Dual Immaculate Quasisymmetric Functions Expand Positively into Young Quasisymmetric Schur Functions

Edward E. Allen, Joshua Hallam, Sarah K. Mason

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Dual Immaculate Quasisymmetric Functions Expand Positively into Young Quasisymmetric Schur Functions

Edward E. Allen, Joshua Hallam, and Sarah K. Mason

Department of Mathematics, Wake Forest University

Abstract. We describe a combinatorial formula for the coefficients when the dual immaculate quasisymmetric functions are decomposed into Young quasisymmetric Schur functions. We prove this using an analogue of Schensted insertion. We also provide a Remmel-Whitney style rule to generate these coefficients algorithmically.

Résumé. Nous décrivons une formule combinatoire pour les coefficients des dual immaculate quasisymmetric functions quand ils sont décomposées en Young quasisymmetric Schur functions. Nous le prouvons en utilisant un analogue d'insertion de Schensted. Nous fournissons également une règle de style Remmel-Whitney pour générer ces coefficients algorithmiquement.

Keywords. quasisymmetric functions, dual immaculate functions, Schensted insertion, Schur functions, tableaux

1 Introduction

The algebra Sym of symmetric functions generalizes to the algebra of quasisymmetric functions $QSym$. Stanley laid the foundation for quasisymmetric functions through his work on P-partitions [Sta72]. Gessel [Ges84] formalized the definition of quasisymmetric functions and introduced the fundamental basis. Ehrenborg [Ehr96] further developed the Hopf algebra structure of $QSym$, which is the Hopf algebra dual to the noncommutative symmetric functions $NSym$. $QSym$ is also the terminal object in the category of combinatorial Hopf algebras [ABS06].

In [HLMvW11a], Haglund *et al* introduced a new basis for the quasisymmetric functions called the *quasisymmetric Schur functions* $\{\check{\mathscr{S}}_\gamma\}_\gamma$. The quasisymmetric Schur functions are the specializations of non-symmetric Macdonald polynomials obtained by setting $q = t = 0$ and summing the resulting Demazure atoms over all weak compositions which collapse to the same strong composition. The quasisymmetric Schur functions are generated by fillings of composition diagrams analogously to how Schur functions are generated by semistandard Young tableaux. The *Young quasisymmetric Schur functions* [LMvW13] are variants of quasisymmetric Schur functions obtained by reversing the entries in composition diagrams. In this paper, we work with the Young quasisymmetric Schur functions.

The *immaculate basis* for $NSym$, introduced in [BBS$^+$14], is constructed using non-commutative Bernstein operators. The forgetful map, see [BBS$^+$14], projects the immaculate basis onto the Schur basis. The *dual immaculate quasisymmetric functions* form the dual to the immaculate basis. Like the quasisymmetric Schur functions, they are generated using fillings of composition diagrams.

In this paper, we investigate the connection between these two quasisymmetric analogues of Schur functions. In particular, we show that the dual immaculate basis, $\{\mathfrak{S}^*{}_\alpha\}_\alpha$, decomposes as a nonnegative sum of Young quasisymmetric Schur functions $\{\hat{\mathscr{S}}_\gamma\}_\gamma$.

Theorem 1.1 *The dual immaculate quasisymmetric functions decompose into Young quasisymmetric Schur functions in the following way:*

$$\mathfrak{S}^*_\alpha = \sum_\beta c_{\alpha,\beta}\hat{\mathscr{S}}_\beta$$

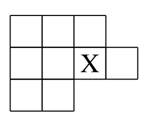
where $c_{\alpha,\beta}$ is the number of DIRTs (See Definition 3.6) of shape β with row strip shape α^{rev} (See Definition 3.2).

The remainder of the paper is organized as follows. In Section 2, we review the background material on compositions and their diagrams. We then define the Young quasisymmetric Schur functions as well as the dual immaculate quasisymmetric functions and explain their decompositions in the fundamental basis. Section 3 describes the insertion algorithm that is used to prove our main result. We then discuss the proof of our main theorem in Section 4. This section includes a Remmel-Whitney style algorithm that computes the coefficients of the decomposition of the dual immaculate quasisymmetric functions without using insertion. We conclude with a section on future directions.

2 Background

A *composition* α of n, written $\alpha \vDash n$, is a finite sequence of positive integers that sum to n. If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l)$, then α_i is the i^{th} *part* of α and $l(\alpha) = l$ is the *length* of α. If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l)$ then we define the reverse of α to be $\alpha^{rev} = (\alpha_l, \alpha_{l-1}, \ldots, \alpha_1)$. A composition β is said to be a *refinement* of a composition α if α can be obtained from β by summing collections of consecutive parts of β. Define $set(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{l-1}\}$.

