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BERRY-ESSEEN BOUND AND PRECISE MODERATE DEVIATIONS FOR PRODUCTS OF RANDOM MATRICES

HUI XIAO, ION GRAMA, AND QUANSHENG LIU

ABSTRACT. Let $(g_n)_{n \geq 1}$ be a sequence of independent and identically distributed (i.i.d.) $d \times d$ real random matrices. Set $G_n = g_n g_{n-1} \dots g_1$ and $X_n^x = G_n x / |G_n x|$, $n \geq 1$, where $|\cdot|$ is an arbitrary norm in \mathbb{R}^d and $x \in \mathbb{R}^d$ is a starting point with $|x| = 1$. For both invertible matrices and positive matrices, under suitable conditions we prove a Berry-Esseen type theorem and an Edgeworth expansion for the couple $(X_n^x, \log |G_n x|)$. These results are established using a brand new smoothing inequality on complex plane, the saddle point method and additional spectral gap properties of the transfer operator related to the Markov chain X_n^x . Cramér type moderate deviation expansions are derived for the couple $(X_n^x, \log |G_n x|)$ with a target function φ on the Markov chain X_n^x . A local limit theorem with moderate deviations is also obtained.

1. INTRODUCTION

1.1. Background and objectives. For any integer $d \geq 2$, denote by $GL(d, \mathbb{R})$ the general linear group of $d \times d$ invertible matrices. Equip \mathbb{R}^d with any norm $|\cdot|$, denote by $\mathbb{P}^{d-1} = \{x \in \mathbb{R}^d, |x| = 1\} / \pm$ the projective space in \mathbb{R}^d , and let $\|g\| = \sup_{x \in \mathbb{P}^{d-1}} |gx|$ be the operator norm for $g \in GL(d, \mathbb{R})$. Let $(g_n)_{n \geq 1}$ be a sequence of i.i.d. $d \times d$ real random matrices of the same law μ on $GL(d, \mathbb{R})$, and consider the product $G_n = g_n g_{n-1} \dots g_1$ and the process $X_n^x = G_n x / |G_n x|$, $n \geq 1$, with starting point $x \in \mathbb{P}^{d-1}$.

The study of the asymptotic properties of the Markov chain $(X_n^x)_{n \geq 1}$ and of the product $(G_n)_{n \geq 1}$ has attracted a good deal of attention since the groundwork of Furstenberg and Kesten [14], where the strong law of large numbers for $\log \|G_n\|$ has been established, which is a fundamental result for the products of random matrices. Furstenberg [15] proved the following version of the law of large numbers: for any $x \in \mathbb{R}^d$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log |G_n x| = \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E} \log |G_n x| = \lambda \quad \mathbb{P}\text{-a.s.},$$

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where the real number λ is called upper Lyapunov exponent associated with the product G_n . Another cornerstone result is the central limit theorem (CLT) for the couple $(X_n^x, \log |G_n x|)$, established under contracting type assumptions by Le Page [29]: for any fixed $y \in \mathbb{R}$ and any Hölder continuous function $\varphi : \mathbb{P}^{d-1} \mapsto \mathbb{R}$, it holds uniformly in $x \in \mathbb{P}^{d-1}$ that

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log |G_n x| - n\lambda}{\sigma \sqrt{n}} \leq y \right\}} \right] = \nu(\varphi) \Phi(y),$$

where ν is the unique stationary probability measure of the Markov chain X_n^x on \mathbb{P}^{d-1} , $\sigma^2 = \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}[(\log |G_n x| - n\lambda)^2]$ is the asymptotic variance independent of x , and Φ is the standard normal distribution function. The optimal conditions for the CLT to hold true have been established recently by Benoist and Quint [2].

A very important topic is the study of large and moderate deviation probabilities, which describe the rate of convergence in the law of large numbers. For an account to the theory of large deviations for sums of independent random variables we refer to Cramér [8], Petrov [30], Strook [34], Varadhan [35] and Dembo and Zeitouni [11]. For products of random matrices, precise large deviations asymptotics have been considered e.g. by Le Page [29], Buraczewski and Mentemeier [6], Guivarc'h [16], Benoist and Quint [3], Sert [33], Xiao, Grama and Liu [37]. For moderate deviations, very little results are known. Benoist and Quint [3] have recently established the asymptotic for the logarithm of probabilities of moderate deviations for reductive groups, which in our setting reads as follows: for any interval $B \subseteq \mathbb{R}$, and positive sequence $(b_n)_{n \geq 1}$ satisfying $\frac{b_n}{n} \rightarrow 0$ and $\frac{b_n}{\sqrt{n}} \rightarrow \infty$, it holds, uniformly in $x \in \mathbb{P}^{d-1}$, that

$$\lim_{n \rightarrow \infty} \frac{n}{b_n^2} \log \mathbb{P} \left(\frac{\log |G_n x| - n\lambda}{b_n} \in B \right) = - \inf_{y \in B} \frac{y^2}{2\sigma^2}. \quad (1.1)$$

A functional moderate deviation principle has been established by Cuny, Dedecker and Jan [7].

The first objective of our paper is to improve on the result (1.1) by establishing a Cramér type moderate deviation expansion for $\log |G_n x|$: we prove that uniformly in $x \in \mathbb{P}^{d-1}$ and $y \in [0, o(\sqrt{n})]$,

$$\frac{\mathbb{P}(\log |G_n x| - n\lambda \geq \sqrt{n}\sigma y)}{1 - \Phi(y)} = e^{\frac{y^3}{\sqrt{n}} \zeta(\frac{y}{\sqrt{n}})} \left[1 + O\left(\frac{y+1}{\sqrt{n}}\right) \right], \quad (1.2)$$

where $t \mapsto \zeta(t)$ is the Cramér series of the logarithm of the eigenfunction related to the transfer operator of the Markov walk associated to the product of random matrices (see Section 2.3). It will be seen that the expansion (1.2) implies the following local limit theorem with moderate deviations: for any bounded Borel set $B \subset \mathbb{R}$ with boundary ∂B satisfying $\ell(\partial B) = 0$, where

ℓ is the Lebesgue measure, and for any positive sequence $(y_n)_{n \geq 1}$ satisfying $y_n \rightarrow 0$ and $\sqrt{ny_n} \rightarrow \infty$, we have, uniformly in $x \in \mathbb{P}^{d-1}$ and $y \in [y_n, o(\sqrt{n})]$,

$$\mathbb{P}(\log |G_n x| - n\lambda \in B + \sqrt{n}\sigma y) \sim \frac{\ell(B)}{\sigma\sqrt{2\pi n}} e^{-\frac{y^2}{2} + \frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})}. \quad (1.3)$$

It is clear that when $y = o(n^{1/6})$ the term $\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})$ tends to 0 and can be removed in (1.3). Thus (1.3) enlarges the range $y = O(\sqrt{\log n})$ in the local limit theorem in [3, Theorem 17.10] established under different assumptions. Local limit theorems of type (1.3) are used for instance in [1] for studying dynamics of group actions on finite volume homogeneous spaces.

It is useful to extend the moderate deviation expansion (1.2) for the couple $(X_n^x, \log |G_n x|)$ which describes completely the random walk $(G_n x)_{n \geq 1}$. We prove that, for any Hölder continuous function φ on \mathbb{P}^{d-1} , uniformly in $x \in \mathbb{P}^{d-1}$ and $y \in [0, o(\sqrt{n})]$,

$$\frac{\mathbb{E}[\varphi(X_n^x) \mathbb{1}_{\{\log |G_n x| - n\lambda \geq \sqrt{n}\sigma y\}}]}{1 - \Phi(y)} = e^{\frac{y^3}{\sqrt{n}}\zeta(\frac{y}{\sqrt{n}})} \left[\nu(\varphi) + O\left(\frac{y+1}{\sqrt{n}}\right) \right], \quad (1.4)$$

see Theorem 2.3. Our second objective, which is also the key point in proving (1.4), is a Berry-Esseen bound for the couple $(X_n^x, \log |G_n x|)$: for any Hölder continuous function φ on \mathbb{P}^{d-1} ,

$$\sup_{x \in \mathbb{P}^{d-1}, y \in \mathbb{R}} \left| \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log |G_n x| - n\lambda}{\sigma\sqrt{n}} \leq y \right\}} \right] - \nu(\varphi)\Phi(y) \right| = O\left(\frac{1}{\sqrt{n}}\right), \quad (1.5)$$

see Theorem 2.1. This extends the result of Le Page [29] established for the particular target function $\varphi = \mathbf{1}$ (see also Jan [26]). We further upgrade (1.5) to an Edgeworth expansion under a non-arithmeticity condition, see Theorem 2.2, which is new even for $\varphi = \mathbf{1}$.

All the results stated above concern invertible matrices, but we also establish analogous theorems for positive matrices. Some limit theorems for $\log |G_n x|$ in case of positive matrices such as central limit theorem and Berry-Esseen theorem have been established earlier by Furstenberg and Kesten [14], Hennion [19], and Hennion and Hervé [21]. Here, we extend the Berry-Esseen theorem of [21] to the couple $(X_n^x, \log |G_n x|)$ with a target function φ on the Markov chain X_n^x . We also complement the results in [14, 19, 21] by giving a Cramér type moderate deviation expansion and a local limit theorem with moderate deviations.

1.2. Key ideas of the approach. For the moderate deviation expansions (1.2) and (1.4), our proof is different from those in [3] and [7]: in [3] the moderate deviation principle (1.1) is obtained by following the strategy of Kolmogorov [28] suited to show the law of iterated logarithm (see also de

Acosta [10] and Wittman [36]); in [7] the proof of the functional moderate deviation principle is based on the martingale approximation method developed in [2].

In order to prove (1.4) we have to rework the spectral gap theory for the transfer operators P_z and $R_{s,z}$, by considering the case when s can take values in the interval $(-\eta, \eta)$ with $\eta > 0$ small, and z belongs to a small complex ball centered at the origin, see Section 3. This allows to define the change of measure \mathbb{Q}_s^x and to extend the Berry-Esseen bound (1.5) for the changed measure \mathbb{Q}_s^x , see Theorem 5.1. The moderate deviation expansion (1.4) is established by adapting the techniques from Petrov [30].

It is surprising that the proof of the Berry-Esseen bound and of the Edgeworth expansion with a non-trivial target function $\varphi \neq \mathbf{1}$ is way more difficult than the analogous results with $\varphi = \mathbf{1}$. This can be seen from the following sketch of the proof.

For simplicity, we assume that $\sigma = 1$. Introduce the transfer operator P_z : for any Hölder continuous function φ on \mathbb{P}^{d-1} and $z \in \mathbb{C}$,

$$P_z \varphi(x) = \mathbb{E}[e^{z \log |g_1 x|} \varphi(X_1^x)], \quad x \in \mathbb{P}^{d-1}. \quad (1.6)$$

Let F be the distribution function of $\frac{\log |G_n x| - n\lambda}{\sqrt{n}}$ and f be its Fourier transform: $f(t) = e^{it\sqrt{n}\lambda} (P_{-it/\sqrt{n}}^n \mathbf{1})(x)$, $t \in \mathbb{R}$. The Berry-Esseen bound (1.5) with target function $\varphi = \mathbf{1}$ is usually proved using Esseen's smoothing inequality: for all $T > 0$,

$$\sup_{y \in \mathbb{R}} |F(y) - \Phi(y)| \leq \frac{1}{\pi} \int_{-T}^T \left| \frac{f(t) - e^{-t^2/2}}{t} \right| dt + \frac{C}{T}. \quad (1.7)$$

Inserting the spectral gap decomposition

$$P_z^n = \kappa^n(z) M_z + L_z^n \quad (n \geq 1) \quad (1.8)$$

into (1.7) allows us to obtain the Berry-Esseen bound (1.5) with $\varphi = \mathbf{1}$: after some straightforward calculations, it reduces to showing that, with $T = c\sqrt{n}$,

$$\int_{-T}^T |(L_{-it/\sqrt{n}}^n \mathbf{1})(x)| / |t| dt < \infty. \quad (1.9)$$

The bound (1.9) is proved using Taylor's expansion $L_z^n \mathbf{1} = L_0^n \mathbf{1} + z \frac{d}{dz} (L_z^n \mathbf{1}) + o(z)$ with $z = -it/\sqrt{n}$, and the fact that $L_0^n \mathbf{1} = 0$. However, when we replace the unit function $\mathbf{1}$ by a target function φ for which in general $L_0^n \varphi \neq 0$, instead of (1.9), we have

$$\int_{-T}^T |L_{-it/\sqrt{n}}^n \varphi(x)| / |t| dt = \infty, \quad (1.10)$$

even though $|L_0^n \varphi(x)|$ decays exponentially fast to 0 as $n \rightarrow \infty$. To overcome it, we have elaborated a new approach based on smoothing inequality on complex contours and on the saddle point method, see Daniels [9] and Fedoryuk [13].

For simplicity, we formulate our smoothing inequality only for $y \leq 0$:

$$\begin{aligned} \sup_{y \leq 0} |F(y) - \Phi(y)| &\leq \frac{1}{\pi} \sup_{y \leq 0} \left| \int_{\mathcal{C}_T^-} \frac{f(z) - e^{-z^2/2}}{z} e^{izy} \widehat{\rho}_T(-z) dz \right| \\ &+ \frac{1}{\pi} \sup_{y \leq 0} \left| \int_{\mathcal{C}_T^-} \frac{f(z) - e^{-z^2/2}}{z} e^{izy} \widehat{\rho}_T(z) dz \right| + \frac{C}{T}, \end{aligned} \quad (1.11)$$

where the integration is taken over the complex contour $\mathcal{C}_T^- = \{z \in \mathbb{C} : |z| = T, \Im z < 0\}$ and $\widehat{\rho}_T$ is the analytic extension of the Fourier transform of a smoothing density ρ_T on the real line (see Section 4). An important issue is to construct the density function ρ_T such that $\widehat{\rho}_T$ has a compact support on the real line \mathbb{R} and can be extended analytically on a domain containing \mathcal{C}_T^- . This enables us to use Cauchy's integral theorem for establishing (1.11) and also for the estimation of the integrals therein.

The smoothing inequality (1.11) together with the spectral gap property (1.8) leads to the estimation of the following integrals:

$$\int_{\mathcal{C}_T^-} \frac{\kappa^n(z) M_z \varphi(x) - e^{-z^2/2}}{z} e^{izy} \widehat{\rho}_T(-z) dz, \quad (1.12)$$

$$\int_{\mathcal{C}_T^-} \frac{L_z^n \varphi(x)}{z} e^{izy} \widehat{\rho}_T(-z) dz. \quad (1.13)$$

The integral (1.12) is handled by using the saddle point method choosing a suitable path for the integration in Section 5.2, which is one of the challenging parts of the proof. For the integral (1.13) we use the facts that $|L_z^n \varphi(x)|$ decays exponentially fast as $n \rightarrow \infty$ and that $|\frac{e^{izy}}{z}| \leq \frac{1}{T}$ on the contour \mathcal{C}_T^- for $y \leq 0$, where $T = c\sqrt{n}$. In contrast to (1.10), this shows that (1.13) is bounded by Ce^{-cn} uniformly in y .

The case $y > 0$ is treated similarly, which allows us to establish (1.5). Note that the non-arithmeticity condition is not needed for the validity of (1.5). Under the non-arithmeticity condition, in Theorem 2.2 we obtain an Edgeworth expansion for $(X_n^x, |G_n x|)$ with the target function φ on X_n^x , which is of independent interest.

2. MAIN RESULTS

2.1. Notation and conditions. Let $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. The real part, imaginary part and the conjugate of a complex number z are denoted by $\Re z$, $\Im z$ and \bar{z} respectively. For $y \in \mathbb{R}$, we write $\phi(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$

and $\Phi(y) = \int_{-\infty}^y \phi(t) dt$. For any $\eta > 0$, set $B_\eta(0) = \{z \in \mathbb{C} : |z| < \eta\}$ for the ball with center 0 and radius η in the complex plane \mathbb{C} . We denote by c, C , positive absolute constants whose values may change from line to line. By c_α, C_α we mean positive constants depending only on the index α . We write $\mathbb{1}_A$ for the indicator function of an event A . For a measure ν and a function φ we denote $\nu(\varphi) = \int \varphi d\nu$.

For $d \geq 2$, let $M(d, \mathbb{R}) := \mathcal{M}$ be the set of $d \times d$ matrices with entries in \mathbb{R} . We shall work with products of invertible or non-negative matrices. Denote by $\mathcal{G} = GL(d, \mathbb{R})$ the group of invertible matrices of \mathcal{M} . A non-negative matrix $g \in \mathcal{M}$ is said to be *allowable*, if every row and every column of g has a strictly positive entry. Denote by \mathcal{G}_+ the multiplicative semigroup of allowable non-negative matrices of \mathcal{M} , which will be called simply positive. We write \mathcal{G}_+° for the subsemigroup of \mathcal{G}_+ with strictly positive entries.

The space \mathbb{R}^d is equipped with any given norm $|\cdot|$. Denote by $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d, |x| = 1\}$ the unit sphere, and by $\mathbb{S}_+^{d-1} = \{x \geq 0 : |x| = 1\}$ the intersection of the unit sphere with the positive quadrant. It will be convenient to consider the projective space $\mathbb{P}^{d-1} = \mathbb{S}^{d-1}/\pm$ by identifying x with $-x$. To unify the exposition, we use the symbol \mathcal{S} to denote \mathbb{P}^{d-1} in case of invertible matrices and \mathbb{S}_+^{d-1} in case of positive matrices. For $x \in \mathcal{S}$ and $g \in \mathcal{G}$ or $g \in \mathcal{G}_+$, we write $g \cdot x = \frac{gx}{|gx|}$ for the projective action of g on \mathcal{S} . The space \mathcal{S} is endowed with the metric \mathbf{d} : for invertible matrices, \mathbf{d} is the angular distance, i.e., for any $x, y \in \mathbb{P}^{d-1}$, $\mathbf{d}(x, y) = |\sin \theta(x, y)|$, where $\theta(x, y)$ is the angle between x and y ; for positive matrices, \mathbf{d} is the Hilbert cross-ratio metric, i.e., for any $x = (x_1, \dots, x_d)$ and $y = (y_1, \dots, y_d)$ in \mathbb{S}_+^{d-1} , $\mathbf{d}(x, y) = \frac{1 - m(x, y)m(y, x)}{1 + m(x, y)m(y, x)}$, where $m(x, y) = \sup\{\lambda > 0 : \lambda y_i \leq x_i, \forall i = 1, \dots, d\}$. In both cases, there exists a constant $C > 0$ such that

$$|x - y| \leq C \mathbf{d}(x, y), \quad \text{for any } x, y \in \mathcal{S}. \quad (2.1)$$

We refer to [17] and [19] for more details.

Let $\mathcal{C}(\mathcal{S})$ be the space of continuous complex-valued functions on \mathcal{S} and $\mathbf{1}$ be the constant function with value 1. Let $\gamma > 0$. For any $\varphi \in \mathcal{C}(\mathcal{S})$, set

$$\|\varphi\|_\gamma := \|\varphi\|_\infty + [\varphi]_\gamma, \quad \|\varphi\|_\infty := \sup_{x \in \mathcal{S}} |\varphi(x)|, \quad [\varphi]_\gamma := \sup_{x, y \in \mathcal{S}} \frac{|\varphi(x) - \varphi(y)|}{\mathbf{d}^\gamma(x, y)}.$$

Introduce the Banach space $\mathcal{B}_\gamma := \{\varphi \in \mathcal{C}(\mathcal{S}) : \|\varphi\|_\gamma < +\infty\}$.

Let $(g_n)_{n \geq 1}$ be a sequence of i.i.d. random matrices with the same law μ , defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Set $G_n = g_n \dots g_1$, $n \geq 1$, then for any starting point $x \in \mathcal{S}$, the process

$$X_0^x = x, \quad X_n^x = G_n \cdot x, \quad n \geq 1$$

forms a Markov chain on \mathcal{S} . The goal of the present paper is to establish a Berry-Esseen bound and a Cramér type moderate deviation expansion for

the couple $(X_n^x, \log |G_n x|)$ with a target function φ on the Markov chain (X_n^x) , for both invertible matrices and positive matrices.

For any $g \in \mathcal{M}$, set $\|g\| = \sup_{x \in \mathcal{S}} |gx|$ and $\iota(g) = \inf_{x \in \mathcal{S}} |gx| > 0$, where $\iota(g) > 0$ for both $g \in \mathcal{G}$ and $g \in \mathcal{G}_+$. In the following we use the notation $N(g) = \max\{\|g\|, \iota(g)^{-1}\}$. From the Cartan decomposition it follows that the norm $\|g\|$ coincides with the largest singular value of g , i.e. $\|g\|$ is the square root of the largest eigenvalue of $g^T g$, where g^T denotes the transpose of g . For an invertible matrix $g \in \mathcal{G}$, $\iota(g) = \|g^{-1}\|^{-1}$, hence $\iota(g)$ is the smallest singular value of g and $N(g) = \max\{\|g\|, \|g^{-1}\|\}$. We need the two-sided exponential moment condition:

A1. *There exists a constant $\eta_0 \in (0, 1)$ such that $\mathbb{E}[N(g_1)^{\eta_0}] < +\infty$.*

We denote by $\Gamma_\mu := [\text{supp } \mu]$ the smallest closed semigroup of \mathcal{M} generated by $\text{supp } \mu$, the support of μ .

For invertible matrices, we will need the strong irreducibility and proximality conditions. Recall that a matrix g is said to be *proximal* if g has an eigenvalue λ_g satisfying $|\lambda_g| > |\lambda'_g|$ for all other eigenvalues λ'_g of g . The normalized eigenvector v_g ($|v_g| = 1$) corresponding to the eigenvalue λ_g is called the dominant eigenvector. It is easy to verify that $\lambda_g \in \mathbb{R}$.

A2. (i)(Strong irreducibility) *No finite union of proper subspaces of \mathbb{R}^d is Γ_μ -invariant.*

(ii)(Proximality) *Γ_μ contains at least one proximal matrix.*

For positive matrices, we will use the allowability and positivity conditions:

A3. (i) (Allowability) *Every $g \in \Gamma_\mu$ is allowable.*

(ii) (Positivity) *Γ_μ contains at least one matrix belonging to \mathcal{G}_+° .*

It follows from the Perron-Frobenius theorem that every $g \in \mathcal{G}_+^\circ$ has a dominant eigenvalue $\lambda_g > 0$, with the corresponding eigenvector $v_g \in \mathbb{S}_+^{d-1}$.

Under conditions **A1** and **A2** for invertible matrices, or conditions **A1** and **A3** for positive matrices, there exists a unique μ -stationary probability measure ν on \mathcal{S} ([17, 5]): for any $\varphi \in \mathcal{C}(\mathcal{S})$,

$$(\mu * \nu)(\varphi) = \int_{\mathcal{S}} \int_{\Gamma_\mu} \varphi(g_1 \cdot x) \mu(dg_1) \nu(dx) = \int_{\mathcal{S}} \varphi(x) \nu(dx) = \nu(\varphi). \quad (2.2)$$

Moreover, for invertible matrices, $\text{supp } \nu$ (the support of ν) is given by

$$V(\Gamma_\mu) = \overline{\{v_g \in \mathbb{P}^{d-1} : g \in \Gamma_\mu, g \text{ is proximal}\}}; \quad (2.3)$$

for positive matrices, $\text{supp } \nu$ is given by

$$V(\Gamma_\mu) = \overline{\{v_g \in \mathbb{S}_+^{d-1} : g \in \Gamma_\mu, g \in \mathcal{G}_+^\circ\}}. \quad (2.4)$$

In addition, for both cases, $V(\Gamma_\mu)$ is the unique minimal Γ_μ -invariant subset (see [17] and [5]).

