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McKay Centralizer Algebras

Georgia Benkart^{1†} and Tom Halverson^{2‡}

¹University of Wisconsin – Madison, Madison, WI 53706 USA

²Macalester College, Saint Paul, MN 55105 USA

Abstract. For a finite subgroup G of the special unitary group SU_2 , we study the centralizer algebra $Z_k(G) = \text{End}_G(V^{\otimes k})$ of G acting on the k -fold tensor product of its defining representation $V = \mathbb{C}^2$. The McKay correspondence relates the representation theory of these groups to an associated affine Dynkin diagram, and we use this connection to study the structure and representation theory of $Z_k(G)$ via the combinatorics of the Dynkin diagram. When G equals the binary tetrahedral, octahedral, or icosahedral group, we exhibit remarkable connections between $Z_k(G)$ and the Martin-Jones set partition algebras.

Résumé. Pour un sous-groupe fini G du groupe unitaire spéciale SU_2 , nous étudions la centralisateur algèbre $Z_k(G) = \text{End}_G(V^{\otimes k})$ de G agissant sur le produit k -fold de tenseur de sa représentation définissant $V = \mathbb{C}^2$. La correspondance de McKay concerne la théorie des représentations de ces groupes à une associé Dynkin diagramme, et nous utiliser cette connexion pour étudier la structure et la théorie des représentations de $Z_k(G)$ par l'intermédiaire de la combinatoire du diagramme de Dynkin. Quand G est égale à la groupe tétraédrique binaire, octaédre binaire, ou icosaédrique binaire, nous exhibons connexions remarquables entre $Z_k(G)$ et les algèbres de partitions de Martin-Jones.

Keywords. Schur-Weyl duality, McKay correspondence, partition algebra, Temperley-Lieb algebra, Dyck paths

1 Introduction

In 1980, John McKay [13] discovered that there is a natural one-to-one correspondence between the finite subgroups of the special unitary group SU_2 and the simply-laced affine Dynkin diagrams. Let $V = \mathbb{C}^2$ be the defining representation of SU_2 , and let G be a finite subgroup of SU_2 with irreducible modules G^λ , $\lambda \in \Lambda(G)$. The representation graph $\mathcal{R}_V(G)$ (also known as the McKay graph or McKay quiver) has vertices indexed by the $\lambda \in \Lambda(G)$ and $a_{\lambda,\mu}$ edges from λ to μ if G^μ occurs in $G^\lambda \otimes V$ with multiplicity $a_{\lambda,\mu}$. Almost a century earlier, Felix Klein had determined that a finite subgroup of SU_2 must be one of the following: (a) a cyclic group C_n of order n , (b) a binary dihedral group D_n of order $4n$, or (c) one of the 3 exceptional groups: the binary tetrahedral group T of order 24, the binary octahedral group O of order 48, or the binary icosahedral group I of order 120. McKay's observation was that the representation graph of C_n, D_n, T, O, I corresponds exactly to the affine Dynkin diagram $\hat{A}_{n-1}, \hat{D}_{n+2}, \hat{E}_6, \hat{E}_7, \hat{E}_8$.

[†]Email: benkart@math.wisc.edu. G. Benkart is grateful to the Simons Foundation for the support she received as a Simons Visiting Professor at MSRI.

[‡]Email: halverson@macalester.edu. T. Halverson was partially supported by National Science Foundation grant DMS-0800085 and partially supported by Simons Foundation grant 283311.

In the papers [1], [2], [7], we examine the McKay correspondence from the point of view of Schur-Weyl duality. Since the McKay graph provides a way to encode the rules for tensoring by V , it is natural to consider the k -fold tensor product module $V^{\otimes k}$ and to study the centralizer algebra $Z_k(G) = \text{End}_G(V^{\otimes k})$ of endomorphisms that commute with the action of G on $V^{\otimes k}$. The algebra $Z_k(G)$ provides essential information about the structure of $V^{\otimes k}$ as a G -module, as the projection maps from $V^{\otimes k}$ onto its irreducible G -summands are idempotents in $Z_k(G)$, and the multiplicity of G^λ in $V^{\otimes k}$ is the dimension of the $Z_k(G)$ -irreducible module corresponding to λ .

The main points of this article are:

- The irreducible $Z_k(G)$ -modules are labeled by vertices of the representation graph $\mathcal{R}_V(G)$.
- The dimension of the $Z_k(G)$ -module labeled by λ is the number of k -walks from 0 to λ on $\mathcal{R}_V(G)$.
- The dimension of $Z_k(G)$ equals the number of $2k$ -walks on $\mathcal{R}_V(G)$ starting and ending at 0.
- $Z_k(G)$ has generators corresponding to the nodes in $\mathcal{R}_V(G)$, as well as generators corresponding to each embedding $Z_i(G) \subseteq Z_{i+1}(G)$, and the relations are determined by the $\mathcal{R}_V(G)$ edge structure.
- $Z_k(G)$ has a basis of words in these generators that correspond to $2k$ -walks on $\mathcal{R}_V(G)$.
- When G is one of the exceptional groups $\mathbf{T}, \mathbf{O}, \mathbf{I}$, the centralizer $Z_k(G)$ can be described using the Martin-Jones partition algebras and their analogs.
- New formulas for the dimensions of the irreducible representations of partition algebras are given.

When G is a subgroup of SU_2 , the centralizer algebras satisfy the reverse inclusion $Z_k(SU_2) \subseteq Z_k(G)$. It is well known that $Z_k(SU_2)$ is isomorphic to the Temperley-Lieb algebra $TL_k(2)$. Thus, the centralizer algebras constructed here all contain a Temperley-Lieb subalgebra. The dimension of $TL_k(2)$ is the Catalan number $\mathcal{C}_k = \frac{1}{k+1} \binom{2k}{k}$, which counts walks of $2k$ steps that begin and end at 0 on the representation graph of SU_2 , i.e. the Dynkin diagram $A_{+\infty}$. In this case, the walks correspond to Dyck paths.

2 McKay Centralizer Algebras

The special unitary group SU_2 is the group of 2×2 complex matrices $\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$ satisfying $\alpha\bar{\alpha} + \beta\bar{\beta} = 1$. For each $r \geq 0$, SU_2 has an irreducible module $V(r)$ of dimension $r + 1$. The module $V = V(1) = \mathbb{C}^2$ with basis $v_{-1} = (1, 0)^t, v_1 = (0, 1)^t$ corresponds to the natural two-dimensional representation on which SU_2 acts by matrix multiplication. These modules satisfy the Clebsch-Gordan formula,

$$V(r) \otimes V = V(r - 1) \oplus V(r + 1), \tag{1}$$

where $V(-1) = 0$. The representation graph $\mathcal{R}_V(SU_2)$ is the infinite graph shown in Figure 1.