Given a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l)$, the *diagram* D_α is constructed by placing boxes (or *cells*) into left-justified rows so that the i^{th} row from the bottom contains α_i cells. The shape of D_α is denoted by α. Note that this notation is analogous to the French notation for the Young diagram of a partition. Position (i, j) in D_α refers to the cell in the i^{th} column (reading from left to right) and the j^{th} row (reading from bottom to top). For example, the diagram D_α pictured below corresponds to a diagram of shape $\alpha = (2, 4, 3)$ with an X in position $(3, 2)$.



A *quasisymmetric function* is a bounded degree formal power series $f(x) \in \mathbb{Q}[[x_1, x_2, \ldots]]$ such that for all compositions $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l)$, the coefficient of $\prod x_i^{\alpha_i}$ is equal to the coefficient of $\prod x_{i_j}^{\alpha_i}$ for all $i_1 < i_2 < \ldots < i_l$. Let $QSym$ denote the ring of quasisymmetric functions and $QSym_n$ denote the space of homogeneous quasisymmetric functions of degree n so that $QSym = \oplus_{n \geq 0} QSym_n$.

$$T \;=\; \begin{array}{|c|c|c|}\cline{1-2} 4 & 6 \\ \hline 2 & 3 & 5 \\ \hline 1 \\ \cline{1-1}\end{array}\,, \quad \bar{T} = \begin{array}{|c|c|c|c|}\cline{1-3} 4 & 6 & \infty \\ \hline 2 & 3 & 5 & \infty \\ \hline 1 & \infty \\ \cline{1-2}\end{array}\,, \quad rw_{\mathscr{S}}(\bar{T}) = \infty\ \infty\ 5\ 6\ 3\ \infty\ \ 4\ 2\ 1,\ rw_{\mathfrak{S}^*}(T) = 4\ 6\ 2\ 3\ 5\ 1$$

Fig. 2.1: As an augmentation of a Young composition tableau, the reading word of $\bar{T}$ is $\infty\ \infty\ 5\ 6\ 3\ \infty\ \ 4\ 2\ 1$. The reading word of T as an immaculate tableau is $4\ 6\ 2\ 3\ 5\ 1$.

A natural basis for $QSym_n$ is the *monomial quasisymmetric basis*, given by the collection $\{M_\alpha\}_{\alpha \models n}$ where

$$M_\alpha = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}.$$

Gessel's *fundamental basis for quasisymmetric functions* [Ges84] can be expressed by

$$F_\alpha = \sum_{\beta \preceq \alpha} M_\beta,$$

where $\beta \preceq \alpha$, means that β is a refinement of α.

A *filling of* D_α is a function $T : D_\alpha \to \mathbb{Z}_+$ on the cells of D_α. Here $T(i,j)$ denotes the image of the cell (i,j) and is called the *entry* of cell (i,j).

Definition 2.1 *The filling $T : D_\alpha \to \mathbb{Z}_+$ is a* semistandard Young composition tableau (SSYCT) *of shape α if it satisfies the following conditions:*

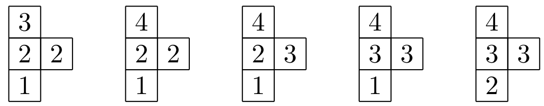
1. *Row entries are weakly increasing from left to right.*
2. *The entries in the leftmost column are strictly increasing from bottom to top.*
3. *(Young composition triple rule) Augment T by adding an ∞ to the first empty cell (from left to right) in each row; denote this by $\bar{T}$. For any subarray in $\bar{T}$ (shown below), if $b \leq a$, then $c < a$.*

$$\begin{array}{cc} \boxed{b} & \!\!\boxed{c} \\ \\ & \boxed{a} \end{array}$$

The following definition will be useful in Section 3.

Definition 2.2 *Read the entries in the columns of a Young composition tableau T (or its augmentation $\bar{T}$) from top to bottom, beginning with the rightmost column of T and working right to left. This ordering of the cells is called the* reading order. *When the entries of the cells are read in reading order, the resulting word is called the* reading word *of T, denoted $rw_{\mathscr{S}}(T)$. See Figure 2.1 for an example.*

The *weight* of a SSYCT T of shape α is the monomial $x^T = \prod_i x_i^{v_i}$ where v_i is the number of times the entry i appears in T as seen in Fig. 2.2. A *standard Young composition tableau* (SYCT) of shape $\alpha \models n$ is a semistandard Young composition tableau in which each of the numbers $\{1, \ldots, n\}$ appears exactly once.



$$\hat{\mathscr{S}}_{121}(x_1, x_2, x_3, x_4) = x_1x_2^2x_3 + x_1x_2^2x_4 + x_1x_2x_3x_4 + x_1x_3^2x_4 + x_2x_3^2x_4$$

Fig. 2.2: The SSYCT that generate $\hat{\mathscr{S}}_{(1,2,1)}(x_1, x_2, x_3, x_4)$.

