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► **To cite this version:**

Scott Andrews, Nathaniel Thiem. The generalized Gelfand–Graev characters of  $GL_n(\mathbb{F}_q)$ . 28-th International Conference on Formal Power Series and Algebraic Combinatorics, Simon Fraser University, Jul 2016, Vancouver, Canada. 10.46298/dmtcs.6406 . hal-02173756

**HAL Id: hal-02173756**

**<https://hal.science/hal-02173756>**

Submitted on 4 Jul 2019

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# *The generalized Gelfand–Graev characters of $GL_n(\mathbb{F}_q)$ (extended abstract)*

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**Abstract.** Introduced by Kawanaka in order to find the unipotent representations of finite groups of Lie type, generalized Gelfand–Graev characters have remained somewhat mysterious. Even in the case of the finite general linear groups, the combinatorics of their decompositions has not been worked out. This paper re-interprets Kawanaka’s definition in type  $A$  in a way that gives far more flexibility in computations. We use these alternate constructions to show how to obtain generalized Gelfand–Graev representations directly from the maximal unipotent subgroups. We also explicitly decompose the corresponding generalized Gelfand–Graev characters in terms of unipotent representations, thereby recovering the Kostka–Foulkes polynomials as multiplicities.

**Résumé.** Introduits par Kawanaka pour trouver des représentations unipotentes de groupes finis de type Lie, les caractères généralisés de Gelfand–Graev sont restés en quelque sorte mystérieux. Même dans le cas des groupes généraux linéaires finis, la combinatoire de leurs décompositions n’a pas été étudiée. Cet article réinterprète la définition de Kawanaka en type  $A$  d’une façon qui offre bien plus de flexibilité pour les calculs. Nous utilisons ces constructions alternatives pour montrer comment obtenir les représentations de Gelfand–Graev directement depuis les sous-groupes unipotents maximaux. Nous décomposons aussi explicitement les caractères généralisés de Gelfand–Graev correspondants en termes de représentations unipotentes, et retrouvons ainsi comme multiplicités les polynômes de Kostka–Foulkes.

**Keywords.** unipotent representation, supercharacter, Kostka polynomial

## 1 Introduction

There has been considerable progress in recent years on the combinatorial representation theory of finite unipotent groups. For example, the representation theory of the maximal unipotent subgroup  $UT_n(\mathbb{F}_q)$  of the finite general linear group  $GL_n(\mathbb{F}_q)$  has developed from a wild problem to a combinatorial theory based on set partitions [And95, Yan10]. Furthermore, by gluing together these theories we get a Hopf structure analogous to the representation theory of the symmetric groups  $S_n$  (where we replace the symmetric functions of  $S_n$  with symmetric functions in non-commuting variables for  $UT_n(\mathbb{F}_q)$ ) [AAB<sup>+</sup>12].

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An underlying philosophy of this paper is that the Bruhat decomposition of a finite group of Lie type

$$G = \bigsqcup_{\substack{w \in W \\ t \in T}} UtwU$$

gives a factorization into a maximal unipotent  $U$ -part, a torus  $T$ -part, and a Weyl group  $W$ -part. The traditional approach to studying the representation theory of these groups has been to tease out the influence of the representation theory of  $W$  and  $T$ . With the representation theory of  $U$  well-known to be wild, this approach seemed natural and eventually led to Lusztig's classification of the irreducible representations of  $G$  [Lus84].

Lusztig's indexing, however, is not overly constructive. In particular, we would like combinatorial constructions of the unipotent representations of  $G$ , and here the representation theory of  $U$  has untapped potential. The most natural way to find unipotent representations of  $G$  is to induce representations from  $U$ . There are a number of known approaches:

- (GG)** The Gelfand–Graev representation is obtained by inducing a linear representation of  $U$  in general position;
- (DGG)** The degenerate Gelfand–Graev representations generalize the GG representations by inducing arbitrary linear representations of  $U$ ;
- (GGG)** The generalized Gelfand–Graev representations provide a different generalization by instead inducing certain linear representations in general position from specified subgroups of  $U$ .

The GG representations were introduced to find cuspidal representations of  $G$ , and it was hoped that the DGG representations could identify all of the unipotent representations of  $G$ . While this works for  $\mathrm{GL}_n(\mathbb{F}_q)$  [Zel81], in general the DGG representations are insufficient. Kawanaka introduced the GGG representations [Kaw85] as a more effective method; the trade-off is that they appear to be more difficult to work with. Even for  $\mathrm{GL}_n(\mathbb{F}_q)$  these representations are not particularly well-understood, and this paper hopes to develop this example as a model for tackling other types.

Each GGG representation  $\mathrm{Ind}_{U'}^{\mathrm{GL}_n(\mathbb{F}_q)}(\gamma)$  is induced from a linear representation  $\gamma$  of a subgroup  $U' \subseteq \mathrm{UT}_n(\mathbb{F}_q)$ . The construction given by Kawanaka uses the root combinatorics of the corresponding Lie algebra to identify  $U'$  and  $\gamma$ ; however, inducing makes these specific choices somewhat artificial. Our main result of Section 3 gives a more direct method of choosing the pairs  $(\gamma, U')$  that induce to GGG representations.