For positive matrices, it will be shown in Proposition 3.14 that under conditions **A1** and **A3**, the asymptotic variance

$$\sigma^2 = \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}[(\log |G_n x| - n\lambda)^2]$$

exists with value in $[0, \infty)$. To establish the Berry-Esseen theorem and the moderate deviation expansion, we need the following condition:

A4. *The asymptotic variance σ^2 satisfies $\sigma^2 > 0$.*

We say that the measure μ is *arithmetic*, if there exist $t > 0$, $\beta \in [0, 2\pi)$ and a function $\vartheta : \mathcal{S} \rightarrow \mathbb{R}$ such that $\exp[it \log |gx| - i\beta + i\vartheta(g \cdot x) - i\vartheta(x)] = 1$ for any $g \in \Gamma_\mu$ and $x \in V(\Gamma_\mu)$. To establish the Edgeworth expansion for positive matrices, we impose the following condition:

A5. (Non-arithmeticity) *The measure μ is non-arithmetic.*

A simple sufficient condition introduced in [27] for the measure μ to be non-arithmetic is that the additive subgroup of \mathbb{R} generated by the set $\{\log \lambda_g : g \in \Gamma_\mu, g \in \mathcal{G}_+^\circ\}$ is dense in \mathbb{R} , see [6, Lemma 2.7].

We end this subsection by giving some implications among the above conditions. For invertible matrices, it was proved in [18] that condition **A2** implies condition **A5**. For positive matrices, conditions **A1**, **A3** and **A5** imply condition **A4**, see Proposition 3.14.

2.2. Berry-Esseen bound and Edgeworth expansion. In this subsection we formulate the Berry-Esseen theorem and the Edgeworth expansion for $(X_n^x, \log |G_n x|)$. We first state the Berry-Esseen theorem with a target function on X_n^x . Through the rest of the paper we assume that $\gamma > 0$ is a fixed small enough constant so that the spectral properties stated in Proposition 3.1 hold true.

Theorem 2.1. *Assume either conditions **A1** and **A2** for invertible matrices, or conditions **A1**, **A3** and **A4** for positive matrices. Then, there exists a constant $C > 0$ such that for all $n \geq 1$, $x \in \mathcal{S}$, $y \in \mathbb{R}$ and $\varphi \in \mathcal{B}_\gamma$,*

$$\left| \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log |G_n x| - n\lambda}{\sigma \sqrt{n}} \leq y \right\}} \right] - \nu(\varphi) \Phi(y) \right| \leq C \frac{\|\varphi\|_\gamma}{\sqrt{n}}. \quad (2.5)$$

The proof of this theorem follows the same line as the proof of the Edgeworth expansion in Theorem 2.2 formulated below, and will be sketched at the end of Section 5. The presence of the target function in Theorem 2.1 turns out to be crucial in the study of the asymptotic of moderate deviations of the scalar product $\log |\langle f, G_n x \rangle|$, which will be done in a forthcoming paper.

Theorem 2.1 extends the Berry-Esseen bounds from [29, 26] for invertible matrices, and [21] for positive matrices to versions with target functions on X_n^x . Note that the results in [26, 21] have been established under some polynomial moment conditions. However, proving (2.5) with the target function $\varphi \neq \mathbf{1}$ under the polynomial moments is still an open problem.

The following result gives an Edgeworth expansion for $\log |G_n x|$ with the target function φ on X_n^x . To formulate the result, we introduce the necessary notation. Consider the following transfer operator: for any $s \in (-\eta, \eta)$ with $\eta > 0$ small, and $\varphi \in \mathcal{C}(\mathcal{S})$,

$$P_s \varphi(x) = \mathbb{E}[e^{s \log |g_1 x|} \varphi(g_1 \cdot x)] \quad x \in \mathcal{S}.$$

It will be shown in Proposition 3.1 that there exist a measure ν_s and a Hölder continuous function r_s on \mathcal{S} such that

$$\nu_s P_s = \kappa(s) \nu_s \quad \text{and} \quad P_s r_s = \kappa(s) r_s, \quad (2.6)$$

where $\kappa(s)$ is the unique dominant eigenvalue of P_s . Set $\Lambda(s) = \log \kappa(s)$. It is shown in Lemma 3.10 that for any $\varphi \in \mathcal{B}_\gamma$, the function

$$b_\varphi(x) = \lim_{n \rightarrow \infty} \mathbb{E}[(\log |G_n x| - n\lambda)\varphi(X_n^x)], \quad x \in \mathcal{S} \quad (2.7)$$

is well defined, belongs to \mathcal{B}_γ and has an equivalent expression (3.39) in terms of derivative of the projection operator $\Pi_{0,z}$, see Proposition 3.8.

Theorem 2.2. *Assume either conditions A1 and A2 for invertible matrices, or conditions A1, A3 and A5 for positive matrices. Then, as $n \rightarrow \infty$, uniformly in $x \in \mathcal{S}$, $y \in \mathbb{R}$ and $\varphi \in \mathcal{B}_\gamma$,*

$$\begin{aligned} & \left| \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log |G_n x| - n\lambda}{\sigma\sqrt{n}} \leq y \right\}} \right] \right. \\ & \left. - \nu(\varphi) \left[\Phi(y) + \frac{\Lambda'''(0)}{6\sigma^3\sqrt{n}} (1 - y^2)\phi(y) \right] + \frac{b_\varphi(x)}{\sigma\sqrt{n}} \phi(y) \right| = \|\varphi\|_{\gamma o} \left(\frac{1}{\sqrt{n}} \right). \end{aligned} \quad (2.8)$$

The proof of this theorem is postponed to Section 5 and is based on a new smoothing inequality (Proposition 4.1) and the saddle point method. Even for $\varphi = \mathbf{1}$, Theorem 2.2 is new.

2.3. Moderate deviation expansions. Denote $\gamma_k = \Lambda^{(k)}(0)$, $k \geq 1$, where $\Lambda = \log \kappa$ with the function κ defined in (2.6). In particular, $\gamma_1 = \lambda$ and $\gamma_2 = \sigma^2$, see Propositions 3.12 and 3.14, where we give also an expression for γ_3 . Throughout the paper, we write ζ for the Cramér series of Λ (see [8] and [30]):

$$\zeta(t) = \frac{\gamma_3}{6\gamma_2^{3/2}} + \frac{\gamma_4\gamma_2 - 3\gamma_3^2}{24\gamma_2^3} t + \frac{\gamma_5\gamma_2^2 - 10\gamma_4\gamma_3\gamma_2 + 15\gamma_3^3}{120\gamma_2^{9/2}} t^2 + \dots \quad (2.9)$$

which converges for $|t|$ small enough.

We start by formulating a Cramér type moderate deviation expansion for the couple $(X_n^x, \log |G_n x|)$ with target function on X_n^x , for both invertible matrices and positive matrices.

Theorem 2.3. *Assume either conditions **A1** and **A2** for invertible matrices, or conditions **A1**, **A3** and **A4** for positive matrices. Then, uniformly in $x \in \mathcal{S}$, $y \in [0, o(\sqrt{n})]$ and $\varphi \in \mathcal{B}_\gamma$, as $n \rightarrow \infty$,*

$$\frac{\mathbb{E}[\varphi(X_n^x) \mathbb{1}_{\{\log |G_n x| - n\lambda \geq \sqrt{n}\sigma y\}}]}{1 - \Phi(y)} = e^{\frac{y^3}{\sqrt{n}} \zeta(\frac{y}{\sqrt{n}})} \left[\nu(\varphi) + \|\varphi\|_\gamma O\left(\frac{y+1}{\sqrt{n}}\right) \right],$$

$$\frac{\mathbb{E}[\varphi(X_n^x) \mathbb{1}_{\{\log |G_n x| - n\lambda \leq -\sqrt{n}\sigma y\}}]}{\Phi(-y)} = e^{-\frac{y^3}{\sqrt{n}} \zeta(-\frac{y}{\sqrt{n}})} \left[\nu(\varphi) + \|\varphi\|_\gamma O\left(\frac{y+1}{\sqrt{n}}\right) \right].$$

Note that the above asymptotic expansions remain valid even when $\nu(\varphi) = 0$. In this case, for example, the first expansion becomes

$$\mathbb{E}[\varphi(X_n^x) \mathbb{1}_{\{\log |G_n x| - n\lambda \geq \sqrt{n}\sigma y\}}] = (1 - \Phi(y)) e^{\frac{y^3}{\sqrt{n}} \zeta(\frac{y}{\sqrt{n}})} \|\varphi\|_\gamma O\left(\frac{y+1}{\sqrt{n}}\right).$$

It is an open question to extend the results of Theorem 2.3 to higher order expansions under the additional condition of non-arithmeticity. We refer to Saulis [32] and Rozovsky [31] for relevant results in the i.i.d. real-valued case. In the case of products of random matrices this problem seems to us challenging because of the presence of the derivatives in s of the eigenfunction r_s and of the eigenmeasure ν_s in the higher order terms.

In particular, under conditions of Theorem 2.3, with $\varphi = \mathbf{1}$ we obtain:

$$\frac{\mathbb{P}\left(\frac{\log |G_n x| - n\lambda}{\sigma\sqrt{n}} \geq y\right)}{1 - \Phi(y)} = e^{\frac{y^3}{\sqrt{n}} \zeta(\frac{y}{\sqrt{n}})} \left[1 + O\left(\frac{y+1}{\sqrt{n}}\right) \right],$$

$$\frac{\mathbb{P}\left(\frac{\log |G_n x| - n\lambda}{\sigma\sqrt{n}} \leq -y\right)}{\Phi(-y)} = e^{-\frac{y^3}{\sqrt{n}} \zeta(-\frac{y}{\sqrt{n}})} \left[1 + O\left(\frac{y+1}{\sqrt{n}}\right) \right].$$

When $\varphi \in \mathcal{B}_\gamma$ is a real-valued function satisfying $\nu(\varphi) > 0$, Theorem 2.3 clearly implies the following moderate deviation principle for $\log |G_n x|$ with target function on X_n^x : for any Borel set $B \subseteq \mathbb{R}$, and positive sequence $(b_n)_{n \geq 1}$ satisfying $\frac{b_n}{n} \rightarrow 0$ and $\frac{b_n}{\sqrt{n}} \rightarrow \infty$, uniformly in $x \in \mathbb{P}^{d-1}$,

$$-\inf_{y \in B^\circ} \frac{y^2}{2\sigma^2} \leq \liminf_{n \rightarrow \infty} \frac{n}{b_n^2} \log \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log |G_n x| - n\lambda}{b_n} \in B \right\}} \right]$$

$$\leq \limsup_{n \rightarrow \infty} \frac{n}{b_n^2} \log \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log |G_n x| - n\lambda}{b_n} \in B \right\}} \right] \leq -\inf_{y \in \bar{B}} \frac{y^2}{2\sigma^2}, \quad (2.10)$$

where B° and \bar{B} are respectively the interior and the closure of B . In fact it is enough to show (2.10) only for the case where B is an interval, the result for general B can be established using Lemma 4.4 of Huang and Liu [24]. With

$\varphi = \mathbf{1}$, (2.10) implies the moderate deviation principle (1.1) established in [3, Proposition 12.12] for invertible matrices. The moderate deviation principle (2.10) with target function on X_n^x is new for both invertible matrices and positive matrices; (1.1) is new for positive matrices. Note that in (2.10) the function φ is not necessarily positive.

2.4. Local limit theorem with moderate deviations. In this subsection we state a local limit theorem with moderate deviations for $\log |G_n x|$, which is a consequence of Theorem 2.3. Recall that ℓ denotes the Lebesgue measure on \mathbb{R} , and ∂B denotes the boundary of a set B on the real line.

Theorem 2.4. *Assume either conditions A1 and A2 for invertible matrices, or conditions A1, A3 and A4 for positive matrices. Let B be a bounded Borel set on \mathbb{R} such that $\ell(\partial B) = 0$. Let $(y_n)_{n \geq 1}$ be a positive sequence satisfying $y_n \rightarrow 0$ and $\sqrt{n}y_n \rightarrow \infty$. Then, as $n \rightarrow \infty$, uniformly in $x \in \mathcal{S}$, $y \in [y_n, o(\sqrt{n})]$ and $\varphi \in \mathcal{B}_\gamma$,*

$$\mathbb{E}[\varphi(X_n^x) \mathbb{1}_{\{\log |G_n x| - n\lambda \in B + \sqrt{n}\sigma y\}}] = \frac{e^{-\frac{y^2}{2} + \frac{y^3}{\sqrt{n}} \zeta(\frac{y}{\sqrt{n}})}}{\sigma \sqrt{2\pi n}} \left[\ell(B) \nu(\varphi) + \|\varphi\|_\gamma o(1) \right].$$

Taking $\varphi = \mathbf{1}$, we have, uniformly in $x \in \mathcal{S}$ and $y \in [y_n, o(\sqrt{n})]$,

$$\lim_{n \rightarrow \infty} \sigma \sqrt{2\pi n} e^{\frac{y^2}{2} - \frac{y^3}{\sqrt{n}} \zeta(\frac{y}{\sqrt{n}})} \mathbb{P}(\log |G_n x| - n\lambda \in B + \sqrt{n}\sigma y) = \ell(B).$$

In the case of invertible matrices, a similar local limit theorem has been established in [3] in a more general setting and plays an important role in studying dynamics of group actions on finite volume homogeneous spaces, see [1, Proposition 4.7]. Specifically, from [3, Theorem 17.10], by simple calculations we deduce that for any $a_1 < a_2$, it holds uniformly in $x \in \mathbb{P}^{d-1}$ and $y \in [0, O(\sqrt{\log n})]$ that, as $n \rightarrow \infty$,

$$\mathbb{P}(\log |G_n x| - n\lambda \in [a_1, a_2] + \sqrt{n}\sigma y) = \frac{e^{-\frac{y^2}{2}}}{\sigma \sqrt{2\pi n}} \left[a_2 - a_1 + o(1) \right], \quad (2.11)$$

Theorem 2.4 extends the range of y in (2.11) beyond $O(\sqrt{\log n})$ and moreover, allows a target function φ on the Markov chain X_n^x . Note also that in [3] the group $SL(d, \mathbb{R})$ is considered instead of $GL(d, \mathbb{R})$, and the proximality condition A2(ii) is replaced by the condition that Γ_μ is unbounded. For positive matrices, Theorem 2.4 and its consequence (2.11) are new.

3. SPECTRAL GAP THEORY

This section is devoted to investigating the spectral gap properties of some operators to be introduced below: the transfer operator P_z , its normalization Q_s which is a Markov operator, and the perturbed operator $R_{s,z}$, for real-valued s and complex-valued z . The properties for these operators have

been intensively studied in recent years, for instance in [29, 5, 17, 6, 3], where various results have been established under different restrictions on s and z , which are not enough for obtaining the results of the paper. We shall complete these results by investigating the case when $s \in (-\eta, \eta)$ with $\eta > 0$ small, and z belongs to a small ball of the complex plane centered at the origin. The case of $s < 0$ turns out to be more difficult than the case $s \geq 0$ and requires a deeper analysis. We also complement the previous results with some new properties to be used in the proofs of the main results of the paper.

3.1. Properties of the transfer operator P_z . Recall that the Banach space \mathcal{B}_γ consists of all the γ -Hölder continuous complex-valued functions on \mathcal{S} . We write \mathcal{B}'_γ for the topological dual of \mathcal{B}_γ endowed with the norm $\|\nu\|_{\mathcal{B}'_\gamma} = \sup_{\|\varphi\|_\gamma=1} |\nu(\varphi)|$, for any linear functional $\nu \in \mathcal{B}'_\gamma$. Let $\mathcal{L}(\mathcal{B}_\gamma, \mathcal{B}_\gamma)$ be the set of all bounded linear operators from \mathcal{B}_γ to \mathcal{B}_γ equipped with the operator norm $\|\cdot\|_{\mathcal{B}_\gamma \rightarrow \mathcal{B}_\gamma}$. Denote by $\varrho(Q)$ the spectral radius of an operator $Q \in \mathcal{L}(\mathcal{B}_\gamma, \mathcal{B}_\gamma)$, and by $Q|_E$ its restriction to the subspace $E \subseteq \mathcal{B}_\gamma$.

For any $z \in \mathbb{C}$ with $|z| < \eta_0$, where η_0 is given in condition **A1**, define the transfer operator P_z as follows: for any $\varphi \in \mathcal{C}(\mathcal{S})$,

$$P_z \varphi(x) = \mathbb{E}[e^{z \log |g_1 x|} \varphi(g_1 \cdot x)] \quad x \in \mathcal{S}. \quad (3.1)$$

The transfer operator P_z acts from $\mathcal{C}(\mathcal{S})$ to the space of bounded functions on \mathcal{S} . The following proposition gives the spectral gap properties of the operator P_z for z in a small enough neighborhood of 0 in the complex plane.

Proposition 3.1. *Assume that μ satisfies either conditions **A1** and **A2** for invertible matrices, or conditions **A1** and **A3** for positive matrices. Then, $P_z \in \mathcal{L}(\mathcal{B}_\gamma, \mathcal{B}_\gamma)$ for any $z \in B_{\frac{\eta_0}{2}}(0)$, and the mapping $z \mapsto P_z : B_{\frac{\eta_0}{2}}(0) \rightarrow \mathcal{L}(\mathcal{B}_\gamma, \mathcal{B}_\gamma)$ is analytic for $\gamma > 0$ small enough, where η_0 is given in condition **A1**. Moreover, there exists a small $\eta > 0$ such that for any $z \in B_\eta(0)$ and $n \geq 1$, we have the decomposition*

$$P_z^n = \kappa^n(z) M_z + L_z^n, \quad (3.2)$$

where the operator $M_z := \nu_z \otimes r_z$ is a rank one projection on \mathcal{B}_γ defined by $M_z \varphi = \frac{\nu_z(\varphi)}{\nu_z(r_z)} r_z$ for any $\varphi \in \mathcal{B}_\gamma$, and the mappings on $B_\eta(0)$

$$z \mapsto \kappa(z) \in \mathbb{C}, \quad z \mapsto r_z \in \mathcal{B}_\gamma, \quad z \mapsto \nu_z \in \mathcal{B}'_\gamma, \quad z \mapsto L_z \in \mathcal{L}(\mathcal{B}_\gamma, \mathcal{B}_\gamma)$$

are unique under the normalizing conditions $\nu(r_z) = 1$ and $\nu_z(\mathbf{1}) = 1$, where ν is defined in (2.2); all these mappings are analytic in $B_\eta(0)$, and possess the following properties:

- (a) for any $z \in B_\eta(0)$, it holds that $M_z L_z = L_z M_z = 0$;
- (b) for any $z \in B_\eta(0)$, $P_z r_z = \kappa(z) r_z$ and $\nu_z P_z = \kappa(z) \nu_z$;

- (c) $\kappa(0) = 1$, $r_0 = \mathbf{1}$, $\nu_0 = \nu$, and $\kappa(s)$ and r_s are real-valued and satisfy $\kappa(s) > 0$ and $r_s(x) > 0$ for any $s \in (-\eta, \eta)$ and $x \in \mathcal{S}$;
- (d) for any $k \in \mathbb{N}$, there exist $0 < a_1 < a_2 < 1$ such that $|\kappa(z)| > 1 - a_1$ and $\|\frac{d^k}{dz^k} L_z^n\|_{\mathcal{B}_\gamma \rightarrow \mathcal{B}_\gamma} \leq C_k(1 - a_2)^n$ for all $z \in B_\eta(0)$.

Let us point out the differences between Proposition 3.1 and the previous results in [29, 5, 3]. Firstly, we complement the results in [29, 3] by giving the explicit formula $M_z \varphi = \frac{\nu_z(\varphi)}{\nu_z(r_z)} r_z$ in (3.2), for $z \in B_\eta(0)$, which is one of the crucial points in the proofs of the results of the paper. Basically, it permits us to deduce the spectral gap properties of the operators Q_s and $R_{s,z}$ from those of P_z . In particular this will enable us to obtain an explicit formula for the operators N_s and $N_{s,z}$ in Propositions 3.4 and 3.8, and the uniformity of the bounds (3.36) and (3.37). Secondly, for positive matrices, some points of Proposition 3.1 have been obtained in [5] only for real $z \geq 0$. The difficulty here is the case when $z \in \mathbb{R}$ is negative and when z is not real, so Proposition 3.1 is new for positive matrices when $|z| \leq \eta$. Thirdly, we show that $\kappa(z)$ and r_z take real positive values when z is real, which allows to define the change of measure \mathbb{Q}_s^x for real s , for both invertible matrices and positive matrices. Previously it was shown in [3] that $\kappa(z)$ is real-valued for real $z \in (-\eta, \eta)$ for invertible matrices.

In the sequel, without explicitly stated, we always assume that $\gamma > 0$ is a sufficiently small constant.

Remark 3.2. Define the conjugate transfer operator P_z^* by

$$P_z^* \varphi(x) = \mathbb{E} \left[e^{z \log |g_1^T x|} \varphi(g_1^T \cdot x) \right] \quad x \in \mathcal{S},$$

where $z \in \mathbb{C}$ with $\Re z \in (-\eta_0, \eta_0)$, and g_1^T denotes the transpose of the matrix g_1 . One can verify that P_z^* satisfies all the properties of Proposition 3.1: under conditions of Proposition 3.1, we have the decomposition

$$P_z^{*n} = \kappa^{*n}(z) \nu_z^* \otimes r_z^* + L_z^{*n}, \quad z \in B_\eta(0), \quad n \geq 1, \quad (3.3)$$

and all the assertions in Proposition 3.1 hold for P_z^* , $\kappa^*(z)$, ν_z^* , r_z^* , L_z^* instead of P_z , $\kappa(z)$, ν_z , r_z , L_z .

Proof of Proposition 3.1. We split the proof into three steps. In steps 1 and 2 we concentrate on the case of positive matrices, since for invertible matrices the results of these steps have been proved in [29, 3]. In step 1 we follow the same lines as in [29, 3]. In step 2 we follow [22] to prove the spectral gap property of the operator P_0 and we use the perturbation theory to extend it to P_z . In step 3 the proof is new and is provided for both invertible and positive matrices by complementing the results in [29, 5, 3].