Definition 2.3 *[LMvW13] Let α be a composition. Then the Young quasisymmetric Schur function $\hat{\mathscr{S}}_\alpha$ is given by*

$$\hat{\mathscr{S}}_\alpha = \sum_T x^T,$$

summed over all semistandard Young composition tableaux T of shape α. See Figure 2.2 for an example.

We now describe the method given in Proposition 5.2.2 of [LMvW13] for writing a Young quasisymmetric Schur function as a nonnegative sum of Gessel's fundamental quasisymmetric functions.

Definition 2.4 *The descent set, $Des_{\hat{\mathscr{S}}}(T)$, of a standard Young composition tableau T is the subset of $\{1, \ldots, n-1\}$ consisting of all entries i of T such that $i+1$ appears weakly to the left of i in T.*

We note that we are using a subscript for the descent set $Des_{\hat{\mathscr{S}}}(T)$, which is not usually done. We do this because we will use two different types of descent sets.

Proposition 2.5 *[LMvW13] Let α, β be compositions. Then*

$$\hat{\mathscr{S}}_\alpha = \sum_\beta d_{\alpha,\beta} F_\beta,$$

where $d_{\alpha,\beta}$ is equal to the number of standard Young composition tableaux T of shape α such that $Des_{\hat{\mathscr{S}}}(T) = set(\beta)$.

The example in Figure 2.2 shows that there is only one SYCT of shape $(1,2,1)$. It has descent set $\{1,3\}$ and therefore $\hat{\mathscr{S}}_{(1,2,1)} = F_{(1,2,1)}$.

In [BBS$^+$14], the authors introduce a new basis of NSym called the immaculate basis. Since QSym and NSym are dual, this gives rise to a dual basis of QSym called the *dual immaculate basis*. One can define the dual immaculate quasisymmetric functions using *immaculate tableaux*.

Note that the dual immaculate basis was originally introduced using English notation. In the following definition we use the French notation for our tableaux; this is why in condition 2 below the entries in the leftmost column increase from bottom to top rather than top to bottom. Our definition of descent also reflects this modification. None of the underlying mathematics is impacted in any way by this cosmetic convention.

Definition 2.6 *[BBS$^+$14] A filling $T : D_\alpha \to \mathbb{Z}_+$ is an immaculate tableau of shape α if it satisfies the following conditions:*

1. *Row entries are weakly increasing from left to right.*

2. *The entries in the leftmost column are strictly increasing from bottom to top.*

$$\begin{array}{|c|c|}\hline 3 & \\ \hline 2 & 2 \\ \hline 1 & \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 4 & \\ \hline 2 & 2 \\ \hline 1 & \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 4 & \\ \hline 2 & 3 \\ \hline 1 & \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 4 & \\ \hline 3 & 3 \\ \hline 1 & \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 4 & \\ \hline 3 & 3 \\ \hline 2 & \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 3 & \\ \hline 2 & 3 \\ \hline 1 & \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 3 & \\ \hline 2 & 4 \\ \hline 1 & \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 4 & \\ \hline 2 & 4 \\ \hline 1 & \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 4 & \\ \hline 3 & 4 \\ \hline 2 & \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 4 & \\ \hline 3 & 4 \\ \hline 1 & \\ \hline\end{array}$$

$$\begin{aligned}\mathfrak{S}^*_{121}(x_1, x_2, x_3, x_4) = {} & x_1x_2^2x_3 + x_1x_2^2x_4 + 2x_1x_2x_3x_4 + x_1x_3^2x_4 + x_2x_3^2x_4 + x_1x_2x_3^2 + x_1x_2x_4^2 \\ & + x_2x_3x_4^2 + x_1x_3x_4^2\end{aligned}$$

Fig. 2.3: The immaculate tableaux that generate $\mathfrak{S}^*_{(1,2,1)}(x_1, x_2, x_3, x_4)$.

Note that every SSYCT is also an immaculate tableau since the definition is the same except that we do not require the Young composition triple rule for the immaculate tableaux. We also have a reading word for immaculate tableaux. Note that it is not the same as the reading word for the Young composition tableaux.

Definition 2.7 *Read the entries in the rows of an immaculate tableau T, from left to right, beginning with the highest row of T and working top to bottom. The resulting word is called the* reading word *of T, denoted $rw_{\mathfrak{S}^*}(T)$. See Figure 2.1 for an example.*

We say that an immaculate tableau is a *standard immaculate tableau* if the numbers $\{1, \ldots, n\}$ each appear exactly once. Just as with Young composition tableaux, the *weight* of an immaculate tableau T of shape α is the monomial $x^T = \prod_i x_i^{v_i}$, where v_i is the number of times the entry i appears in T as seen in Fig. 2.3.