An alternative approach is to identify representations of  $\mathrm{UT}_n(\mathbb{F}_q)$  that induce to GGG representations; that is, are there choices for  $(\gamma, U')$  such that we already know the  $\mathrm{UT}_n(\mathbb{F}_q)$ -module  $\mathrm{Ind}_{U'}^{\mathrm{UT}_n(\mathbb{F}_q)}(\gamma)$ ? Our main results of Section 4 (Corollaries 4.3 and 4.4) show that these induced representations may in fact be chosen so that they afford supercharacters of a natural supercharacter theory [DI08] of  $\mathrm{UT}_n(\mathbb{F}_q)$ ; this supercharacter theory, which is built on non-nesting set partitions, is described by Andrews in [And15a]. From this point of view, we could conduct all our constructions using known monomial representations of  $\mathrm{UT}_n(\mathbb{F}_q)$ .

Given our understanding of GGG representations from the previous sections, we decompose the corresponding characters into unipotent characters of  $\mathrm{GL}_n(\mathbb{F}_q)$  using Green's symmetric function description [Gre55, Mac95]. Our main result of Section 5 (Theorem 5.1) is that the multiplicities of the unipotent

characters are given exactly by Kostka–Foulkes polynomials; this effectively makes the GGG characters  $q$ -analogues of the DGG characters, and gives another representation theoretic interpretation for these polynomials.

We consider this paper to be the first steps in a larger program of constructing unipotent modules for finite groups of Lie type. In [And15b], Andrews uses the constructions of this paper to explicitly construct the unipotent modules for  $\mathrm{GL}_n(\mathbb{F}_q)$ . However, for other types there is more work to do. For type  $C$  we have an idea of what the analogues of Corollaries 4.3 and 4.4 should be, but even for type  $B$  there is again more work. We hope this paper can give a road map for future constructions.

## 2 Preliminaries

This section introduces some of the background topics for the paper. In particular, we give brief introductions to the representation theory of the finite general linear groups, our set partition combinatorics, the unipotent subgroups of greatest interest, and the notion of a supercharacter theory.

### 2.1 The combinatorial representation theory of $\mathrm{GL}_n(\mathbb{F}_q)$

In this section we present an indexing of the irreducible characters of  $\mathrm{GL}_n(\mathbb{F}_q)$  and describe these characters in terms of symmetric functions. This result is initially due to Green [Gre55] and is also found in [Mac95].

If  $n = mk$ ,  $\varphi \in \mathrm{Hom}(\mathbb{F}_{q^k}^\times, \mathbb{C}^\times)$ , and  $x \in \mathbb{F}_{q^n}^\times$ , define an injective homomorphism

$$\mathrm{Hom}(\mathbb{F}_{q^k}^\times, \mathbb{C}^\times) \longrightarrow \mathrm{Hom}(\mathbb{F}_{q^n}^\times, \mathbb{C}^\times) \quad \text{where} \quad N_{\mathbb{F}_{q^n}/\mathbb{F}_{q^k}} : \mathbb{F}_{q^n} \longrightarrow \mathbb{F}_{q^k}$$

$$\varphi \longmapsto \varphi \circ N_{\mathbb{F}_{q^n}/\mathbb{F}_{q^k}} \quad t \longmapsto t^{1+q^m+q^{2m}+\dots+q^{(k-1)m}}.$$

With these identifications, let  $\hat{\mathbb{F}}_q^\times = \bigcup_{n \geq 1} \mathrm{Hom}(\mathbb{F}_{q^n}^\times, \mathbb{C}^\times)$ . Let  $\sigma : \hat{\mathbb{F}}_q^\times \rightarrow \hat{\mathbb{F}}_q^\times$  be the Frobenius map defined by  $\sigma(x) = x^q$ , and let

$$\Theta = \{\langle \sigma \rangle\text{-orbits in } \hat{\mathbb{F}}_q^\times\} \quad \text{and} \quad \Phi = \{\langle \sigma \rangle\text{-orbits in } \bar{\mathbb{F}}_q^\times\}.$$

If  $\mathcal{P}$  is the set of integer partitions and  $\mathcal{X}$  is a set, define the set of  $\mathcal{X}$ -partitions  $\mathcal{P}^\mathcal{X}$  to be the set

$$\mathcal{P}^\mathcal{X} = \left\{ \begin{array}{l} \lambda : \mathcal{X} \longrightarrow \mathcal{P} \\ \varphi \longmapsto \lambda^{(\varphi)} \end{array} \right\}.$$

For  $\mathcal{X} \in \{\Theta, \Phi\}$ , the size of  $\lambda \in \mathcal{P}^\mathcal{X}$  is  $|\lambda| = \sum_{\varphi \in \mathcal{X}} |\varphi| |\lambda^{(\varphi)}|$ .

