

$$-u_1''(x) = u_1(x) + n^2 \cos(nx) \cos x.$$

We write

$$\begin{aligned} \cos(nx) \cos x &= \frac{1}{4} (e^{inx} + e^{-inx}) (e^{ix} + e^{-ix}) \\ &= \frac{1}{4} (e^{i(n+1)x} + e^{i(n-1)x} + e^{-i(n-1)x} + e^{-i(n+1)x}) \end{aligned}$$

and look for a particular solution of the differential equation in the form

$$u(x) = \frac{n^2}{4} \left( \alpha_{-(n+1)} e^{-i(n+1)x} + \alpha_{-(n-1)} e^{-i(n-1)x} + \alpha_{(n-1)} e^{i(n-1)x} + \alpha_{(n+1)} e^{i(n+1)x} \right).$$

We get

$$\alpha_{-(n+1)} = \frac{1}{(n+1)^2-1}, \alpha_{-(n-1)} = \frac{1}{(n-1)^2-1}, \\ \alpha_{(n-1)} = \frac{1}{(n-1)^2-1}, \alpha_{(n+1)} = \frac{1}{(n+1)^2-1}.$$

The general solution of the differential equation for  $u_1$  is given by  $\alpha \cos(x) + \beta \sin(x) + u$ , and since  $u_1$  is even and orthogonal to  $\cos(x)$ , we find that  $u_1 = u$ .

We now compute of  $\sigma_2$ , our main concern being its sign. We have,

$$(4.24) \quad -u_2''(x) = u_2(x) + n^2 \cos(nx) u_1(x) - n^2 \cos^2(nx) \cos x + \sigma_2 \cos x.$$

Taking the scalar product with  $\cos x$ , we have

$$n^2 \int \cos^2(nx) (\cos x)^2 dx - n^2 \int \cos(nx) u_1(x) \cos x dx = \sigma_2 \int (\cos x)^2 dx.$$

The sign of  $\sigma_2$  is the same as the sign of

$$A_n := \int_{-\pi}^{+\pi} \cos^2(nx)^2 (\cos x)^2 dx - \int_{-\pi}^{+\pi} \cos(nx) u_1(x) \cos x dx.$$

Computing each term of the sum, we get

$$\int \cos^2(nx)^2 (\cos x)^2 dx = \frac{\pi}{2},$$

and

$$\int \cos(nx) u_1(x) \cos x dx = \frac{\pi n^2}{4} (\alpha_{n+1} + \alpha_{n-1}).$$

Finally,

$$\frac{2}{\pi} A_n = 1 - \frac{n^2}{2} (\alpha_{n+1} + \alpha_{n-1}) = -\frac{4}{n^2 - 4} < 0.$$

Since  $\sigma_2 < 0$ , the eigenvalue  $\sigma_a$  satisfies  $\sigma_a < 1$ , and the eigenvalue 1 has at least labelling 3 for  $a = a_n$  small enough.

**Remark 4.8.** We could continue the construction at any order, but we do not need it for our purposes.

**Odd case.** The reasoning is the same, except that we work with odd functions. We look for the expansions of the first eigenvalue  $\tau_a$  of (4.20) restricted to odd functions, and of a corresponding eigenfunction  $v_a$ , in the form

$$\tau_a \sim 1 + \sum_{j>0} \tau_j a^j, \quad v_a(x) \sim \sin x + \sum_{j>0} v_j(x) a^j,$$

with  $v_a$  odd, and the functions  $v_j$  orthogonal to  $\sin x$ . We now expand in powers of  $a$  in (4.20). The coefficients of  $a$  in (4.20) give this time:

$$(4.25) \quad -v_1''(x) = v_1(x) + n^2 \cos(nx) \sin x + \tau_1 \sin x.$$

The orthogonality condition reads

$$(4.26) \quad \int_{-\pi}^{\pi} v_1(x) \sin x \, dx = 0.$$

Taking the  $L^2$  scalar product with  $\sin x$  in (4.25), we get (for  $n > 2$ )

$$(4.27) \quad \tau_1 = 0.$$

We have now to solve (with the parity and orthogonality conditions)

$$-v_1''(x) = v_1(x) + n^2 \cos(nx) \sin x.$$

We write

$$\begin{aligned} \cos(nx) \sin x &= \frac{1}{4i} (e^{inx} + e^{-inx})(e^{ix} - e^{-ix}) \\ &= \frac{1}{4i} (e^{i(n+1)x} - e^{i(n-1)x} + e^{-i(n-1)x} - e^{-i(n+1)x}), \end{aligned}$$

and we look for a particular solution in the form

$$\begin{aligned} v(x) = \frac{n^2}{4i} &\left( \beta_{-(n+1)} e^{-i(n+1)x} + \beta_{-(n-1)} e^{-i(n-1)x} \right. \\ &\left. + \beta_{(n-1)} e^{i(n-1)x} + \beta_{(n+1)} e^{i(n+1)x} \right). \end{aligned}$$

We get

$$\begin{aligned} \beta_{-(n+1)} &= -\frac{1}{(n+1)^2-1}, \quad \beta_{-(n-1)} = \frac{1}{(n-1)^2-1}, \\ \beta_{(n-1)} &= -\frac{1}{(n-1)^2-1}, \quad \beta_{(n+1)} = \frac{1}{(n+1)^2-1}. \end{aligned}$$

As in the preceding case, taking parity and orthogonality into account, we find that  $v_1 = v$ .

We now compute  $\tau_2$ . We have

$$(4.28) \quad -v_2''(x) = v_2(x) + n^2 \cos(nx)v_1(x) - n^2 \cos^2(nx) \sin x + \tau_2 \sin x.$$

Taking the scalar product with  $\sin x$ , we have

$$n^2 \int \cos^2(nx)(\sin x)^2 dx - n^2 \int \cos(nx)v_1(x) \sin x \, dx = \tau_2 \int (\sin x)^2 dx.$$

Computing the terms in the left-hand side, we get

$$\int \cos(nx)^2 (\sin x)^2 dx = \frac{\pi}{2},$$

and

$$\int \cos(nx)v_1(x) \sin x \, dx = -\frac{\pi n^2}{4} (\beta_{-(n+1)} + \beta_{n-1}).$$

We conclude that  $\tau_2 < 0$  as  $\sigma_2$ .