Definition 2.8 *Let α be a composition. The* dual immaculate function $\mathfrak{S}^*_\alpha$ *is given by*

$$\mathfrak{S}^*_\alpha = \sum_T x^T,$$

where the sum is over all immaculate tableaux of shape α. See Figure 2.3 for an example.

Just as Young quasisymmetric Schur functions decompose into a positive sum of fundamental quasisymmetric functions, the dual immaculate quasisymmetric functions decompose into the fundamental basis using descents sets. Now we define the descent set of a standard immaculate tableau.

Definition 2.9 *The descent set, $Des_{\mathfrak{S}^*}(T)$, of a standard immaculate tableau T is the subset of $\{1, \ldots, n-1\}$ consisting of all entries i of T such that $i+1$ appears strictly above i in T.*

As an example, consider the tableau

$$T = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 & 4 \\ \hline\end{array}$$

We see that $Des_{\mathfrak{S}^*}(T) = \{1\}$. Note that the descent set of a standard immaculate tableau is not the same as the descent set of a standard Young composition tableau. In fact, the tableau T is both a standard immaculate tableau and a standard Young composition tableau. However, $Des_{\hat{\mathscr{S}}}(T) = \{1, 3\}$ and so the two descents sets for the same tableau need not be the same.

We now explain how the dual immaculate functions decompose into the fundamental basis.

$$5 \;\rightarrow\; \begin{array}{|c|c|c|}\cline{1-2} 6 & 8 \\ \hline 3 & 4 & 7 \\ \hline 2 \\ \cline{1-1}\end{array} \;=\; \begin{array}{|c|c|c|}\cline{1-2} 6 & \mathbf{7} \\ \hline 3 & 4 & \mathbf{5} \\ \hline 2 & \mathbf{8} \\ \cline{1-2}\end{array}$$

Fig. 3.1: The insertion of 5 into a Young composition tableau of shape $(1,3,2)$.

Proposition 2.10 *[BBS$^+$14] Let α, β be compositions. Then*

$$\mathfrak{S}^*_\alpha = \sum_\beta d_{\alpha,\beta} F_\beta,$$

where $d_{\alpha,\beta}$ equals the number of standard immaculate tableaux T of shape α with $Des_{\mathfrak{S}^*}(T) = set(\beta)$.

As an example, consider the decomposition of $\mathfrak{S}^*_{(1,2,1)}$ into the fundamental basis. Figure 2.3 shows that there are two standard immaculate tableaux of shape $(1,2,1)$. Their descent sets are $\{1,3\}$ and $\{1,2\}$. Thus, $\mathfrak{S}^*_{(1,2,1)} = F_{(1,2,1)} + F_{(1,1,2)}$.

3 An insertion and recording algorithm

We recall the insertion procedure $k \to C$ given in [HLMvW11a] which maps a positive integer k into a composition tableau C. We describe an analogous procedure that maps a positive integer k into a Young composition tableau T. Our procedure is obtained from the procedure for composition tableaux in the same way that Young composition tableaux are obtained from composition tableaux. Therefore the fact that the procedure $k \to C$ produces a composition tableau immediately implies that our procedure produces a Young composition tableau.

Recall that $\bar{T}$ is the augmentation of T. (See Definition 2.6.) Let $(c_1,d_1),(c_2,d_2),\ldots$ be the cells of this extended diagram listed in reading order. Formally, we define the insertion procedure of k into the sequence of cells $(c_1,d_1),(c_2,d_2),\ldots$.

Procedure 3.1 *Set $k_0 := k$ and let i be the smallest positive integer such that $\bar{T}(c_i - 1, d_i) < k_0 < \bar{T}(c_i, d_i)$. If such an i exists, there are two cases.*

Case 1. *If $\bar{T}(c_i,d_i) = \infty$, then place k_0 in cell (c_i,d_i) and terminate the procedure.*

Case 2. *If $\bar{T}(c_i,d_i) \neq \infty$, then set $k := \bar{T}(c_i,d_i)$, place k_0 in cell (c_i,d_i), and repeat the procedure by inserting k into the sequence of cells $(c_{i+1},d_{i+1}),(c_{i+2},d_{i+2}),\ldots$. In such a situation, we say that $\bar{T}(c_i,d_i)$ is* bumped.

If no such i exists, append k_0 to the bottom of the leftmost column and terminate the procedure.

The sequence of cells that contain elements which are bumped in the insertion $k \to T$ plus the final cell which is added when the procedure is terminated is called the *bumping path* of the insertion. Note that when an entry reaches the leftmost column during the insertion procedure described in [HLMvW11a] it might not always be placed at the bottom of the column. However, the situations in which an entry is placed in a higher row of the leftmost column do not occur in this paper, so we provide this more simplified statement of the insertion for our purposes.