**Theorem 2.1 ([Gre55, Theorem 14])** *The complex irreducible characters of  $\mathrm{GL}_n(\mathbb{F}_q)$  are indexed by the functions  $\lambda \in \mathcal{P}^\Theta$  such that  $|\lambda| = n$ , and the conjugacy classes of  $\mathrm{GL}_n(\mathbb{F}_q)$  are indexed by the functions  $\mu \in \mathcal{P}^\Phi$  such that  $|\mu| = n$ .*

The  $\mathbb{C}$ -vector space

$$\mathrm{cf}(\mathrm{GL}) = \bigoplus_{n \geq 1} \mathrm{cf}(\mathrm{GL}_n), \quad \text{where} \quad \mathrm{cf}(\mathrm{GL}_n) = \{\text{class functions of } \mathrm{GL}_n(\mathbb{F}_q)\},$$

has a graded commutative  $\mathbb{C}$ -algebra structure with multiplication given by parabolic induction.

For each  $f \in \Phi$ , let  $X^{(f)} = \{X_1^{(f)}, X_2^{(f)}, \dots\}$  be a countably infinite set of variables. We define  $\text{Sym}(\text{GL}) = \bigotimes_{f \in \Phi} \text{Sym}(X^{(f)})$ , where  $\text{Sym}(X^{(f)})$  is the  $\mathbb{C}$ -algebra of symmetric functions in the variables  $X^{(f)}$ .

For  $\lambda = (\lambda_1, \dots, \lambda_\ell)$  a partition of  $k$  and  $X$  an infinite set of variables, define  $\tilde{P}_\lambda(X; q) = q^{-n(\lambda)} P_\lambda(X; q^{-1})$ , where  $n(\lambda) = \sum_{i=1}^\ell (i-1)\lambda_i$  and  $P_\lambda(X; q)$  is the Hall–Littlewood symmetric function. For  $\mu \in \mathcal{P}^\Phi$ , define  $\tilde{P}_\mu(X; q) = \prod_{f \in \Phi} \tilde{P}_{\mu^{(f)}}(X^{(f)}; q^{|f|})$ .

Each  $\mu \in \mathcal{P}^\Phi$  corresponds to a conjugacy class of  $\text{GL}_{|\mu|}(\mathbb{F}_q)$ ; we define the indicator functions  $\delta_\mu : \text{GL}_{|\mu|}(\mathbb{F}_q) \rightarrow \mathbb{C}$  by

$$\delta_\mu(g) = \begin{cases} 1 & \text{if } g \text{ has conjugacy type } \mu, \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 2.2 ([Mac95, IV.4.1])** *The characteristic function*

$$\begin{aligned} \text{ch} : \text{cf}(\text{GL}) &\longrightarrow \text{Sym}(\text{GL}) \\ \delta_\mu &\mapsto \tilde{P}_\mu(X; q). \end{aligned}$$

is an isomorphism of graded  $\mathbb{C}$ -algebras.

To describe the images of the irreducible characters of  $\text{GL}_n(\mathbb{F}_q)$  under the characteristic map, we introduce a new set of variables. For each  $\varphi \in \Theta$ , let  $Y^{(\varphi)} = \{Y_1^{(\varphi)}, Y_2^{(\varphi)}, \dots\}$  be the countably infinite set of variables completely determined by

$$p_k(Y^{(\varphi)}) = (-1)^{k|\varphi|-1} \sum_{\substack{f \in \Phi \\ f \subseteq \varphi \\ q^{k|\varphi|}}} \left( \sum_{x \in f} \varphi(x) \right) p_{\frac{k|\varphi|}{|f|}}(X^{(f)}), \quad \text{for } k \in \mathbb{Z}_{\geq 1}. \quad (2.1)$$

For  $\lambda$  a partition of  $n$ , let  $s_\lambda(Y)$  denote the Schur function. For  $\nu \in \mathcal{P}^\Theta$ , let  $s_\nu = \prod_{\varphi \in \Theta} s_{\nu^{(\varphi)}}(Y^{(\varphi)})$ .

**Theorem 2.3 ([Mac95, IV.6.8])** *The set  $\{\text{ch}^{-1}(s_\nu) \mid \nu \in \mathcal{P}, |\nu| = n\}$  is exactly the set of irreducible characters of  $\text{GL}_n(\mathbb{F}_q)$ .*

## 2.2 Set partition combinatorics

In this section we introduce some background and terminology regarding set partitions. A *set partition*  $\eta$  of  $\{1, 2, \dots, n\}$  is a subset

$$\eta \subseteq \{i \frown j \mid 1 \leq i < j \leq n\}$$

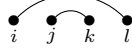
such that if  $i \frown k, j \frown l \in \eta$ , then  $i = j$  if and only if  $k = l$ . Let

$$\mathcal{S}_n = \{\text{set partitions of } \{1, 2, \dots, n\}\}. \quad (2.2)$$

We can represent these set partitions by an arc diagram where we line up  $n$  nodes and connect the  $i$ th to the  $j$ th if  $i \frown j \in \eta$ . For example,

$$\begin{array}{cccccccc} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \quad \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \end{array} = \{1 \frown 4, 3 \frown 6, 6 \frown 8\} \in \mathcal{S}_8.$$

We say that a set partition is *nonnesting* if it contains no pair of arcs  $i \frown l, j \frown k$  with  $i < j < k < l$ . In other words, the relative positioning of arcs



never occurs in the arc diagram of the set partition. Let

$$\mathcal{S}_n^{\mathrm{nn}} = \{\eta \in \mathcal{S}_n \mid \eta \text{ nonnesting}\}. \quad (2.3)$$

We often will be interested in the integer partition consisting of the block sizes of a set partition  $\eta$ , which we will denote  $\mathrm{bl}(\eta)$ .