Now let G be a subgroup of SU_2 . Then G acts on the natural two-dimensional representation $V = \mathbb{C}^2$ by restriction. Let $\{G^\lambda \mid \lambda \in \Lambda(G)\}$ denote a complete set of pairwise non-isomorphic irreducible finite-dimensional G -modules occurring in some $V^{\otimes k}$ for $k = 0, 1, \dots$. By convention $V^{\otimes 0} = G^{(0)}$ is the trivial G -module. The *representation graph* $\mathcal{R}_V(G)$ is the graph with vertices labeled by elements of $\Lambda(G)$ with $a_{\lambda,\mu}$ edges between λ and μ if the decomposition of $G^\lambda \otimes V$ into irreducible G -modules is given by

$$G^\lambda \otimes V = \bigoplus_{\mu \in \Lambda(G)} a_{\lambda,\mu} G^\mu. \tag{2}$$

For finite subgroups $G \subseteq SU_2$, the representation graph $\mathcal{R}_V(G)$ is an undirected, simple graph (see [15]). Since V is faithful (being the defining module for G) and G is finite, all irreducible G -modules

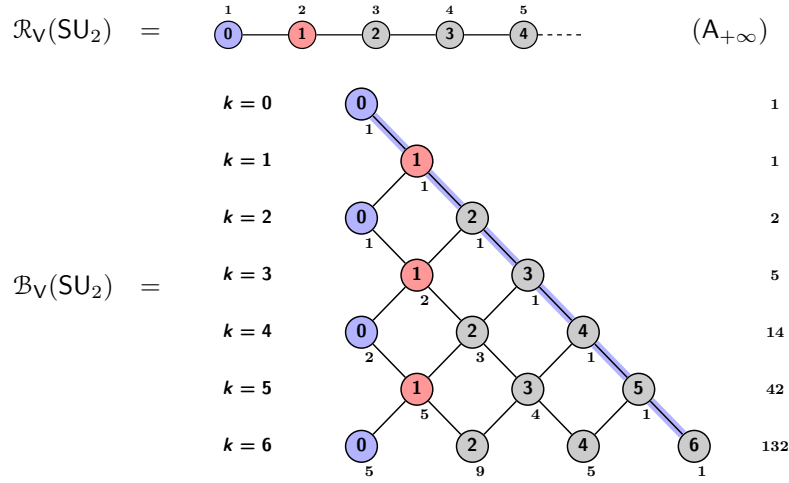


Fig. 1: The representation graph $\mathcal{R}_V(\text{SU}_2)$, which is the Dynkin diagram $A_{+\infty}$, and the first 6 levels of the corresponding Bratteli diagram $\mathcal{B}_V(\text{SU}_2)$. In $\mathcal{R}_V(\text{SU}_2)$, the label on the node is the index of the SU_2 -module, and the label above the node is its dimension. The trivial module is shown in blue and the defining module V in red. In $\mathcal{B}_V(\text{SU}_2)$ the label below vertex r on level k gives the number of paths (Dyck paths in this case) from the top of the diagram to r , which is multiplicity of $V(r)$ in $V^{\otimes k}$. These numbers also count the walks of length k from 0 to r on $\mathcal{R}_V(G)$. The column to the right contains the sum of the squares of the multiplicities, which at level k is the Catalan number \mathcal{C}_k and equals the dimension of the centralizer algebra $Z_k(\text{SU}_2)$.

occur in some $V^{\otimes k}$, and thus $\mathcal{R}_V(G)$ is connected. The representation graphs $\mathcal{R}_V(G)$ corresponding to $G = C_n, D_n, T, O, I$, are displayed in Figure 2. McKay observed that these graphs correspond exactly to the affine Dynkin diagrams of type $\hat{A}_{n-1}, \hat{D}_{n+2}, \hat{E}_6, \hat{E}_7, \hat{E}_8$, respectively. The trivial module $G^{(0)}$ corresponds to the affine node in those cases.

For $k \geq 1$, the k -fold tensor power $V^{\otimes k}$ is 2^k -dimensional and has a basis of simple tensors $V^{\otimes k} = \text{span}_{\mathbb{C}} \{ v_{r_1} \otimes v_{r_2} \otimes \dots \otimes v_{r_k} \mid r_j \in \{-1, 1\} \}$. Group elements $g \in G$ act on simple tensors by the diagonal action $g(v_{r_1} \otimes v_{r_2} \otimes \dots \otimes v_{r_k}) = gv_{r_1} \otimes gv_{r_2} \otimes \dots \otimes gv_{r_k}$. Let

$$\Lambda_k(G) = \{ \lambda \in \Lambda(G) \mid G^\lambda \text{ appears as a summand in the decomposition of } V^{\otimes k} \}. \tag{3}$$

Then, $\Lambda_k(G)$ is the set of vertices in $\mathcal{R}_V(G)$ that can be reached by paths of length k starting from 0. Furthermore, $\Lambda_k(G) \subseteq \Lambda_{k+2}(G)$, for all $k \geq 0$, since if a node can be reached in k steps, then it can also be reached in $k + 2$ steps. The *Bratteli diagram* $\mathcal{B}_V(G)$ is the infinite graph with vertices labeled by $\Lambda_k(G)$ on level k and $a_{\lambda, \mu}$ edges from vertex $\lambda \in \Lambda_k(G)$ to vertex $\mu \in \Lambda_{k+1}(G)$. The Bratteli diagram for $G = \text{SU}_2$ is shown in Figure 1, and the Bratteli diagrams for $G = C_n, D_n, T, O, I$ are shown in Figure 3.

A *walk* of length k on the representation graph $\mathcal{R}_V(G)$ from 0 to $\lambda \in \Lambda(G)$, is a sequence $(0, \lambda^1, \lambda^2, \dots, \lambda^k = \lambda)$ starting at $\lambda^0 = 0$, such that $\lambda^j \in \Lambda(G)$ for each $1 \leq j \leq k$, and λ^{j-1} is connected to λ^j by an edge in $\mathcal{R}_V(G)$. Such a walk is equivalent to a unique *path* of length k on the Bratteli diagram $\mathcal{B}_V(G)$ from $0 \in \Lambda_0(G)$ to $\lambda \in \Lambda_k(G)$. Let $\mathcal{W}_k^\lambda(G)$ denote the set of walks on $\mathcal{R}_V(G)$ of length k from $0 \in \Lambda(G)$