Figure 3.1 shows the insertion of 5 into a Young composition tableau of shape $(1,3,2)$. The first element bumped is the 7 in column 3. This 7 is replaced by 5 and 7 is then inserted into the remaining sequence of cells. The 7 then bumps the 8 in column 2 and 8 is inserted into the remaining cells. The 8 is placed to the right of the 2 and the procedure terminates. The bumping path is therefore the sequence of cells $\{(3,2),(2,3),(2,1)\}$. Notice that the entries of $\bar{T}$ in the bumping path must strictly increase as we proceed in reading order.

Let D be a standard immaculate tableau of shape α and let $rw_{\mathfrak{S}^*}(D)$ be the reading word of D. We define a procedure f that maps $rw_{\mathfrak{S}^*}(D)$ to a pair (P,Q) consisting of a standard Young composition tableau P (the "insertion Young composition tableau") and a recording filling Q.

Begin with $(P,Q) = (\emptyset,\emptyset)$, where $\emptyset$ is the empty filling. Let k_1 be the first letter in the word $rw_{\mathfrak{S}^*}(D)$. Insert k_1 into P using the insertion procedure described above and let P_1 be the resulting Young composition tableau. Record the location in P where the new cell was created by placing a "1" in Q in the corresponding location and let Q_1 be the resulting filling. Next assume the first $j-1$ letters of $rw_{\mathfrak{S}^*}(D)$ have been inserted. Let k_j be the j^{th} letter in $rw_{\mathfrak{S}^*}(D)$. Insert k_j into P_{j-1} and let P_j be the resulting diagram. Place the number j in the cell of Q_{j-1} corresponding to the new cell in P_j created from this insertion and let Q_j be the resulting filling.

Notice that P is a standard Young composition tableau since the insertion procedure produces a Young composition tableau. Note that the recording filling Q has the same shape as P but is not a Young composition tableau. We now describe the properties of Q. We begin with a definition.

Definition 3.2 *Let Q be a filling of a diagram for $\beta \vDash n$ with the integers $\{1,\dots,n\}$. A* row strip *of Q is a maximal sequence of consecutive integers, none of which are in the same column of Q. The* row strip shape *of Q is the composition $(\alpha_1,\alpha_2,\dots,\alpha_l)$ where α_i is the length of the row strip sequence which starts with the number $\alpha_1+\alpha_2+\cdots+\alpha_{i-1}+1$.*

For an example, consider the filling

$$Q = \begin{array}{|c|c|cc}\hline 1 & 6 & \multicolumn{2}{c}{} \\ \hline 2 & 3 & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{7} \\ \hline 5 & \multicolumn{3}{c}{} \\ \cline{1-1}\end{array}.$$

The first row strip is 1, the next row strip is $2,3,4$, and finally we have $5,6,7$. It follows that the row strip shape is $(1,3,3)$. We now describe the row strips of the recording tableaux obtained from inserting reading words of standard immaculate tableaux.

Proposition 3.3 *Let Q be any recording tableau obtained from inserting the reading word of a standard immaculate tableau of shape α. Then the row strips start in the leftmost column and the row strip shape is α^{rev}.*

To verify this proposition, suppose that $x < y$ and x is inserted into P immediately before y. If the bumping paths are $(b_1,c_1),(b_2,c_2),\dots,(b_k,c_k)$ and $(d_1,e_1),(d_2,e_2),\dots,(d_m,e_m)$, respectively, then (b_k,c_k) must be strictly left of (d_m,e_m). It follows that as we insert a row of a standard immaculate tableau, the corresponding elements of the recording filling form a row strip. In addition, when we start inserting a new row of the standard immaculate tableau, the element we are inserting is the smallest element that has been inserted so far and thus it must placed in a new row at the bottom of the leftmost column. The row strip shape is α^{rev} rather than α since insertion of the reading word of a standard immaculate tableau begins at the top and hence we insert a row of length α_l, then α_{l-1} and so on.

The recording filling we obtain from insertion of a standard immaculate tableau also satisfies a triple rule. We describe it next.

Definition 3.4 *Let Q be a filling of a diagram D_α for $\alpha \vDash n$ with the integers $\{1, \dots, n\}$. We say that Q satisfies the* recording triple rule *if whenever $a > b$ then $a > c$ in the subarray pictured below.*

$$\begin{array}{cc} \boxed{a} & \\ \boxed{b} & \boxed{c} \end{array}$$

The next proposition justifies the name recording triple rule.

Proposition 3.5 *Let Q be a recording filling obtained by inserting a word from a standard immaculate tableau. Then Q satisfies the recording triple rule.*

The fact that recording fillings satisfy the recording triple rule is a consequence of the Young composition triple rule. Essentially if the recording triple rule is violated at any time during insertion, then the Young composition triple rule was violated earlier during insertion.