### 2.3 Unipotent subgroups of $\mathrm{GL}_n(\mathbb{F}_q)$

A *unipotent subgroup*  $U \subseteq \mathrm{GL}_n(\mathbb{F}_q)$  is a subgroup that is conjugate to a subgroup of the group of unipotent upper-triangular matrices

$$\mathrm{UT}_n(\mathbb{F}_q) = \{g \in \mathrm{GL}_n(\mathbb{F}_q) \mid (g - \mathrm{Id})_{ij} \neq 0 \text{ implies } i < j\}.$$

Since every  $\mathrm{char}(\mathbb{F}_q)$ -group is isomorphic to a unipotent subgroup of  $\mathrm{GL}_n(\mathbb{F}_q)$  for some  $n$ , these subgroups can get quite messy. We will therefore focus on the set of *normal pattern subgroups*

$$\mathcal{U}_n = \{U \leq \mathrm{UT}_n(\mathbb{F}_q) \mid T_n \subseteq \mathbf{N}_{\mathrm{GL}_n(\mathbb{F}_q)}(U)\},$$

where  $T_n \subseteq \mathrm{GL}_n(\mathbb{F}_q)$  is the subgroup of diagonal matrices. Note that  $\mathrm{UT}_n(\mathbb{F}_q) \in \mathcal{U}_n$ .

The following special cases will be of particular importance.

**Integer compositions.** For a composition  $\alpha = (\alpha_1, \dots, \alpha_\ell)$  of  $n$ , let

$$U_\alpha = \left[ \begin{array}{c|c|c|c} \mathrm{Id}_{\alpha_1} & * & \cdots & * \\ \hline 0 & \mathrm{Id}_{\alpha_2} & \ddots & \vdots \\ \hline \vdots & \ddots & \ddots & * \\ \hline 0 & \cdots & 0 & \mathrm{Id}_{\alpha_\ell} \end{array} \right] \subseteq \mathrm{UT}_n(\mathbb{F}_q).$$

**Set partitions.** For a non-nesting set partition  $\eta \in \mathcal{S}_n^{\mathrm{nn}}$ , let

$$U_\eta = \{u \in \mathrm{UT}_n(\mathbb{F}_q) \mid u_{jk} = 0 \text{ if } i \leq j < k \leq l \text{ with } i \frown l \in \eta - \{j \frown k\}\}.$$

We will be particularly interested in the case where  $U_\eta \subseteq U_{\mathrm{bl}(\eta)}$ .

### 2.4 Supercharacter theories

The notion of a supercharacter theory for a finite group  $G$  was introduced by Diaconis–Isaacs in [DI08]. The basic idea is to treat linear combinations of irreducible characters as the “irreducible characters” of the theory, and have a corresponding partition of  $G$  whose blocks (called superclasses) are unions of conjugacy classes. From a slightly different point of view, this gives us a Schur ring [Hen12], and we will define a supercharacter theory from that point of view.

A *supercharacter theory*  $\mathrm{scf}(G)$  of a finite group  $G$  is a subspace  $\mathrm{scf}(G)$  of the  $\mathbb{C}$ -space of class functions  $\mathrm{cf}(G)$  such that  $\mathrm{scf}(G)$  is a subalgebra of  $\mathrm{cf}(G)$  with respect to the ring structures

(R1)  $(\chi \circ \psi)(g) = \chi(g)\psi(g)$ , for  $\chi, \psi \in \text{cf}(G)$ ,  $g \in G$ ; and

(R2)  $(\chi \circ \psi)(g) = \sum_{h \in G} \chi(h)\psi(h^{-1}g)$ , for  $\chi, \psi \in \text{cf}(G)$ ,  $g \in G$ .

Each ring structure gives rise to a  $\mathbb{C}$ -basis of orthogonal idempotents, one consisting of orthogonal characters (with respect to (R2)) and the other consists of set identifier functions that identify the superclasses (with respect to (R1)).

### 3 Generalized Gelfand–Graev representation construction and characterization

This section gives the definition and construction of the generalized Gelfand–Graev representations and presents a characterization of the generalized Gelfand–Graev characters.

#### 3.1 A combinatorial version of Kawanaka’s construction

The generalized Gelfand–Graev representations were introduced by Kawanaka [Kaw85] as a source for cuspidal representations of finite groups of Lie type. In the case of  $\text{GL}_n(\mathbb{F}_q)$ , the GGG characters form a basis for the space of class functions of unipotent support

$$\text{cf}_{\text{supp}}^{\text{un}}(\text{GL}) \cong \text{Sym}(X^{(1)}) \quad (\text{in the notation of Section 2.1}).$$

It follows that the GGG representations are indexed by integer partitions; Kawanaka constructs them from the nilpotent  $\text{GL}_n(\mathbb{F}_q)$ -orbits of the corresponding Lie algebra  $\mathfrak{gl}_n(\mathbb{F}_q)$ . In this section we present a different construction that is more combinatorial in nature.