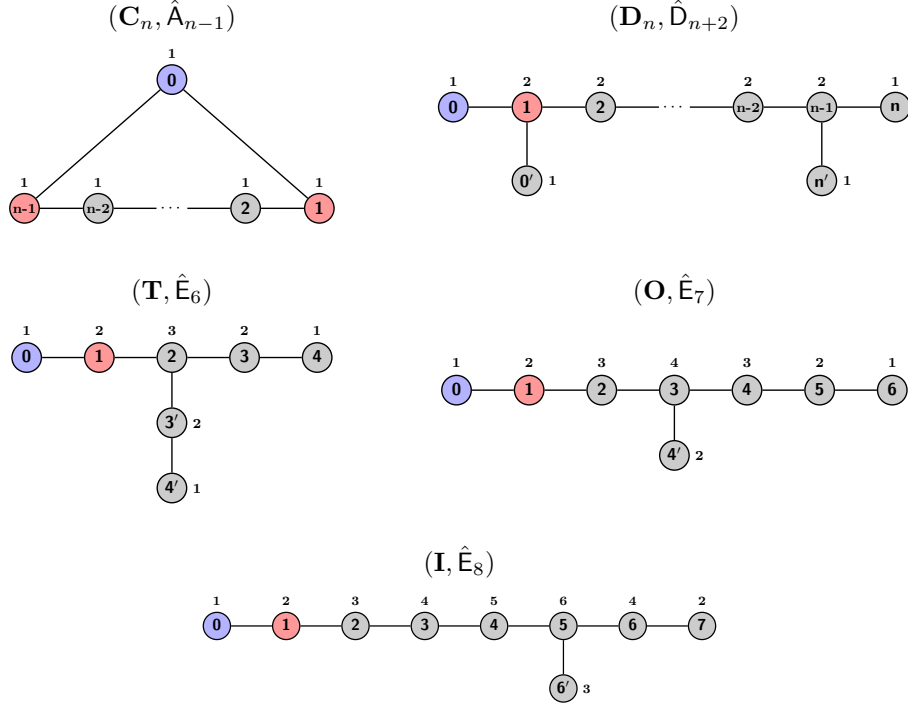


Fig. 2: The representation graphs $\mathcal{R}_V(G)$ for the finite subgroups $G = C_n, D_n, T, O, I$ correspond to the affine Dynkin diagrams of type $\hat{A}_{n-1}, \hat{D}_{n+1}, \hat{E}_6, \hat{E}_7, \hat{E}_8$. The label on the node is the index of the G -module, and the label above the node is its dimension. The trivial module (affine node) is blue and the defining module V is red.

to $\lambda \in \Lambda(G)$, and let $\mathcal{P}_k^\lambda(G)$ denote the set of paths on $\mathcal{B}_V(G)$ of length k from $0 \in \Lambda_0(G)$ to $\lambda \in \Lambda_k(G)$. Let m_k^λ denote the multiplicity of G^λ in $V^{\otimes k}$. Then, by induction on (2) we have

$$\begin{aligned}
 m_k^\lambda &= |\mathcal{W}_k^\lambda(G)| = \#(\text{walks on } \mathcal{R}_V(G) \text{ of length } k \text{ from } 0 \text{ to } \lambda) \\
 &= |\mathcal{P}_k^\lambda(G)| = \#(\text{paths in } \mathcal{B}_V(G) \text{ of length } k \text{ from } 0 \in \Lambda_0(G) \text{ to } \lambda \in \Lambda_k(G)).
 \end{aligned}
 \tag{4}$$

The centralizer of G on $V^{\otimes k}$ is the algebra

$$Z_k(G) = \text{End}_G(V^{\otimes k}) = \{ a \in \text{End}(V^{\otimes k}) \mid a(gw) = ga(w) \text{ for all } g \in G, w \in V^{\otimes k} \}.
 \tag{5}$$

If the group G is apparent from the context, we will simply write Z_k for $Z_k(G)$. Since $V^{\otimes 0} = G^{(0)}$, we have $Z_0(G) = \mathbb{C}1$. There is a natural embedding $Z_k(G) \hookrightarrow Z_{k+1}(G)$ given by $a \mapsto a \otimes 1$, where $a \otimes 1$ acts as a on the first k tensor factors and 1 acts as the identity in the $(k + 1)$ st tensor position. Iterating this embedding gives an infinite tower of algebras $Z_0(G) \subseteq Z_1(G) \subseteq Z_2(G) \subseteq \dots$.

By classical double-centralizer theory (see for example [6, 3B,68]), we have:

- $Z_k(G)$ is a semisimple \mathbb{C} -algebra with irreducible modules $\{ Z_k^\lambda \mid \lambda \in \Lambda_k(G) \}$ labeled by $\Lambda_k(G)$.

- $\dim Z_k^\lambda = m_k^\lambda = |\mathcal{W}_k^\lambda(\mathbb{G})| = |\mathcal{P}_k^\lambda(\mathbb{G})|$.
- Edges from level k to $k-1$ in $\mathcal{B}_V(\mathbb{G})$ represent restriction and induction rules for $Z_{k-1}(\mathbb{G}) \subseteq Z_k(\mathbb{G})$.
- If $d^\lambda = \dim \mathbb{G}^\lambda$, then the tensor space $V^{\otimes k}$ has the following decomposition

$$V^{\otimes k} \cong \underbrace{\bigoplus_{\lambda \in \Lambda_k(\mathbb{G})} m_k^\lambda \mathbb{G}^\lambda}_{\text{as a } \mathbb{G}\text{-module}} \cong \underbrace{\bigoplus_{\lambda \in \Lambda_k(\mathbb{G})} d^\lambda Z_k^\lambda}_{\text{as a } Z_k(\mathbb{G})\text{-module}} \cong \underbrace{\bigoplus_{\lambda \in \Lambda_k(\mathbb{G})} (\mathbb{G}^\lambda \otimes Z_k^\lambda)}_{\text{as a } (\mathbb{G}, Z_k(\mathbb{G}))\text{-bimodule}}. \quad (6)$$

- By general Wedderburn theory, the dimension of $Z_k(\mathbb{G})$ is

$$\dim Z_k(\mathbb{G}) = \sum_{\lambda \in \Lambda_k(\mathbb{G})} (m_k^\lambda)^2 = \sum_{\lambda \in \Lambda_k(\mathbb{G})} |\mathcal{W}_k^\lambda(\mathbb{G})|^2 = |\mathcal{W}_{2k}^0(\mathbb{G})| = \dim Z_{2k}^{(0)}, \quad (7)$$

which equals the number of walks of length $2k$ that begin and end at 0 on $\mathcal{R}_V(\mathbb{G})$. The third equality follows from the property that a pair of walks of length k from 0 to λ corresponds uniquely (by reversing the second walk) to a walk of length $2k$ beginning and ending at 0 .

3 Paths and Dimensions

Using an inductive argument on the structure of the Bratteli diagram, we compute the dimensions of the irreducible $Z_k(\mathbb{G})$ -modules Z_k^λ for $\lambda \in \Lambda_k(\mathbb{G})$. They are given explicitly in [1]. This dimension also equals the multiplicity of \mathbb{G}^λ in $V^{\otimes k}$. The dimension of the centralizer algebra $Z_k(\mathbb{G})$ is then the sum of the squares of these dimensions: $\dim Z_k(\mathbb{G}) = \sum_{\lambda} \dim(Z_k^\lambda)^2$.