Definition 3.6 *Let β be a composition of n. A filling of a diagram of shape β with exactly $\{1, \dots, n\}$ is a* dual immaculate recording tableau *(DIRT) if it has the following properties:*

1. *The rows increase from left to right.*
2. *The row strips start in the first column.*
3. *The leftmost column increases from top to bottom.*
4. *The recording triple rule is satisfied.*

Combining Proposition 3.3 and 3.5, we get the next corollary.

Corollary 3.7 *If T is a standard immaculate tableau and Q is the recording filling obtained by insertion of T, then Q is a DIRT.*

We now explain how the descent sets behave under insertion. This will be critical in the proof of Theorem 1.1 because it implies that the fundamental quasisymmetric functions associated to the standard immaculate tableaux and the fundamental quasisymmetric functions associated to the standard Young composition tableaux obtained from insertion are the same.

Proposition 3.8 *Let T be a standard immaculate tableau and let P be the Young composition tableau obtained from insertion of the reading word of T. Then $Des_{\mathfrak{S}^*}(T) = Des_{\hat{\mathscr{P}}}(P)$.*

To see this, observe that if $i \in Des_{\mathfrak{S}^*}(T)$ then i is initially inserted into a column weakly to the right of $i+1$. If i is inserted into the same column as $i+1$ and is below $i+1$, then it is in the leftmost column. If i is bumped during the insertion process, then it remains weakly right of $i+1$. Therefore $i \in Des_{\hat{\mathscr{P}}}(P)$. Similar observations imply that if $i \notin Des_{\mathfrak{S}^*}(T)$ then $i \notin Des_{\hat{\mathscr{P}}}(P)$.

We now define an algorithm which turns out to be the inverse of the insertion algorithm which we call *uninsertion*.

Procedure 3.9 *Given a pair (P, Q) with P a Young composition tableau and Q a DIRT both of shape β, augment P by appending ∞ to the right of each row. Suppose that the largest entry in Q occurs at position (i, j). Set $x = P(i, j)$. Then x will start the uninsertion algorithm. Proceeding in reverse reading order, if we get to a position which has b in it and c immediately to the right such that $x > b$ and $x < c$, then replace b with x and continue. We call this unbumping. If we get to a position which contains infinity and the element immediately to the left is less than x, then place x in this position.*

Standard Immaculate Tableau	SYCT	DIRT	Fundamental
$\begin{array}{cc} 3 & 4 \\ 1 & 2 \end{array}$	$\begin{array}{cc} 3 & 4 \\ 1 & 2 \end{array}$	$\begin{array}{cc} 1 & 2 \\ 3 & 4 \end{array}$	$F_{(2,2)}$
$\begin{array}{cc} 2 & 4 \\ 1 & 3 \end{array}$	$\begin{array}{cc} 2 & 3 \\ 1 & 4 \end{array}$	$\begin{array}{cc} 1 & 2 \\ 3 & 4 \end{array}$	$F_{(1,2,1)}$
$\begin{array}{cc} 2 & 3 \\ 1 & 4 \end{array}$	$\begin{array}{ccc} 2 & 3 & 4 \\ 1 & & \end{array}$	$\begin{array}{ccc} 1 & 2 & 4 \\ 3 & & \end{array}$	$F_{(1,3)}$

Fig. 4.1: The three standard immaculate tableau of shape $(2,2)$ with the standard Young composition tableaux, and dual immaculate recording tableaux obtained from insertion and the fundamental quasisymmetric function associated with both the standard immaculate tableau and standard Young composition tableau.

Uninsertion will always produce a word provided the recording filling Q satisfies the recording triple rule. Therefore uninsertion is a well-defined procedure. Moreover, we have the following lemma.

Lemma 3.10 *Suppose P is a Young composition tableau and Q is a DIRT of the same shape as P with row strip shape α whose row strips start in the leftmost column. If we uninsert (P,Q) we get the reading word of an immaculate tableau of shape α^{rev}. Moreover, uninsertion is the inverse of insertion.*

4 Proof of Main Theorem

Before we prove Theorem 1.1, we compute a small example that shows how the insertion algorithm gives the corresponding decomposition. We decompose $\mathfrak{S}^*_{(2,2)}$. Figure 4.1 contains the three standard immaculate tableaux of shape $(2,2)$, the three standard Young composition tableaux obtained from insertion, their respective dual immaculate recording tableaux, and the associated fundamental quasisymmetric functions.