Given an integer partition  $\lambda \vdash n$ , we construct a unipotent subgroup  $U_{\text{ctr}(\lambda')} \subseteq \text{UT}_n(\mathbb{F}_q)$  and a linear representation  $\gamma_\lambda : U_{\text{ctr}(\lambda')} \rightarrow \text{GL}_1(\mathbb{C})$  such that the generalized Gelfand–Graev representation  $\Gamma_\lambda$  is given by

$$\Gamma_\lambda = \text{Ind}_{U_{\text{ctr}(\lambda')}}^{\text{GL}_n(\mathbb{F}_q)}(\gamma_\lambda).$$

##### 3.1.1 The unipotent subgroup $U_{\text{ctr}(\lambda')}$

Fix an integer partition  $\lambda = (\lambda_1, \dots, \lambda_\ell)$ . We define a permutation  $\text{ctr} \in S_{\lambda_1}$  by

$$\text{ctr}(j) = \begin{cases} \lfloor \lambda_1/2 + 1 \rfloor + \frac{j-1}{2} & \text{if } j \notin 2\mathbb{Z}, \\ \lceil \lambda_1/2 + 1 \rceil - j/2 & \text{if } j \in 2\mathbb{Z}. \end{cases}$$

That is,  $\text{ctr}$  pushes all of the odd elements to the end and the even ones to the beginning; the odd elements stay in the same relative order and the even ones get placed in reverse order. We will use  $\text{ctr}$  to permute the parts of  $\lambda'$  (or equivalently the columns of the Ferrers diagram of  $\lambda$ ).

We are interested in three Ferrers shapes corresponding to this partition:

- (F0) the usual left-justified Ferrers shape,
- (F1) the shape obtained by centering the rows of (F0),
- (F2) the shape obtained by applying  $\text{ctr}$  to the columns of (F0) to get nearly centered rows without offsets.

For example, if  $\lambda = (4, 3, 2, 2, 1)$ , then we write

$$(F0) = \begin{array}{cccc} \square & \square & \square & \square \\ \square & \square & \square & \\ \square & \square & \square & \\ \square & \square & \square & \end{array}, \quad (F1) = \begin{array}{cccc} \square & \square & \square & \square \\ \square & \square & \square & \\ \square & \square & \square & \\ \square & \square & \square & \end{array}, \quad \text{and} \quad (F2) = \begin{array}{cccc} \square & \square & \square & \square \\ \square & \square & \square & \\ \square & \square & \square & \\ \square & \square & \square & \end{array}.$$

We will need (F1) to define  $\gamma_\lambda$ ; the composition  $\mathrm{ctr}(\lambda')$  determined by the columns of (F2) gives us the subgroup  $U_{\mathrm{ctr}(\lambda')}$  (as in Section 2.3).

**Remark 3.1** *The group  $U_{\mathrm{ctr}(\lambda')}$  is the group  $U_{1,5}$  in Kawanaka [Kaw85]. There are some choices to be made in the construction of  $U_{1,5}$ , and our choice of column permutation to get (F2) makes these choices.*

### 3.1.2 $\Gamma_\lambda$ from the linear representation $\gamma_\lambda$

Consider the column reading tableau on (F1) obtained by numbering in order down consecutive half-columns. Let  $C_\lambda$  be the corresponding tableau of shape (F2) by viewing (F2) as a row shift from (F1). In our example,

$$\begin{array}{cccc} 1 & 3 & 8 & 12 \\ 2 & 6 & 11 & \\ 4 & 9 & & \\ 5 & 10 & & \\ 7 & & & \end{array} \quad \text{becomes} \quad C_\lambda = \begin{array}{cccc} 1 & 3 & 8 & 12 \\ 2 & 6 & 11 & \\ 4 & 9 & & \\ 5 & 10 & & \\ 7 & & & \end{array}.$$

Given  $\lambda \vdash n$ , we obtain a set-partition

$$\mathrm{ggg}(\lambda) = \{i \sim j \mid i < j \text{ are in the same row and consecutive columns of } C_\lambda\},$$

whose block sizes are the parts of  $\lambda$ . In our running example,

$$\mathrm{ggg}(\lambda) = \begin{array}{cccccccccccccc} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & & \end{array}$$

Fix a nontrivial homomorphism  $\vartheta : \mathbb{F}_q^+ \rightarrow \mathrm{GL}_1(\mathbb{C})$ . Define

$$\begin{aligned} \gamma_\lambda : U_{\mathrm{ctr}(\lambda')} &\longrightarrow \mathrm{GL}_1(\mathbb{C}) \\ u &\longmapsto \prod_{i \sim j \in \mathrm{ggg}(\lambda)} \vartheta(u_{ij}). \end{aligned}$$

Note that the commutator subgroup of  $U_{\mathrm{ctr}(\lambda')}$  is given by

$$[U_{\mathrm{ctr}(\lambda')}, U_{\mathrm{ctr}(\lambda')}] = \left\{ u \in \mathrm{UT}_n(\mathbb{F}_q) \mid (u - \mathrm{Id}_n)_{ij} \neq 0 \text{ implies } i \text{ is at least two columns left of } j \text{ in } C_\lambda \right\}.$$

Since  $[U_{\mathrm{ctr}(\lambda')}, U_{\mathrm{ctr}(\lambda')}] \subseteq \ker(\gamma_\lambda)$ , the function  $\gamma_\lambda$  is a representation.