We have shown that 1 has exactly labelling 4 for  $n > 4$ , and  $a$  small enough.  $\square$

**Remark 4.9.** When  $n \equiv 0$  modulo 4, the 1D-operator commutes with  $x \mapsto \frac{\pi}{2} + x$ . In this case,  $\sigma_a = \tau_a$ , and it follows that 1 has labelling 4 for  $a$  small enough.

#### 4.6. Comparison with a result of Gladwell and Zhu.

In [8], the authors prove the following result for a bounded domain in  $\mathbb{R}^d$ .

**Proposition 4.10.** *Let  $\Omega \subset \mathbb{R}^d$  be a connected bounded domain. Call  $(\delta_j, u_j)$  the eigenpairs of the Dirichlet eigenvalue problem in  $\Omega$ ,*

$$(4.29) \quad \begin{cases} -\Delta u &= \delta u \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \end{cases}$$

where the eigenvalues  $\delta_1 < \delta_2 \leq \delta_3 \leq \dots$  are listed in non-decreasing order, with multiplicities. Assume that the first eigenfunction  $u_1$  is positive. For  $n \geq 2$ , let  $v = u_n + cu_1$ , for some positive constant  $c$ . Then, the function  $v$  has at most  $(n - 1)$  positive sign domains, i.e., the super-level set  $\{v > 0\}$  has at most  $(n - 1)$  connected components.

The same result is true if instead of the Dirichlet boundary condition, one considers the Neumann boundary condition (assuming in this case that  $\partial\Omega$  is smooth enough), or if one considers a closed real analytic Riemannian surface<sup>3</sup>.

A more convenient formulation, is as follows. For a function  $w$ , and  $\varepsilon \in \{-, +\}$ , define  $\beta_0^\varepsilon(w)$  to be the number of nodal domains of  $w$ , on which  $\varepsilon w$  is positive. Proposition 4.29 can be restated as follows. For any  $n \geq 2$ , and any real nonzero constant  $c$ ,

$$(4.30) \quad \beta_0^{\text{sign}(cu_1)}(u_n + cu_1) \leq (n - 1).$$

Proposition 4.10 is weaker than Conjecture 1.2. Indeed, it only gives control on the number of nodal domains where the function  $u_n + cu_1$  has the sign of  $\text{sign}(cu_1)$ . Propositions 4.3 and 4.5 show that one can a priori not control  $\beta_0^{-\text{sign}(cu_1)}(w)$ , at least in the case of the Neumann (or empty) boundary condition. However, one can observe that, fixing  $n_0$ , it is easy to construct examples for which Conjecture 1.2 is true for all linear combinations of the  $n$  first eigenfunctions,  $w \in \mathcal{L}_n$ , with  $n \leq n_0$ . Indeed, for  $L$  large, the rectangle  $]0, 1[ \times ]0, L[$  provides such an example for the Dirichlet or Neumann boundary conditions. More generally, one can consider manifolds which collapse on a lower dimensional manifold for which the ECP is true.

## 5. COUNTEREXAMPLES ON $\mathbb{S}^2$

**5.1. Results and general approach.** In this section, we prove the following results.

**Proposition 5.1.** *There exist  $C^\infty$  functions  $\Phi$  and  $G$  on  $\mathbb{S}^2$ , with the following properties.*

<sup>3</sup>It might be necessary to use a real analytic surface in order to apply Green's theorem to the nodal sets of a linear combination of eigenfunctions.

- (1) *The super-level set  $\{\Phi > 1\}$  has infinitely many connected components.*
- (2) *The function  $G$  is positive, and defines a conformal metric  $g_G = G g_0$  on  $\mathbb{S}^2$  with associated Laplace-Beltrami operator  $\Delta_G = G^{-1} \Delta_0$ .*
- (3)  *$-\Delta_G \Phi = 2 \Phi$ .*
- (4) *The eigenvalue 2 of  $-\Delta_G$  has labelling at most 9.*

**Proposition 5.2.** *There exists  $M > 0$  such that, for any  $m \geq M$ , there exist  $C^\infty$  functions  $\Phi_m$  and  $G_m$  on  $\mathbb{S}^2$  with the following properties.*

- (1) *The super-level set  $\{\Phi_m > 1\}$  has infinitely many connected components.*
- (2) *The function  $G_m$  is positive, and defines a conformal metric  $g_m = G_m g_0$  on  $\mathbb{S}^2$  with associated Laplace-Beltrami operator  $\Delta_{g_m} = G_m^{-1} \Delta_0$ .*
- (3) *For  $m \geq M$ ,  $\left(1 - \frac{2}{m}\right) \leq G_m \leq \left(1 + \frac{2}{m}\right)$ , and*
- (4)  *$-\Delta_{g_m} \Phi_m = m(m+1) \Phi_m$ .*

These propositions provide counterexamples to Conjecture 1.2 and to Questions 1.3.

**Remark 5.3.** The eigenfunctions on  $\mathbb{S}^2$  constructed in the above propositions have infinitely many isolated critical points. For a similar result on  $\mathbb{T}^2$ , see Remark 4.4(ii) which is a particular case of [4, Theorem 1].

The approach is inspired by Section 4, with the following steps.

- (1) Start from a special spherical harmonic  $Y$  of the standard sphere  $(\mathbb{S}^2, g_0)$ , with eigenvalue  $m(m+1)$ .
- (2) Modify  $Y$  into a smooth function  $F$ , whose super-level set  $\{F > 1\}$  has infinitely many connected components.
- (3) Construct a conformal metric  $g_Q = Q g_0$  on  $\mathbb{S}^2$ , whose associated Laplace-Beltrami has  $F$  as eigenfunction, with eigenvalue  $m(m+1)$ .

The proofs of Propositions 5.1 and 5.2, following the above steps is split into the next subsections.