Step 1. We only need to consider the case of positive matrices. We will show that there exists $\gamma \in (0, \frac{\eta_0}{6})$ such that $P_z \in \mathcal{L}(\mathcal{B}_\gamma, \mathcal{B}_\gamma)$, and that the

mapping $z \mapsto P_z$ is analytic on $B_{\frac{\eta_0}{2}}(0)$. For any $m \geq 0$, $z \in B_{\frac{\eta_0}{2}}(0)$ and $\varphi \in \mathcal{B}_\gamma$, let

$$P_z^{(m)}\varphi(x) = \mathbb{E}\left[(\log |g_1 x|)^m |g_1 x|^z \varphi(g_1 \cdot x)\right], \quad x \in \mathbb{S}_+^{d-1}.$$

It suffices to show that for $z \in B_{\frac{\eta_0}{2}}(0)$ and $\theta \in B_{\frac{\eta_0}{6}}(0)$,

$$P_{z+\theta} = \sum_{m=0}^{\infty} \frac{\theta^m}{m!} P_z^{(m)}, \quad (3.4)$$

and that there exists a constant $C > 0$ not depending on θ and z such that

$$\sum_{m=0}^{\infty} \frac{|\theta|^m}{m!} \|P_z^{(m)}\varphi\|_\gamma \leq C \|\varphi\|_\gamma. \quad (3.5)$$

From (3.5) we deduce that $P_z^{(0)} = P_z \in \mathcal{L}(\mathcal{B}_\gamma, \mathcal{B}_\gamma)$. Moreover, the bound (3.5) ensures the validity of (3.4) which implies the analyticity of the mapping $z \mapsto P_z$ on $B_{\frac{\eta_0}{2}}(0)$.

It remains to prove (3.5). We first give a control of $\|P_z^{(m)}\varphi\|_\infty$. Since $|\log |gx|| \leq \log N(g)$ for $g \in \Gamma_\mu$ and $x \in \mathbb{S}_+^{d-1}$, we get

$$\sum_{m=0}^{\infty} \frac{|\theta|^m}{m!} \|P_z^{(m)}\varphi\|_\infty \leq \|\varphi\|_\infty \mathbb{E}\left[e^{(|\theta| + |\Re z|) \log N(g_1)}\right] \leq C \|\varphi\|_\infty. \quad (3.6)$$

To control $[P_z^{(m)}\varphi]_\gamma$, note that for any $\varphi \in \mathcal{B}_\gamma$,

$$\begin{aligned} [P_z^{(m)}\varphi]_\gamma &\leq \sup_{x,y \in \mathbb{S}_+^{d-1}, x \neq y} \left| \mathbb{E}\left[\frac{(\log |g_1 x|)^m - (\log |g_1 y|)^m}{\mathbf{d}^\gamma(x,y)} |g_1 x|^z \varphi(g_1 \cdot x)\right] \right| \\ &\quad + \sup_{x,y \in \mathbb{S}_+^{d-1}, x \neq y} \left| \mathbb{E}\left[(\log |g_1 y|)^m \frac{|g_1 x|^z - |g_1 y|^z}{\mathbf{d}^\gamma(x,y)} \varphi(g_1 \cdot x)\right] \right| \\ &\quad + \sup_{x,y \in \mathbb{S}_+^{d-1}, x \neq y} \left| \mathbb{E}\left[(\log |g_1 y|)^m |g_1 y|^z \frac{\varphi(g_1 \cdot x) - \varphi(g_1 \cdot y)}{\mathbf{d}^\gamma(x,y)}\right] \right| \\ &= I_{1,m} + I_{2,m} + I_{3,m}. \end{aligned} \quad (3.7)$$

We then control each of the three terms $I_{1,m}, I_{2,m}, I_{3,m}$.

Control of $I_{1,m}$. Since for any $a, b \in \mathbb{C}$, $m \in \mathbb{N}$ and $0 < \gamma < 1$,

$$|a^m - b^m| \leq 2m \max\{|a|^{m-\gamma}, |b|^{m-\gamma}\} |a - b|^\gamma, \quad (3.8)$$

we get

$$I_{1,m} \leq 2m \|\varphi\|_\infty \sup_{x,y \in \mathbb{S}_+^{d-1}, x \neq y} \mathbb{E}\left[\frac{(\log N(g_1))^{m-\gamma} N(g_1)^{|\Re z|}}{\mathbf{d}^\gamma(x,y)} \left|\log \frac{|g_1 x|}{|g_1 y|}\right|^\gamma\right].$$

Using (2.1), we deduce that for any $g \in \Gamma_\mu$,

$$\left| \log \frac{|gx|}{|gy|} \right| \leq \frac{|g(x-y)|}{|gy|} \leq \|g\| \iota(g)^{-1} |x-y| \leq C \|g\| \iota(g)^{-1} \mathbf{d}(x, y), \quad (3.9)$$

and hence

$$\sum_{m=0}^{\infty} \frac{|\theta|^m}{m!} I_{1,m} \leq 2 \|\varphi\|_\infty \mathbb{E} \left[(\log N(g_1))^{1-\gamma} e^{(|\theta| + |\Re z| + 2\gamma) \log N(g_1)} \right]. \quad (3.10)$$

Control of $I_{2,m}$. Using (3.8), we deduce that for any $z_1, z_2 \in \mathbb{C}$,

$$|e^{z_1} - e^{z_2}| \leq 2 \max\{|z_1|^{1-\gamma}, |z_2|^{1-\gamma}\} \max\{e^{\Re z_1}, e^{\Re z_2}\} |z_1 - z_2|^\gamma. \quad (3.11)$$

By this inequality, we find that for any $g \in \Gamma_\mu$,

$$\left| e^{z \log |gx|} - e^{z \log |gy|} \right| \leq 2 \log N(g)^{1-\gamma} e^{|\Re z| \log N(g)} |\log |gx| - \log |gy||^\gamma.$$

Combining this with (3.9) implies that

$$\sum_{m=0}^{\infty} \frac{|\theta|^m}{m!} I_{2,m} \leq 2 \|\varphi\|_\infty \mathbb{E} \left[(\log N(g_1))^{1-\gamma} e^{(|\theta| + |\Re z| + 2\gamma) \log N(g_1)} \right]. \quad (3.12)$$

Control of $I_{3,m}$. Since $\varphi \in \mathcal{B}_\gamma$ and $\mathbf{d}(g \cdot x, g \cdot y) \leq \mathbf{d}(x, y)$ for any $g \in \Gamma_\mu$, we get

$$\sum_{m=0}^{\infty} \frac{|\theta|^m}{m!} I_{3,m} \leq \|\varphi\|_\gamma \mathbb{E} \left[e^{(|\theta| + |\Re z| + 2\gamma) \log N(g_1)} \right].$$

Combining this with (3.6), (3.7), (3.10) and (3.12), we obtain (3.5).

Step 2. Again we need only to consider the case of positive matrices. We will prove the decomposition formula (3.2) together with parts (a), (b) and (d). Our proof follows closely [22]. Define the operator M on \mathcal{B}_γ by $M\varphi = \nu(\varphi)\mathbf{1}$, $\varphi \in \mathcal{B}_\gamma$. Set $E = \ker M \cap \mathcal{B}_\gamma$. We first show that $\|\varphi\|_\infty \leq [\varphi]_\gamma$ for any $\varphi \in E$. Since $\nu(\varphi) = 0$ for any $\varphi \in E$, there exist $x_1, x_2 \in \mathbb{S}_+^{d-1}$ such that $\Re\varphi(x_1) = \Im\varphi(x_2) = 0$. Since $\mathbf{d}(x, y) \in [0, 1]$, it follows that

$$\|\varphi\|_\infty \leq \sup_{x \in \mathbb{S}_+^{d-1}} |\Re\varphi(x) - \Re\varphi(x_1)| + \sup_{x \in \mathbb{S}_+^{d-1}} |\Im\varphi(x) - \Im\varphi(x_2)| \leq 2[\varphi]_\gamma. \quad (3.13)$$

We next show that $\varrho(P|_E) < 1$, where $P = P_0$ (see (3.1)). For any $x, y \in \mathbb{S}_+^{d-1}$, $x \neq y$, and $\varphi \in \mathcal{B}_\gamma$, there exists $a \in (0, 1)$ such that for large $n \geq 1$,

$$\frac{|P^n \varphi(x) - P^n \varphi(y)|}{\mathbf{d}^\gamma(x, y)} \leq \|\varphi\|_\gamma \mathbb{E} \left[\frac{\mathbf{d}^\gamma(G_n \cdot x, G_n \cdot y)}{\mathbf{d}^\gamma(x, y)} \right] \leq \|\varphi\|_\gamma a^n,$$

where for the last inequality we use [19, Lemma 3.2]. Observe that for any $\varphi \in \mathcal{B}_\gamma$, we have $\varphi - M\varphi \in E$, thus $P^n(\varphi - M\varphi) \in E$ for any $n \geq 1$ since $\nu P = \nu$. Combining this with (3.13) and the above inequality, we get

$$\|P^n(\varphi - M\varphi)\|_\gamma \leq 2[P^n(\varphi - M\varphi)]_\gamma \leq 2a^n [\varphi]_\gamma \leq 2a^n \|\varphi\|_\gamma,$$

which implies $\varrho(P|_E) < 1$. This, together with the definition of E and the fact that $P\mathbf{1} = \mathbf{1}$, shows that 1 is the isolated dominant eigenvalue of the operator P . Using this and the analyticity of $P_z \in \mathcal{L}(\mathcal{B}_\gamma, \mathcal{B}_\gamma)$, and applying the perturbation theorem (see [20, Theorem III.8]), we obtain the decomposition formula (3.2) with $M_z(\varphi) = c_1 \nu_z(\varphi) r_z$ for some constant $c_1 \neq 0$, as well as parts (a), (b) and (d). Using $P_z r_z = \kappa(z) r_z$, we get $c_1 = 1/\nu_z(r_z)$ and thus $M_z \varphi = \frac{\nu_z(\varphi)}{\nu_z(r_z)} r_z$ for any $\varphi \in \mathcal{B}_\gamma$.

Step 3. We prove part (c) for invertible matrices and positive matrices. From $P\mathbf{1} = \mathbf{1}$, we see that $\kappa(0) = 1$ and $r_0 = \mathbf{1}$. Letting $z = 0$ in $\nu_z P_z = \kappa(z) \nu_z$, we get $\nu_0 P = \nu_0$ and thus $\nu_0 = \nu$ since ν is the unique μ -stationary probability measure. Now we fix $z \in (-\eta, \eta)$ and we show that $\kappa(z)$ and r_z are real-valued. Taking the conjugate in the equality $P_z r_z = \kappa(z) r_z$, we get $P_z \bar{r}_z = \overline{\kappa(z)} \bar{r}_z$, so that $\overline{\kappa(z)}$ is an eigenvalue of the operator P_z . By the uniqueness of the dominant eigenvalue of P_z , it follows that $\overline{\kappa(z)} = \kappa(z)$, showing that $\kappa(z)$ is real-valued for $z \in (-\eta, \eta)$. We now prove that r_z is real-valued. Write r_z in the form $r_z = u_z + i v_z$, where u_z and v_z are real-valued functions on \mathcal{S} . From the normalization condition $\nu(r_z) = 1$, we get $\nu(u_z) = 1$ and $\nu(v_z) = 0$. From the equation $P_z r_z = \kappa(z) r_z$ and the fact that $\kappa(z)$ is real-valued, we get that $P_z u_z = \kappa(z) u_z$ and $P_z v_z = \kappa(z) v_z$. By part (a), the space of eigenvectors corresponding to the eigenvalue $\kappa(z)$ is one dimensional. Therefore, we have either $u_z = c v_z$ for some constant $c \in \mathbb{R}$, or $v_z = 0$. However, the equality $u_z = c v_z$ is impossible because we have seen that $\nu(u_z) = 1$ and $\nu(v_z) = 0$. Hence $v_z = 0$ and r_z is real-valued for $z \in (-\eta, \eta)$. The positivity of $\kappa(z)$ and r_z then follows from $\kappa(0) = 1$, $r_0 = \mathbf{1}$ and the analyticity of the mappings $z \mapsto \kappa(z)$ and $z \mapsto r_z$. This ends the proof of part (c), as well as the proof of Proposition 3.1. \square

3.2. Definition of the change of measure \mathbb{Q}_s^x . Proposition 3.1 allows us to perform a change of measure. Note that this change of measure for positive s has been extensively studied in [5, 6, 17]; however, for negative s it is new. For any $s \in (-\eta, \eta)$, $x \in \mathcal{S}$ and $g \in \Gamma_\mu$, denote

$$q_n^s(x, g) = \frac{|gx|^s}{\kappa^n(s)} \frac{r_s(g \cdot x)}{r_s(x)}, \quad n \geq 1. \quad (3.14)$$

Note that (q_n^s) verifies the cocycle property: for any $n, m \geq 1$ and $g_1, g_2 \in \Gamma_\mu$,

$$q_n^s(x, g_1) q_m^s(g_1 \cdot x, g_2) = q_{n+m}^s(x, g_2 g_1). \quad (3.15)$$

Since $\kappa(s)$ and r_s are strictly positive, $q_n^s(x, G_n) \mu(dg_1) \dots \mu(dg_n)$, $n \geq 1$, is a sequence of probability measures, and forms a projective system on $\mathcal{M}^{\mathbb{N}^*}$. By the Kolmogorov extension theorem, there is a unique probability measure \mathbb{Q}_s^x on $\mathcal{M}^{\mathbb{N}^*}$ with marginals $q_n^s(x, G_n) \mu(dg_1) \dots \mu(dg_n)$. Denote by $\mathbb{E}_{\mathbb{Q}_s^x}$ the corresponding expectation. For any $n \in \mathbb{N}$ and any bounded measurable

function h on $(\mathcal{S} \times \mathbb{R})^{n+1}$, it holds that

$$\begin{aligned} & \mathbb{E} \left[\frac{r_s(X_n^x) |G_n x|^s}{\kappa^n(s) r_s(x)} h(X_0^x, \log |x|, \dots, X_n^x, \log |G_n x|) \right] \\ &= \mathbb{E}_{\mathbb{Q}_s^x} [h(X_0^x, \log |x|, \dots, X_n^x, \log |G_n x|)]. \end{aligned} \quad (3.16)$$

3.3. Properties of the Markov operator Q_s . For any $s \in (-\eta, \eta)$ and $\varphi \in \mathcal{B}_\gamma$, define the Markov operator Q_s by

$$Q_s \varphi(x) = \frac{1}{\kappa(s) r_s(x)} P_s(\varphi r_s)(x), \quad x \in \mathcal{S}.$$

Under the changed measure \mathbb{Q}_s^x , the process $(X_n^x)_{n \in \mathbb{N}}$ is a Markov chain with the transition operator given by Q_s .

The following assertion will be useful to prove that the function κ is strictly convex (see Lemma 3.15). Recall that $V(\Gamma_\mu)$ is the support of the measure ν (cf. (2.3), (2.4)).

Lemma 3.3. *Assume the conditions of Proposition 3.1. Let $s \in (-\eta, \eta)$ where η is small. If $\varphi \leq Q_s \varphi$ for some real-valued $\varphi \in \mathcal{C}(\mathcal{S})$, then $\varphi(x) = \sup_{y \in \mathcal{S}} \varphi(y)$ for any $x \in V(\Gamma_\mu)$.*

Proof. We use the approach developed in [17]. Set $\mathcal{M} = \sup_{y \in \mathcal{S}} \varphi(y)$ and $\mathcal{S}^+ = \{x \in \mathcal{S} : \varphi(x) = \mathcal{M}\}$. From the condition $\varphi \leq Q_s \varphi$ and the fact that $\int q_1^s(x, g_1) \mu(dg_1) = 1$, we get that if $x \in \mathcal{S}^+$, then $g \cdot x \in \mathcal{S}^+$ for any $g \in \Gamma_\mu$, so that $\Gamma_\mu \mathcal{S}^+ \subseteq \mathcal{S}^+$. Since $V(\Gamma_\mu)$ is the unique minimal Γ_μ -invariant set (see [17] and [5]), we obtain $V(\Gamma_\mu) \subseteq \mathcal{S}^+$ and the claim follows. \square

We state the spectral gap property of the Markov operator Q_s , whose proof is postponed to Section 3.5.

Proposition 3.4. *Assume the conditions of Proposition 3.1. Then there exists $\eta > 0$ such that for any $s \in (-\eta, \eta)$ and $n \geq 1$, we have*

$$Q_s^n = \Pi_s + N_s^n,$$

where the mappings $s \mapsto \Pi_s$, $s \mapsto N_s \in \mathcal{L}(\mathcal{B}_\gamma, \mathcal{B}_\gamma)$ are analytic and satisfy the following properties:

(a) with $\pi_s(\varphi) := \frac{\nu_s(\varphi r_s)}{\nu_s(r_s)}$, we have for any $\varphi \in \mathcal{B}_\gamma$,

$$\Pi_s(\varphi)(x) = \pi_s(\varphi) \mathbf{1}, \quad N_s^n(\varphi)(x) = \frac{1}{\kappa^n(s)} \frac{L_s^n(\varphi r_s)(x)}{r_s(x)}, \quad x \in \mathcal{S}$$

where ν_s, r_s, L_s are given in Proposition 3.1;

(b) $\Pi_s N_s = N_s \Pi_s = 0$, and for each $k \in \mathbb{N}$, there exist constants $C_k > 0$ and $a \in (0, 1)$ such that

$$\sup_{s \in (-\eta, \eta)} \left\| \frac{d^k}{ds^k} N_s^n \right\|_{\mathcal{B}_\gamma \rightarrow \mathcal{B}_\gamma} \leq C_k a^n. \quad (3.17)$$

3.4. Quasi-compactness of the operator Q_{s+it} . For $s \in (-\eta, \eta)$ and $t \in \mathbb{R}$, define the operator Q_{s+it} as follows: for any $\varphi \in \mathcal{B}_\gamma$,

$$\begin{aligned} Q_{s+it}\varphi(x) &= \frac{1}{\kappa(s)r_s(x)} P_{s+it}(\varphi r_s)(x) \\ &= \frac{1}{\kappa(s)r_s(x)} \mathbb{E} \left[|g_1 x|^{s+it} \varphi(g_1 \cdot x) r_s(g_1 \cdot x) \right], \quad x \in \mathcal{S}. \end{aligned}$$

The spectral gap properties of the operator Q_{s+it} for $|t|$ small enough can be deduced from Proposition 3.1. However, this approach does not work for large $|t|$. In order to investigate the spectral gap properties of the operator Q_{s+it} for $t \in \mathbb{R}$, we first prove the Doeblin-Fortet inequality and then we apply the theorem of Ionescu-Tulcea and Marinescu [25] to establish the quasi-compactness of the operator Q_{s+it} . Based on this property, we shall use the non-arithmeticity condition A5 to prove that the spectral radius of Q_{s+it} is strictly less than 1 when t is different from 0.

The following is the Doeblin-Fortet inequality for the operator Q_{s+it} .

Lemma 3.5. *Assume the conditions of Proposition 3.1. Then, there exist constants $0 < a < 1$, and $\eta > 0$ small enough, such that for any $s \in (-\eta, \eta)$, $t \in \mathbb{R}$, $n \geq 1$ and $\varphi \in \mathcal{B}_\gamma$, we have*

$$[Q_{s+it}^n \varphi]_\gamma \leq C_{s,t,n} \|\varphi\|_\infty + C_s a^n [\varphi]_\gamma. \quad (3.18)$$

For positive-valued s , analogous results can be found in [17] for invertible matrices and in [6] for positive matrices. The proofs in [17, 6] rely essentially on the Hölder continuity of the mapping $x \mapsto q_n^s(x, g)$ defined in (3.14). However, this property doesn't hold any more in the case when s is negative. Our proof of Lemma 3.5 is carried out using the Hölder inequality and the spectral gap properties of the operator P_s established in Proposition 3.1.

Proof of Lemma 3.5. Using the definition of Q_{s+it} and (3.15), we get that for any $n \geq 1$,

$$Q_{s+it}^n \varphi(x) = \frac{1}{\kappa^n(s)r_s(x)} P_{s+it}^n(\varphi r_s)(x), \quad x \in \mathcal{S}.$$

It follows that

$$\sup_{x,y \in \mathcal{S}, x \neq y} \frac{|Q_{s+it}^n \varphi(x) - Q_{s+it}^n \varphi(y)|}{\mathbf{d}^\gamma(x, y)} \leq J_1(n) + J_2(n), \quad (3.19)$$

where

$$\begin{aligned} J_1(n) &= \sup_{x,y \in \mathcal{S}, x \neq y} \frac{1}{\mathbf{d}^\gamma(x, y) \kappa^n(s)} \left| \frac{1}{r_s(x)} - \frac{1}{r_s(y)} \right| |P_{s+it}^n(\varphi r_s)(x)|, \\ J_2(n) &= \sup_{x,y \in \mathcal{S}, x \neq y} \frac{1}{r_s(y) \mathbf{d}^\gamma(x, y) \kappa^n(s)} |P_{s+it}^n(\varphi r_s)(x) - P_{s+it}^n(\varphi r_s)(y)|. \end{aligned}$$

Note that by Proposition 3.1, for any $s \in (-\eta, \eta)$, we have $\min_{x \in \mathcal{S}} r_s(x) > 0$, $\max_{x \in \mathcal{S}} r_s(x) < \infty$ and $\kappa(s) > 0$.

Control of $J_1(n)$. Observe that uniformly in $x \in \mathcal{S}$,

$$|P_{s+it}^n(\varphi r_s)(x)| \leq P_s^n(|\varphi| r_s)(x) \leq \|\varphi\|_\infty \kappa^n(s) \|r_s\|_\infty \leq C_s \|\varphi\|_\infty \kappa^n(s).$$

Since $r_s \in \mathcal{B}_\gamma$, this implies that for any $s \in (-\eta, \eta)$ and $t \in \mathbb{R}$,

$$J_1(n) \leq C_s \|\varphi\|_\infty. \quad (3.20)$$

Control of $J_2(n)$. Using the definition of P_{s+it} and taking into account that r_s is strictly positive and bounded on \mathcal{S} , we have

$$J_2(n) \leq C_s (J_{21}(n) + J_{22}(n) + J_{23}(n)), \quad (3.21)$$

where

$$\begin{aligned} J_{21}(n) &= \sup_{x,y \in \mathcal{S}, x \neq y} \frac{1}{\mathbf{d}^\gamma(x,y) \kappa^n(s)} \left| \mathbb{E} \left[(|G_n x|^{s+it} - |G_n y|^{s+it}) \varphi(X_n^x) \right] \right|, \\ J_{22}(n) &= \sup_{x,y \in \mathcal{S}, x \neq y} \frac{1}{\mathbf{d}^\gamma(x,y) \kappa^n(s)} \left| \mathbb{E} \left[|G_n y|^{s+it} (\varphi(X_n^x) - \varphi(X_n^y)) \right] \right|, \\ J_{23}(n) &= \sup_{x,y \in \mathcal{S}, x \neq y} \frac{1}{\mathbf{d}^\gamma(x,y) \kappa^n(s)} \left| \mathbb{E} \left\{ |G_n y|^{s+it} \varphi(X_n^y) [r_s(X_n^x) - r_s(X_n^y)] \right\} \right|. \end{aligned}$$

Control of $J_{21}(n)$. Using (3.11) and the inequality $\log u \leq u^\varepsilon$, $u > 1$, for $\varepsilon > 0$ small enough, we obtain

$$\left| |G_n x|^{s+it} - |G_n y|^{s+it} \right| \leq 2(N(G_n))^{|s|+\varepsilon} \left| \log |G_n x| - \log |G_n y| \right|^\gamma. \quad (3.22)$$

From the inequality (2.1), by arguing as in the estimate of (3.9), we get

$$\left| \log |G_n x| - \log |G_n y| \right|^\gamma \leq C \|G_n\|^\gamma \iota(G_n)^{-\gamma} \mathbf{d}^\gamma(x,y).$$

Using first (3.22) and then the last bound, we deduce that

$$J_{21}(n) \leq \frac{C \|\varphi\|_\infty}{\kappa^n(s)} \left\{ \mathbb{E} \left[(N(g_1))^{|s|+\varepsilon} \|g_1\|^\gamma \iota(g_1)^{-\gamma} \right] \right\}^n \leq C_{s,t,n} \|\varphi\|_\infty, \quad (3.23)$$

where the last inequality holds by condition A1.