Define the *generalized Gelfand–Graev* representation (or GGG representation) corresponding to the integer partition  $\lambda \vdash n$  to be the induced representation

$$\Gamma_\lambda = \mathrm{Ind}_{U_{\mathrm{ctr}(\lambda')}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\gamma_\lambda).$$

**Remark 3.2** *There are a number of choices made in this construction, but none of them matter once we induce to  $\mathrm{GL}_n(\mathbb{F}_q)$ .*



### 3.2 A characterization of generalized Gelfand-Graev characters

Using our construction, we obtain an elementary proof for a known characterization of the generalized Gelfand–Graev characters.

**Theorem 3.3 (Geck–Héazard [GH08])** *Suppose that  $f \in \text{cf}_{\text{supp}}^{\text{un}}(\text{GL}_n)$  and  $\lambda \vdash n$ . Then  $f$  satisfies*

(G1)  $\langle f, \chi^{\mu^{(1)}} \rangle = 0$  unless  $\mu \succeq \lambda$ , and

(G2)  $f(u_\mu) = 0$  unless  $\mu \preceq \lambda$ ,

if and only if  $f = c\Gamma_\lambda$  for some constant  $c \in \mathbb{C}$ . In particular,  $\{\Gamma_\lambda \mid \lambda \vdash n\}$  is a basis for the space of unipotently supported class functions of  $\text{GL}_n(\mathbb{F}_q)$ .

## 4 GGG characters from non-nesting supercharacters of $\text{UT}_n(\mathbb{F}_q)$

Since  $\text{UT}_n(\mathbb{F}_q) \subseteq \text{GL}_n(\mathbb{F}_q)$  contains all of the subgroups that we are inducing from, it is natural to try to classify the representations of  $\text{UT}_n(\mathbb{F}_q)$  that induce to GGG characters. This section shows that even though the representation theory of  $\text{UT}_n(\mathbb{F}_q)$  is wild, we already know the representations that induce to GGG characters from the study of supercharacters.

We fix a nontrivial homomorphism

$$\vartheta : \mathbb{F}_q^+ \rightarrow \mathbb{C}^\times.$$

### 4.1 A supercharacter theory from non-nesting set partitions

Retaining the notation from Section 2.3, for a non-nesting set partition  $\eta \in \mathcal{S}_n^{\text{nn}}$ , the group

$$U_\eta = \{u \in \text{UT}_n(\mathbb{F}_q) \mid u_{jk} = 0 \text{ if there exists } i \frown l \in \eta - \{j \frown k\} \text{ with } i \leq j < k \leq l\},$$

has an associated nilpotent  $\mathbb{F}_q$ -algebra  $\mathfrak{u}_\eta = U_\eta - \text{Id}_n$ . For any function

$$\begin{aligned} \eta^\times : \eta &\longrightarrow \mathbb{F}_q^\times \\ i \frown j &\mapsto \eta_{ij}, \end{aligned}$$

we define a linear character  $\gamma_{\eta^\times}$  of  $U_\eta$  by

$$\gamma_{\eta^\times}(u) = \prod_{i \frown j \in \eta} \vartheta(\eta_{ij} u_{ij}).$$

Define

$$\chi_{\text{nn}}^{\eta^\times} = \text{Ind}_{U_\eta}^{\text{UT}_n(\mathbb{F}_q)}(\gamma_{\eta^\times}).$$

These characters are in fact the supercharacters of a supercharacter theory of  $\text{UT}_n(\mathbb{F}_q)$  [And15a], which we will refer to as the *non-nesting supercharacter theory* of  $\text{UT}_n(\mathbb{F}_q)$ .

**Theorem 4.1 ([And15a, Theorem 5.4])** *The subspace*

$$\text{scf}_{\text{nn}}(\text{UT}_n(\mathbb{F}_q)) = \mathbb{C}\text{-span}\{\chi_{\text{nn}}^{\eta^\times} \mid \eta \in \mathcal{S}_n^{\text{nn}}, \eta^\times : \eta \rightarrow \mathbb{F}_q^\times\} \subseteq \text{cf}(\text{UT}_n(\mathbb{F}_q))$$

is a supercharacter theory of  $\text{UT}_n(\mathbb{F}_q)$ .