The figure shows that $\mathfrak{S}^*_{(2,2)} = F_{(2,2)} + F_{(1,2,1)} + F_{(1,3)}$. Moreover, every standard Young composition tableau of shape $(2,2)$ and $(1,3)$ appears exactly once. Since the the insertion map preserves the descent sets, this implies that $\hat{\mathscr{S}}_{(2,2)} + \hat{\mathscr{S}}_{(1,3)} = F_{(2,2)} + F_{(1,2,1)} + F_{(1,3)}$. Therefore, $\mathfrak{S}^*_{(2,2)} = \hat{\mathscr{S}}_{(2,2)} + \hat{\mathscr{S}}_{(1,3)}$. Note also that each recording tableau we obtain is a DIRT with row strip shape $(2,2)^{rev}$. Moreover, each DIRT of shape β with row strip shape $(2,2)^{rev}$ appears with each standard Young composition tableau of shape β exactly once. Thus we just need to count the number of DIRTs of shape β to get the coefficient of $\hat{\mathscr{S}}_\beta$ in $\mathfrak{S}^*_{(2,2)}$. There is one DIRT of shape $(2,2)$ and one of shape $(1,3)$, both with row strip shape $(2,2)^{rev}$. Therefore, we see again that $\mathfrak{S}^*_{(2,2)} = \hat{\mathscr{S}}_{(2,2)} + \hat{\mathscr{S}}_{(1,3)}$.

We are now ready to prove Theorem 1.1. Recall that Theorem 1.1 states that

$$\mathfrak{S}^*_\alpha = \sum_\beta c_{\alpha,\beta} \hat{\mathscr{S}}_\beta, \tag{4.1}$$

where $c_{\alpha,\beta}$ is the number of DIRTs of shape β with row strip shape α^{rev}.

Proof of Theorem 1.1: For $\alpha \vDash n$, let $I(\alpha)$ be the set of standard immaculate tableaux of shape α. Additionally, let $Y(\alpha)$ be the set of pairs (P,Q) such that P is a SYCT, Q is a DIRT with row strip shape α^{rev}, and P and Q have the same shape.

$$\begin{array}{|c|c|c|c|}\hline 4 & 6 & 9 \\ \hline 2 & 8 \\ \hline 1 & 3 & 5 & 7 \\ \hline \end{array} \longrightarrow \; 4\,6\,9\,2\,8\,1\,3\,5\,7 \; \longrightarrow \left(\begin{array}{|c|c|c|c|}\hline 4 & 6 & 8 \\ \hline 2 & 3 & 5 & 7 \\ \hline 1 & 9 \\ \hline \end{array} \, , \, \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 8 & 9 \\ \hline 6 & 7 \\ \hline \end{array} \right)$$

Fig. 4.2: The map from a standard immaculate tableau to a SYCT and the corresponding DIRT.

We claim that there is a bijection, φ, from $I(\alpha)$ to $Y(\alpha)$ such that if $\varphi(T) = (P, Q)$ then $Des_{\mathfrak{S}^*}(T) = Des_{\hat{\mathscr{S}}}(P)$. Assume for now that such a bijection φ exists. It follows that

$$\sum_{T \in I(\alpha)} F_{Des_{\mathfrak{S}^*}(T)} = \sum_{(P,Q) \in Y(\alpha)} F_{Des_{\hat{\mathscr{S}}}(P)}. \tag{4.2}$$

By Proposition 2.5, we know that the right hand side of equation (4.2) is the right hand side of equation (4.1). Moreover, by Proposition 2.10, the left hand side of equation (4.2) is the left hand side of equation (4.1). It follows that if such a bijection exists, equation (4.1) holds.

To see that our desired bijection φ exists, begin with an arbitrary composition α and let A be a standard immaculate tableau of shape α. Recall the reading word of A is given by reading the rows of A from left to right, beginning at the top row and working from top to bottom. See Figure 4.2 for an example. Note that the rows of A appear as the longest consecutive increasing subsequences in this reading word $rw_{\mathfrak{S}^*}(A)$, since the leftmost column entries are strictly decreasing from top to bottom. Let φ be the map that sends this reading word to a pair (P, Q) consisting of a standard Young composition tableau P and a dual immaculate recording tableau Q using the insertion algorithm described in Procedure 3.1. See Figure 4.2 for an example. We know P is a standard Young composition tableau because P was obtained using the insertion procedure, and Corollary 3.7 implies that Q is a dual immaculate recording tableau.

To see that φ is a bijection, note that its inverse is given by repeated iteration of Procedure 3.9. Here, we record the resulting output from the uninsertion procedure each time to form a word. Lemma 3.10 implies that the resulting word is in fact the reading word of the unique standard immaculate tableau of shape α which mapped to P under insertion. Therefore the map is a bijection, as desired. $\square$

Since the coefficient of $\hat{\mathscr{S}}_\beta$ is the number of DIRTs of shape β with row strip shape α^{rev}, we can decompose $\mathfrak{S}^*_\alpha$ without actually implementing the insertion algorithm. Instead, we only need to find the the number of DIRTs of the correct shape and row strip shape. We now explain how to find the DIRTs using an algorithm similar to the Remmel-Whitney method [RW84] used to multiply Schur functions. The algorithm is recursive and produces a rooted tree where each leaf is a DIRT.