Control of $J_{22}(n)$. Since $\varphi \in \mathcal{B}_\gamma$, applying the Hölder inequality leads to

$$\begin{aligned} J_{22}(n) &\leq \frac{C_s [\varphi]_\gamma}{\kappa^n(s)} \sup_{x,y \in \mathcal{S}, x \neq y} \mathbb{E} \left[|G_n y|^s \frac{\mathbf{d}^\gamma(X_n^x, X_n^y)}{\mathbf{d}^\gamma(x,y)} \right] \\ &\leq C_s [\varphi]_\gamma \sup_{x,y \in \mathcal{S}, x \neq y} \frac{\{\mathbb{E} [|G_n y|^{2s}]\}^{1/2}}{\kappa^n(s)} \left\{ \mathbb{E} \frac{\mathbf{d}^{2\gamma}(X_n^x, X_n^y)}{\mathbf{d}^{2\gamma}(x,y)} \right\}^{1/2}. \end{aligned} \quad (3.24)$$

Since $\gamma > 0$ is small enough, by [29, Theorem 1] for invertible matrices and [19, Lemma 3.2] for positive matrices, there exists a constant $a_0 \in (0, 1)$

such that for sufficiently large n ,

$$\sup_{x,y \in \mathcal{S}, x \neq y} \left\{ \mathbb{E} \frac{\mathbf{d}^{2\gamma}(X_n^x, X_n^y)}{\mathbf{d}^{2\gamma}(x, y)} \right\}^{1/2} \leq a_0^n. \quad (3.25)$$

In view of Proposition 3.1, we have

$$\mathbb{E}[|G_n y|^{2s}] = \kappa^n(2s)(M_{2s} \mathbf{1})(y) + L_{2s}^n \mathbf{1}(y), \quad y \in \mathcal{S}.$$

Since, by Proposition 3.1(d), $\|M_{2s} \mathbf{1}\|_\infty$ is bounded by some constant C_s , and $\|L_{2s}^n \mathbf{1}\|_\infty$ is bounded by $C_s \kappa^n(2s)$ uniformly in $n \geq 1$, it follows that

$$\sup_{n \geq 1} \sup_{y \in \mathcal{S}} \frac{\mathbb{E}[|G_n y|^{2s}]}{\kappa^n(2s)} \leq C_s. \quad (3.26)$$

As κ is continuous in the neighborhood of 0 and $\kappa(0) = 1$, one can choose $\eta > 0$ small enough and a constant $\alpha \in (0, 1/a_0)$ such that $\kappa^{n/2}(2s)/\kappa^n(s) \leq \alpha^n$, uniformly in $s \in (-\eta, \eta)$. Substituting this inequality together with (3.25) and (3.26) into (3.24), we obtain that for any $s \in (-\eta, \eta)$ with $\eta > 0$ small, there exists $0 < a < 1$ such that uniformly in $n \geq 1$,

$$J_{22}(n) \leq C_s a^n [\varphi]_\gamma, \quad (3.27)$$

Control of $J_{23}(n)$. Using (3.26) and the fact that $r_s \in \mathcal{B}_\gamma$, and applying similar techniques as in the control of $J_{22}(n)$, one can verify that there exists a constant $0 < a < 1$ such that uniformly in $n \geq 1$,

$$J_{23}(n) \leq C_s a^n \|\varphi\|_\infty [r_s]_\gamma \leq C_s a^n \|\varphi\|_\infty. \quad (3.28)$$

Inserting (3.23), (3.27) and (3.28) into (3.21), we conclude that

$$J_2(n) \leq C_{s,t,n} \|\varphi\|_\infty + C_s a^n [\varphi]_\gamma.$$

Combining this with (3.20) and (3.19), we obtain the inequality (3.18). \square

From Lemma 3.5 and the fact that $\|Q_{s+it}\varphi\|_\infty \leq C_s \|\varphi\|_\infty$ for any $s \in (-\eta, \eta)$ and $t \in \mathbb{R}$, we can deduce that $Q_{s+it} \in \mathcal{L}(\mathcal{B}_\gamma, \mathcal{B}_\gamma)$. We next prove that the operator Q_{s+it} is quasi-compact. Recall that an operator $Q \in \mathcal{L}(\mathcal{B}_\gamma, \mathcal{B}_\gamma)$ is *quasi-compact* if \mathcal{B}_γ can be decomposed into two Q invariant closed subspaces $\mathcal{B}_\gamma = E \oplus F$, such that $\dim E < \infty$, each eigenvalue of $Q|_E$ has modulus $\varrho(Q)$, and $\varrho(Q|_F) < \varrho(Q)$ (see [20] for more details).

Proposition 3.6. *Assume the conditions of Proposition 3.1. Then, there exists a small $\eta > 0$ such that for any $s \in (-\eta, \eta)$ and $t \in \mathbb{R}$, the operator Q_{s+it} is quasi-compact.*

Proof. The proof consists of verifying the conditions of the theorem of Ionescu-Tulcea and Marinescu [25]. We follow the formulation in [20, Theorem II.5].

Firstly, by the definition of Q_{s+it} , there exists a constant $C_s > 0$ such that $\|Q_{s+it}\varphi\|_\infty \leq C_s \|\varphi\|_\infty$ for any $s \in (-\eta, \eta)$, $t \in \mathbb{R}$ and $\varphi \in \mathcal{B}_\gamma$.

Secondly, by Lemma 3.5, the Doeblin-Fortet inequality (3.18) holds for the operator Q_{s+it} .

Thirdly, denoting $K = Q_{s+it}\{\varphi : \|\varphi\|_\gamma \leq 1\}$, we claim that for any $s \in (-\eta, \eta)$ and $t \in \mathbb{R}$, the set K is conditionally compact in $(\mathcal{B}_\gamma, \|\cdot\|_\infty)$. Since $\|Q_{s+it}\varphi\|_\infty \leq C_s\|\varphi\|_\infty$ for any $\varphi \in \mathcal{B}_\gamma$, we conclude that K is uniformly bounded in $(\mathcal{B}_\gamma, \|\cdot\|_\infty)$. Moreover, by taking $n = 1$ in (3.18), we get that uniformly in $\varphi \in \mathcal{B}_\gamma$ with $\|\varphi\|_\gamma \leq 1$,

$$|Q_{s+it}\varphi(x) - Q_{s+it}\varphi(y)| \leq C_{s,t}\mathbf{d}^\gamma(x, y).$$

This shows that K is equicontinuous in $(\mathcal{B}_\gamma, \|\cdot\|_\infty)$. Therefore, we obtain the claim by the Arzelà-Ascoli theorem.

The assertion of the proposition now follows from the theorem of Ionescu-Tulcea and Marinescu. \square

The following proposition shows that the spectral radius of the operator Q_{s+it} is strictly less than 1 when t is different from 0. The proof which relies on the non-arithmeticity condition A5, follows the standard pattern in [17, 6]; it is included for the commodity of the reader.

Proposition 3.7. *Assume either conditions A1 and A2 for invertible matrices, or conditions A1, A3 and A5 for positive matrices. Then, for any $s \in (-\eta, \eta)$ with small $\eta > 0$, and any $t \in \mathbb{R} \setminus \{0\}$, we have $\varrho(Q_{s+it}) < 1$.*

Proof. By the definition of Q_{s+it} , we have $\varrho(Q_{s+it}) \leq \varrho(Q_s) = 1$. Suppose that $\varrho(Q_{s+it}) = 1$ for some $t \neq 0$. Then, applying Proposition 3.6, there exist $\varphi \in \mathcal{B}_\gamma$ and $\beta \in \mathbb{R}$ such that $Q_{s+it}\varphi = e^{i\beta}\varphi$. From this equation, we deduce that $|\varphi| \leq Q_s|\varphi|$. Using Lemma 3.3, this implies that $|\varphi(x)| = \sup_{y \in \mathcal{S}} |\varphi(y)|$ for any $x \in V(\Gamma_\mu)$, so that $\varphi(x) = ce^{i\vartheta(x)}$, where $c \neq 0$ is a constant and ϑ is a real-valued continuous function on \mathcal{S} . Substituting this into the equation $Q_{s+it}\varphi = e^{i\beta}\varphi$ gives that for any $x \in V(\Gamma_\mu)$,

$$\mathbb{E}_{\mathbb{Q}_s^x} \exp[it \log |g_1 x| - i\beta + i\vartheta(g_1 \cdot x) - i\vartheta(x)] = 1.$$

Since ϑ is real-valued, this implies $\exp[it \log |gx| - i\beta + i\vartheta(g \cdot x) - i\vartheta(x)] = 1$ for any $x \in V(\Gamma_\mu)$ and μ -a.e. $g \in \Gamma_\mu$, which contradicts to condition A5. Therefore $\varrho(Q_{s+it}) < 1$ for any $t \neq 0$. Recalling that condition A2 implies condition A5, the proof of Proposition 3.7 is complete. \square

3.5. Spectral gap properties of the perturbed operator $R_{s,z}$. For any $s \in (-\eta, \eta)$ and $z \in \mathbb{C}$ such that $s + \Re z \in (-\eta_0, \eta_0)$, define the perturbed operator $R_{s,z}$ as follows: for any $\varphi \in \mathcal{B}_\gamma$,

$$R_{s,z}\varphi(x) = \mathbb{E}_{\mathbb{Q}_s^x} \left[e^{z(\log |g_1 x| - \Lambda'(s))} \varphi(X_1^x) \right], \quad x \in \mathcal{S}. \quad (3.29)$$

With some calculations using (3.15), it follows that for any $n \geq 1$,

$$R_{s,z}^n \varphi(x) = \mathbb{E}_{\mathbb{Q}_s^x} \left[e^{z(\log |G_n x| - n\Lambda'(s))} \varphi(X_n^x) \right], \quad x \in \mathcal{S}. \quad (3.30)$$

The following formula relates the operator $R_{s,z}^n$ to the operator P_{s+z}^n and is of independent interest: for any $\varphi \in \mathcal{B}_\gamma$, $n \geq 1$, $s \in (-\eta, \eta)$ and $z \in B_\eta(0)$,

$$R_{s,z}^n(\varphi) = e^{-nz\Lambda'(s)} \frac{P_{s+z}^n(\varphi r_s)}{\kappa^n(s) r_s}. \quad (3.31)$$

The identity (3.31) is obtained by the definitions of $R_{s,z}$ and P_z using the change of measure (3.16).

There are two ways to establish spectral gap properties of the operator $R_{s,z}$: one is to use the perturbation theory of operators [20, Theorem III.8], another is based on the Ionescu-Tulcea and Marinescu theorem [25] about the quasi-compactness of operators. The representation (3.31) allows us to deduce the spectral gap properties of $R_{s,z}$ directly from the properties of the operator P_z . This has some advantages: it ensures the uniformity in $s \in (-\eta, \eta)$, allows to deal with negative-valued s and provides an explicit formula for the projection operator $\Pi_{s,z}$ and the remainder operator $N_{s,z}^n$ defined below.

Recall that $\Lambda = \log \kappa$, where κ is defined in (2.6).

Proposition 3.8. *Assume the conditions of Proposition 3.1. Then, there exist $\eta > 0$ and $\delta \in (0, \eta)$ such that for any $s \in (-\eta, \eta)$ and $z \in B_\delta(0)$,*

$$R_{s,z}^n = \lambda_{s,z}^n \Pi_{s,z} + N_{s,z}^n, \quad n \geq 1, \quad (3.32)$$

$$\lambda_{s,z} = e^{\Lambda(s+z) - \Lambda(s) - \Lambda'(s)z} \quad (3.33)$$

and for $\varphi \in \mathcal{B}_\gamma$,

$$\Pi_{s,z}(\varphi) = \frac{\nu_{s+z}(\varphi r_s)}{\nu_{s+z}(r_{s+z})} \frac{r_{s+z}}{r_s}, \quad (3.34)$$

$$N_{s,z}^n(\varphi) = e^{-n[\Lambda(s) + \Lambda'(s)z]} \frac{L_{s+z}^n(\varphi r_s)}{r_s}, \quad (3.35)$$

where r_z , ν_z and L_z are given in Proposition 3.1. In addition, we have:

- (a) for fixed s , the mappings $z \mapsto \Pi_{s,z} : B_\delta(0) \rightarrow \mathcal{L}(\mathcal{B}_\gamma, \mathcal{B}_\gamma)$, $z \mapsto N_{s,z} : B_\delta(0) \rightarrow \mathcal{L}(\mathcal{B}_\gamma, \mathcal{B}_\gamma)$ and $z \mapsto \lambda_{s,z} : B_\delta(0) \rightarrow \mathbb{C}$ are analytic,
- (b) for fixed s and z , $\Pi_{s,z}$ is a rank-one projection with $\Pi_{s,0}(\varphi)(x) = \pi_s(\varphi)$ for any $\varphi \in \mathcal{B}_\gamma$ and $x \in \mathcal{S}$, and $\Pi_{s,z} N_{s,z} = N_{s,z} \Pi_{s,z} = 0$,
- (c) for $k \in \mathbb{N}$, there exist $0 < a < 1$ and $C_k > 0$ such that

$$\sup_{s \in (-\eta, \eta)} \sup_{z \in B_\delta(0)} \left\| \frac{d^k}{dz^k} \Pi_{s,z} \right\|_{\mathcal{B}_\gamma \rightarrow \mathcal{B}_\gamma} \leq C_k, \quad (3.36)$$

$$\sup_{s \in (-\eta, \eta)} \sup_{z \in B_\delta(0)} \left\| \frac{d^k}{dz^k} N_{s,z}^n \right\|_{\mathcal{B}_\gamma \rightarrow \mathcal{B}_\gamma} \leq C_k a^n. \quad (3.37)$$

Note that, for $s > 0$, similar results have been obtained in [6]. The novelty here is that s can account for negative values $s \in (-\eta, 0]$ and that the bounds

(3.36) and (3.37) hold uniformly in $s \in (-\eta, \eta)$. This plays a crucial role in establishing Theorem 2.3.

Proof of Proposition 3.8. The proof is divided into three steps.

Step 1. By Proposition 3.1,

$$P_{s+z}^n(\varphi r_s) = \kappa^n(s+z) \frac{\nu_{s+z}(\varphi r_s)}{\nu_{s+z}(r_{s+z})} r_{s+z} + L_{s+z}^n(\varphi r_s).$$

Substituting this into (3.31) shows (3.32), (3.33), (3.34) and (3.35).

Step 2. We prove parts (a) and (b). The assertion in part (a) follows from the expressions (3.33), (3.34) and (3.35), and the analyticity of the mappings $z \mapsto \kappa(z)$, $z \mapsto r_z$, $z \mapsto \nu_z$ and $z \mapsto L_z$ defined in Proposition 3.1. To show part (b), by (3.34), we have that $\Pi_{s,z}$ is a rank-one projection on the subspace $\{w \frac{r_{s+z}}{r_s} : w \in \mathbb{C}\}$. The identity $\Pi_{s,0}(\varphi)(x) = \pi_s(\varphi)$ follows from (3.34) and the fact that $\pi_s(\varphi) = \frac{\nu_s(\varphi r_s)}{\nu_s(r_s)}$. Using Proposition 3.1, we get that $L_z r_z = 0$ and $\nu_z(L_z \varphi) = 0$ for any $\varphi \in \mathcal{B}_\gamma$. This, together with (3.34) and (3.35), shows $\Pi_{s,z} N_{s,z} = N_{s,z} \Pi_{s,z} = 0$.

Step 3. We prove part (c). By Proposition 3.1, there exists $\eta > 0$ such that the mappings $z \mapsto \kappa(z)$, $z \mapsto r_z$, $z \mapsto \nu_z$ are analytic and uniformly bounded on $B_{2\eta}(0)$. Combining this with (3.34), we obtain (3.36). We now prove (3.37). Since the function r_s is strictly positive on the compact set \mathcal{S} , by Proposition 3.1(d), we deduce that there exists $0 < a_0 < 1$ such that uniformly in $\varphi \in \mathcal{B}_\gamma$,

$$\sup_{s \in (-\eta, \eta)} \sup_{z \in B_\eta(0)} \left\| \frac{L_{s+z}^n(\varphi r_s)}{r_s} \right\|_\gamma \leq C \|\varphi\|_\gamma a_0^n. \quad (3.38)$$

Using the fact that the function Λ is continuous and $\Lambda(0) = 0$, there exist a small $\eta > 0$, $\delta \in (0, \eta)$ and a constant $a_1 < \frac{1}{a_0}$ such that

$$\sup_{s \in (-\eta, \eta)} \sup_{z \in B_\delta(0)} \left| e^{-n[\Lambda(s) + \Lambda'(s)z]} \right| \leq C a_1^n.$$

Combining this with (3.38) proves (3.37) with $k = 0$. The proof of (3.37) when $k \geq 1$ can be carried out in the same way as in the case of $k = 0$. \square

Proof of Proposition 3.4. The assertion is obtained from Proposition 3.8 taking $z = 0$. \square

Proposition 3.9. *Assume the conditions of Proposition 3.7. For any compact set $K \subseteq \mathbb{R} \setminus \{0\}$, there exist a constant $C_K > 0$ and small $\eta > 0$ such that for any $n \geq 1$ and $\varphi \in \mathcal{B}_\gamma$,*

$$\sup_{s \in (-\eta, \eta)} \sup_{t \in K} \sup_{x \in \mathcal{S}} |R_{s,it}^n \varphi(x)| \leq e^{-nC_K} \|\varphi\|_\gamma.$$

Proof. Set $\rho_n(s, t) = \|R_{s,it}^n \varphi\|_\infty^{1/n}$. Using the inequality $|\rho_n(s_1, t_1) - \rho_n(s_2, t_2)| \leq \|R_{s_1,it_1}^n \varphi - R_{s_2,it_2}^n \varphi\|_\infty^{1/n}$, Proposition 3.1 and the definition of the operator $R_{s,it}$, we get that for any fixed $n \in \mathbb{N}$ and $\varphi \in \mathcal{B}_\gamma$, the function ρ_n is continuous on the compact set $I_\eta \times K$, where $I_\eta = [-\eta, \eta]$. This implies that the function $\rho := \limsup_{n \rightarrow \infty} \rho_n$ is upper semicontinuous on $I_\eta \times K$, so that ρ attains the maximum at the point $(s_0, t_0) \in I_\eta \times K$. By Proposition 3.7, we have $\rho(s_0, t_0) \leq \varrho(R_{s_0+it_0}) = \varrho(Q_{s_0+it_0}) < 1$ and thus the assertion follows. \square

We now give some properties of the function $b_{s,\varphi}$ defined as follows: for any $s \in (-\eta, \eta)$ and $\varphi \in \mathcal{B}_\gamma$,

$$b_{s,\varphi}(x) = \lim_{n \rightarrow \infty} \mathbb{E}_{\mathbb{Q}_s^x} [(\log |G_n x| - n\lambda)\varphi(X_n^x)], \quad x \in \mathcal{S}.$$

In particular, with $s = 0$, $b_{0,\varphi} = b_\varphi$, which is defined in (2.7).

Lemma 3.10. *Assume the conditions of Proposition 3.1. Then the function $b_{s,\varphi}$ is well-defined, $b_{s,\varphi} \in \mathcal{B}_\gamma$ and*

$$b_{s,\varphi}(x) = \left. \frac{d\Pi_{s,z}}{dz} \right|_{z=0} \varphi(x), \quad x \in \mathcal{S}. \quad (3.39)$$

Proof. In view of Proposition 3.8, we have that for any $\varphi \in \mathcal{B}_\gamma$,

$$\mathbb{E}_{\mathbb{Q}_s^x} \left[e^{z(\log |G_n x| - n\Lambda'(s))} \varphi(X_n^x) \right] = \lambda_{s,z}^n \Pi_{s,z} \varphi(x) + N_{s,z}^n \varphi(x), \quad x \in \mathcal{S}.$$

From (3.33), we have $\lambda_{s,0} = 1$ and $\left. \frac{d\lambda_{s,z}}{dz} \right|_{z=0} = 0$. Differentiating both sides of the above equation w.r.t. z at the point 0 gives that for any $x \in \mathcal{S}$,

$$\mathbb{E}_{\mathbb{Q}_s^x} [(\log |G_n x| - n\Lambda'(s))\varphi(X_n^x)] = \left. \frac{d\Pi_{s,z}}{dz} \right|_{z=0} \varphi(x) + \left. \frac{dN_{s,z}^n}{dz} \right|_{z=0} \varphi(x). \quad (3.40)$$

Using the bounds (3.36) and (3.37), we find that the first term on the right-hand side of (3.40) belongs to \mathcal{B}_γ , and the second term converges to 0 exponentially fast as $n \rightarrow \infty$. Hence, letting $n \rightarrow \infty$ in (3.40), we obtain (3.39). This shows that the function $b_{s,\varphi}$ is well-defined and $b_{s,\varphi} \in \mathcal{B}_\gamma$. \square

For any $s \in (-\eta, \eta)$ with $\eta > 0$ small, define $\mathbb{Q}_s = \int_{\mathcal{S}} \mathbb{Q}_s^x \pi_s(dx)$. The following result will be used to prove the strong law of large numbers for $\log |G_n x|$ under the changed measure \mathbb{Q}_s .

Lemma 3.11. *Assume the conditions of Proposition 3.1. There exist $\eta > 0$ and $c, C > 0$ such that uniformly in $s \in (-\eta, \eta)$, $\varphi \in \mathcal{B}_\gamma$ and $n \geq 1$,*

$$\left| \mathbb{E}_{\mathbb{Q}_s} [(\log |G_n x| - n\Lambda'(s))\varphi(X_n^x)] \right| \leq C \|\varphi\|_\gamma e^{-cn}. \quad (3.41)$$

Proof. We follow the proof of the previous lemma. Integrating both sides of the identity (3.40) w.r.t. π_s , we get, for any $\varphi \in \mathcal{B}_\gamma$,

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}_s}[(\log |G_n x| - n\Lambda'(s))\varphi(X_n^x)] \\ &= \pi_s\left(\frac{d\Pi_{s,z}}{dz}\Big|_{z=0}\varphi\right) + \pi_s\left(\frac{dN_{s,z}^n}{dz}\Big|_{z=0}\varphi\right). \end{aligned} \quad (3.42)$$

Since $\Pi_{s,z}^2\varphi = \Pi_{s,z}\varphi$, it follows that $2\Pi_{s,0}(\frac{d\Pi_{s,z}}{dz}\Big|_{z=0}\varphi) = \frac{d\Pi_{s,z}}{dz}\Big|_{z=0}\varphi$. Integrating both sides of this equation w.r.t. π_s and using the fact that $\Pi_{s,0} = \pi_s$, we find that

$$\pi_s\left(\frac{d\Pi_{s,z}}{dz}\Big|_{z=0}\varphi\right) = 0. \quad (3.43)$$

It follows from (3.37) that uniformly in $\varphi \in \mathcal{B}_\gamma$ and $s \in (-\eta, \eta)$, the second term on the right-hand side of (3.42) is bounded by $C\|\varphi\|_\gamma e^{-cn}$. Therefore, from (3.42) and (3.43) we obtain (3.41). \square

We now establish the strong laws of large numbers for $\log |G_n x|$ under the measures \mathbb{Q}_s^x and \mathbb{Q}_s , which are of independent interest.