**Remark 4.2** In relating these supercharacters to GGG characters, our choice of  $\eta^\times$  becomes immaterial; that is, we might as well send all of the arcs to  $1 \in \mathbb{F}_q^\times$ . Therefore, we will use the convention that for  $\eta \in \mathcal{S}_n^{\text{nn}}$ ,

$$\gamma_\eta(u) = \prod_{i \sim j \in \eta} \vartheta(u_{ij}).$$

#### 4.2 From non-nesting supercharacters to GGG characters

Let  $\alpha \vDash n$  be a composition. An  $\alpha$ -column tableau  $T$  is a filling of the Ferrers shape with ordered column lengths  $\alpha$ , such that

- (T1) Each number  $\{1, \dots, n\}$  appears exactly once,
- (T2) For  $1 \leq i < j \leq \ell(\alpha)$ , the entries of column  $i$  are strictly less than the entries of column  $j$ ,
- (T3) If row  $i$  and row  $j$  in  $T$  have the same length, then  $i < j$  implies the first entry of row  $i$  is less than the first entry of row  $j$ .

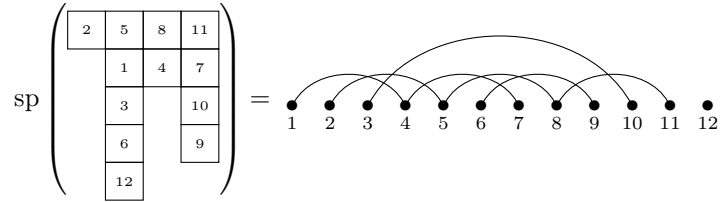
Let

$$\mathcal{T}_\alpha = \{\alpha\text{-column tableaux}\}.$$

We can obtain set partitions from such tableaux by letting the rows give the connected components; more formally, we have a map

$$\begin{aligned} \text{sp} : \mathcal{T}_\alpha &\longrightarrow \mathcal{S}_{|\alpha|} \\ T &\mapsto \bigcup_{\text{row } (i_1, \dots, i_r) \text{ of } T} \{i_1 \frown i_2, i_2 \frown i_3, \dots, i_{r-1} \frown i_r\}. \end{aligned}$$

For example,



Let

$$\mathcal{T}_\alpha^{\text{nn}} = \{T \in \mathcal{T}_\alpha \mid \text{sp}(T) \in \mathcal{S}_{|\alpha|}^{\text{nn}}\}.$$

For  $T \in \mathcal{T}_\alpha^{\text{nn}}$ , let  $U_T = V_T \rtimes U_{\text{sp}(T)}$ , where

$$V_T = \left\{ u \in \text{UT}_n(\mathbb{F}_q) \mid \begin{array}{l} (u - \text{Id}_n)_{ij} \neq 0 \text{ implies } i \frown k \in \text{sp}(T) \text{ with} \\ i < j < k, i \text{ strictly North of } j \text{ in } T \end{array} \right\}.$$

Note that while  $U_T \not\cong U_\alpha$ , it will follow from the proof of Corollary 4.3 that  $|U_T| = |U_\alpha|$ .

The following result allows us to obtain generalized Gelfand–Graev characters from the supercharacters  $\chi_{\text{nn}}^{\text{sp}(T)}$ .

**Corollary 4.3** *Let  $\mu \vdash n$ ,  $\alpha \vDash n$  be a rearrangement of  $\mu'$  and  $T \in \mathcal{T}_\alpha^{\text{nn}}$ . Then*

$$\text{Ind}_{\text{UT}_n(\mathbb{F}_q)}^{\text{GL}_n(\mathbb{F}_q)}(\chi_{\text{nn}}^{\text{sp}(T)}) = |V_T| \Gamma_\mu.$$

Note that if  $\alpha = \mu'$ , then we may set  $T$  equal to the column reading tableau to satisfy the hypotheses of the corollary. From this we can conclude that we get all the generalized Gelfand–Graev characters by inducing from non-nesting supercharacters.

**Corollary 4.4** *Let  $\mu \vdash n$  and  $T$  be the column reading tableau of the standard Ferrer diagram. Then*

$$\frac{1}{|V_T|} \chi_{\text{nn}}^{\text{sp}(T)}$$

*is a character of  $\text{UT}_n(\mathbb{F}_q)$  that induces to  $\Gamma_\mu$  (without scaling).*

## 5 Symmetric functions

In this section we study the images of the generalized Gelfand–Graev characters under the characteristic map. In particular, we calculate the multiplicities of the irreducible characters of  $\text{GL}_n(\mathbb{F}_q)$  in the generalized Gelfand–Graev characters.

### 5.1 The multiplicities of the unipotent characters

We first use our construction to calculate the multiplicities of the irreducible unipotent characters of  $\text{GL}_n(\mathbb{F}_q)$  in the generalized Gelfand–Graev characters.

**Theorem 5.1** *We have that*

$$\langle \Gamma_\lambda, \chi^{\mu^{(1)}} \rangle = K_{\mu\lambda}(q),$$

*where  $\chi^{\mu^{(1)}}$  is the unipotent character of  $\text{GL}_n(\mathbb{F}_q)$  corresponding to the partition  $\mu$ .*

As a corollary, we determine the image of  $\Gamma_\lambda$  under the characteristic map.