Theorem 8 ([1] Dimension Formulas) *For $k \geq 1$, the following formulas give the dimension $\dim Z_k(\mathbb{G})$ of the McKay centralizer algebra, which also equals the number of $2k$ -walks on the representation graph $\mathcal{R}_V(\mathbb{G})$ from 0 to 0 .*

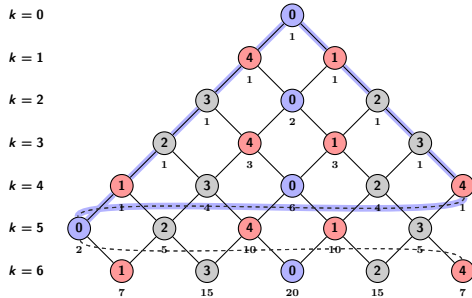
(a) $\dim Z_k(\mathbf{C}_n) = 2 \dim Z_k(\mathbf{D}_n) = \sum_{\substack{0 \leq a, b \leq k \\ a \equiv b \pmod{\tilde{n}}} \binom{k}{a} \binom{k}{b} = \text{coefficient of } z^k \text{ in } (1+z)^{2k} \Big|_{z^{\tilde{n}}=1}$ which equals the $2k$ - k coefficient in Pascal's triangle on a cylinder of "diameter" \tilde{n} (Fig. 3), where $\tilde{n} = n$, if n is odd, and $\tilde{n} = \frac{1}{2}n$, if n is even.

(b) $\dim Z_k(\mathbf{T}) = \frac{4^k + 8}{12}$ ([14] OEIS sequence A047849).

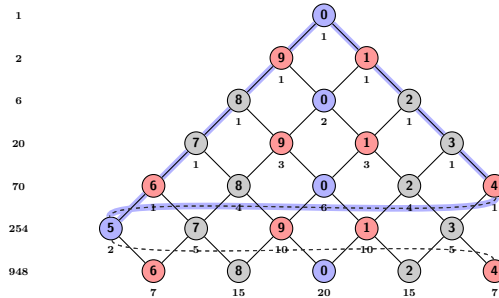
(c) $\dim Z_k(\mathbf{O}) = \frac{4^k + 6 \cdot 2^k + 8}{24}$ ([14] OEIS sequence A007581).

(d) $\dim Z_k(\mathbf{I}) = \frac{4^k + 12L_{2k} + 20}{60}$, where L_n is the Lucas number defined by $L_0 = 2, L_1 = 1$, and $L_{n+2} = L_{n+1} + L_n$.

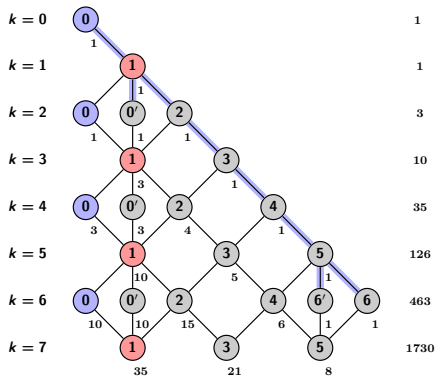
C_5 :



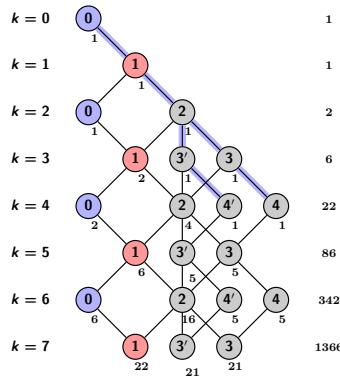
C_{10} :



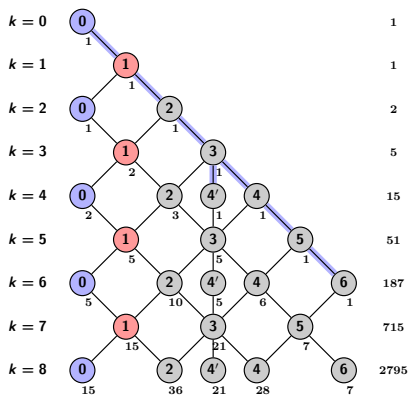
D_6 :



T :



O :



I :

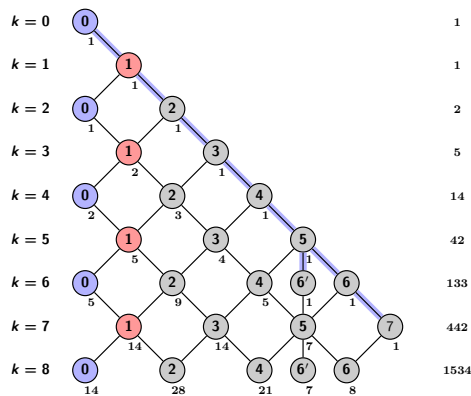


Fig. 3: The first several rows of the Bratteli diagrams $\mathcal{B}_V(G)$ for $G = C_5, C_{10}, D_6, T, O, I$. The representation graph $\mathcal{R}_V(G)$ is embedded as the shaded edges. The unshaded edges are reflections from the row above and correspond to the Jones basic construction ideal $Z_k e_k Z_k$ (see Sec. 4). The Bratteli diagrams for C_5 and C_{10} are isomorphic. The label below vertex r on level k gives the number of k -paths from the top of the diagram to r , which is also multiplicity of $G^{(r)}$ in $V^{\otimes k}$. These numbers also give the number of k -walks from 0 to r on the representation graph $\mathcal{R}_V(G)$. The column to the right contains the sum of the squares of the multiplicities which equals $\dim Z_k(G)$.

4 Basic Construction

In this section we define two kinds of (essential) idempotents $\{e_i \mid 1 \leq i \leq k-1\}$ and $\{f_\nu \mid \nu \in \Lambda(\mathbb{G})\}$ which together with 1 generate $Z_k(\mathbb{G})$. For $1 \leq i \leq k-1$, define an endomorphism e_i on $V^{\otimes k}$ by

$$e_i = \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes e \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}, \quad (9)$$

where $\mathbf{1}$ is the 2×2 identity matrix, and $e : V \otimes V \rightarrow V \otimes V$ acts in tensor positions i and $i+1$ by $e(v_j \otimes v_\ell) = v_j \otimes v_\ell - v_\ell \otimes v_j$, for $j, \ell \in \{-1, 1\}$. Thus, $e : V^{\otimes 2} \rightarrow V^{\otimes 2}$ projects onto the antisymmetric tensors in $V^{\otimes 2}$, and it is easy to confirm that $e_i \in Z_k(\mathbb{G})$.

For $\nu \in \Lambda(\mathbb{G})$, we let $|\nu|$ equal the distance from 0 to ν in the representation graph $\mathcal{R}_V(\mathbb{G})$. Thus by the way we have chosen our labels in Figure 3, $|\ell| = |\ell'| = \ell$. Correspondingly, the module G^ν first appears as a constituent of $V^{\otimes k}$, when $k = |\nu|$, and it appears in that tensor product with multiplicity exactly 1. Define

$f_\nu :=$ the \mathbb{G} -module homomorphism projecting onto the unique copy of G^ν in $V^{\otimes |\nu|}$.