Proposition 3.12. *Assume the conditions of Proposition 3.1. Then, there exists $\eta > 0$ such that for any $s \in (-\eta, \eta)$ and $x \in \mathcal{S}$,*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log |G_n x| = \Lambda'(s), \quad \mathbb{Q}_s^x\text{-a.s.}$$

Proof. By the Borel-Cantelli lemma, it suffices to show that for any $\varepsilon > 0$, $s \in (-\eta, \eta)$ and $x \in \mathcal{S}$, we have

$$\sum_{n=1}^{\infty} \mathbb{Q}_s^x\left(|\log |G_n x| - n\Lambda'(s)| \geq n\varepsilon\right) < \infty. \quad (3.44)$$

Now let us prove (3.44). By Markov's inequality, we have for small $\delta > 0$,

$$\begin{aligned} & \mathbb{Q}_s^x\left(|\log |G_n x| - n\Lambda'(s)| \geq n\varepsilon\right) \\ & \leq e^{-n\delta\varepsilon} \mathbb{E}_{\mathbb{Q}_s^x}\left(e^{\delta(\log |G_n x| - n\Lambda'(s))}\right) + e^{-n\delta\varepsilon} \mathbb{E}_{\mathbb{Q}_s^x}\left(e^{-\delta(\log |G_n x| - n\Lambda'(s))}\right). \end{aligned}$$

From (3.30) and Proposition 3.8, we deduce that there exist positive constants c, C independent of s, x, δ such that

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}_s^x}\left(e^{\delta(\log |G_n x| - n\Lambda'(s))}\right) + \mathbb{E}_{\mathbb{Q}_s^x}\left(e^{-\delta(\log |G_n x| - n\Lambda'(s))}\right) \\ & \leq C e^{n[\Lambda(s+\delta) - \Lambda(s) - \Lambda'(s)\delta]} + C e^{n[\Lambda(s-\delta) - \Lambda(s) + \Lambda'(s)\delta]} + C e^{-cn}. \end{aligned}$$

Using Taylor's formula and taking $\delta > 0$ small enough, we conclude that

$$\mathbb{Q}_s^x\left(|\log |G_n x| - n\Lambda'(s)| \geq n\varepsilon\right) \leq C e^{-n\frac{\delta}{2}\varepsilon},$$

which implies the assertion (3.44). \square

Proposition 3.13. *Assume the conditions of Proposition 3.1. Then, there exists $\eta > 0$ such that for any $s \in (-\eta, \eta)$ and $x \in \mathcal{S}$,*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log |G_n x| = \Lambda'(s), \quad \mathbb{Q}_s\text{-a.s.}$$

Proof. Taking $\varphi = \mathbf{1}$ in (3.41) leads to

$$\lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}_{\mathbb{Q}_s} \log |G_n x| = \Lambda'(s). \quad (3.45)$$

Let $\Omega = M(d, \mathbb{R})^{\mathbb{N}^*}$ and $\widehat{\Omega} = \mathcal{S} \times \Omega$. Following [17, Theorem 3.10], define the shift operator $\widehat{\theta}$ on $\widehat{\Omega}$ by $\widehat{\theta}(x, \omega) = (g_1 \cdot x, \theta\omega)$, where $\omega \in \Omega$ and θ is the shift operator on Ω . For any $x \in \mathcal{S}$ and $\omega \in \Omega$, set $h(x, \omega) = \log |g_1(\omega)x|$. Then h is \mathbb{Q}_s -integrable. Since $\log |G_n x| = \sum_{k=0}^{n-1} (h \circ \widehat{\theta}^k)(x, \omega)$ and \mathbb{Q}_s is $\widehat{\theta}$ -ergodic, it follows from Birkhoff's ergodic theorem that $\frac{1}{n} \log |G_n x|$ converges \mathbb{Q}_s -a.s. to some constant c_s as $n \rightarrow \infty$. If we suppose that c_s is different from $\Lambda'(s)$, then this contradicts to (3.45). Thus $c_s = \Lambda'(s)$ and the assertion of the lemma follows. \square

Now we give the third-order Taylor expansion of $\lambda_{s,z}$ defined by (3.33), w.r.t. z at the origin in the complex plane.

Proposition 3.14. *Assume the conditions of Proposition 3.1. Then, there exist $\eta > 0$ and $\delta > 0$ such that for any $s \in (-\eta, \eta)$ and $z \in B_\delta(0)$,*

$$\lambda_{s,z} = 1 + \frac{\sigma_s^2}{2} z^2 + \frac{\Lambda'''(s)}{6} z^3 + o(|z|^3) \quad \text{as } |z| \rightarrow 0, \quad (3.46)$$

where

- (a) $\sigma_s^2 = \Lambda''(s) \geq 0$ and $\Lambda'''(s) \in \mathbb{R}$;
- (b) for invertible matrices, $\sigma_s > 0$ under the stated conditions; for positive matrices, $\sigma_s > 0$ if additionally $\sigma = \sigma_0 > 0$ or if the measure μ is non-arithmetic;
- (c) uniformly in $s \in (-\eta, \eta)$ and $x \in \mathcal{S}$,

$$\begin{aligned} \sigma_s^2 &= \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}_{\mathbb{Q}_s^x} [\log |G_n x| - n\Lambda'(s)]^2 \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}_{\mathbb{Q}_s} [\log |G_n x| - n\Lambda'(s)]^2; \end{aligned}$$

- (d) uniformly in $s \in (-\eta, \eta)$,

$$\Lambda'''(s) = \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}_{\mathbb{Q}_s} [\log |G_n x| - n\Lambda'(s)]^3.$$

The proof of Proposition 3.14 is based on the following lemma.

Lemma 3.15. *Assume the conditions of Proposition 3.1. Then the functions Λ and κ are convex on $(-\eta, \eta)$ for $\eta > 0$ small enough. Moreover, Λ*

and κ are strictly convex for invertible matrices under the given conditions, and for positive matrices under the additional condition **A5**.

Proof. The proof follows [17]. Since $\Lambda = \log \kappa$, it suffices to prove Lemma 3.15 for the function Λ . For any $t \in (0, 1)$, $s_1, s_2 \in (-\eta, \eta)$, set $s' = ts_1 + (1-t)s_2$. Using Hölder's inequality and the fact that $P_s r_s = \kappa(s)r_s$,

$$P_{s'}(r_{s_1}^t r_{s_2}^{1-t}) \leq \kappa^t(s_1) \kappa^{1-t}(s_2) r_{s_1}^t r_{s_2}^{1-t}. \quad (3.47)$$

Since $\kappa(s')$ is the dominant eigenvalue of the operator $P_{s'}$, we obtain $\kappa(s') \leq \kappa^t(s_1) \kappa^{1-t}(s_2)$ and thus Λ is convex.

To show that the function Λ is strictly convex, we suppose, by absurd, that there exist $s_1 \neq s_2$ and some $t \in (0, 1)$ such that $\kappa(s') = \kappa^t(s_1) \kappa^{1-t}(s_2)$. Using this equality, the definition of Q_s and (3.47), we get $Q_{s'}(r_{s_1}^t r_{s_2}^{1-t}/r_{s'}) \leq r_{s_1}^t r_{s_2}^{1-t}/r_{s'}$. By Lemma 3.3, this implies that $r_{s_1}^t r_{s_2}^{1-t} = cr_{s'}$ on $V(\Gamma_\mu)$ for some constant $c > 0$. Substituting this equality and the identity $\kappa(s') = \kappa^t(s_1) \kappa^{1-t}(s_2)$ into (3.47), we see that the Hölder inequality in (3.47) is actually an equality. This yields that there exists a function $c(x) > 0$ such that for any $g \in \Gamma_\mu$ and $x \in V(\Gamma_\mu)$, we have

$$|gx|^{s_1} r_{s_1}(g \cdot x) = c(x) |gx|^{s_2} r_{s_2}(g \cdot x). \quad (3.48)$$

Integrating both sides of the equation (3.48) w.r.t. μ gives $c(x) = \frac{\kappa(s_1)r_{s_1}(x)}{\kappa(s_2)r_{s_2}(x)}$. Substituting this into (3.48) and noting that $s_1 \neq s_2$, we find that there exist a constant $c_1 > 0$ and a real-valued function φ on \mathcal{S} such that $|gx| = c_1 \frac{\varphi(g \cdot x)}{\varphi(x)}$ for any $g \in \Gamma_\mu$ and $x \in V(\Gamma_\mu)$. This contradicts to condition **A5**. Recall that condition **A2** implies condition **A5** for invertible matrices. Hence Λ is strictly convex for invertible matrices under stated conditions. \square

Proof of Proposition 3.14. The expansion (3.46) follows from the identity (3.33) and Taylor's formula.

For part (a), by Lemma 3.15, we have $\Lambda''(s) \geq 0$. Since $\Lambda = \log \kappa$ and it is shown in Proposition 3.1 that the function κ is real-valued and strictly positive on $(-\eta, \eta)$, we get $\Lambda'''(s) \in \mathbb{R}$.

For part (b), recall that it was shown in [6] that $\sigma_0 > 0$ for invertible matrices under the stated conditions, and for positive matrices under the additional condition of non-arithmeticity. Hence, using the continuity of the function Λ'' , we obtain that $\sigma_s > 0$.

For part (c), by Proposition 3.8, we get that for $|z|$ small,

$$\mathbb{E}_{\mathbb{Q}_s^z} \left[e^{z(\log |G_n x| - n\Lambda(s))} \right] = \lambda_{s,z}^n (\Pi_{s,z} \mathbf{1})(x) + (N_{s,z}^n \mathbf{1})(x). \quad (3.49)$$

It follows from (3.46) that for $|z| = o(n^{-1/3})$,

$$\lambda_{s,z}^n = 1 + n\sigma_s^2 \frac{z^2}{2} + n\Lambda'''(s) \frac{z^3}{6} + o(n|z|^3). \quad (3.50)$$

Using Taylor's formula, the bound (3.36) and the fact $\Pi_{s,0}\mathbf{1} = \mathbf{1}$, we obtain

$$(\Pi_{s,z}\mathbf{1})(x) = 1 + c_{s,x,1}z + c_{s,x,2}z^2 + c_{s,x,3}z^3 + o(|z|^3), \quad (3.51)$$

where the constants $c_{s,x,1}, c_{s,x,2}, c_{s,x,3} \in \mathbb{C}$ are bounded as functions of s and x . Similarly, using the fact $N_{s,0}\mathbf{1} = 0$ and the bound (3.37), there exist constants $C_{s,x,n,1}, C_{s,x,n,2}, C_{s,x,n,3} \in \mathbb{C}$ which are bounded as functions of s, x and n such that

$$(N_{s,z}^n\mathbf{1})(x) = C_{s,x,n,1}z + C_{s,x,n,2}z^2 + C_{s,x,n,3}z^3 + o(|z|^3). \quad (3.52)$$

Taking the second derivative on both sides of the equation (3.49) with respect to z at 0, and using the expansions (3.50)-(3.52), we deduce that

$$\mathbb{E}_{\mathbb{Q}_s^x} [\log |G_n x| - n\Lambda'(s)]^2 = n\sigma_s^2 + 2c_{s,x,2} + 2C_{s,x,n,2}. \quad (3.53)$$

This, together with the definition of \mathbb{Q}_s and the fact that the constants $c_{s,x,2}, C_{s,x,n,2}$ are bounded as functions of s, x, n , concludes the proof of part (c).

For part (d), integrating both sides of the equations (3.49), (3.51) and (3.52) with respect to the invariant measure π_s , and using the property (3.43) with $\varphi = \mathbf{1}$ (thus the second term on the right-hand side of (3.51) vanishes), in the same way as in the proof of (3.53), we get

$$\mathbb{E}_{\mathbb{Q}_s} [\log |G_n x| - n\Lambda'(s)]^3 = n\Lambda'''(s) + 6c_{s,3} + 6C_{s,n,3}.$$

This implies the assertion in part (d). \square

Remark 3.16. Inspecting the proof of Proposition 3.14, it is easy to see that the results in parts (c) and (d) can be reinforced to the following bounds:

$$\begin{aligned} \sup_{s \in (-\eta, \eta)} \sup_{x \in \mathcal{S}} \left| \frac{1}{n} \mathbb{E}_{\mathbb{Q}_s^x} [\log |G_n x| - n\Lambda'(s)]^2 - \sigma_s^2 \right| &\leq \frac{C}{n}, \\ \sup_{s \in (-\eta, \eta)} \left| \frac{1}{n} \mathbb{E}_{\mathbb{Q}_s} [\log |G_n x| - n\Lambda'(s)]^3 - \Lambda'''(s) \right| &\leq \frac{C}{n}. \end{aligned}$$

The first bound above also holds with the measure \mathbb{Q}_s^x replaced by \mathbb{Q}_s .

4. SMOOTHING INEQUALITY ON THE COMPLEX PLANE

In this section we aim to establish a new smoothing inequality, which plays a crucial role in proving the Berry-Esseen theorem and Edgeworth expansion with a target function φ on X_n^x ; see Theorems 2.1, 2.2, 5.1 and 5.3.

From now on, for any integrable function $h : \mathbb{R} \rightarrow \mathbb{C}$, denote its Fourier transform by $\widehat{h}(t) = \int_{\mathbb{R}} e^{-ity} h(y) dy$, $t \in \mathbb{R}$. If \widehat{h} is integrable on \mathbb{R} , then using the inverse Fourier transform gives $h(y) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ity} \widehat{h}(t) dt$, for almost

all $y \in \mathbb{R}$ with respect to the Lebesgue measure on \mathbb{R} . Denote by $h_1 * h_2$ the convolution of the functions h_1, h_2 on the real line. For any $r > 0$, set

$$D_r = \{z \in \mathbb{C} : |z| < r, \Im z \neq 0\} \quad \text{and} \quad \bar{D}_r = \{z \in \mathbb{C} : |z| \leq r\}.$$

We have the decomposition $D_r = D_r^+ \cup D_r^-$, where

$$D_r^+ = \{z \in \mathbb{C} : |z| < r, \Im z > 0\} \quad \text{and} \quad D_r^- = \{z \in \mathbb{C} : |z| < r, \Im z < 0\}.$$

Following [37], we construct a density function ρ_T which plays an important role in establishing a new smoothing inequality. Specifically, on the real line define $\hat{\zeta}(t) = e^{-\frac{1}{1-t^2}}$ if $t \in [-1, 1]$, and $\hat{\zeta} = 0$ elsewhere. Let $\rho = 2\pi[\hat{\zeta} * \hat{\zeta}(0)]^{-1}\zeta^2$. By the inversion formula,

$$\rho(y) = \frac{1}{2\pi} \left(\int_{-1}^1 e^{-\frac{2}{1-x^2}} dx \right)^{-1} \left(\int_{-1}^1 \cos(yx) e^{-\frac{1}{1-x^2}} dx \right)^2, \quad y \in \mathbb{R}.$$

Then ρ is a non-negative Schwartz function with $\int_{\mathbb{R}} \rho(y) dy = 1$. Its Fourier transform $\hat{\rho}$ is given by

$$\hat{\rho}(t) = \left(\int_{-1}^1 e^{-\frac{2}{1-x^2}} dx \right)^{-1} \left(\int_{-1}^1 e^{-\frac{1}{1-(t-y)^2}} e^{-\frac{1}{1-y^2}} \mathbb{1}_{\{|t-y| \leq 1\}} dy \right), \quad t \in \mathbb{R}.$$

We see that $\hat{\rho}$ is compactly supported on $[-2, 2]$. Moreover, it is proved in [37] that $\hat{\rho}$ has an analytic extension on the domain $D_1 := \{z \in \mathbb{C} : |z| < 1, \Im z \neq 0\}$ and has a continuous extension on the domain $\bar{D}_1 = \{z \in \mathbb{C} : |z| \leq 1\}$. The Hölder inequality implies that ρ is bounded by $\frac{1}{\pi}$. Since ρ is a density function on \mathbb{R} and $\hat{\rho}$ is non-negative, we have $0 \leq \hat{\rho} \leq 1$ on \mathbb{R} .

For any $T > 0$ and the fixed constant $b > 0$ satisfying $\int_{-b}^b \rho(y) dy = 3/4$, define the density function

$$\rho_T(y) = \frac{T}{2} \rho\left(\frac{T}{2}y - b\right), \quad y \in \mathbb{R},$$

whose Fourier transform $\hat{\rho}_T$ is given by $\hat{\rho}_T(t) = e^{-2ibt/T} \hat{\rho}(2t/T)$, $t \in \mathbb{R}$. We see that on the real line $\hat{\rho}_T$ is compactly supported on $[-T, T]$ since $\hat{\rho}$ is compactly supported on $[-2, 2]$. Since $\hat{\rho}$ has an analytic extension on the domain D_1 , we can extend the function $\hat{\rho}_T$ analytically to the domain D_T as follows:

$$\hat{\rho}_T(z) = e^{-2ibz/T} \hat{\rho}(2z/T), \quad z \in D_T. \quad (4.1)$$

Note that $\hat{\rho}_T$ has a continuous extension on the domain \bar{D}_T since $\hat{\rho}$ has a continuous extension on \bar{D}_1 .

Now we are ready to establish a new smoothing inequality. Its proof is based on the properties of the density function ρ_T , Cauchy's integral theorem and some techniques from [12, 30].

Proposition 4.1. *Assume that F is non-decreasing on \mathbb{R} , and that H is differentiable of bounded variation on \mathbb{R} such that $\sup_{y \in \mathbb{R}} |H'(y)| < \infty$. Suppose that $F(-\infty) = H(-\infty)$ and $F(\infty) = H(\infty)$. Let*

$$f(t) = \int_{\mathbb{R}} e^{-ity} dF(y) \quad \text{and} \quad h(t) = \int_{\mathbb{R}} e^{-ity} dH(y), \quad t \in \mathbb{R}.$$

Suppose that $r > 0$ and that f and h have analytic extensions on D_r , and have continuous extensions on \overline{D}_r . Then, for any $T \geq r$,

$$\begin{aligned} \sup_{y \in \mathbb{R}} |F(y) - H(y)| \leq & \frac{1}{\pi} \sup_{y \leq 0} \left| \int_{\mathcal{C}_r^-} \frac{f(z) - h(z)}{z} e^{izy} \widehat{\rho}_T(-z) dz \right| \\ & + \frac{1}{\pi} \sup_{y > 0} \left| \int_{\mathcal{C}_r^+} \frac{f(z) - h(z)}{z} e^{izy} \widehat{\rho}_T(-z) dz \right| \\ & + \frac{1}{\pi} \sup_{y \leq 0} \left| \int_{\mathcal{C}_r^-} \frac{f(z) - h(z)}{z} e^{izy} \widehat{\rho}_T(z) dz \right| \\ & + \frac{1}{\pi} \sup_{y > 0} \left| \int_{\mathcal{C}_r^+} \frac{f(z) - h(z)}{z} e^{izy} \widehat{\rho}_T(z) dz \right| \\ & + \frac{1}{\pi} \int_{r \leq |t| \leq T} \left| \frac{f(t) - h(t)}{t} \right| dt + \frac{6b}{T} \sup_{y \in \mathbb{R}} |H'(y)|, \end{aligned}$$

where $b > 0$ is a fixed constant satisfying $\int_{-b}^b \rho(y) dy = 3/4$, and the semi-circles \mathcal{C}_r^- and \mathcal{C}_r^+ are given by

$$\mathcal{C}_r^- = \{z \in \mathbb{C} : |z| = r, \Im z < 0\}, \quad \mathcal{C}_r^+ = \{z \in \mathbb{C} : |z| = r, \Im z > 0\}. \quad (4.2)$$

Proof. Let $T \geq r$. From the definition of ρ_T and the choice of the constant b , we have $\int_0^{4b/T} \rho_T(y) dy = 3/4$. Since $\rho \leq 1/\pi$, the function ρ_T is bounded by $T/2\pi$. The proof of Proposition 4.1 consists in establishing first an upper bound and then a lower bound.

Upper bound. Since the function F is non-decreasing on \mathbb{R} and ρ_T is a density function on \mathbb{R} , we find that for any $y \in \mathbb{R}$,

$$\begin{aligned} F(y) & \leq \frac{4}{3} \int_y^{y+\frac{4b}{T}} F(u) \rho_T(u-y) du \\ & = H(y) + \frac{4}{3} \int_y^{y+\frac{4b}{T}} \left[(F(u) - H(u)) \rho_T(u-y) + (H(u) - H(y)) \rho_T(u-y) \right] du \\ & \leq H(y) + \frac{4}{3} \int_y^{y+\frac{4b}{T}} (F(u) - H(u)) \rho_T(u-y) du + \frac{4b}{T} \sup_{y \in \mathbb{R}} |H'(y)|. \end{aligned} \quad (4.3)$$

Let $F_1(y) = \int_{\mathbb{R}} F(u) \rho_T(u-y) du$, and $H_1(y) = \int_{\mathbb{R}} H(u) \rho_T(u-y) du$, $y \in \mathbb{R}$. Elementary calculations lead to

$$\int_{\mathbb{R}} e^{-ity} dF_1(y) = f(t) \widehat{\rho}_T(-t), \quad \int_{\mathbb{R}} e^{-ity} dH_1(y) = h(t) \widehat{\rho}_T(-t), \quad t \in \mathbb{R}.$$

Restricted on the real line, the function $\widehat{\rho}_T$ is supported on $[-T, T]$. By the inversion formula we get

$$\begin{aligned} F_1(y) - F_1(v) &= \frac{1}{2\pi} \int_{-T}^T \frac{e^{ity} - e^{itv}}{it} f(t) \widehat{\rho}_T(-t) dt, & y, v \in \mathbb{R}, \\ H_1(y) - H_1(v) &= \frac{1}{2\pi} \int_{-T}^T \frac{e^{ity} - e^{itv}}{it} h(t) \widehat{\rho}_T(-t) dt & y, v \in \mathbb{R}. \end{aligned}$$

We shall use Cauchy's integral theorem to change the integration path $[-T, T]$ to a contour in the complex plane. In order to estimate the difference $|F_1(y) - H_1(y)|$, we are led to consider two cases: $y \leq 0$ and $y > 0$.