**Corollary 5.2** *For  $\lambda \in \mathcal{P}$ , we have that*

$$\text{ch}(\Gamma_\lambda) = (-1)^{|\lambda|} Q_\lambda(X^{(1)}; q),$$

*where the  $Q_\lambda(X^{(1)}; q)$  are the Hall–Littlewood symmetric functions [Mac95, III.2.11].*

### 5.2 Multiplicities of the irreducible characters of $\text{GL}_n(\mathbb{F}_q)$

By Theorem 5.1, we have that for  $\chi^\nu$  a unipotent character,

$$\langle \Gamma_\lambda, \chi^\nu \rangle = K_{\nu\lambda}(q).$$

A class function with unipotent support is uniquely determined by the multiplicities of the unipotent characters, thus we can use the above result to determine a formula for the multiplicities of arbitrary irreducible characters of  $\text{GL}_n(\mathbb{F}_q)$  in  $\Gamma_\lambda$ .

For  $\mu \in \mathcal{P}^\Theta$ , we define  $ss(\mu), un(\mu) \in \mathcal{P}^\Theta$  by

$$ss(\mu)^{(\varphi)} = \left(1^{|\mu^{(\varphi)}|}\right) \quad \text{and} \quad un(\mu)^{(\varphi)} = \begin{cases} \bigcup_{\varphi \in \Theta} |\varphi| \mu^{(\varphi)} & \text{if } \varphi = \{1\}, \\ \emptyset & \text{otherwise.} \end{cases}$$

For  $\mu$  and  $\lambda$  partitions of  $n$ , let  $\psi_\mu^\lambda$  be the value of the irreducible character  $\psi^\lambda$  of  $S_n$  on the conjugacy class  $C_\mu$ , and let  $z_\mu$  be the size of the centralizer of an element of  $C_\mu$ . Then

$$s_\lambda = \sum_{\mu \vdash n} \frac{\psi_\mu^\lambda}{z_\mu} p_\mu \quad \text{and} \quad p_\mu = \sum_{\lambda \vdash n} \psi_\mu^\lambda s_\lambda.$$

For  $\nu, \mu \in \mathcal{P}^\Theta$  with  $ss(\nu) = ss(\mu)$ , define

$$\psi_\mu^\nu = \prod_{\varphi \in \Theta} \psi_{\mu^{(\varphi)}}^{\nu^{(\varphi)}} \quad \text{and} \quad z_\mu = \prod_{\varphi \in \Theta} z_{\mu^{(\varphi)}}.$$

**Proposition 5.3** For  $\nu \in \mathcal{P}^\Theta$ ,

$$\langle \Gamma_\lambda, \chi^\nu \rangle = \sum_{\substack{\mu \in \mathcal{P}^\Theta \\ ss(\mu) = ss(\nu)}} \frac{\psi_\mu^\nu}{z_\mu} X_{un(\mu)}^\lambda(q),$$

where  $X_{un(\mu)}^\lambda(q)$  is as in [Mac95, III.7.1].

When the irreducible character is a product of cuspidal characters, this equation simplifies nicely.

**Corollary 5.4** If  $|\nu^{(\varphi)}| \leq 1$  for all  $\varphi \in \Theta$ , then

$$\langle \Gamma_\lambda, \chi^\nu \rangle = X_{un(\nu)}^\lambda(q).$$

In particular, if  $\lambda$  is cuspidal, then [Mac95, III.7.E2] gives the following result.

**Corollary 5.5** If  $\nu \in \mathcal{P}_N^\Theta$  satisfies  $\nu^{(\varphi)} = (1)$  for some  $\varphi \in \Theta$  with  $|\varphi| = N$ , then

$$\langle \Gamma_\lambda, \chi^\nu \rangle = q^{n(\lambda)} \prod_{i=1}^{\ell(\lambda)-1} (1 - q^{-i}).$$

We can also consider the opposite extreme, where  $\nu \in \mathcal{P}^\Theta$  satisfies  $\nu^{(\varphi)} = \emptyset$  for  $|\varphi| > 1$ . Recall that if  $\mu$  and  $\nu$  are partitions, the *Littlewood–Richardson coefficients*  $c_{\mu\nu}^\lambda$  are defined by

$$s_\mu s_\nu = \sum_{\lambda \vdash n} c_{\mu\nu}^\lambda s_\lambda,$$

as in [Mac95, I.9.1]. For  $\nu \in \mathcal{P}^\Theta$  with  $\nu^{(\varphi)} = \emptyset$  for  $|\varphi| > 1$ , we define  $c_\nu^\mu$  by

$$\prod_{\varphi \in \Theta} s_{\nu^{(\varphi)}} = \sum_{\mu \vdash |\nu|} c_\nu^\mu s_\mu.$$

**Proposition 5.6** If  $\nu \in \mathcal{P}^\Theta$  satisfies  $\nu^{(\varphi)} = \emptyset$  for  $|\varphi| > 1$ , then

$$\langle \Gamma_\lambda, \chi^\nu \rangle = \sum_{\mu \vdash |\lambda|} c_\nu^\mu K_{\mu\lambda}(q).$$

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