**5.2. Metrics on  $\mathbb{S}^2$  with a prescribed eigenfunction.** Let  $g_0$  be the standard metric on the sphere

$$\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}.$$

The spherical coordinates are  $(\theta, \phi) \mapsto (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ , with  $(\theta, \phi) \in ]0, \pi[ \times ]0, 2\pi[$ . In these coordinates,

$$g_0 = d\theta^2 + \sin^2 \theta d\phi^2,$$

the associated measure is  $\sin \theta d\theta d\phi$ , and the Laplace-Beltrami operator of  $g_0$  is given by

$$\Delta_0 = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2}.$$

We consider conformal metrics on  $\mathbb{S}^2$ , in the form  $g_Q = Q g_0$ , where  $Q$  is  $C^\infty$  and positive. We denote by

$$\Delta_Q = Q^{-1} \Delta_0$$

the associated Laplace-Beltrami operator. We assume that  $Q$  is invariant under the rotations with respect to the  $z$ -axis, i.e., that  $Q$  only depends on the variable  $\theta$ .

Assume that  $F$  is a smooth function on  $\mathbb{S}^2$ , given in spherical coordinates by  $F(\theta, \phi) = T(\theta)P(\phi)$ . If  $F$  is an eigenfunction of  $-\Delta_Q$  associated with the eigenvalue  $\lambda$ , then the functions  $T$  and  $P$  satisfy the equations,

$$(5.1) \quad P''(\phi) + m^2 P(\phi) = 0,$$

$$(5.2) \quad (\lambda Q(\theta) \sin^2(\theta) - m^2)T(\theta) + \sin(\theta) (\sin(\theta)T'(\theta))' = 0,$$

where  $m$  is an integer.

When  $Q \equiv 1$ , we get the spherical harmonics

$$(\theta, \phi) \mapsto \cos(m\phi) Y_m^\ell(\theta) \text{ and } (\theta, \phi) \mapsto \sin(m\phi) Y_m^\ell(\theta),$$

with  $\lambda = \ell(\ell + 1)$ ,  $\ell \in \mathbb{N}$ , and  $\ell \geq m \geq 0$ .

For  $m \geq 1$ , we consider the special spherical harmonics

$$Y_{m,m}(\theta, \phi) = \sin^m(\theta) \cos(m\phi)$$

with  $\ell = m$  (we could consider  $\cos^m(\theta) \sin(m\phi)$  as well). For later purposes, we introduce the linear differential operator  $\mathcal{K}_m$ , defined by

$$(5.3) \quad \begin{aligned} T \mapsto (\mathcal{K}_m T)(\theta) &= \sin^2(\theta) T''(\theta) + \sin(\theta) \cos(\theta) T'(\theta) \\ &\quad + (m(m+1) \sin^2(\theta) - m^2) T(\theta). \end{aligned}$$

With the notation  $Y_{(m)}(\theta) = \sin^m(\theta)$ , we have

$$(5.4) \quad \mathcal{K}_m Y_{(m)} = 0.$$

Given  $Q$  a smooth positive function, which only depends on  $\theta$ , a necessary and sufficient condition for the function  $F(\theta, \phi) = T(\theta) \cos(m\phi)$  to satisfy  $-\Delta_Q F = m(m+1) F$ , is that

$$(5.5) \quad Q(\theta) = \frac{m^2 T(\theta) - \sin(\theta) \cos(\theta) T'(\theta) - \sin^2(\theta) T''(\theta)}{m(m+1) \sin^2(\theta) T(\theta)},$$

or, equivalently,

$$(5.6) \quad 1 - Q(\theta) = \frac{(\mathcal{K}_m T)(\theta)}{m(m+1) \sin^2(\theta) T(\theta)}$$

In particular, taking  $F(\theta, \phi) = Y_{(m)}(\theta) \cos(m\phi)$ , we find that  $Q \equiv 1$ .

**Remark 5.4.** The main difficulty in prescribing the eigenfunction  $F$ , and hence the function  $T$ , is to show that function  $Q$  defined by (5.5) is actually smooth and positive.

### 5.3. Constructing perturbations of the function $\sin^m(\theta)$ .

Given  $m \geq 1$ , we start from the spherical harmonic  $\sin^m(\theta) \cos(m\phi)$  in spherical coordinates. We look for functions  $F$  of the form  $F(\theta, \phi) = T(\theta) \cos(m\phi)$ . To determine  $T$ , we construct a family  $T_{m,n,\alpha}$  of perturbations of the function  $Y_{(m)}(\theta) = \sin^m(\theta)$ , in the form,

$$(5.7) \quad T_{m,n,\alpha}(\theta) = \sin^m(\theta) + P_{m,n,\alpha}(\theta) + u_{m,n,\alpha}(\theta),$$

with  $n \in \mathbb{N}$  (to be chosen large), and  $\alpha \in (0, \frac{1}{4}]$  (to be chosen small). The function  $P_{m,n,\alpha}$  is constructed such that

$$\sin^m(\theta) + P_{m,n,\alpha}(\theta) \equiv 1$$

in an interval around  $\frac{\pi}{2}$ , and  $u_{m,n,\alpha}$  is a rapidly oscillating function in the same interval. The construction of the family  $T_{m,n,\alpha}$  is explained in the following paragraphs.

#### 5.3.1. Construction of $P_{m,n,\alpha}$ .

**Proposition 5.5** (Construction of  $P_{m,n,\alpha}$ ). *For all  $m \geq 1$  and  $\alpha \in (0, \frac{1}{4}]$ , there exist  $N(m, \alpha) \in \mathbb{N}$ , and a sequence of functions  $(P_{m,n,\alpha})_{n \geq 1}$ ,  $P_{m,n,\alpha} : [0, \pi] \rightarrow \mathbb{R}$ , such that  $P_{m,n,\alpha}(\pi - \theta) = P_{m,n,\alpha}(\theta)$  for all  $\theta \in [0, \pi]$ , with the following properties for all  $n \geq N(m, \alpha)$ .*

- (1)  $P_{m,n,\alpha}$  is smooth;
- (2) for  $\theta \in (\frac{\pi}{2} + \frac{1}{mn}, \pi)$ ,  $P_{m,n,\alpha}(\theta) = 0$ ;
- (3) for  $\theta \in (\frac{\pi}{2}, \frac{\pi}{2} + \frac{\alpha}{(mn)^2})$ ,  $P_{m,n,\alpha}(\theta) = 1 - \sin^m(\theta)$ ;
- (4) for  $\theta \in [\frac{\pi}{2}, \pi]$ ,  $|P_{m,n,\alpha}(\theta)| \leq \frac{2m}{(mn)^3}$ , and  $|P'_{m,n,\alpha}(\theta)| \leq \frac{2m}{(mn)^2}$ ;
- (5) for  $\theta \in [\frac{\pi}{2}, \pi]$ ,  $-m(1 + 5\alpha) \leq P''_{m,n,\alpha}(\theta) \leq m(1 + \alpha)$ .