Control of $|F_1(y) - H_1(y)|$ when $y \leq 0$. Let $\mathcal{C}_- = \mathcal{C}_{r,T} \cup \mathcal{C}_r^-$, where $\mathcal{C}_{r,T} = [-T, -r] \cup [r, T]$ and the lower semicircle \mathcal{C}_r^- is given in (4.2). Since the functions f , h and $\widehat{\rho}_T$ are analytic on the domain D_r , and have continuous extensions on its closure \overline{D}_r , applying Cauchy's integral theorem gives

$$\begin{aligned} F_1(y) - H_1(y) - (F_1(v) - H_1(v)) &= \\ \frac{1}{2\pi} \left[\int_{\mathcal{C}_-} \frac{f(z) - h(z)}{iz} e^{izy} \widehat{\rho}_T(-z) dz - \int_{\mathcal{C}_-} \frac{f(z) - h(z)}{iz} e^{izv} \widehat{\rho}_T(-z) dz \right], \end{aligned} \quad (4.4)$$

where the integration is over the complex curve \mathcal{C}_- oriented from $-T$ to T . The second integral in (4.4) converges to 0 as $v \rightarrow -\infty$, by using the Riemann-Lebesgue lemma on the real segment $\mathcal{C}_{r,T}$ and by applying the Lebesgue convergence theorem on the semicircle \mathcal{C}_r^- . Note that $F_1(-\infty) = H_1(-\infty)$ since $F(-\infty) = H(-\infty)$. Consequently, letting $v \rightarrow -\infty$ in (4.4), we get

$$F_1(y) - H_1(y) = \frac{1}{2\pi} \int_{\mathcal{C}_-} \frac{f(z) - h(z)}{iz} e^{izy} \widehat{\rho}_T(-z) dz,$$

and hence

$$\begin{aligned} \sup_{y \leq 0} |F_1(y) - H_1(y)| &\leq \frac{1}{2\pi} \int_{\mathcal{C}_{r,T}} \left| \frac{f(t) - h(t)}{t} \right| dt \\ &+ \frac{1}{2\pi} \sup_{y \leq 0} \left| \int_{\mathcal{C}_r^-} \frac{f(z) - h(z)}{z} e^{izy} \widehat{\rho}_T(-z) dz \right|. \end{aligned} \quad (4.5)$$

Control of $|F_1(y) - H_1(y)|$ when $y > 0$. Let $\mathcal{C}_+ = \mathcal{C}_{r,T} \cup \mathcal{C}_r^+$, where $\mathcal{C}_{r,T} = [-T, -r] \cup [r, T]$ and the upper semicircle \mathcal{C}_r^+ is given in (4.2). In an analogous way as in (4.4), we have

$$\begin{aligned} F_1(y) - H_1(y) - (F_1(v) - H_1(v)) &= \\ \frac{1}{2\pi} \left[\int_{\mathcal{C}_+} \frac{f(z) - h(z)}{iz} e^{izy} \widehat{\rho}_T(-z) dz - \int_{\mathcal{C}_+} \frac{f(z) - h(z)}{iz} e^{izv} \widehat{\rho}_T(-z) dz \right], \end{aligned} \quad (4.6)$$

where the integration is over the complex curve \mathcal{C}_+ also oriented from $-T$ to T . The second integral in (4.6) converges to 0 as $v \rightarrow +\infty$, by using again

the Riemann-Lebesgue lemma on the real segment $\mathcal{C}_{r,T}$ and by applying the Lebesgue convergence theorem on the upper semicircle \mathcal{C}_r^+ . Note that $F_1(\infty) = H_1(\infty)$ since $F(\infty) = H(\infty)$. Hence, letting $v \rightarrow +\infty$ in (4.6), similarly to (4.5), we obtain

$$\begin{aligned} \sup_{y>0} |F_1(y) - H_1(y)| &\leq \frac{1}{2\pi} \int_{\mathcal{C}_{r,T}} \left| \frac{f(t) - h(t)}{t} \right| dt \\ &+ \frac{1}{2\pi} \sup_{y>0} \left| \int_{\mathcal{C}_r^+} \frac{f(z) - h(z)}{z} e^{izy} \widehat{\rho}_T(-z) dz \right|. \end{aligned} \quad (4.7)$$

As a result, putting together (4.5) and (4.7) leads to

$$\begin{aligned} \sup_{y \in \mathbb{R}} |F_1(y) - H_1(y)| &\leq \frac{1}{2\pi} \int_{\mathcal{C}_{r,T}} \left| \frac{f(t) - h(t)}{t} \right| dt \\ &+ \frac{1}{2\pi} \sup_{y \leq 0} \left| \int_{\mathcal{C}_r^-} \frac{f(z) - h(z)}{z} e^{izy} \widehat{\rho}_T(-z) dz \right| \\ &+ \frac{1}{2\pi} \sup_{y > 0} \left| \int_{\mathcal{C}_r^+} \frac{f(z) - h(z)}{z} e^{izy} \widehat{\rho}_T(-z) dz \right|. \end{aligned} \quad (4.8)$$

Denote $\Delta = \sup_{y \in \mathbb{R}} |F(y) - H(y)|$. Then, taking into account that ρ_T is a density function on \mathbb{R} , using (4.8) and the fact that $\int_0^{4b/T} \rho_T(y) dy = 3/4$, we get

$$\begin{aligned} &\left| \int_y^{y+\frac{4b}{T}} (F(u) - H(u)) \rho_T(u-y) du \right| \\ &\leq |F_1(y) - H_1(y)| + \Delta \left(1 - \int_0^{4b/T} \rho_T(u) du \right) \\ &\leq \frac{1}{2\pi} \int_{\mathcal{C}_{r,T}} \left| \frac{f(t) - h(t)}{t} \right| dt + \frac{1}{2\pi} \sup_{y \leq 0} \left| \int_{\mathcal{C}_r^-} \frac{f(z) - h(z)}{z} e^{izy} \widehat{\rho}_T(-z) dz \right| \\ &\quad + \frac{1}{2\pi} \sup_{y > 0} \left| \int_{\mathcal{C}_r^+} \frac{f(z) - h(z)}{z} e^{izy} \widehat{\rho}_T(-z) dz \right| + \frac{\Delta}{4}. \end{aligned}$$

Substituting this inequality into (4.3), we obtain the following upper bound:

$$\begin{aligned} F(y) - H(y) &\leq \frac{2}{3\pi} \int_{\mathcal{C}_{r,T}} \left| \frac{f(t) - h(t)}{t} \right| dt \\ &+ \frac{2}{3\pi} \sup_{y \leq 0} \left| \int_{\mathcal{C}_r^-} \frac{f(z) - h(z)}{z} e^{izy} \widehat{\rho}_T(-z) dz \right| \\ &+ \frac{2}{3\pi} \sup_{y > 0} \left| \int_{\mathcal{C}_r^+} \frac{f(z) - h(z)}{z} e^{izy} \widehat{\rho}_T(-z) dz \right| + \frac{\Delta}{3} + \frac{4b}{T} \sup_{y \in \mathbb{R}} |H'(y)|. \end{aligned} \quad (4.9)$$

Lower bound. Similarly to (4.3), we have for any $y \in \mathbb{R}$,

$$\begin{aligned} F(y) &\geq \frac{4}{3} \int_{y-\frac{4b}{T}}^y F(u) \rho_T(y-u) du \\ &\geq H(y) + \frac{4}{3} \int_{y-\frac{4b}{T}}^y (F(u) - H(u)) \rho_T(y-u) du - \frac{4b}{T} \sup_{y \in \mathbb{R}} |H'(y)|. \end{aligned}$$

Let $F_2(y) = (F * \rho_T)(y)$ and $H_2(y) = (H * \rho_T)(y)$, $y \in \mathbb{R}$, then

$$\int_{\mathbb{R}} e^{-ity} dF_2(y) = f(t) \hat{\rho}_T(t), \quad \int_{\mathbb{R}} e^{-ity} dH_2(y) = h(t) \hat{\rho}_T(t), \quad t \in \mathbb{R}.$$

Proceeding in the same way as in the proof of (4.8), we get

$$\begin{aligned} \sup_{y \in \mathbb{R}} |F_2(y) - H_2(y)| &\leq \frac{1}{2\pi} \int_{C_{r,T}} \left| \frac{f(t) - h(t)}{t} \right| dt \\ &\quad + \frac{1}{2\pi} \sup_{y \leq 0} \left| \int_{C_r^-} \frac{f(z) - h(z)}{z} e^{izy} \hat{\rho}_T(z) dz \right| \\ &\quad + \frac{1}{2\pi} \sup_{y > 0} \left| \int_{C_r^+} \frac{f(z) - h(z)}{z} e^{izy} \hat{\rho}_T(z) dz \right|. \end{aligned}$$

Following the proof of (4.9), we obtain the lower bound:

$$\begin{aligned} &F(y) - H(y) \\ &\geq -\frac{2}{3\pi} \int_{C_{r,T}} \left| \frac{f(t) - h(t)}{t} \right| dt - \frac{2}{3\pi} \sup_{y \leq 0} \left| \int_{C_r^-} \frac{f(z) - h(z)}{z} e^{izy} \hat{\rho}_T(z) dz \right| \\ &\quad - \frac{2}{3\pi} \sup_{y > 0} \left| \int_{C_r^+} \frac{f(z) - h(z)}{z} e^{izy} \hat{\rho}_T(z) dz \right| - \frac{\Delta}{3} - \frac{4b}{T} \sup_{y \in \mathbb{R}} |H'(y)|. \quad (4.10) \end{aligned}$$

Combining (4.9) and (4.10), we conclude the proof of Proposition 4.1. \square

5. PROOFS OF BERRY-ESSEEN BOUND AND EDGEWORTH EXPANSION

5.1. Berry-Esseen bound and Edgeworth expansion under the changed measure. We first formulate a Berry-Esseen bound under the changed measure \mathbb{Q}_s^x .

Theorem 5.1. *Assume either conditions **A1** and **A2** for invertible matrices, or conditions **A1**, **A3** and **A4** for positive matrices. Then there exist constants $\eta > 0$ and $C > 0$ such that for all $n \geq 1$, $s \in (-\eta, \eta)$, $x \in \mathcal{S}$, $y \in \mathbb{R}$ and $\varphi \in \mathcal{B}_\gamma$,*

$$\left| \mathbb{E}_{\mathbb{Q}_s^x} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log |G_n x| - n\Lambda'(s)}{\sigma_s \sqrt{n}} \leq y \right\}} \right] - \pi_s(\varphi) \Phi(y) \right| \leq C \frac{\|\varphi\|_\gamma}{\sqrt{n}}. \quad (5.1)$$

The following result gives an Edgeworth expansion for $\log |G_n x|$ with the target function φ on X_n^x under \mathbb{Q}_s^x . The function $b_{s,\varphi}(x)$, $x \in \mathcal{S}$, which will be used in the formulation of this result, is defined in Lemma 3.10 and

has an equivalent expression (3.39) in terms of derivative of the projection operator $\Pi_{s,z}$, see Proposition 3.8.

Theorem 5.2. *Assume either conditions A1 and A2 for invertible matrices, or conditions A1, A3 and A5 for positive matrices. Then there exists $\eta > 0$ such that as $n \rightarrow \infty$, uniformly in $s \in (-\eta, \eta)$, $x \in \mathcal{S}$, $y \in \mathbb{R}$ and $\varphi \in \mathcal{B}_\gamma$,*

$$\left| \mathbb{E}_{\mathbb{Q}_s^x} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log |G_{nx}| - n\Lambda'(s)}{\sigma_s \sqrt{n}} \leq y \right\}} \right] - \mathbb{E}_{\mathbb{Q}_s^x} [\varphi(X_n^x)] \left[\Phi(y) + \frac{\Lambda'''(s)}{6\sigma_s^3 \sqrt{n}} (1-y^2) \phi(y) \right] + \frac{b_{s,\varphi}(x)}{\sigma_s \sqrt{n}} \phi(y) \right| = \|\varphi\|_\gamma o\left(\frac{1}{\sqrt{n}}\right).$$

The following asymptotic expansion is slightly different from that in Theorem 5.2, with the term $\mathbb{E}_{\mathbb{Q}_s^x} [\varphi(X_n^x)]$ replaced by $\pi_s(\varphi)$.

Theorem 5.3. *Under the conditions of Theorem 5.2, there exists $\eta > 0$ such that, as $n \rightarrow \infty$, uniformly in $s \in (-\eta, \eta)$, $x \in \mathcal{S}$, $y \in \mathbb{R}$ and $\varphi \in \mathcal{B}_\gamma$,*

$$\left| \mathbb{E}_{\mathbb{Q}_s^x} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log |G_{nx}| - n\Lambda'(s)}{\sigma_s \sqrt{n}} \leq y \right\}} \right] - \pi_s(\varphi) \left[\Phi(y) + \frac{\Lambda'''(s)}{6\sigma_s^3 \sqrt{n}} (1-y^2) \phi(y) \right] + \frac{b_{s,\varphi}(x)}{\sigma_s \sqrt{n}} \phi(y) \right| = \|\varphi\|_\gamma o\left(\frac{1}{\sqrt{n}}\right). \quad (5.2)$$

With fixed $s > 0$ and $\varphi = \mathbf{1}$, the expansion (5.2) has been established earlier in [6].

The assertion of Theorem 5.3 follows from Theorem 5.2, since the bound (3.17) implies that there exist constants $c, C > 0$ such that uniformly in $\varphi \in \mathcal{B}_\gamma$,

$$\sup_{s \in (-\eta, \eta)} \sup_{x \in \mathcal{S}} |\mathbb{E}_{\mathbb{Q}_s^x} [\varphi(X_n^x)] - \pi_s(\varphi)| \leq C e^{-cn} \|\varphi\|_\gamma. \quad (5.3)$$

Theorems 2.1 and 2.2 follow from the above theorems taking $s = 0$ and recalling the fact that $\Lambda'(0) = \lambda$, $\sigma_0 = \sigma$ and $b_{0,\varphi} = b_\varphi$.

5.2. Proof of Theorem 5.2. Without loss of generality, we assume that φ is non-negative. Denote

$$F(y) = \mathbb{E}_{\mathbb{Q}_s^x} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log |G_{nx}| - n\Lambda'(s)}{\sigma_s \sqrt{n}} \leq y \right\}} \right], \quad y \in \mathbb{R},$$

$$H(y) = \mathbb{E}_{\mathbb{Q}_s^x} [\varphi(X_n^x)] \left[\Phi(y) + \frac{\Lambda'''(s)}{6\sigma_s^3 \sqrt{n}} (1-y^2) \phi(y) \right] - \frac{b_{s,\varphi}(x)}{\sigma_s \sqrt{n}} \phi(y), \quad y \in \mathbb{R}.$$

Define

$$f(t) = \int_{\mathbb{R}} e^{-ity} dF(y), \quad h(t) = \int_{\mathbb{R}} e^{-ity} dH(y), \quad t \in \mathbb{R}.$$

By straightforward calculations we have

$$f(t) = \mathbb{E}_{\mathbb{Q}_s^x} \left[\varphi(X_n^x) e^{-it \frac{\log |G_n x| - n\Lambda'(s)}{\sigma_s \sqrt{n}}} \right] = R_{s, \frac{-it}{\sigma_s \sqrt{n}}}^n \varphi(x), \quad t \in \mathbb{R}, \quad (5.4)$$

$$h(t) = e^{-\frac{t^2}{2}} \left\{ \left[1 - (it)^3 \frac{\Lambda'''(s)}{6\sigma_s^3 \sqrt{n}} \right] R_{s,0} \varphi(x) - it \frac{b_{s,\varphi}(x)}{\sigma_s \sqrt{n}} \right\}, \quad t \in \mathbb{R}. \quad (5.5)$$

It is clear that $F(-\infty) = H(-\infty) = 0$ and $F(\infty) = H(\infty)$. Moreover, one can verify that the functions F, H and their corresponding Fourier-Stieltjes transforms f, h satisfy the conditions of Proposition 4.1 for $r = \delta_1 \sqrt{n}$, with some $\delta_1 > 0$ sufficiently small. Then, for any real $T \geq r$,

$$\sup_{y \in \mathbb{R}} |F(y) - H(y)| \leq \frac{1}{\pi} (I_1 + I_2 + I_3), \quad (5.6)$$

where

$$\begin{aligned} I_1 &= \frac{6\pi b}{T} \sup_{y \in \mathbb{R}} |H'(y)|, \quad I_2 = \int_{r \leq |t| \leq T} \left| \frac{f(t) - h(t)}{t} \right| dt, \\ I_3 &= \sup_{y \leq 0} \left| \int_{\mathcal{C}_r^-} \frac{f(z) - h(z)}{z} e^{izy} \widehat{\rho}_T(-z) dz \right| \\ &\quad + \sup_{y > 0} \left| \int_{\mathcal{C}_r^+} \frac{f(z) - h(z)}{z} e^{izy} \widehat{\rho}_T(-z) dz \right| \\ &\quad + \sup_{y \leq 0} \left| \int_{\mathcal{C}_r^-} \frac{f(z) - h(z)}{z} e^{izy} \widehat{\rho}_T(z) dz \right| \\ &\quad + \sup_{y > 0} \left| \int_{\mathcal{C}_r^+} \frac{f(z) - h(z)}{z} e^{izy} \widehat{\rho}_T(z) dz \right| \\ &= I_{31} + I_{32} + I_{33} + I_{34}, \end{aligned} \quad (5.7)$$

with the constant $b > 0$ and the complex contours $\mathcal{C}_r^-, \mathcal{C}_r^+$ defined in (4.2).

By virtue of (5.6), in order to prove Theorem 5.2 it suffices to show that uniformly in $s \in (-\eta, \eta)$, $x \in \mathcal{S}$ and $\varphi \in \mathcal{B}_\gamma$,

$$I_1 + I_2 + I_3 = \|\varphi\|_\gamma o\left(\frac{1}{\sqrt{n}}\right). \quad (5.8)$$

Control of I_1 . From (5.3) we deduce that uniformly in $\varphi \in \mathcal{B}_\gamma$,

$$\sup_{s \in (-\eta, \eta)} \sup_{x \in \mathcal{S}} |\mathbb{E}_{\mathbb{Q}_s^x} [\varphi(X_n^x)]| \leq C \|\varphi\|_\gamma. \quad (5.9)$$

By the formula (3.39) and the bound (3.36), we get that uniformly in $\varphi \in \mathcal{B}_\gamma$,

$$\sup_{s \in (-\eta, \eta)} \sup_{x \in \mathcal{S}} |b_{s,\varphi}(x)| \leq C \|\varphi\|_\gamma. \quad (5.10)$$

Using the bounds (5.9) and (5.10), and taking into account that $\sigma_s^2 > 0$ and $\Lambda'''(s) \in \mathbb{R}$ are bounded by a constant independent of $s \in (-\eta, \eta)$, we obtain

that $|H'(y)|$ is bounded by $c_1\|\varphi\|_\gamma$, uniformly in $s \in (-\eta, \eta)$, $x \in \mathcal{S}$, $y \in \mathbb{R}$ and $\varphi \in \mathcal{B}_\gamma$. Hence, for any $\varepsilon > 0$, we can choose $a > 0$ large enough, such that, for $T = a\sqrt{n}$, uniformly in $\varphi \in \mathcal{B}_\gamma$, we have

$$\sup_{s \in (-\eta, \eta)} \sup_{x \in \mathcal{S}} I_1 \leq \frac{6\pi bc_1}{T} \|\varphi\|_\gamma < \frac{\varepsilon}{\sqrt{n}} \|\varphi\|_\gamma. \quad (5.11)$$

Control of I_2 . Since $\sigma_m := \inf_{s \in (-\eta, \eta)} \sigma_s > 0$, we can pick δ_1 small enough, such that $0 < \delta_1 < \min\{a, \delta\sigma_m/2\}$, where the constant $\delta > 0$ is given in Proposition 3.8. Then with $r = \delta_1\sqrt{n}$ we bound I_2 as follows:

$$I_2 \leq \int_{\delta_1\sqrt{n} < |t| \leq a\sqrt{n}} \left| \frac{f(t)}{t} \right| dt + \int_{\delta_1\sqrt{n} < |t| \leq a\sqrt{n}} \left| \frac{h(t)}{t} \right| dt. \quad (5.12)$$

Let $\sigma_M := \sup_{s \in (-\eta, \eta)} \sigma_s < \infty$. On the right-hand side of (5.12), using Proposition 3.9 with $K = \{t \in \mathbb{R} : \delta_1/\sigma_M \leq |t| \leq a/\sigma_m\}$, the first integral is bounded by $Ce^{-cn}\|\varphi\|_\gamma$, uniformly in $s \in (-\eta, \eta)$, $x \in \mathcal{S}$ and $\varphi \in \mathcal{B}_\gamma$; the second integral, by the bounds (5.9) and (5.10) and direct calculations, is bounded by $Ce^{-c\sqrt{n}}\|\varphi\|_\gamma$, also uniformly in $s \in (-\eta, \eta)$, $x \in \mathcal{S}$ and $\varphi \in \mathcal{B}_\gamma$. Consequently, we conclude that uniformly in $\varphi \in \mathcal{B}_\gamma$,

$$\sup_{s \in (-\eta, \eta)} \sup_{x \in \mathcal{S}} I_2 \leq Ce^{-c\sqrt{n}} \|\varphi\|_\gamma. \quad (5.13)$$

Control of I_3 . Recall that the term I_3 is decomposed into four terms in (5.7). We will only deal with I_{31} , since I_{32}, I_{33}, I_{34} can be treated in a similar way. In view of (5.4) and (5.5), by the spectral gap decomposition (3.32), we get

$$f(z) - h(z) = J_1(z) + J_2(z) + J_3(z) + J_4(z), \quad (5.14)$$

where

$$J_1(z) = \pi_s(\varphi) \left\{ \lambda_{s, \frac{-iz}{\sigma_s\sqrt{n}}}^n - e^{-\frac{z^2}{2}} \left[1 - (iz)^3 \frac{\Lambda'''(s)}{6\sigma_s^3\sqrt{n}} \right] \right\}, \quad (5.15)$$

$$J_2(z) = \lambda_{s, \frac{-iz}{\sigma_s\sqrt{n}}}^n \left[\Pi_{s, \frac{-iz}{\sigma_s\sqrt{n}}} \varphi(x) - \pi_s(\varphi) - iz \frac{b_{s, \varphi}(x)}{\sigma_s\sqrt{n}} \right], \quad (5.16)$$

$$J_3(z) = iz \frac{b_{s, \varphi}(x)}{\sigma_s\sqrt{n}} \left(\lambda_{s, \frac{-iz}{\sigma_s\sqrt{n}}}^n - e^{-\frac{z^2}{2}} \right), \quad (5.17)$$

$$J_4(z) = N_{s, \frac{-iz}{\sigma_s\sqrt{n}}}^n \varphi(x) + N_{s, 0}^n \varphi(x) e^{-\frac{z^2}{2}} \left[1 - (iz)^3 \frac{\Lambda'''(s)}{6\sigma_s^3\sqrt{n}} \right]. \quad (5.18)$$

With the above notation, we use the decomposition (5.14) to bound I_{31} in (5.7) as follows:

$$I_{31} \leq \sum_{k=1}^4 A_k, \quad \text{where} \quad A_k = \sup_{y \leq 0} \left| \int_{\mathcal{C}_r^-} \frac{J_k(z)}{z} e^{izy} \widehat{\rho}_T(-z) dz \right|. \quad (5.19)$$

We now give bounds of A_k , $1 \leq k \leq 4$, in a series of lemmata. Let us start by giving an elementary inequality, which will be used repeatedly in the sequel. Let $[z_1, z_2] = \{z_1 + \theta(z_2 - z_1) : 0 \leq \theta \leq 1\}$ be the complex segment with the endpoints z_1 and z_2 .