In [1] we show how to explicitly construct f_ν for each $\nu \in \Lambda(\mathbb{G})$. In particular, we show that

$$f_\nu - \frac{d^{\nu-1}}{d^\nu} f_\nu e_{|\nu|} f_\nu = \sum_{\mu=\nu+1} f_\mu, \quad (10)$$

where $\nu-1 \in \Lambda(\mathbb{G})$ is the unique neighbor of ν in $\mathcal{R}_V(\mathbb{G})$ that is closer to 0 (i.e., $|\nu-1| = |\nu|-1$), where the sum is over the neighbors $\mu = \nu+1$ of ν in $\mathcal{R}_V(\mathbb{G})$ that are farther from 0 (i.e., $|\mu| = |\nu|+1$), and where $d^\lambda = \dim G^\lambda$. In most cases, ν has degree 2, and there is a unique node of the form $\mu = \nu+1$, and thus (10) defines f_μ uniquely. If ν has degree greater than 2, then we use other methods [1, Sec. 1.8] to decompose the sum into the constituent f_μ . This recursive construction is a generalization of the construction of the Jones-Wenzl idempotent for SU_2 .

A *branch node* in the representation graph $\mathcal{R}_V(\mathbb{G})$ is any vertex of degree greater than 2. Let $\text{br}(\mathbb{G})$ denote the set of branch nodes in $\mathcal{R}_V(\mathbb{G})$. In the special case of $\mathcal{R}_V(\mathbb{C}_n)$ for $n \leq \infty$, we consider the affine node to be the branch node. Let the *diameter* of $\mathcal{R}_V(\mathbb{G})$, denoted by $\text{diam}(\mathbb{G})$, be the maximum distance between any vertex $\lambda \in \Lambda(\mathbb{G})$ and $0 \in \Lambda(\mathbb{G})$. For $\mathbb{G} = \mathbb{C}_n$, we let $\text{diam}(\mathbb{G}) = \tilde{n}$ as in (11).

\mathbb{G}	SU_2	\mathbb{C}_n	\mathbb{D}_n	\mathbb{T}	\mathbb{O}	\mathbb{I}	\mathbb{C}_∞	\mathbb{D}_∞	$\tilde{n} = \begin{cases} \frac{1}{2}n, & \text{if } n \text{ is even,} \\ n, & \text{if } n \text{ is odd.} \end{cases} \quad (11)$
$\text{diam}(\mathbb{G})$	∞	\tilde{n}	n	4	6	7	∞	∞	
$\text{br}(\mathbb{G})$	\emptyset	$\{0\}$	$\{1, n\}$	$\{2\}$	$\{3\}$	$\{5\}$	$\{0\}$	$\{1\}$	

The *Jones basic construction* (see [8], [16], [9]) uses the ideal $Z_k e_k Z_k \subseteq Z_{k+1}$ to recursively study the structure of Z_{k+1} . We use it to prove the following theorem.

Theorem 12 ([1] Basic Construction) *Let $Z_k = Z_k(\mathbb{G})$ and let $Z_0 = \mathbb{C}1$. Then for $k \geq 1$, except for the two special cases in part (d), we have*

- (a) $Z_{k+1} = Z_k e_k Z_k \oplus N_{k+1}$, where $Z_k e_k Z_k$ is a two-sided basic construction ideal, and N_{k+1} is a commutative subalgebra spanned by $\{f_\nu \mid |\nu| = k+1\}$. In particular, $N_{k+1} = 0$ if $k \geq \text{diam}(\mathbb{G})$.
- (b) $Z_{k+1} = \langle Z_k, e_k \rangle$ if $k \notin \text{br}(\mathbb{G})$, and $Z_{k+1} = \langle Z_k, e_k, f_{(k+1)} \rangle$ if $k \in \text{br}(\mathbb{G})$.

- (c) Z_{k+1} is generated by $\{1, e_1, \dots, e_k\} \cup \{f_{(\ell+1)} \mid \ell \in \text{br}(\mathbf{G}), \ell \leq k\}$.
- (d) $\text{TL}_k(2) \subseteq Z_k$ for all $k \geq 0$, and $\text{TL}_k(2) = Z_k$ for $0 \leq k \leq \min(\text{br}(\mathbf{G}))$.
- (e) Two special cases: (i) if $\mathbf{G} = \mathbf{C}_n$, then $Z_{\bar{n}} = \langle Z_{\bar{n}-1}, e_{\bar{n}-1}, E_{p,q}, \text{ for } p, q \in \{-1, 1\} \rangle$, where $E_{p,q}$ are matrix units. (ii) if $\mathbf{G} = \mathbf{D}_2$, then $Z_2 = \langle Z_1, e_1, f_{\mu_1}, f_{\mu_2} \rangle$ where $\mu_1, \mu_2 \in \{(0'), (2), (2')\}$, $\mu_1 \neq \mu_2$.

Example 13 Compare these examples with their representation graphs $\mathcal{R}_V(\mathbf{G})$ in Figure 2.

- (a) If $\mathbf{G} = \mathbf{O}$, then $\text{br}(\mathbf{O}) = \{3\}$, $\text{diam}(\mathbf{O}) = 6$, and

$$\begin{aligned} Z_1 &= \mathbf{C1} = \mathbf{Cf}_{(1)} = Z_0 \cong \text{TL}_1(2) & Z_5 &= Z_4 e_4 Z_4 \oplus \mathbf{Cf}_{(5)} = \langle Z_4, e_4 \rangle \\ Z_2 &= Z_1 e_1 Z_1 \oplus \mathbf{Cf}_{(2)} = \langle Z_1, e_1 \rangle \cong \text{TL}_2(2) & Z_6 &= Z_5 e_5 Z_5 \oplus \mathbf{Cf}_{(6)} = \langle Z_5, e_5 \rangle \\ Z_3 &= Z_2 e_2 Z_2 \oplus \mathbf{Cf}_{(3)} = \langle Z_2, e_2 \rangle \cong \text{TL}_3(2) & Z_7 &= Z_6 e_6 Z_6 = \langle Z_6, e_6 \rangle \\ Z_4 &= Z_3 e_3 Z_3 \oplus \mathbf{Cf}_{(4)} \oplus \mathbf{Cf}_{(4')} = \langle Z_3, e_3, f_{(4)} \rangle & Z_{k+1} &= Z_k e_k Z_k = \langle Z_k, e_k \rangle, k \geq 6. \end{aligned}$$