The idea is to construct  $P_{m,n,\alpha}$  as

$$(5.8) \quad P_{m,n,\alpha}(\theta) = \int_{\frac{\pi}{2}}^{\theta} R_{m,n,\alpha}(t) dt,$$

for  $\theta \in [\frac{\pi}{2}, \pi]$ , and to extend it so that  $P_{m,n,\alpha}(\pi - \theta) = P_{m,n,\alpha}(\theta)$ . We first construct a sequence  $S_{m,n,\alpha}$  (Lemma 5.6), and then the sequence  $R_{m,n,\alpha} = S_{m,n,\alpha} + s_{m,n,\alpha}$  (Lemma 5.7).

**Lemma 5.6** (Construction of  $S_{m,n,\alpha}$ ). *For any  $m \geq 1$ , and any  $\alpha \in (0, \frac{1}{4}]$ , there exists a sequence of functions  $(S_{m,n,\alpha})_{n \geq 1}$ ,  $S_{m,n,\alpha} : [0, \pi] \rightarrow \mathbb{R}$ , such that  $S_{m,n,\alpha}(\pi - \theta) = -S_{m,n,\alpha}(\theta)$  for all  $\theta \in [0, \pi]$ , with the following properties.*

- (1)  $S_{m,n,\alpha}$  is smooth;
- (2) for  $\theta \in (\frac{\pi}{2} + \frac{1}{(mn)^2}, \pi)$ ,  $S_{m,n,\alpha}(\theta) = 0$ ;

- (3) for  $\theta \in (\frac{\pi}{2}, \frac{\pi}{2} + \frac{\alpha}{(mn)^2})$ ,  $S_{m,n,\alpha}(\theta) = -m \cos(\theta) \sin^{m-1}(\theta)$ ;  
(4) for  $\theta \in [\frac{\pi}{2}, \pi]$ ,  $|S_{m,n,\alpha}(\theta)| \leq \frac{1}{mn^2}$ ;  
(5) for  $\theta \in [\frac{\pi}{2}, \pi]$ ,  $-m(1 + 4\alpha) \leq S'_{m,n,\alpha}(\theta) \leq m$ .

*Proof.* We construct  $S_{m,n,\alpha}$  on  $[\frac{\pi}{2}, \pi]$ , and extend it to  $[0, \pi]$  so that  $S_{m,n,\alpha}(\pi - \theta) = -S_{m,n,\alpha}(\theta)$ .

Choose a function  $\chi_\alpha : \mathbb{R} \rightarrow [0, 1]$ , such that  $\chi_\alpha$  is smooth and even,  $\chi_\alpha(\theta) = 1$  on  $[-\alpha, \alpha]$ ,  $\text{Supp}(\chi_\alpha) \subset [-1, 1]$ , and

$$(5.9) \quad -1 - 4\alpha \leq -\frac{1}{1 - 2\alpha} \leq \chi'_\alpha(t) \leq 0, \quad \forall t \geq 0.$$

A natural Lipschitz candidate would be a piecewise linear function  $\xi_\alpha$  which is equal to 1 in  $[0, \alpha]$ , and to  $\theta \mapsto 1 - (\theta - \alpha)/(1 - \alpha)$  in  $[\alpha, 1]$ . To get  $\chi_\alpha$ , we can regularize a function  $\xi_\beta$ , keeping the other properties at the price of a small loss in the control of the derivative in (5.9).

For  $\theta \in [\frac{\pi}{2}, \pi]$ , we introduce  $\hat{\theta} = \theta - \frac{\pi}{2}$ , and we take  $S_{m,n,\alpha}$  in the form

$$(5.10) \quad \begin{cases} S_{m,n,\alpha}(\theta) &= -m \chi_\alpha((mn)^2 \hat{\theta}) \cos(\theta) \sin^{m-1}(\theta) \\ &= m \chi_\alpha((mn)^2 \hat{\theta}) \sin(\hat{\theta}) \sin^{m-1}(\theta). \end{cases}$$

Properties (1), (2) and (3) are clear. Property (4) follows from the inequality  $|\sin(\hat{\theta})| \leq |\hat{\theta}|$  and (2). To prove (5), we introduce

$$h(\theta) := -m \cos(\theta) \sin^{m-1}(\theta).$$

We have

$$h'(\theta) = m^2 \sin^{m-2}(\theta) \left( \frac{1}{m} - \sin^2(\hat{\theta}) \right),$$

and hence,

$$m \sin^{m-2}(\theta) \geq h'(\theta) \geq \frac{m}{2} \sin^{m-2}(\theta),$$

in the set  $\{\theta \mid 0 \leq (mn)^2 \hat{\theta} \leq 1\}$ , as soon as  $n \geq 2$ .