Lemma 5.4. *Let f be an analytic function on the open convex domain $D \subseteq \mathbb{C}$. Then for any $z_1, z_2 \in D$, and $n \geq 1$,*

$$\left| f(z_2) - \sum_{k=0}^{n-1} \frac{f^{(k)}(z_1)}{k!} (z_2 - z_1)^k \right| \leq \frac{\sup_{z \in [z_1, z_2]} |f^{(n)}(z)|}{n!} |z_2 - z_1|^n.$$

Proof. The proof of this inequality can be carried out by induction. The inequality clearly holds for $n = 1$ since for any $z_1, z_2 \in D$,

$$|f(z_2) - f(z_1)| = \left| \int_{[z_1, z_2]} f'(z) dz \right| \leq \sup_{z \in [z_1, z_2]} |f'(z)| |z_2 - z_1|. \quad (5.20)$$

For $n \geq 2$, applying (5.20) to $F(z) = f(z) - \sum_{k=1}^{n-1} \frac{f^{(k)}(z_1)}{k!} (z - z_1)^k$, $z \in D$, leads to the desired assertion. \square

Now we are ready to establish a bound of each term A_k . The proof is based on the saddle point method. To be more precise, we deform the integration path, which passes through a suitable point related to the saddle point, to minimise the integral in A_k (see (5.19)).

Lemma 5.5. *Let \mathcal{C}_r^- be defined by (4.2) with $r = \delta_1 \sqrt{n}$ and $\delta_1 > 0$ small enough. Then, for $T = a\sqrt{n}$ with $a > 0$ large enough, uniformly in $x \in \mathcal{S}$, $s \in (-\eta, \eta)$ and $\varphi \in \mathcal{B}_\gamma$,*

$$A_1 = \sup_{y \leq 0} \left| \int_{\mathcal{C}_r^-} \frac{J_1(z)}{z} e^{izy} \widehat{\rho}_T(-z) dz \right| \leq \frac{c}{n} \|\varphi\|_\infty.$$

Proof. In view of (3.33), using $\Lambda = \log \kappa$ and Taylor's formula, we have

$$\lambda_{s, \frac{-iz}{\sigma_s \sqrt{n}}}^n = e^{-\frac{z^2}{2}} e^n \sum_{k=3}^{\infty} \frac{\Lambda^{(k)}(s)}{k!} \left(-\frac{iz}{\sigma_s \sqrt{n}}\right)^k. \quad (5.21)$$

For brevity, for any $z \in \mathcal{C}_r^-$, denote

$$h_1(z) = \frac{1}{z} \left[e^n \sum_{k=3}^{\infty} \frac{\Lambda^{(k)}(s)}{k!} \left(-\frac{iz}{\sigma_s \sqrt{n}}\right)^k - 1 - (-iz)^3 \frac{\Lambda'''(s)}{6\sigma_s^3 \sqrt{n}} \right] \widehat{\rho}_T(-z). \quad (5.22)$$

Then, in view of (5.15), the term A_1 can be rewritten as

$$A_1 = \pi_s(\varphi) \sup_{y \leq 0} \left| \int_{\mathcal{C}_r^-} e^{-\frac{z^2}{2} + izy} h_1(z) dz \right|. \quad (5.23)$$

The main contribution to the integral in (5.23) is given by the saddle point $z = iy$ which is the solution of the equation $\frac{d}{dz} \left(-\frac{z^2}{2} + izy\right) = 0$. Denote by

$D_{2r}^- = \{z \in \mathbb{C} : |z| < 2r, \Im z < 0\}$ the domain on analyticity of h_1 , where $r = \delta_1 \sqrt{n}$. Set

$$y_n = \min\{-y, \delta_1 \sqrt{n}\}. \quad (5.24)$$

When $-\delta_1 \sqrt{n} \leq y \leq 0$, the saddle point iy belongs to D_{2r}^- . By Cauchy's integral theorem, we change the integration in (5.23) to a rectangular path inside the domain on analyticity D_{2r}^- which passes through the saddle point. When $y < -\delta_1 \sqrt{n}$ is large, the saddle point iy is outside the domain D_{2r}^- . In this case we choose a rectangular path inside D_{2r}^- which passes through the point $-iy_n = -i\delta_1 \sqrt{n}$. Note that $\pi_s(\varphi)$ is bounded by $c_1 \|\varphi\|_\infty$ uniformly in $s \in (-\eta, \eta)$ and $\varphi \in \mathcal{B}_\gamma$. Since the function h_1 has an analytic extension on the domain D_{2r}^- with $r = \delta_1 \sqrt{n}$, applying Cauchy's integral theorem, we deduce that

$$\begin{aligned} A_1 &\leq c_1 \|\varphi\|_\infty \sup_{y \leq 0} \left| \left\{ \int_{-\delta_1 \sqrt{n}}^{-\delta_1 \sqrt{n} - iy_n} + \int_{\delta_1 \sqrt{n} - iy_n}^{\delta_1 \sqrt{n}} \right\} e^{-\frac{z^2}{2} + izy} h_1(z) dz \right| \\ &\quad + c_1 \|\varphi\|_\infty \sup_{y \leq 0} \left| \int_{-\delta_1 \sqrt{n} - iy_n}^{\delta_1 \sqrt{n} - iy_n} e^{-\frac{z^2}{2} + izy} h_1(z) dz \right| \\ &= c_1 \|\varphi\|_\infty (A_{11} + A_{12}). \end{aligned} \quad (5.25)$$

Control of A_{11} . Using a change of variable, we get

$$\begin{aligned} A_{11} &= e^{-\frac{\delta_1^2}{2}n} \sup_{y \leq 0} \left| \int_0^{y_n} e^{\frac{t^2}{2} + ty - i\delta_1 \sqrt{n}(t+y)} h_1(-\delta_1 \sqrt{n} - it) dt \right. \\ &\quad \left. - \int_0^{y_n} e^{\frac{t^2}{2} + ty + i\delta_1 \sqrt{n}(t+y)} h_1(\delta_1 \sqrt{n} - it) dt \right| \\ &\leq e^{-\frac{\delta_1^2}{2}n} \sup_{y \leq 0} \left| \int_0^{y_n} e^{\frac{t^2}{2} + ty} \left\{ |h_1(-\delta_1 \sqrt{n} - it)| + |h_1(\delta_1 \sqrt{n} - it)| \right\} dt \right|. \end{aligned} \quad (5.26)$$

We first bound $|h_1(\pm\delta_1 \sqrt{n} - it)|$. Since $t \in [0, y_n]$ and $y_n \leq \delta_1 \sqrt{n}$, direct calculations give

$$\Re[(-i)^3 (\pm\delta_1 \sqrt{n} - it)^3] = 3\delta_1^2 n t - t^3 \leq 2\delta_1^3 n^{3/2},$$

which implies that for $\delta_1 > 0$ sufficiently small,

$$\Re \left\{ n \sum_{k=3}^{\infty} \frac{\Lambda^{(k)}(s)}{k!} \frac{(-i)^k (\pm\delta_1 \sqrt{n} - it)^k}{(\sigma_s \sqrt{n})^k} \right\} \leq \frac{1}{4} \delta_1^2 n. \quad (5.27)$$

Observe that there exists a constant $c > 0$ such that uniformly in $t \in [0, y_n]$ and $s \in (-\eta, \eta)$,

$$\left| \frac{1}{z} \right| = \left| \frac{1}{\pm\delta_1 \sqrt{n} - it} \right| \leq \frac{c}{\delta_1 \sqrt{n}}, \quad \left| i^3 (\pm\delta_1 \sqrt{n} - it)^3 \frac{\Lambda'''(s)}{6\sigma_s^3 \sqrt{n}} \right| \leq cn. \quad (5.28)$$

Since the function $\widehat{\rho}_T$ has a continuous extension on the domain \overline{D}_T , we infer that $|\widehat{\rho}_T(\pm\delta_1\sqrt{n} + it)|$ is bounded uniformly in $t \in [0, y_n]$ and $n \geq 1$. Combining this with the bounds (5.27) and (5.28), uniformly in $s \in (-\eta, \eta)$,

$$|h_1(-\delta_1\sqrt{n} - it)| + |h_1(\delta_1\sqrt{n} - it)| \leq \frac{c}{\delta_1\sqrt{n}} \left(e^{\frac{\delta_1^2}{4}n} + cn \right) \leq \frac{c\delta_1}{\sqrt{n}} e^{\frac{\delta_1^2}{4}n}.$$

In view of (5.24), we have $t \leq y_n \leq -y$ and thus $e^{\frac{t^2}{2} + ty} \leq 1$ for any $t \in [0, y_n]$. Note that $y_n \leq \delta_1\sqrt{n}$ by (5.24). Consequently, we obtain the bound:

$$\sup_{s \in (-\eta, \eta)} A_{11} \leq c\delta_1 \frac{y_n}{\sqrt{n}} e^{-\frac{\delta_1^2}{2}n} e^{\frac{\delta_1^2}{4}n} \leq c\delta_1 e^{-\frac{\delta_1^2}{4}n}. \quad (5.29)$$

Control of A_{12} . Using a change of variable $z = t - iy_n$ leads to

$$\begin{aligned} A_{12} &= \sup_{y \leq 0} \left| e^{\frac{1}{2}y_n^2 + y_n y} \int_{-\delta_1\sqrt{n}}^{\delta_1\sqrt{n}} e^{-\frac{t^2}{2} + it(y_n + y)} h_1(t - iy_n) dt \right| \\ &\leq \sup_{y \leq 0} \left| e^{\frac{1}{2}y_n^2 + y_n y} \int_{-\delta_1\sqrt{n}}^{\delta_1\sqrt{n}} e^{-\frac{t^2}{2}} |h_1(t - iy_n)| dt \right|, \end{aligned} \quad (5.30)$$

where the function h_1 is defined by (5.22). To estimate the term A_{12} , the main task is to give a control of $|h_1(t - iy_n)|$. It follows from Lemma 5.4 that $|e^{z_1} - e^{z_2}| \leq e^{\max\{\Re z_1, \Re z_2\}} |z_1 - z_2|$ and $|e^{z_2} - 1 - z_2| \leq \frac{1}{2}|z_2|^2 e^{|z_2|}$ for any $z_1, z_2 \in \mathbb{C}$, and hence

$$|e^{z_1} - 1 - z_2| \leq e^{\max\{\Re z_1, \Re z_2\}} |z_1 - z_2| + \frac{1}{2}|z_2|^2 e^{|z_2|}. \quad (5.31)$$

We shall make use of the inequality (5.31) to derive a bound of $|h_1(t - iy_n)|$. Since $\frac{y_n}{\sqrt{n}} \leq \delta_1$ where $\delta_1 > 0$ can be sufficiently small, for any $|t| \leq \delta_1\sqrt{n}$, we get that uniformly in $s \in (-\eta, \eta)$,

$$\Re \left\{ [-i(t - iy_n)]^3 \frac{\Lambda^{(3)}(s)}{6\sigma_s^3\sqrt{n}} \right\} = \frac{y_n}{\sqrt{n}} \frac{(3t^2 - y_n^2)\Lambda^{(3)}(s)}{6\sigma_s^3} \leq \frac{1}{4}t^2, \quad (5.32)$$

$$\Re \left[n \sum_{k=3}^{\infty} \frac{\Lambda^{(k)}(s)}{k!} \left(-\frac{i(t - iy_n)}{\sigma_s\sqrt{n}} \right)^k \right] \leq \frac{y_n}{\sqrt{n}} \frac{(6t^2 - \frac{1}{2}y_n^2)\Lambda^{(3)}(s)}{6\sigma_s^3} \leq \frac{1}{4}t^2. \quad (5.33)$$

Moreover, elementary calculations yield that there exists a constant $c > 0$ such that uniformly in $s \in (-\eta, \eta)$,

$$\begin{aligned} &\left| n \sum_{k=3}^{\infty} \frac{\Lambda^{(k)}(s)}{k!} \left(-\frac{i(t - iy_n)}{\sigma_s\sqrt{n}} \right)^k - [-i(t - iy_n)]^3 \frac{\Lambda^{(3)}(s)}{6\sigma_s^3\sqrt{n}} \right| \\ &= \left| n \sum_{k=4}^{\infty} \frac{\Lambda^{(k)}(s)}{k!} \left(-\frac{i(t - iy_n)}{\sigma_s\sqrt{n}} \right)^k \right| \leq c \frac{t^4 + y_n^4}{n}. \end{aligned} \quad (5.34)$$

It is clear that

$$\sup_{s \in (-\eta, \eta)} \left| [-i(t - iy_n)]^3 \frac{\Lambda^{(3)}(s)}{6\sigma_s^3 \sqrt{n}} \right|^2 \leq c \frac{t^6 + y_n^6}{n}. \quad (5.35)$$

Taking into account that both $|t|$ and y_n are less than $\delta_1 \sqrt{n}$, and the fact $\delta_1 > 0$ can be small enough, it follows that

$$\sup_{s \in (-\eta, \eta)} \exp \left\{ \left| [-i(t - iy_n)]^3 \frac{\Lambda^{(3)}(s)}{6\sigma_s^3 \sqrt{n}} \right| \right\} \leq e^{\frac{1}{4}(t^2 + y_n^2)}.$$

Combining this with the bounds (5.32), (5.33), (5.34) and (5.35), and using the inequality (5.31), we conclude that

$$\begin{aligned} & \sup_{s \in (-\eta, \eta)} \left| e^n \sum_{k=3}^{\infty} \frac{\Lambda^{(k)}(s)}{k!} \left(-\frac{iz}{\sigma_s \sqrt{n}}\right)^k - 1 - (-iz)^3 \frac{\Lambda^{(3)}(s)}{6\sigma_s^3 \sqrt{n}} \right| \\ & \leq c \frac{t^4 + y_n^4}{n} e^{\frac{1}{4}t^2} + c \frac{t^6 + y_n^6}{n} e^{\frac{1}{4}(t^2 + y_n^2)} \leq c \frac{t^4 + y_n^4 + t^6 + y_n^6}{n} e^{\frac{1}{4}(t^2 + y_n^2)}. \end{aligned} \quad (5.36)$$

Since the function $\hat{\rho}_T$ has a continuous extension on the domain \bar{D}_T , we get that $|\hat{\rho}_T(-t + iy_n)|$ is bounded uniformly in $|t| \leq \delta_1 \sqrt{n}$ and $n \geq 1$. Combining this with (5.36) and the fact $|\frac{1}{t - iy_n}| = 1/\sqrt{t^2 + y_n^2}$ leads to

$$\sup_{s \in (-\eta, \eta)} |h_1(t - iy_n)| \leq c \frac{|t|^3 + y_n^3 + |t|^5 + y_n^5}{n} e^{\frac{1}{4}(t^2 + y_n^2)}.$$

Therefore, noting that $y \leq -y_n$ and $0 \leq y_n \leq \delta_1 \sqrt{n}$, we obtain

$$\begin{aligned} \sup_{s \in (-\eta, \eta)} A_{12} & \leq \frac{c}{n} \sup_{y \leq 0} \left| e^{\frac{3}{4}y_n^2 + y_n y} \int_{-\delta_1 \sqrt{n}}^{\delta_1 \sqrt{n}} e^{-\frac{t^2}{4}} (|t|^3 + y_n^3 + |t|^5 + y_n^5) dt \right| \\ & \leq \frac{c}{n} \sup_{y_n \in [0, \delta_1 \sqrt{n}]} e^{-\frac{1}{4}y_n^2} (1 + y_n^3 + y_n^5) \leq \frac{c}{n}. \end{aligned}$$

Substituting this and (5.29) into (5.25), we conclude the proof. \square

Lemma 5.6. *Let $J_2(z)$ be defined by (5.16), and \mathcal{C}_r^- be defined by (4.2) with $r = \delta_1 \sqrt{n}$ and $\delta_1 > 0$ small enough. Then, for $T = a\sqrt{n}$ with $a > 0$ large enough, uniformly in $x \in \mathcal{S}$, $s \in (-\eta, \eta)$ and $\varphi \in \mathcal{B}_\gamma$,*

$$A_2 = \sup_{y \leq 0} \left| \int_{\mathcal{C}_r^-} \frac{J_2(z)}{z} e^{izy} \hat{\rho}_T(-z) dz \right| \leq \frac{c}{n} \|\varphi\|_\gamma.$$

Proof. Denote

$$h_2(z) = e^n \sum_{k=3}^{\infty} \frac{\Lambda^{(k)}(s)}{k!} \left(-\frac{iz}{\sigma_s \sqrt{n}}\right)^k \left[\Pi_{s, \frac{-iz}{\sigma_s \sqrt{n}}} \varphi(x) - \pi_s(\varphi) - iz \frac{b_{s, \varphi}(x)}{\sigma_s \sqrt{n}} \right] \frac{\hat{\rho}_T(-z)}{z}.$$

Using (5.21), we rewrite A_2 as

$$A_2 = \sup_{y \leq 0} \left| \int_{C_r^-} e^{-\frac{z^2}{2} + izy} h_2(z) dz \right|.$$

As in the estimation of Lemma 5.5, the solution of the saddle point equation $\frac{d}{dz}(-\frac{z^2}{2} + izy) = 0$ is $z = iy$. Set $y_n = \min\{-y, \delta_1 \sqrt{n}\}$. Since $y_n \in D_{2r}^-$, where $r = \delta_1 \sqrt{n}$, and the function h_2 is analytic on the domain D_{2r}^- , by Cauchy's integral theorem we obtain

$$\begin{aligned} A_2 &\leq \sup_{y \leq 0} \left| \left\{ \int_{-\delta_1 \sqrt{n}}^{-\delta_1 \sqrt{n} - iy_n} + \int_{\delta_1 \sqrt{n} - iy_n}^{\delta_1 \sqrt{n}} \right\} e^{-\frac{z^2}{2} + izy} h_2(z) dz \right| \\ &\quad + \sup_{y \leq 0} \left| \int_{-\delta_1 \sqrt{n} - iy_n}^{\delta_1 \sqrt{n} - iy_n} e^{-\frac{z^2}{2} + izy} h_2(z) dz \right| =: A_{21} + A_{22}. \end{aligned}$$

Control of A_{21} . Similarly to (5.26), we use a change of variable to get

$$A_{21} \leq e^{-\frac{\delta_1^2}{2} n} \sup_{y \leq 0} \left| \int_0^{y_n} e^{\frac{t^2}{2} + ty} \left[|h_2(-\delta_1 \sqrt{n} - it)| + |h_2(\delta_1 \sqrt{n} - it)| \right] dt \right|.$$

Using Lemma 5.4, the formula (3.39) and the bound (3.36), for any $z = \pm \delta_1 \sqrt{n} - it$ with $t \in [0, y_n]$, we get that uniformly in $s \in (-\eta, \eta)$, $x \in \mathcal{S}$ and $\varphi \in \mathcal{B}_\gamma$,

$$\left| \frac{1}{z} \right| \left| \Pi_{s, \frac{-iz}{\sigma_s \sqrt{n}}} \varphi(x) - \pi_s(\varphi) - iz \frac{b_{s, \varphi}(x)}{\sigma_s \sqrt{n}} \right| \leq c \frac{|z|}{n} \|\varphi\|_\gamma \leq \frac{c}{\sqrt{n}} \|\varphi\|_\gamma. \quad (5.37)$$

Recall that the function $\widehat{\rho}_T$ is continuous on the domain \overline{D}_T , so that $|\widehat{\rho}_T(-z)|$ is bounded uniformly in $z = \pm \delta_1 \sqrt{n} - it$, where $t \in [0, y_n]$. Therefore, taking into account of the bound (5.27), we get that uniformly in $s \in (-\eta, \eta)$, $x \in \mathcal{S}$ and $\varphi \in \mathcal{B}_\gamma$,

$$|h_2(-\delta_1 \sqrt{n} - it)| + |h_2(\delta_1 \sqrt{n} - it)| \leq \frac{c}{\sqrt{n}} e^{\frac{\delta_1^2}{4} n} \|\varphi\|_\gamma.$$

Since $y \leq 0$, for any $t \in [0, y_n]$, it follows that $\frac{t^2}{2} + ty \leq 0$ and thus $e^{\frac{t^2}{2} + ty} \leq 1$. Combining this with the above inequality yields that uniformly in $\varphi \in \mathcal{B}_\gamma$,

$$\sup_{s \in (-\eta, \eta)} \sup_{x \in \mathcal{S}} A_{21} \leq c e^{-\frac{\delta_1^2}{2} n} \frac{y_n}{\sqrt{n}} e^{\frac{\delta_1^2}{4} n} \|\varphi\|_\gamma \leq c e^{-\frac{\delta_1^2}{4} n} \|\varphi\|_\gamma. \quad (5.38)$$

Control of A_{22} . Similarly to (5.30), we use a change of variable to get

$$A_{22} \leq \sup_{y \leq 0} \left| e^{\frac{1}{2} y_n^2 + y_n y} \int_{-\delta_1 \sqrt{n}}^{\delta_1 \sqrt{n}} e^{-\frac{t^2}{2}} |h_2(t - iy_n)| dt \right|.$$

We first estimate $|h_2(t - iy_n)|$. In the same way as in (5.37), with $z = t - iy_n$, we obtain that uniformly in $s \in (-\eta, \eta)$, $x \in \mathcal{S}$ and $\varphi \in \mathcal{B}_\gamma$,

$$\left| \frac{1}{z} \left| \Pi_{s, \frac{-iz}{\sigma_s \sqrt{n}}} \varphi(x) - \pi_s(\varphi) - iz \frac{b_{s, \varphi}(x)}{\sigma_s \sqrt{n}} \right| \right| \leq c \frac{|z|}{n} \|\varphi\|_\gamma \leq c \frac{|t| + y_n}{n} \|\varphi\|_\gamma.$$

Combining this with the bound (5.33), we get that uniformly in $\varphi \in \mathcal{B}_\gamma$,

$$\begin{aligned} \sup_{s \in (-\eta, \eta)} \sup_{x \in \mathcal{S}} A_{22} &\leq \frac{c}{n} \|\varphi\|_\gamma \sup_{y \leq 0} \left| e^{\frac{1}{2} y_n^2 + y_n y} \int_{-\delta_1 \sqrt{n}}^{\delta_1 \sqrt{n}} e^{-\frac{t^2}{4}} (|t| + y_n) dt \right| \\ &\leq \frac{c}{n} \|\varphi\|_\gamma \sup_{y_n \in [0, \delta_1 \sqrt{n}]} e^{-\frac{1}{2} y_n^2} (1 + y_n) \leq \frac{c}{n} \|\varphi\|_\gamma. \end{aligned} \quad (5.39)$$