- (b) If $\mathbf{G} = \mathbf{D}_6$, then $\text{br}(\mathbf{D}_6) = \{1, 5\}$, $\text{diam}(\mathbf{D}_6) = 6$, and

$$\begin{aligned} Z_1 &= \mathbf{C1} = \mathbf{Cf}_{(1)} = Z_0 \cong \text{TL}_1(2) & Z_5 &= Z_4 e_4 Z_4 \oplus \mathbf{Cf}_{(5)} = \langle Z_4, e_4 \rangle \\ Z_2 &= Z_1 e_1 Z_1 \oplus \mathbf{Cf}_{(0')} \oplus \mathbf{Cf}_{(2)} = \langle Z_1, e_1, f_{(2)} \rangle & Z_6 &= Z_5 e_5 Z_5 \oplus \mathbf{Cf}_{(6')} \oplus \mathbf{Cf}_{(6)} = \langle Z_5, e_5, f_{(6)} \rangle \\ Z_3 &= Z_2 e_2 Z_2 \oplus \mathbf{Cf}_{(3)} = \langle Z_2, e_2 \rangle & Z_7 &= Z_6 e_6 Z_6 = \langle Z_6, e_6 \rangle \\ Z_4 &= Z_3 e_3 Z_3 \oplus \mathbf{Cf}_{(4)} = \langle Z_3, e_3 \rangle & Z_{k+1} &= Z_k e_k Z_k = \langle Z_k, e_k \rangle, k \geq 6. \end{aligned}$$

5 Linear Bases

Let $\mathcal{P}_{2k}^0(\mathbf{G})$ denote the set of paths on $\mathcal{B}_V(\mathbf{G})$ of length $2k$ from 0 at level 0 to 0 at level $2k$. Then $\dim Z_k(\mathbf{G}) = |\mathcal{P}_{2k}^0(\mathbf{G})|$, and so it is natural to seek a basis $\{w_p \mid p \in \mathcal{P}_{2k}^0(\mathbf{G})\}$ of $Z_k(\mathbf{G})$ where each w_p is a word in the generators $\{1, e_1, \dots, e_{k-1}\} \cup \{f_{(\ell+1)} \mid \ell \in \text{br}(\mathbf{G}), \ell < k\}$. For example, when $\mathbf{G} = \text{SU}_2$, the centralizer is the Temperley-Lieb algebra $Z_k(\text{SU}_2) \cong \text{TL}_k(2)$, the dimension is the Catalan number \mathcal{C}_k , the paths are Dyck paths, and the bijection to a basis of words in e_1, \dots, e_{k-1} is given in [8, 2.8]. Here we generalize this result uniformly to the finite subgroups $\mathbf{G} \subseteq \text{SU}_2$.

For $\ell = \min(\text{br}(\mathbf{G}))$, i.e., the first branch point, define

$$b_i = \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{(i-1) \text{ factors}} \otimes f_{(\ell+1)} \otimes \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{(k-i-\ell) \text{ factors}}, \quad 1 \leq i \leq k - \ell, \quad \text{for each } \mathbf{G},$$

and define

$$c_i = \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{(i-1) \text{ factors}} \otimes f_{(n)} \otimes \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{(k-i-n+1) \text{ factors}}, \quad 1 \leq i \leq k - n + 1, \quad \text{for } \mathbf{G} = \mathbf{D}_n.$$

A path $p \in \mathcal{P}_{2k}^0$ is a sequence $p = (0 = p_0, p_1, \dots, p_{2k-1}, p_{2k} = 0)$, where each $p_i \in \Lambda_i(\mathbf{G})$ is a label of an irreducible \mathbf{G} -module that appears in $V^{\otimes i}$. A *peak* in a path p is a label p_i such that $|p_{i-1}| < |p_i|$ and $|p_i| > |p_{i+1}|$. If a peak p_i is marked with the prime symbol (e.g., $0'_i$ or $4'_i$, etc), then p_i is a *nonstandard peak*. Otherwise it is said to be *standard*. To each peak p_i we associate a product of generators called *block* as follows

$$\mathcal{B}(p_i) = e_\alpha e_{\alpha-1} e_{\alpha-2} \cdots e_\beta, \quad \text{with } \alpha = \frac{i + |p_i| - 2}{2} \text{ and } \beta = \frac{i - |p_i| + 2}{2}, \quad (14)$$

$$\mathcal{B}(p'_i) = b_\beta, \quad \text{with } \beta = \frac{i - |p_i| + 2}{2}, \quad (15)$$

where the b_β is replaced by a c_β in \mathbf{D}_n if $p'_i = n'_i$. When $G = \mathbf{T}$, we have the following special case,

$$(G = \mathbf{T}) \quad \mathcal{B}(4'_i) = \begin{cases} b_{\alpha-2}e_\alpha e_{\alpha-1}e_{\alpha-2}, & \text{with } \alpha = \frac{i+2}{2}, & \text{if } p_{i-2} = 2, \\ e_\alpha e_{\alpha-1}e_{\alpha-2}, & \text{with } \alpha = \frac{i+2}{2}, & \text{if } p_{i-2} = 4. \end{cases} \quad (16)$$

If $G = \mathbf{C}_n$ with $n \leq \infty$ then there are further special cases for $\mathcal{B}(p'_i)$ and $\mathcal{B}(\tilde{n}_i^\pm)$ described in [7].

If $p \in \mathcal{P}_{2k}^0(G)$ is a path in the Bratteli diagram, then we define the *word* w_p of p as the product blocks for each peak in p (see Example 19):

$$w_p = \mathcal{B}(p_{i_1})\mathcal{B}(p_{i_2}) \cdots \mathcal{B}(p_{i_\ell}), \quad \text{where } p_{i_1}, p_{i_2}, \dots, p_{i_\ell} \text{ are the peaks in } p. \quad (17)$$

Theorem 18 ([7] Basis Theorem) *For $G = \mathbf{SU}_2, \mathbf{C}_n, \mathbf{D}_n, \mathbf{C}_\infty, \mathbf{D}_\infty, \mathbf{T}, \mathbf{O}$, or \mathbf{I} , and $k \geq 0$, the set $\{w_p \mid p \in \mathcal{P}_{2k}^0(G)\}$ is a basis for $Z_k(G)$.*

Example 19 (Paths and their Corresponding Words) *The following are examples of paths p in the Bratteli diagram $\mathcal{B}_V(G)$ with their peaks circled and the corresponding words w_p .*

(a) In $Z_{10}(\mathbf{O})$

$$\begin{aligned} p &= (0_0, 1_1, 2_2, 3_3, 4_4, \textcircled{5_5}, 4_6, 3_7, 4_8, 5_9, \textcircled{6_{10}}, 5_{11}, 4_{12}, 3_{13}, 2_{14}, 1_{15}, 2_{16}, \textcircled{3_{17}}, 2_{18}, 1_{19}, 0_{20}) \\ w_p &= (e_4 e_3 e_2 e_1) (e_7 e_6 e_5 e_3 e_3) (e_9 e_8) \\ p &= (0_0, 1_1, 2_2, 3_3, \textcircled{4_4}, 3_5, 4_6, 5_7, \textcircled{6_8}, 5_9, 4_{10}, 3_{11}, \textcircled{4'_{12}}, 3_{13}, 2_{14}, 1_{15}, 0_{16}, 1_{17}, \textcircled{2_{18}}, 1_{19}, 0_{20}) \\ w_p &= (e_3 e_2 e_1) (e_6 e_5 e_4 e_3 e_2) (b_3) (e_9) \end{aligned}$$