We have,

$$S'_{m,n,\alpha}(\theta) = \chi_\alpha((mn)^2 \hat{\theta}) h'(\theta) + (mn)^2 \chi'_\alpha((mn)^2 \hat{\theta}) h(\theta).$$

Using the inequality  $|\cos \theta| = |\sin \hat{\theta}| \leq |\hat{\theta}|$ , and (5.9) for  $0 \leq \hat{\theta} \leq \frac{1}{(mn)^2}$ , we obtain

$$(5.11) \quad -m(1 + 4\alpha) \leq S'_{m,n,\alpha}(\theta) \leq m.$$

The lemma is proved.  $\square$

**Lemma 5.7** (Construction of  $R_{m,n,\alpha}$ ). *For all  $m \geq 1$  and  $\alpha \in (0, \frac{1}{4}]$ , there exist  $N(m, \alpha) \in \mathbb{N}$ , and a sequence of functions  $(R_{m,n,\alpha})_{n \geq 1}$ ,  $R_{m,n,\alpha} : [0, \pi] \rightarrow \mathbb{R}$ , such that  $R_{m,n,\alpha}(\pi - \theta) = -R_{m,n,\alpha}(\theta)$  for all  $\theta \in [0, \pi]$ , with the following properties for all  $n \geq N(m, \alpha)$ .*

- (1)  $R_{m,n,\alpha}$  is smooth;
- (2) for  $\theta \in (\frac{\pi}{2} + \frac{1}{mn}, \pi)$ ,  $R_{m,n,\alpha}(\theta) = 0$ ;

- (3) for  $\theta \in (\frac{\pi}{2}, \frac{\pi}{2} + \frac{\alpha}{(mn)^2})$ ,  $R_{m,n,\alpha}(\theta) = -m \cos(\theta) \sin^{m-1}(\theta)$  ;  
(4) for  $\theta \in [\frac{\pi}{2}, \pi]$ ,  $|R_{m,n,\alpha}(\theta)| \leq \frac{2}{mn^2}$  ;  
(5) for  $\theta \in [\frac{\pi}{2}, \pi]$ ,  $-m(1 + 5\alpha) \leq R'_{m,n,\alpha}(\theta) \leq m(1 + \alpha)$  ;  
(6)  $\int_{\frac{\pi}{2}}^{\frac{\pi}{2} + \frac{1}{mn}} R_{m,n,\alpha}(\theta) d\theta = 0$  .  
(7)  $\int_{\frac{\pi}{2}}^{\frac{\pi}{2} + \frac{1}{mn}} |R_{m,n,\alpha}(\theta)| d\theta \leq \frac{2m}{(mn)^3}$  .

*Proof.* We construct  $R_{m,n,\alpha}$  on  $[\frac{\pi}{2}, \pi]$ , and extend it to  $[0, \pi]$  so that  $R_{m,n,\alpha}(\pi - \theta) = -R_{m,n,\alpha}(\theta)$ . Define

$$\beta_{m,n,\alpha} := \int_{\frac{\pi}{2}}^{\frac{\pi}{2} + \frac{1}{(mn)^2}} S_{m,n,\alpha}(t) dt.$$

Using (5.10), we find that  $\beta_{m,n,\alpha}$  satisfies,

$$\begin{cases} \beta_{m,n,\alpha} \geq \int_{\frac{\pi}{2}}^{\frac{\pi}{2} + \frac{\alpha}{(mn)^2}} (-m \cos(t) \sin^{m-1}(t)) dt, \text{ and} \\ \beta_{m,n,\alpha} \leq \int_{\frac{\pi}{2}}^{\frac{\pi}{2} + \frac{1}{(mn)^2}} (-m \cos(t) \sin^{m-1}(t)) dt. \end{cases}$$

Using the inequalities  $\frac{2}{\pi}t \leq \sin(t) \leq t$  for  $t \in [0, \frac{\pi}{2}]$ , we obtain,

$$(5.12) \quad \frac{\alpha^2}{\pi^2} \frac{m}{(mn)^4} \leq \frac{\alpha^2}{\pi^2} \frac{m}{(mn)^4} 2 \cos^{m-1}\left(\frac{\alpha}{(mn)^2}\right) \leq \beta_{m,n,\alpha} \leq \frac{1}{2} \frac{m}{(mn)^4},$$

where the first inequality holds provided that  $n$  is larger than some  $N_1(m)$ .

Choose a  $C^\infty$  function  $\phi$ , such that  $0 \leq \phi \leq 1$ ,  $\text{Supp}(\phi) \subset (\frac{1}{2}, 1)$ , and  $\int_{\mathbb{R}} \phi(t) dt = 1$ . Note that for  $n \geq 3$ ,  $[0, \frac{1}{(mn)^2}] \cap [\frac{1}{2mn}, \frac{1}{mn}] = \emptyset$ .

Define  $s_{m,n,\alpha}$  by,

$$(5.13) \quad s_{m,n,\alpha}(\theta) = -\gamma_{m,n,\alpha} \phi(mn\hat{\theta}),$$

and note that for  $n \geq 3$ ,  $\text{Supp}(S_{m,n,\alpha}) \cap \text{Supp}(s_{m,n,\alpha}) = \emptyset$ .

Define

$$R_{m,n,\alpha} = S_{m,n,\alpha} + s_{m,n,\alpha}.$$

Property (2) is satisfied. Choose  $\gamma_{m,n,\alpha}$  such that Property (6) is satisfied,

$$\int_{\frac{\pi}{2}}^{\pi} R_{n,\alpha}(t) dt = \int_{\frac{\pi}{2}}^{\pi} (S_{n,\alpha}(t) + s_{n,\alpha}(t)) dt = 0,$$

so that,

$$\gamma_{m,n,\alpha} = mn \beta_{m,n,\alpha}.$$

We get

$$(5.14) \quad \frac{\alpha^2}{\pi^2} \frac{m}{(mn)^3} \leq \gamma_{m,n,\alpha} \leq \frac{1}{2} \frac{m}{(mn)^3}, \text{ for } n \geq N_1(m).$$

Properties (1) and (3) are clear. Using the properties of  $S_{m,n,\alpha}$  given by Lemma 5.6, inequality (5.14), and the fact that  $\frac{1}{2} \leq mn\hat{\theta} \leq 1$  when  $\phi' \neq 0$ , we obtain

$$|R_{m,n,\alpha}(\theta)| \leq \frac{2}{mn^2},$$

and

$$-m \left( 1 + 4\alpha + \frac{1}{2(mn)^2} \|\phi'\|_\infty \right) \leq R'_{m,n,\alpha}(\theta) \leq m \left( 1 + \frac{1}{2(mn)^2} \|\phi'\|_\infty \right).$$

Properties (4) and (5) follow by taking  $n$  larger than some  $N(m, \alpha)$ . Property (7) follows from (4).  $\square$