Putting together (5.38) and (5.39) completes the proof. \square

Lemma 5.7. *Let $J_3(z)$ be defined by (5.17), and \mathcal{C}_r^- be defined by (4.2) with $r = \delta_1 \sqrt{n}$ and $\delta_1 > 0$ small enough. Then, for $T = a\sqrt{n}$ with $a > 0$ large enough, uniformly in $x \in \mathcal{S}$, $s \in (-\eta, \eta)$ and $\varphi \in \mathcal{B}_\gamma$,*

$$A_3 = \sup_{y \leq 0} \left| \int_{\mathcal{C}_r^-} \frac{J_3(z)}{z} e^{izy} \widehat{\rho}_T(-z) dz \right| \leq \frac{c}{n} \|\varphi\|_\gamma.$$

Proof. Denote

$$h_3(z) = \frac{1}{\sigma_s \sqrt{n}} \left[e^{n \sum_{k=3}^{\infty} \frac{\Lambda^{(k)}(s)}{k!} \left(-\frac{iz}{\sigma_s \sqrt{n}}\right)^k} - 1 \right] \widehat{\rho}_T(-z). \quad (5.40)$$

Using the expansion (5.21) and the bound (5.10), we have that uniformly in $s \in (-\eta, \eta)$, $x \in \mathcal{S}$ and $\varphi \in \mathcal{B}_\gamma$,

$$A_3 \leq c \|\varphi\|_\gamma \sup_{y \leq 0} \left| \int_{\mathcal{C}_r^-} e^{-\frac{z^2}{2} + izy} h_3(z) dz \right|.$$

As in Lemma 5.5, the saddle point equation $\frac{d}{dz}(-\frac{z^2}{2} + izy) = 0$ has the solution $z = iy$. Set $y_n = \min\{-y, \delta_1 \sqrt{n}\}$. It follows from Cauchy's integral theorem that

$$\begin{aligned} A_3 &\leq c \|\varphi\|_\gamma \sup_{y \leq 0} \left| \left\{ \int_{-\delta_1 \sqrt{n}}^{-\delta_1 \sqrt{n} - iy_n} + \int_{\delta_1 \sqrt{n} - iy_n}^{\delta_1 \sqrt{n}} \right\} e^{-\frac{z^2}{2} + izy} h_3(z) dz \right| \\ &\quad + c \|\varphi\|_\gamma \sup_{y \leq 0} \left| \int_{-\delta_1 \sqrt{n} - iy_n}^{\delta_1 \sqrt{n} - iy_n} e^{-\frac{z^2}{2} + izy} h_3(z) dz \right| =: A_{31} + A_{32}. \end{aligned}$$

Control of A_{31} . Similarly to (5.26), we use a change of variable to get

$$A_{31} \leq c \|\varphi\|_\gamma e^{-\frac{\delta_1^2}{2} n} \sup_{y \leq 0} \left| \int_0^{y_n} e^{\frac{t^2}{2} + ty} \left[|h_3(-\delta_1 \sqrt{n} - it)| + |h_3(\delta_1 \sqrt{n} - it)| \right] dt \right|.$$

Using the bounds (5.10) and (5.27), we deduce that uniformly in $s \in (-\eta, \eta)$ and $x \in \mathcal{S}$,

$$|h_3(-\delta_1\sqrt{n} - it)| + |h_3(\delta_1\sqrt{n} - it)| \leq \frac{c}{\sqrt{n}}(e^{\frac{\delta_1^2}{4}n} + 1) \leq \frac{c}{\sqrt{n}}e^{\frac{\delta_1^2}{4}n}.$$

Since $\frac{t^2}{2} + ty \leq 0$ for any $t \in [0, y_n]$ and $y \leq 0$, it follows that $e^{\frac{t^2}{2} + ty} \leq 1$. This, together with the above inequality, implies that uniformly in $\varphi \in \mathcal{B}_\gamma$,

$$\sup_{s \in (-\eta, \eta)} \sup_{x \in \mathcal{S}} A_{31} \leq c \frac{y_n}{\sqrt{n}} e^{-\frac{\delta_1^2}{4}n} \|\varphi\|_\gamma \leq ce^{-\frac{\delta_1^2}{4}n} \|\varphi\|_\gamma. \quad (5.41)$$

Control of A_{32} . Similarly to (5.30), we use a change of variable to get

$$A_{32} \leq c \|\varphi\|_\gamma \sup_{y \leq 0} \left| e^{\frac{1}{2}y_n^2 + y_n y} \int_{-\delta_1\sqrt{n}}^{\delta_1\sqrt{n}} e^{-\frac{t^2}{2}} |h_3(t - iy_n)| dt \right|.$$

We first give a control of $|h_3(t - iy_n)|$. It follows from Lemma 5.4 that $|e^z - 1| \leq e^{\max\{\Re z, 0\}} |z|$ for any $z \in \mathbb{C}$. Using this inequality and taking into account of the bound (5.33), we obtain

$$\sup_{s \in (-\eta, \eta)} \left| e^{n \sum_{k=3}^{\infty} \frac{\Lambda^{(k)}(s)}{k!} \left(-\frac{iz}{\sigma_s \sqrt{n}}\right)^k} - 1 \right| \leq ce^{\frac{1}{4}t^2} \frac{|t|^3 + y_n^3}{\sqrt{n}},$$

and hence

$$\sup_{s \in (-\eta, \eta)} \sup_{x \in \mathcal{S}} |h_3(t - iy_n)| \leq ce^{\frac{1}{4}t^2} \frac{|t|^3 + y_n^3}{n}.$$

It follows that uniformly in $s \in (-\eta, \eta)$, $x \in \mathcal{S}$ and $\varphi \in \mathcal{B}_\gamma$,

$$A_{32} \leq \frac{c}{n} \|\varphi\|_\gamma \sup_{y \leq 0} \left| e^{-\frac{1}{2}y_n^2} \int_{-\delta_1\sqrt{n}}^{\delta_1\sqrt{n}} e^{-\frac{t^2}{4}} (|t|^3 + y_n^3) dt \right| \leq \frac{c}{n} \|\varphi\|_\gamma. \quad (5.42)$$

Putting together (5.41) and (5.42), we conclude the proof. \square

Lemma 5.8. *Let $J_4(z)$ be defined by (5.18), and \mathcal{C}_r^- be defined by (4.2) with $r = \delta_1\sqrt{n}$ and $\delta_1 > 0$ small enough. Then, for $T = a\sqrt{n}$ with $a > 0$ large enough, uniformly in $x \in \mathcal{S}$, $s \in (-\eta, \eta)$ and $\varphi \in \mathcal{B}_\gamma$,*

$$A_4 = \sup_{y \leq 0} \left| \int_{\mathcal{C}_r^-} \frac{J_4(z)}{z} e^{izy} \widehat{\rho}_T(-z) dz \right| \leq ce^{-cn} \|\varphi\|_\gamma.$$

Proof. Since $\Im z \leq 0$ on \mathcal{C}_r^- and $y \leq 0$, we have $|e^{izy}| \leq 1$. Since the function $\widehat{\rho}_T$ has a continuous extension on the domain \overline{D}_T , the function $z \mapsto |\widehat{\rho}_T(-z)|$ is uniformly bounded on \mathcal{C}_r^- . Using the bound (3.37) and the fact that $\delta_1 > 0$ can be sufficiently small, we deduce that $|J_4(z)| \leq ce^{-cn} \|\varphi\|_\gamma$, uniformly in $s \in (-\eta, \eta)$, $x \in \mathcal{S}$ and $\varphi \in \mathcal{B}_\gamma$. Therefore, noting that $|\frac{1}{z}| = (\delta_1\sqrt{n})^{-1}$ and that the length of \mathcal{C}_r^- is $\pi\delta_1\sqrt{n}$, the desired result follows. \square

End of the proof of Theorem 5.2. Combining Lemmata 5.5-5.8, we obtain that $I_{31} \leq \frac{c}{n} \|\varphi\|_\gamma$, uniformly in $s \in (-\eta, \eta)$, $x \in \mathcal{S}$ and $\varphi \in \mathcal{B}_\gamma$.

Now we give a control of the term I_{32} defined in (5.7). Note that $y > 0$ in I_{32} and the integral in I_{32} is taken over the semicircle \mathcal{C}_r^+ , which lies in the upper part of the complex plane. In this case we have the saddle point equation $\frac{d}{dz}(-\frac{z^2}{2} + izy) = 0$ whose solution $z = iy$ also lies in the upper part of the complex plane. Similarly to (5.24), we choose a suitable point $y_n = \min\{y, \delta_1 \sqrt{n}\}$. Proceeding in the same way as for bounding I_{31} we obtain that $I_{32} \leq \frac{c}{n} \|\varphi\|_\gamma$, uniformly in $s \in (-\eta, \eta)$, $x \in \mathcal{S}$ and $\varphi \in \mathcal{B}_\gamma$.

Let us now bound the terms I_{33} and I_{34} defined in (5.7). Since the mapping $z \mapsto \widehat{\rho}_T(z)$ is analytic on \mathcal{C}_r^- and \mathcal{C}_r^+ , the estimates of I_{33} and I_{34} are similar to those of I_{31} and I_{32} , respectively. From these bounds, one concludes that $I_3 \leq c \|\varphi\|_\gamma / n$, uniformly in $s \in (-\eta, \eta)$, $x \in \mathcal{S}$ and $\varphi \in \mathcal{B}_\gamma$. Combining this with the bounds for I_1 and I_2 in (5.11) and (5.13), and using the fact that ε can be arbitrary small, we obtain (5.8), which finishes the proof. \square

5.3. Proof of Theorem 5.1. Since the proof of Theorem 5.1 is quite similar to that of Theorem 5.3, we only sketch the main differences. Denote

$$\begin{aligned} F(y) &= \mathbb{E}_{\mathbb{Q}_s^x} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log |G_n x| - n\Lambda'(s)}{\sigma_s \sqrt{n}} \leq y \right\}} \right], \quad y \in \mathbb{R}, \\ H(y) &= \mathbb{E}_{\mathbb{Q}_s^x} [\varphi(X_n^x)] \Phi(y), \quad y \in \mathbb{R}. \end{aligned}$$

By the definition of the operator $R_{s,z}$ in (3.29), direct calculations lead to

$$\begin{aligned} f(t) &= \int_{\mathbb{R}} e^{-ity} dF(y) = R_{s, \frac{-it}{\sigma_s \sqrt{n}}}^n \varphi(x), \quad t \in \mathbb{R}, \\ h(t) &= \int_{\mathbb{R}} e^{-ity} dH(y) = e^{-\frac{t^2}{2}} R_{s,0} \varphi(x), \quad t \in \mathbb{R}. \end{aligned}$$

One can verify that the functions F, H and their corresponding Fourier-Stieljes transforms f, h satisfy all the conditions stated in Proposition 4.1. Instead of using Proposition 4.1 with $r < T$ in the proof of Theorem 5.3, we apply Proposition 4.1 with $r = T = \delta_1 \sqrt{n}$, where $\delta_1 > 0$ is a sufficiently small constant. Then we obtain a similar inequality as (5.6) but with the term $I_2 = 0$. Since the non-arithmeticity condition A5 is only used in the bound of the term I_2 , following the proof of Theorem 5.3 we show that under the conditions of Theorem 5.1, the terms I_1 and I_3 defined in (5.7) are bounded by $c \|\varphi\|_\gamma / \sqrt{n}$, uniformly in $s \in (-\eta, \eta)$, $x \in \mathcal{S}$ and $\varphi \in \mathcal{B}_\gamma$. We omit the details of the rest of the proof.

6. PROOF OF MODERATE DEVIATION EXPANSIONS

In this section we prove Theorem 2.3. The proof is based on the Berry-Esseen bound in Theorem 5.1 and follows the standard techniques in Petrov [30], and therefore some details will be left to the reader.

We start with the following lemma whose proof uses the analyticity of the eigenfunction r_s and the eigenmeasure ν_s , see Proposition 3.1.

Lemma 6.1. *Assume either conditions A1 and A2 for invertible matrices, or conditions A1 and A3 for positive matrices. Then, there exists $\eta > 0$ such that uniformly in $s \in (-\eta, \eta)$ and $\varphi \in \mathcal{B}_\gamma$,*

$$\|r_s - \mathbf{1}\|_\infty \leq C|s| \quad \text{and} \quad |\nu_s(\varphi) - \nu(\varphi)| \leq C|s|\|\varphi\|_\gamma.$$

Proof. According to Proposition 3.1, we have $r_0 = \mathbf{1}$, $\nu_0 = \nu$. In addition, the mappings $s \mapsto r_s$ and $s \mapsto \nu_s$ are analytic on $(-\eta, \eta)$. The assertions follow using Taylor's formula. \square

Now we prove Theorem 2.3. When $y \in [0, 1]$, Theorem 2.3 is a direct consequence of Theorem 5.1, so it remains to prove Theorem 2.3 in the case when $y > 1$ with $y = o(\sqrt{n})$. We proceed to prove the first assertion Theorem 2.3. Applying the change of measure formula (3.16), we have

$$\begin{aligned} I &:= \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\{\log |G_n x| \geq n\Lambda'(0) + \sqrt{n}\sigma_0 y\}} \right] \\ &= r_s(x) \kappa^n(s) \mathbb{E}_{\mathbb{Q}_s^x} \left[(\varphi r_s^{-1})(X_n^x) e^{-s \log |G_n x|} \mathbb{1}_{\{\log |G_n x| \geq n\Lambda'(0) + \sqrt{n}\sigma_0 y\}} \right]. \end{aligned} \quad (6.1)$$

Under the assumptions of Theorem 2.3, by Proposition 3.14, $\sigma_s^2 = \Lambda''(s) > 0$, for any $s \in (-\eta, \eta)$ and $\eta > 0$ small enough. Denote $W_n^x = \frac{\log |G_n x| - n\Lambda'(s)}{\sigma_s \sqrt{n}}$. Recalling that $\Lambda = \log \kappa$, we rewrite (6.1) as follows:

$$\begin{aligned} I &= r_s(x) e^{-n[s\Lambda'(s) - \Lambda(s)]} \\ &\quad \times \mathbb{E}_{\mathbb{Q}_s^x} \left[(\varphi r_s^{-1})(X_n^x) e^{-s\sigma_s \sqrt{n} W_n^x} \mathbb{1}_{\{W_n^x \geq \frac{\sqrt{n}[\Lambda'(0) - \Lambda'(s)] + \sigma_0 y}{\sigma_s}\}} \right]. \end{aligned} \quad (6.2)$$

By Proposition 3.1, the function Λ is analytic and hence for $s \in (-\eta, \eta)$, $\Lambda(s) = \sum_{k=1}^{\infty} \frac{\gamma_k}{k!} s^k$, where $\gamma_k = \Lambda^{(k)}(0)$. For any $y > 1$ with $y = o(\sqrt{n})$, consider the equation

$$\sqrt{n}[\Lambda'(s) - \Lambda'(0)] = \sigma_0 y. \quad (6.3)$$

Choosing the unique real root s of (6.3), it follows from Petrov [30] that

$$s\Lambda'(s) - \Lambda(s) = \frac{y^2}{2n} - \frac{y^3}{n^{3/2}} \zeta\left(\frac{y}{\sqrt{n}}\right), \quad (6.4)$$

where ζ is the Cramér series defined by (2.9). Substituting (6.3) into (6.2), and using (6.4), we get

$$I = r_s(x) e^{-\frac{y^2}{2} + \frac{y^3}{\sqrt{n}} \zeta(\frac{y}{\sqrt{n}})} \mathbb{E}_{\mathbb{Q}_s^x} \left[(\varphi r_s^{-1})(X_n^x) e^{-s\sigma_s \sqrt{n} W_n^x} \mathbb{1}_{\{W_n^x \geq 0\}} \right]. \quad (6.5)$$

For brevity, denote $F(u) = \mathbb{E}_{\mathbb{Q}_s^x} \left[(\varphi r_s^{-1})(X_n^x) \mathbb{1}_{\{W_n^x \leq u\}} \right]$, $u \in \mathbb{R}$. In view of (6.5), using Fubini's theorem and the integration by parts, we deduce that

$$\begin{aligned} I &= r_s(x) e^{-\frac{y^2}{2} + \frac{y^3}{\sqrt{n}} \zeta(\frac{y}{\sqrt{n}})} \mathbb{E}_{\mathbb{Q}_s^x} \left[(\varphi r_s^{-1})(X_n^x) \int_0^\infty \mathbb{1}_{\{0 \leq W_n^x \leq u\}} s\sigma_s \sqrt{n} e^{-s\sigma_s \sqrt{n} u} du \right] \\ &= r_s(x) e^{-\frac{y^2}{2} + \frac{y^3}{\sqrt{n}} \zeta(\frac{y}{\sqrt{n}})} \int_0^\infty e^{-s\sqrt{n}\sigma_s u} dF(u). \end{aligned} \quad (6.6)$$

Letting $l(u) = F(u) - \pi_s(\varphi r_s^{-1})\Phi(u)$, $u \in \mathbb{R}$, we have

$$\int_0^\infty e^{-s\sqrt{n}\sigma_s u} dF(u) = I_1 + \frac{\pi_s(\varphi r_s^{-1})}{\sqrt{2\pi}} I_2, \quad (6.7)$$

$$I_1 = \int_0^\infty e^{-s\sqrt{n}\sigma_s u} dl(u), \quad I_2 = \int_0^\infty e^{-s\sqrt{n}\sigma_s u - \frac{u^2}{2}} du. \quad (6.8)$$

Estimate of I_1 . Integrating by parts, using the fact that $r_s \in \mathcal{B}_\gamma$ and the Berry-Esseen bound in Theorem 5.1 implies that uniformly in $s \in [0, \eta]$, $x \in \mathcal{S}$ and $\varphi \in \mathcal{B}_\gamma$,

$$|I_1| \leq |l(0)| + s\sqrt{n}\sigma_s \int_0^\infty e^{-s\sqrt{n}\sigma_s u} |l(u)| du \leq \frac{C}{\sqrt{n}} \|\varphi\|_\gamma. \quad (6.9)$$

Estimate of I_2 . The function Λ is analytic on $(-\eta, \eta)$ and $\sigma_s^2 = \Lambda''(s) > 0$. By Taylor's formula, $\Lambda'(s) - \Lambda'(0) = s\sigma_0^2[1 + O(s)]$ and $\sigma_s^2 = \sigma_0^2[1 + O(s)]$. Then, using standard techniques from Petrov [30], we obtain

$$I_2 = I_3 + O\left(\frac{1}{\sqrt{n}}\right), \quad \text{where } I_3 = \int_0^\infty e^{-\frac{\sqrt{n}[\Lambda'(s) - \Lambda'(0)]}{\sigma_0} u - \frac{u^2}{2}} du. \quad (6.10)$$

Since σ_s is strictly positive and bounded uniformly in $s \in (0, \eta)$, using (6.3) and the fact that $y > 1$, for sufficiently large n , we get that $s\sqrt{n}\sigma_s \geq \frac{y}{2\sigma_0}\sigma_s \geq c_1 > 0$. This implies that $C_1 \leq s\sqrt{n}I_2 \leq C_2$ holds for large enough n , where $C_1 < C_2$ are two positive constants independent of n and s . Combining this two-sided bound with (6.7), (6.9) and (6.10),

$$\int_0^\infty e^{-s\sqrt{n}\sigma_s u} dF(u) = I_3 \left[\frac{\pi_s(\varphi r_s^{-1})}{\sqrt{2\pi}} + \|\varphi\|_\gamma O(s) \right]. \quad (6.11)$$

Substituting (6.3) into (6.11), it follows that

$$\int_0^\infty e^{-s\sqrt{n}\sigma_s u} dF(u) = e^{\frac{y^2}{2}} \int_y^\infty e^{-\frac{1}{2}u^2} du \left[\frac{\pi_s(\varphi r_s^{-1})}{\sqrt{2\pi}} + \|\varphi\|_\gamma O(s) \right].$$

Together with (6.6), this implies

$$I = r_s(x) e^{\frac{y^3}{\sqrt{n}} \zeta(\frac{y}{\sqrt{n}})} [1 - \Phi(y)] \left[\pi_s(\varphi r_s^{-1}) + \|\varphi\|_\gamma O(s) \right], \quad (6.12)$$

where $\pi_s(\varphi r_s^{-1}) = \frac{\nu_s(\varphi)}{\nu_s(r_s)}$. By Lemma 6.1, we have $\|r_s - \mathbf{1}\|_\infty \leq Cs$ and $|\pi_s(\varphi r_s^{-1}) - \nu(\varphi)| \leq Cs \|\varphi\|_\gamma$, uniformly in $s \in [0, \eta)$ and $\varphi \in \mathcal{B}_\gamma$. Since $s = O(\frac{y}{\sqrt{n}})$, this concludes the proof of the first assertion of Theorem 2.3.

The proof of the second assertion of Theorem 2.3 can be carried out in a similar way. Instead of (6.3), we consider the equation $\sqrt{n}[\Lambda'(s) - \Lambda'(0)] = -\sigma_0 y$, where $y > 1$ and $s \in (-\eta, 0]$. We then apply the spectral gap properties of operators $P_s, Q_s, R_{s,z}$ (see Section 3) for negative valued s to deduce the second assertion by following the proof of the first one. We omit the details.

7. PROOF OF A LOCAL LIMIT THEOREM

In this section we prove Theorem 2.4. By a general result on the narrow convergence of measures, it is enough to prove the theorems for intervals $B = [a_1, a_2]$, where $a_1, a_2 \in \mathbb{R}$. Without loss of generality, we assume that φ is non-negative. Denote $y_i = y + \frac{a_i}{\sqrt{n}\sigma}$, $i = 1, 2$. From Theorem 2.3, we get that uniformly in $x \in \mathcal{S}$, $y \in [0, o(\sqrt{n})]$ and $\varphi \in \mathcal{B}_\gamma$, as $n \rightarrow \infty$,

$$\begin{aligned} & \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\{\log |G_n x| - n\lambda \in [a_1, a_2] + \sqrt{n}\sigma y\}} \right] \\ &= [1 - \Phi(y_1)] e^{\frac{y_1^3}{\sqrt{n}} \zeta(\frac{y_1}{\sqrt{n}})} \left[\nu(\varphi) + \|\varphi\|_\gamma O\left(\frac{y_1 + 1}{\sqrt{n}}\right) \right] \\ & \quad - [1 - \Phi(y_2)] e^{\frac{y_2^3}{\sqrt{n}} \zeta(\frac{y_2}{\sqrt{n}})} \left[\nu(\varphi) + \|\varphi\|_\gamma O\left(\frac{y_2 + 1}{\sqrt{n}}\right) \right]. \end{aligned} \quad (7.1)$$

Since $y \in [y_n, o(\sqrt{n})]$ and $\sqrt{n}y_n \rightarrow \infty$, straightforward computations yield that uniformly in $y \in [y_n, o(\sqrt{n})]$,

$$\begin{aligned} e^{\frac{y_i^3}{\sqrt{n}} \zeta(\frac{y_i}{\sqrt{n}})} &= e^{\frac{y^3}{\sqrt{n}} \zeta(\frac{y}{\sqrt{n}})} \left[1 + O\left(\frac{y^2}{n}\right) \right], \\ [1 - \Phi(y_1)] - [1 - \Phi(y_2)] &= \frac{a_2 - a_1}{\sqrt{2\pi n\sigma}} e^{-\frac{y^2}{2}} [1 + o(1)]. \end{aligned}$$

Substituting the above asymptotic expansions into (7.1), we get the result.

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