(b) In $Z_{10}(\mathbf{T})$

$$\begin{aligned} p &= (0_0, 1_1, 2_2, \textcircled{3_3}, 2_4, \textcircled{3_5}, 2_6, 1_7, 0_8, 1_9, 2_{10}, 3'_{11}, \textcircled{4'_{12}}, 3'_{13}, 2_{14}, 3_{15}, \textcircled{4_{16}}, 3_{17}, 2_{18}, 1_{19}, 0_{20}) \\ w_p &= (e_2 e_1) (e_3 e_2) (b_5 e_7 e_6 e_5) (e_9 e_8 e_7) \\ p &= (0_0, 1_1, 2_2, 3_3, \textcircled{4_4}, 3_5, \textcircled{4_6}, 3_7, \textcircled{4_8}, 3_9, 2_{10}, 1_{11}, 0_{12}, 1_{13}, 2_{14}, 3'_{15}, \textcircled{4'_{16}}, 3'_{17}, 2_{18}, 1_{19}, 0_{20}) \\ w_p &= (e_3 e_2 e_1) (e_4 e_3 e_2) (e_5 e_4 e_3) (b_7 e_9 e_8 e_7) \end{aligned}$$

(c) In $Z_{10}(\mathbf{D}_5)$

$$\begin{aligned} p &= (0_1, 1_1, \textcircled{2'_2}, 1_3, 2_4, 3_5, 4_6, \textcircled{5_7}, 4_8, \textcircled{5'_9}, 4_{10}, 3_{11}, 2_{12}, 1_{13}, 0_{14}, 1_{15}, \textcircled{2_{16}}, 1_{17}, \textcircled{2'_{18}}, 1_{19}, 0_{20}) \\ w_p &= (b_1) (e_5 e_4 e_3 e_2) (c_3) (e_8) (b_9) \\ p &= (0_0, 1_1, 2_2, 3_3, \textcircled{4_4}, 3_5, 4_6, \textcircled{5_7}, 4_8, 3_9, 2_{10}, 3_{11}, \textcircled{4_{12}}, 3_{13}, 2_{14}, 1_{15}, 2_{16}, \textcircled{3_{17}}, 2_{18}, 1_{19}, 0_{20}) \\ w_p &= (e_3 e_2 e_1) (e_5 e_4 e_3 e_2) (e_7 e_6 e_5) (e_9 e_8) \end{aligned}$$

As above, if $\nu \in \Lambda(G)$, then $|\nu|$ equals the distance from 0 to ν in $\mathcal{R}_V(G)$. Let $e_\nu := e_{|\nu|}$, and let $d^\nu = \dim G^\nu$. For $\nu \neq 0$, $\nu - 1 \in \Lambda(G)$ is the unique node that is connected to ν by an edge with $|\nu - 1| = |\nu| - 1$. We say that $\mu = \nu + 1$ if $\mu \in \Lambda(G)$ is connected to ν by an edge and $|\mu| = |\nu| + 1$. Finally we say that $\mu \prec \nu$ if μ is on the shortest path from 0 to ν .

Theorem 20 (Presentation on Generators and Relations) For $k \geq 1$ and $k < \tilde{n}$ if $G = C_n$, the algebra $Z_k(G)$ is generated by $\{1, e_1, \dots, e_{k-1}\} \cup \{f_\nu \mid \nu \in \Lambda(G), |\nu| \leq k\}$ subject to the following relations:

- (1) $e_i^2 = 2e_i; \quad e_i e_{i\pm 1} e_i = e_i; \quad e_i e_j = e_j e_i, \quad |i - j| > 1;$
- (2) $f_\nu^2 = f_\nu; \quad f_\nu e_i = e_i f_\nu = 0$ for $i < |\nu|; \quad f_\nu e_i = e_i f_\nu$ for $i > |\nu|; \quad e_\nu f_\nu e_\nu = \frac{d^\nu}{d^{\nu-1}} f_{\nu-1} e_\nu;$
- (3) $\sum_{\mu=\nu+1} f_\mu = f_\nu - \frac{d^{\nu-1}}{d^\nu} f_\nu e_{\nu-1} f_\nu$ (and this equals 0 if no such μ exists);
- (4) If $|\mu| \leq |\nu|$, then $f_\mu f_\nu = f_\nu f_\mu = f_\mu$, if $\mu \prec \nu$, and $f_\mu f_\nu = f_\nu f_\mu = 0$, if $\mu \not\prec \nu$.

The case of $k \geq \tilde{n}$, for $G = C_n$ requires some further notation and is handled in [7].

6 Exceptional McKay Centralizers and Partition Algebras

The exceptional groups are referred to as the binary tetrahedral, binary octahedral, and binary icosahedral groups because modulo their centers $Z(G) = \{1, -1\}$, we have the following isomorphisms:

- $T/\{1, -1\} \cong A_4$, the alternating group on 4 letters (rotation group of a tetrahedron),
- $O/\{1, -1\} \cong S_4$, the symmetric group on 4 letters (rotation group of an octahedron),
- $I/\{1, -1\} \cong A_5$, the alternating group on 5 letters (rotation group of an icosahedron).

Group elements act on $V^{\otimes 2} = V \otimes V$ diagonally: $g \cdot (v_i \otimes v_j) = gv_i \otimes gv_j$. If $g \in Z(G) = \{1, -1\}$ then g acts on V as multiplication by 1 and -1 , so it acts trivially on $V^{\otimes 2}$. Thus, $Z(G)$ is in the kernel of the action on tensor powers $V^{\otimes k}$, with k even, and T, O , and I act the same as A_4, S_4 , and A_5 , respectively

Furthermore, $\dim(V^{\otimes 2}) = 4$, and as a module for A_4, S_4 , and A_5 it decomposes in the following way:

$$\begin{aligned} V^{\otimes 2} &\cong A_4^{(4)} \oplus A_4^{(3,1)} \cong M, \text{ the permutation module for } A_4, \\ V^{\otimes 2} &\cong S_4^{(4)} \oplus S_4^{(2,1,1)} \cong \tilde{M}, \text{ a "twisted" permutation module for } S_4, \\ V^{\otimes 2} &\cong A_5^{(5)} \oplus A_5^{(3,1,1)^+} \cong \tilde{\tilde{M}}, \text{ which is not a permutation module for } A_5, \end{aligned}$$

where here we are using usual integer partition notation to label the irreducible modules for S_n and A_n . This means that

$$Z_{2k}(T) \cong \text{End}_{A_4}(M^{\otimes k}), \quad Z_{2k}(O) \cong \text{End}_{S_4}(\tilde{M}^{\otimes k}), \quad Z_{2k}(I) \cong \text{End}_{A_5}(\tilde{\tilde{M}}^{\otimes k}). \quad (21)$$

The Martin-Jones partition algebras $P_k(n)$ maps surjectively onto $\text{End}_{S_n}(M^{\otimes k})$ for all n and isomorphically for $n \geq 2k$, where M is the n -dimensional permutation representation of S_n (see [11], [12], [10]). When M is restricted from S_n to A_n the corresponding partition algebra $\tilde{P}_k(n)$ is studied by Bloss [5]. The partition algebra $P_k(n)$ has a basis indexed by set partitions of $\{1, 2, \dots, 2k\}$ and a multiplication given by set partition diagram concatenation. See [9] for a survey on partition algebras.