*Proof of Proposition 5.5.* Recall that

$$P_{m,n,\alpha}(\theta) = \int_{\pi/2}^{\theta} R_{m,n,\alpha}(z) dz$$

for  $\theta \geq \pi/2$ , and that  $P_{m,n,\alpha}$  is symmetric with respect to  $\frac{\pi}{2}$ . The properties of  $P_{m,n,\alpha}$  follow from Lemma 5.7. For  $|\theta - \frac{\pi}{2}| \leq \frac{\alpha}{(mn)^2}$ ,  $P_{m,n,\alpha}(\theta) = 1 - \sin^m(\theta)$ . We also note that  $P_{m,n,\alpha}(\theta) = 0$  for  $\theta \in (0, \frac{\pi}{2} - \frac{1}{mn}) \cup (\frac{\pi}{2} + \frac{1}{mn}, \pi)$ .  $\square$

### 5.3.2. Construction of $u_{m,n,\alpha}$ .

Define the family of functions  $u_{m,n,\alpha} : [0, \pi] \rightarrow \mathbb{R}$ , such that they are symmetric with respect to  $\frac{\pi}{2}$ , and given by the following formula in the interval  $[\frac{\pi}{2}, \pi]$ .

(5.15)

$$u_{m,n,\alpha}(\theta) = a_{m,n,\alpha} \exp\left(\frac{1}{\hat{\theta}(\hat{\theta} - \alpha(mn)^{-2})}\right) \cos\left(\frac{1}{\hat{\theta}(\hat{\theta} - \alpha(mn)^{-2})}\right),$$

for  $0 < \hat{\theta} < \alpha(mn)^{-2}$ , and 0 otherwise. This function is smooth with compact support, therefore we can choose the constant  $a_{m,n,\alpha}$  such that, for any  $m, n \geq 1$ , and  $\alpha \in (0, \frac{1}{4}]$ ,

$$(5.16) \quad |u_{m,n,\alpha}| + |u'_{m,n,\alpha}| + |u''_{m,n,\alpha}| \leq \alpha.$$

### 5.3.3. Properties of $T_{m,n,\alpha}$ .

From the construction,  $T_{m,n,\alpha}(\theta) - 1$  changes sign infinitely many times on the interval  $[\frac{\pi}{2} - \frac{\alpha}{(mn)^2}, \frac{\pi}{2} + \frac{\alpha}{(mn)^2}]$ . Indeed,  $\sin^m(\theta) + P_{m,n,\alpha}(\theta) = 1$  on that interval, and  $u_{m,n,\alpha}$  changes sign infinitely often on the same interval. Also, since  $\sin^m$ ,  $P_{m,n,\alpha}$  and  $u_{m,n,\alpha}$  are all smooth,  $T_{m,n,\alpha}$  is smooth.

#### 5.4. Non-degeneracy of the metric.

We use Subsection 5.2. To the function  $T_{m,n,\alpha}$  we associate the function  $Q_{m,n,\alpha}$  through the relation (5.5). This function defines a conformal metric  $g_{m,n,\alpha}$  on  $\mathbb{S}^2$  provided that it is positive and smooth. Taking into account the relations (5.4) and (5.6), we have the relation

$$(5.17) \quad 1 - Q_{m,n,\alpha}(\theta) = \frac{N(\theta)}{D(\theta)},$$

where

$$(5.18) \quad \begin{cases} N(\theta) &= (\mathcal{K}_m P_{m,n,\alpha})(\theta) + (\mathcal{K}_m u_{m,n,\alpha})(\theta), \\ D(\theta) &= m(m+1) \sin^2(\theta) T_{m,n,\alpha}(\theta). \end{cases}$$

Because  $P_{m,n,\alpha}$  and  $u_{m,n,\alpha}$  are supported in  $J_{mn} := [\frac{\pi}{2} - \frac{1}{mn}, \frac{\pi}{2} + \frac{1}{mn}]$ , we have  $Q_{m,n,\alpha} \equiv 1$  in  $(0, \pi) \setminus J_{mn}$ .

It therefore suffices to study  $Q_{m,n,\alpha}$  in the interval  $J_{mn}$  and, by symmetry with respect to  $\frac{\pi}{2}$ , in  $J_{+,mn} := [0, \frac{\pi}{2} + \frac{1}{mn}]$ . As above, we set  $\hat{\theta} = \theta - \frac{\pi}{2}$ .

From (5.7) and the definition of  $T_{m,n,\alpha}$ , we deduce

$$|T_{m,n,\alpha}(\theta) - 1| \leq 1 - \sin^m(\theta) + |P_{m,n,\alpha}(\theta)| + |u_{m,n,\alpha}(\theta)|.$$

Using (5.16) and Proposition 5.5(4), we obtain

$$|T_{m,n,\alpha}(\theta) - 1| \leq \frac{3m}{(mn)^2} + \alpha, \text{ for } n \geq N(m, \alpha), \theta \in J_{+,mn}.$$

It follows that for  $\alpha$  small enough and  $n \geq N(m, \alpha)$ ,

$$(5.19) \quad |T_{m,n,\alpha}(\theta) - 1| \leq \frac{1}{2} \text{ in } J_{+,mn}.$$

In particular, this inequality implies that for  $\alpha$  small enough, and  $n$  large enough,  $Q_{m,n,\alpha}$  is well-defined on  $\mathbb{S}^2$  and  $C^\infty$ .