Suppose that we want to decompose $\mathfrak{S}^*_\alpha$ with $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_l)$ into Young quasisymmetric Schur functions. First, we set the root node to be the dual immaculate recording tableau:

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & \cdots & \alpha_l \\ \hline \end{array}.$$

Now we describe how to create *the children* of a node. Given a DIRT T of shape γ with $k < l$ rows, we create a child of T by placing the numbers $\alpha_l + \alpha_{l-1} + \cdots + \alpha_{l-k} + 1, \alpha_l + \alpha_{l-1} + \cdots + \alpha_{l-k} + 2, \dots, \alpha_l + \alpha_{l-1} + \cdots + \alpha_{l-k} + \alpha_{l-k-1}$ one at a time into T using the following rules:

(1) The element $\alpha_l + \alpha_{l-1} + \cdots + \alpha_{l-k} + 1$ is placed in the leftmost column of the DIRT T below its last row.

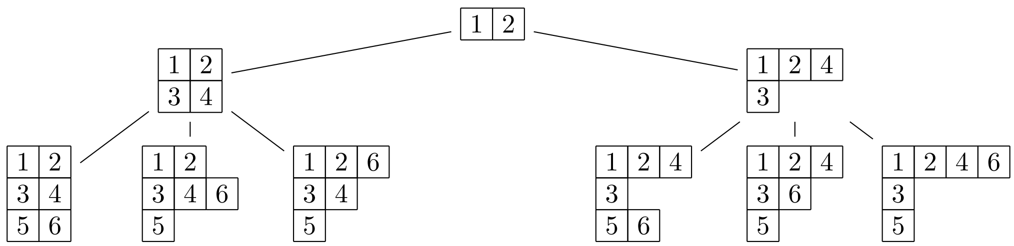


Fig. 4.3: The dual immaculate recording tableaux for the decomposition of $\mathfrak{S}^*_{(2,2,2)}$.

(2) Place each subsequent element at the end of a row strictly to the right of the last element placed.

(3) No element can be placed at the end of a row of length m if there exists a row of length $m+1$ below this row.

This algorithm continues until all the terminal nodes are dual immaculate recording tableaux with l rows.

It is clear that this algorithm forces the rows to increase. Moreover, (1) and (2) force the row strip condition and (3) forces the recording triple rule. Thus, the nodes are DIRTs with row strip shape α^{rev}.

As an example of this algorithm, suppose that we want to decompose $\mathfrak{S}^*_{(2,2,2)}$. The rooted tree in Figure 4.3 shows the output of the algorithm. It follows that

$$\mathfrak{S}^*_{(2,2,2)} = \hat{\mathscr{S}}_{(2,2,2)} + \hat{\mathscr{S}}_{(2,1,3)} + \hat{\mathscr{S}}_{(1,3,2)} + 2\hat{\mathscr{S}}_{(1,2,3)} + \hat{\mathscr{S}}_{(1,1,4)}.$$

From this example, one can see that this algorithm is advantageous in that it does not require knowing what the standard immaculate or standard Young composition tableaux are. However, it is disadvantageous in that it is recursive and so one must find all the smaller DIRTs in order to complete the algorithm.

5 Future Directions

A natural next step is to investigate the coefficients when the Young quasisymmetric Schur functions are expanded into dual immaculate quasisymmetric functions. If $\hat{\mathscr{S}}_\beta$ appears in the decomposition of $\mathfrak{S}^*_\alpha$, then Theorem 1.1 implies there is a DIRT of shape β and row strip shape α^{rev}. It is not hard to see that if we have such a DIRT, β must be lexicographically less than or equal to α. Moreover, there is exactly one DIRT of shape α and row strip shape α^{rev}. Theorem 1.1 therefore implies that

$$\hat{\mathscr{S}}_\alpha = \mathfrak{S}^*_\alpha - \sum_\beta c_{\alpha,\beta}\hat{\mathscr{S}}_\beta, \tag{5.1}$$

where is the sum is now over β which are strictly lexicographically smaller than α. Since $\mathfrak{S}^*_{1^k,n-k} = \hat{\mathscr{S}}_{1^k,n-k}$, equation (5.1) along with induction implies that the Young quasisymmetric functions can be decomposed into the dual immaculate quasisymmetric functions with integer coefficients. Of course the coefficients are not always positive, but they might have a nice combinatorial interpretation. For example, they might be determined using an analogue of the "rim-hook" enumeration used to find the coefficients appearing in the inverse Kostka matrix [ER90].

In [BBS+14], the authors give a formula for the number of standard immaculate tableaux for a fixed shape. Using this formula and our bijection we hope to find a formula for the number of standard Young composition tableaux in terms of the number of standard immaculate tableaux. It would also be interesting to investigate the relationship between these functions and other new bases for quasisymmetric functions. For example the shin basis [CFL+14] is a new basis for $NSym$ whose dual is also in $QSym$.

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