In [3, 4] we study the partition algebras $\text{End}_{S_n}(M^{\otimes k})$ and $\text{End}_{A_n}(M^{\otimes k})$ for $n < 2k$ ("low rank") and for both the permutation module $M = S_n^{(n)} \oplus S_n^{(n-1,1)}$ and its twisted counterpart $\tilde{M} = S_n^{(n)} \oplus S_n^{(2,1^{n-1})}$. This work gives us alternative precise descriptions of the centralizer algebras $Z_{2k}(T)$ and $Z_{2k}(O)$. In [3] we do the following:

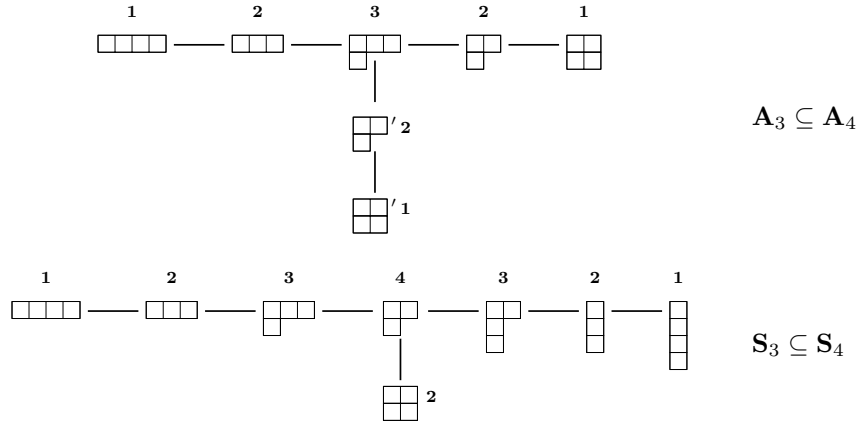


Fig. 4: Restriction-induction graphs for $\mathbf{A}_3 \subseteq \mathbf{A}_4$ and $\mathbf{S}_3 \subseteq \mathbf{S}_4$. These graphs are isomorphic to the representation graphs $\mathcal{R}_V(\mathbf{T})$ and $\mathcal{R}_V(\mathbf{O})$, respectively, which in turn equal the Dynkin diagrams of type \hat{E}_6 and \hat{E}_7 . See Figure 2.

- Explicitly describe the kernel of $P_k(n) \rightarrow \text{End}_G(M^{\otimes k})$, when $n < 2k$, for $G = \mathbf{S}_n, \mathbf{A}_n$ and for either the permutation module $M = \mathbf{S}^{(n)} \oplus \mathbf{S}^{(n-1,1)}$ or its twist $\mathbf{S}^{(n)} \oplus \mathbf{S}^{(2,1^{n-1})}$.
- Give two linear bases for the image $\text{End}_G(M^{\otimes k})$ in terms of a restricted collection of set partitions. Combinatorially describe multiplication in both bases and give the change of basis matrix between them in terms of the refinement ordering in the partition lattice Π_n .

Furthermore, these partition algebras can be realized in terms of restriction and induction (this perspective is emphasized in [9]) in the following way. If U is any \mathbf{S}_n module, then upon restriction from \mathbf{S}_n to \mathbf{S}_{n-1} followed by induction back to \mathbf{S}_n we get $\text{Ind}_{\mathbf{S}_{n-1}}^{\mathbf{S}_n} \text{Res}_{\mathbf{S}_{n-1}}^{\mathbf{S}_n}(U) \cong U \otimes M$. This is an application of the “tensor identity” (see [9, 3.18]). It follows that the module $M^{\otimes k}$ is isomorphic to k iterations of restriction and induction (starting with the trivial module). This process works both for $\mathbf{S}_{n-1} \subseteq \mathbf{S}_n$ and $\mathbf{A}_{n-1} \subseteq \mathbf{A}_n$. In Figure 6 we see this amazing correspondence: the restriction and induction graphs for $\mathbf{A}_3 \subseteq \mathbf{A}_4$ and $\mathbf{S}_3 \subseteq \mathbf{S}_4$ correspond exactly to the representation graphs $\mathcal{R}_V(\mathbf{T})$ and $\mathcal{R}_V(\mathbf{O})$ and thus also to the affine Dynkin diagrams \hat{E}_6 and \hat{E}_7 . In particular, this reveals why the partition algebras also work to describe the centralizer algebras $Z_k(\mathbf{T})$ and $Z_k(\mathbf{O})$ for k odd as well as k even. It remains an open question to fully understand the connections between $Z_k(\mathbf{I})$ and the generalized partition algebra $\text{End}_{\mathbf{A}_5}(\tilde{M}^{\otimes k})$, where \tilde{M} is the \mathbf{A}_5 module $\mathbf{A}_5^{(5)} \oplus \mathbf{A}_5^{(3,1,1)^+}$.

The irreducible representations of $P_k(n)$ are labeled by integer partitions $\lambda \vdash n$ with $|\lambda^\#| \leq k$, where if $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ then $\lambda^\# = (\lambda_2, \dots, \lambda_\ell)$. Using an inductive argument in [4], we give formulas for these dimensions which to our knowledge are new:

$$\dim P_k^\lambda(n) = \sum_{r=1}^{\ell} (-1)^{r-1} F_r^\lambda \left(\sum_{t=n-\lambda_r+r-1}^{n-2} \binom{t}{n-\lambda_r+r-1} \left\{ \begin{matrix} k \\ t \end{matrix} \right\} \right) + f^\lambda \left(\left\{ \begin{matrix} k \\ n-1 \end{matrix} \right\} + \left\{ \begin{matrix} k \\ n \end{matrix} \right\} \right), \tag{22}$$

where f^λ is the number of standard tableaux of shape λ , $\left\{ \begin{smallmatrix} k \\ j \end{smallmatrix} \right\}$ is the Stirling number of the 2nd kind, and F_r^λ is defined by: (1) $F_1^\lambda = f^{\lambda^\#}$, (2) $F_r^\lambda = 0$, if $r > \lambda_r$, and (3) $F_r^\lambda = \sum_{\mu \subseteq \lambda, \mu_{r-1} = \mu_r = \lambda_r} F_{r-1}^\mu \cdot f^{\lambda/\mu}$, if $r > 1$. We have analogous formulas in [4] for the partition algebras corresponding to \mathbf{A}_n .

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