For  $\theta \in J_{+,mn}$ , we have,

$$(5.20) \quad \left| m(m+1) \sin^2(\theta) - m^2 \right| \leq m + \frac{1}{n^2}.$$

For  $n \geq N(m, \alpha)$  and  $\theta \in J_{+,mn}$ , we have,

$$(5.21) \quad D(\theta) \geq m(m+1) \left( 1 - \frac{5}{n^2} - \alpha \right) \geq m(m+1)(1 - 2\alpha).$$

From (5.3), (5.16), and (5.20), we have

$$\begin{aligned} |(\mathcal{K}_m u_{m,n,\alpha})(\theta)| &\leq |u_{m,n,\alpha}''(\theta)| + |u_{m,n,\alpha}'(\theta)| \\ &\quad + \left( m(m+1) \sin^2(\theta) - m^2 \right) |u_{m,n,\alpha}(\theta)| \\ &\leq (m+1)\alpha \end{aligned}$$

We now need to estimate  $|\mathcal{K}_m P_{m,n,\alpha}|$ . We clearly have

$$\begin{aligned} |(\mathcal{K}_m P_{m,n,\alpha})(\theta)| &\leq |P''_{m,n,\alpha}(\theta)| + |P'_{m,n,\alpha}(\theta)| \\ &\quad + \left( m(m+1) \sin^2(\theta) - m^2 \right) |P_{m,n,\alpha}(\theta)| \end{aligned}$$

Using the estimates in Proposition 5.5, and the fact that  $|\sin(t)| \leq |t|$ , we obtain the following inequalities for  $n \geq N(m, \alpha)$  and  $\theta \in J_{+,mn}$ ,

$$\begin{cases} |P''_{m,n,\alpha}(\theta)| &\leq m(1 + 5\alpha), \\ |P'_{m,n,\alpha}(\theta)| &\leq \frac{2m}{(mn)^2}, \\ |P_{m,n,\alpha}(\theta)| &\leq \frac{2m}{(mn)^3}. \end{cases}$$

From these estimates and (5.20), we obtain

$$|(\mathcal{K}_m P_{m,n,\alpha})(\theta)| \leq m(1 + 5\alpha) + \frac{5}{n^2}$$

for  $n \geq N(m, \alpha)$  and  $\theta \in J_{+,mn}$ .

Finally, we arrive at the following estimates in  $J_{+,mn}$ , for  $n \geq N(m, \alpha)$

$$\begin{cases} N(\theta) &\leq m \left( 1 + \frac{5}{n^2} + 5\alpha \right), \\ |1 - Q_{m,n,\alpha}(\theta)| &\leq \frac{1}{m+1} \left( 1 + \frac{10}{n^2} + 10\alpha \right). \end{cases}$$

**Proposition 5.8.** *For any  $m \geq 1$ , and  $\alpha \in (0, \frac{1}{24})$ , there exists  $N_1(m, \alpha)$  such that, for  $n \geq N_1(m, \alpha)$ ,*

$$(5.22) \quad |Q_{m,n,\alpha}(\theta) - 1| \leq \frac{1}{m+1} (1 + 12\alpha).$$

*In particular, the metric  $g_{m,n,\alpha} = Q_{m,n,\alpha}$  is smooth and non-degenerate.*

**5.5. Proof of Propositions 5.1 and 5.2.** We Apply the results of the preceding subsections, with  $m = 1$ , and define the function  $\Phi$  in spherical coordinates by,

$$\Phi(\theta, \phi) = T_{1,n,\alpha}(\theta) \cos(\phi),$$

where  $n$  is large enough, and  $\alpha$  small enough. This function is clearly smooth outside the north and south poles of the sphere. Near the poles  $\Phi$  is equal to the spherical harmonic  $\sin(\theta) \cos(\phi)$ . It follows that  $\Phi$  is smooth. By Proposition 5.8, the function  $Q_{1,n,\alpha}$  associated with  $T_{1,n,\alpha}$  by the relation (5.5) extends to a smooth positive function on  $\mathbb{S}^2$ . Choose  $G = Q_{1,n,\alpha}$ . Then,

$$-\Delta_G \Phi = 2\Phi.$$

The proof of Assertion (4) follows from the min-max. Note that  $G$  can a priori not be chosen close to 1.

Similarly, when  $m \geq 1$ , an appropriate choice of  $(\alpha, n)$  yields a function

$$\Phi_m(\theta, \phi) = T_{m,n,\alpha}(\theta) \cos(m\phi),$$

and a function  $G_m = Q_{m,n,\alpha}$  satisfying (5.22), such that

$$-\Delta_{G_m} \Phi_m = m(m+1)\Phi_m.$$

Choosing  $m$  large enough, the metric  $g_m = G_m g_0$  can be made as close as desired from the standard metric  $g_0$ .

It remains to prove that the eigenfunctions  $\Phi$  and  $\Phi_m$  have the required nodal properties (Assertion (1) in the propositions). We give the argument for  $\Phi_m$ .

The function  $V = T_{m,n,\alpha}$  is of the form  $V(\theta) = \sin^m(\theta) + P_{m,n,\alpha}(\theta) + u_{m,n,\alpha}(\theta)$ . Let  $V_0(\theta) = \sin^m(\theta) + P_{m,n,\alpha}(\theta)$ . We have  $V_0(\frac{\pi}{2}) = 1$ . From the properties of  $P_{m,n,\alpha}$ , we deduce that

$$\begin{aligned} V'_0(\theta) = & m \cos(\theta) \sin^{m-1}(\theta) \left(1 - \chi_\alpha(m^2 n^2 \hat{\theta})\right) \\ & - \gamma \cos(\theta) \sin^{m-1}(\theta) \phi(mn\hat{\theta}), \end{aligned}$$

and it follows that  $V'_0(\theta) \leq 0$ , so that  $0 \leq V_0(\theta) \leq 1$  in  $[0, \pi]$ . With the notation  $u = u_{m,n,\alpha}$ , we have

$$(5.23) \quad \begin{cases} \{u(\theta) < 0\} \times [0, 2\pi] \subset \{\Phi_m < 1\}, \\ \{\Phi_m \geq 1\} \subset \{u(\theta) \geq 0\} \times [0, 2\pi], \\ \{u(\theta) \geq 0\} \times \{0\} \subset \{\Phi_m \geq 1\}. \end{cases}$$

This means that the set  $\{\Phi_m \geq 1\}$  has at least one connected component in each band  $\{u(\theta) \geq 0\} \times [0, 2\pi]$ .  $\square$

**Remark 5.9.** If we restrict the eigenfunction  $\Phi$  (Proposition 5.1) or  $\Phi_m$  (Proposition 5.2) to the hemisphere  $\{(\theta, \phi) \in [0, \frac{\pi}{2}] \times [0, 2\pi]\}$ , we obtain counterexamples to Conjecture 1.2 for the hemisphere, with a metric conformal to the standard metric  $g_0$ , and Neumann boundary condition.

## APPENDIX A. BOUNDS ON THE NUMBER OF NODAL DOMAINS ON $\mathbb{S}^2$ WITH THE ROUND METRIC

The following result can be found in [5]:

**Proposition A.1.** *Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a polynomial of degree  $d$ . Then, the number of nodal domains of its restriction to  $\mathbb{S}^{n-1}$  is bounded by  $2^{2n-1}d^{n-1}$ .*

In the case of  $\mathbb{S}^2$  with the round metric, every eigenfunction is the restriction of a harmonic homogeneous polynomial to the sphere. Also, for such a polynomial of degree  $\ell$ , the eigenfunction on the sphere is  $\ell(\ell+1)$ , with multiplicity  $2\ell+1$ . For a sum  $w$  of spherical harmonics of degree less than or equal to  $\ell$ , Conjecture 1.2 would give  $\beta_0(w) \leq \ell^2+1$ . Using Proposition A.1, we get the following weaker estimate.

**Corollary A.2.** *Let  $g_0$  be the round metric for  $\mathbb{S}^2$ . Then, the sum  $w$  of spherical harmonics of degree less than or equal to  $\ell$  has at most  $8\ell^2$  nodal domains.*

However, this property is highly unstable in the case of  $C^\infty$  metrics, as we have shown in Section 5.

## APPENDIX B. ISOTROPIC QUANTUM HARMONIC OSCILLATOR IN DIMENSION 2

In this section, we will show that Conjecture 1.2 is true for the harmonic oscillator  $H : L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)$ ,  $H = -\Delta + x^2 + y^2$ .

**Proposition B.1.** *Let  $f_i$  be the eigenfunctions of  $H$  with eigenvalues ordered in increasing order with multiplicities. Then, for any linear combination  $f = \sum_{i=1}^n a_i f_i$ , we have  $\beta_0(f) \leq n$ .*

A basis  $f_n$  of eigenfunctions of  $H$  is given by

$$H_{a,b}(x, y) := e^{-\frac{x^2+y^2}{2}} H_a(x) H_b(y), \quad 0 \leq a, b \in \mathbb{Z},$$

where  $H_n$  refers to the  $n$ -th Hermite polynomial.

The associated eigenvalue is given by  $2(a + b + 1)$ , with multiplicity  $a + b + 1$ . Therefore, counting multiplicities, for each  $n$  in the interval  $[\frac{k(k+1)}{2} + 1, \frac{(k+1)(k+2)}{2}]$  for some positive integer  $k$ ,  $f_n$  is a polynomial of degree  $k$ .

For a polynomial  $f$  of degree  $k$  in 2 variables, we have the following upper bound on the number of its nodal domains:

**Lemma B.2.** *For any polynomial  $f$  of degree  $k$  in  $\mathbb{R}^2$ ,*

$$\beta_0(f) \leq k(k + 1)/2 + 1.$$

*The upper bound is achieved by  $k$  non-parallel lines.*

To prove this, we first note that the number of nodal domains is bounded from above by  $U(f) + S(f) + 1$ , where  $U(f)$  is the number of connected components of the nodal set and  $S(f) = \sum (s_i - 1)$ , where the sum is taken over all singular points  $a_i$  and  $s_i$  is the order of the singularity at  $a_i$  (the lowest homogeneous order term in the Taylor expansion of  $f$  around  $a_i$ ).

Now, we use a classical theorem by Harnack:

**Theorem B.3** (Harnack's curve theorem). *Let  $f$  be a real irreducible polynomial in two variables, of degree  $k$ . Let  $a_i$  be the singular points of the nodal set, with order  $s_i$ . We have the following inequality<sup>4</sup> for the number of connected components of its nodal set:*

<sup>4</sup>In fact, the original theorem as stated in [3] deals with algebraic curves in  $\mathbb{R}\mathbb{P}^2$ . However, it is easily adapted to  $\mathbb{R}^2$  by adding at most  $k$  unbounded components.

$$U(f) \leq \frac{(k-1)k}{2} - \sum_i \frac{s_i(s_i-1)}{2} + 1.$$

Now, we proceed by induction. For  $k = 1$ , the lemma is trivial. Now, consider a polynomial  $f$  of degree  $k > 1$ . It can be either irreducible or the product of two smaller degree polynomials.

If  $f$  is irreducible, then by Harnack's theorem we have

$$\beta_0(f) \leq (k-1)k/2 + 2,$$

since for all  $a \geq 1$ ,  $a-1 \leq a(a-1)/2$ .

If  $f = PQ$  with  $\deg P = j$  and  $\deg Q = k-j$ , the number of nodal domains is bounded by  $\beta_0(P) + \beta_0(Q) + j(n-j) - 1$ . Indeed, every intersection between  $P$  and  $Q$  adds the same number of nodal domains as the degree of their intersection, and this number can be bounded by Bezout's theorem. We need to subtract 1 to remove the initial original domain of  $\mathbb{R}^2$  (otherwise, multiplying two linear functions would give 5 nodal domains.)

By induction, we have the following inequality:

$$\begin{aligned} \beta_0(f) &\leq \frac{j(j+1)}{2} + \frac{(n-j)(n-j+1)}{2} + j(n-j) + 1 \\ &\leq \frac{n(n+1)}{2} + 1. \end{aligned}$$

Now, since this was achieved by  $P$  and  $Q$  being the product of linear factors, then  $f$  is a product of linear factors. This proves lemma B.2.  $\square$

We can now complete the proof of proposition B.1.

Let  $n \in [k(k+1)/2 + 1, (k+1)(k+2)/2]$ . Then, any linear combination of  $f_1, f_2, \dots, f_n$  will be a polynomial of degree at most  $k$ . Any such polynomial has at most  $k(k+1)/2 + 1$  nodal domains. Therefore, Conjecture 1.2 is true in the case of the isotropic two-dimensional quantum harmonic oscillator.

**Remark B.4.** It is still unclear if this upper bound can be reached for any  $k > 2$ .

**Remark B.5.** Considering the results of this paper, it seems likely that a small perturbation of either the metric in  $\mathbb{R}^2$  or the potential could break this upper bound.

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