Precise large deviation asymptotics for products of random matrices

Hui Xiao, Ion Grama, Quansheng Liu

▶ **To cite this version:**

Hui Xiao, Ion Grama, Quansheng Liu. Precise large deviation asymptotics for products of random matrices. Stochastic Processes and their Applications, 2020, 130, pp.5213-5242. hal-02173735

HAL Id: hal-02173735
https://hal.science/hal-02173735

Submitted on 4 Jul 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

PRECISE LARGE DEVIATION ASYMPTOTICS FOR PRODUCTS OF RANDOM MATRICES

HUI XIAO[1], ION GRAMA[1,2], AND QUANSHENG LIU[1]

ABSTRACT. Let $(g_n)_{n\geqslant 1}$ be a sequence of independent identically distributed $d\times d$ real random matrices with Lyapunov exponent γ. For any starting point x on the unit sphere in $\mathbb{R}^d$, we deal with the norm $|G_nx|$, where $G_n := g_n \dots g_1$. The goal of this paper is to establish precise asymptotics for large deviation probabilities $\mathbb{P}(\log|G_nx| \geqslant n(q+l))$, where $q>\gamma$ is fixed and l is vanishing as $n\to\infty$. We study both invertible matrices and positive matrices and give analogous results for the couple $(X_n^x, \log|G_nx|)$ with target functions, where $X_n^x = G_nx/|G_nx|$. As applications we improve previous results on the large deviation principle for the matrix norm $\|G_n\|$ and obtain a precise local limit theorem with large deviations.

1. INTRODUCTION

1.1. **Background and main objectives.** One of the fundamental results in the probability theory is the law of large numbers. The large deviation theory describes the rate of convergence in the law of large numbers. The most important results in this direction are the Bahadur-Rao and the Petrov precise large deviation asymptotics that we recall below for independent and identically distributed (i.i.d.) real-valued random variables $(X_i)_{i\geqslant 1}$. Let $S_n = \sum_{i=1}^n X_i$. Denote by I_Λ the set of real numbers $s \geqslant 0$ such that $\Lambda(s) := \log \mathbb{E}[e^{sX_1}] < +\infty$ and by I_Λ° the interior of I_Λ. Let Λ^* be the Frenchel-Legendre transform of Λ. Assume that $s\in I_\Lambda^\circ$ and q are related by $q=\Lambda'(s)$. Set $\sigma_s^2 = \Lambda''(s)$. From the results of Bahadur and Rao [1] and Petrov [31] it follows that if the law of X_1 is non-lattice, then the following

[1]Université de Bretagne-Sud, LMBA UMR CNRS 6205, Vannes, France.
[2]Corresponding author: ion.grama@univ-ubs.fr

Date: July 3, 2019.
2010 *Mathematics Subject Classification.* Primary 60F10, 60B20; Secondary 60J05.
Key words and phrases. Product of random matrices; Random walk on the general linear group; Random walk on the semigroup of positive matrices; spectral gap; large deviation; Bahadur-Rao theorem.

large deviation asymptotic holds true:

$$\mathbb{P}(S_n \geqslant n(q+l)) \sim \frac{\exp(-n\Lambda^*(q+l))}{s\sigma_s\sqrt{2\pi n}},\ n\to\infty, \tag{1.1}$$

where $\Lambda^*(q+l)=\Lambda^*(q)+sl+\frac{l^2}{2\sigma_s^2}+O(l^3)$ and l is a vanishing perturbation as $n\to\infty$. Bahadur and Rao [1] have established the equivalence (1.1) with $l=0$. Petrov improved it by showing that (1.1) holds uniformly in $|l|\leqslant l_n\to 0$ as $n\to\infty$. Actually, Petrov's result is also uniform in q and is therefore stronger than Bahadur-Rao's theorem even with $l=0$. The relation (1.1) with $l=0$ and its extension to $|l|\leqslant l_n\to 0$ have multiple implications in various domains of probability and statistics. The main goal of the present paper is to establish an equivalence similar to (1.1) for products of i.i.d. random matrices.

Let $(g_n)_{n\geqslant 1}$ be a sequence of i.i.d. $d\times d$ real random matrices defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ with common law μ. Denote by $\|\cdot\|$ the operator norm of a matrix and by $|\cdot|$ the Euclidean norm in $\mathbb{R}^d$. Set for brevity $G_n:=g_n\dots g_1$, $n\geqslant 1$. The study of asymptotic behavior of the product G_n attracted much attention, since the fundamental work of Furstenberg and Kesten [15], where the strong law of large numbers for $\log\|G_n\|$ has been established. Under additional assumptions, Furstenberg [14] extended it to $\log|G_nx|$, for any starting point x on the unit sphere $\mathbb{S}^{d-1}=\{x\in\mathbb{R}^d:|x|=1\}$. A number of noteworthy results in this area can be found in Kesten [28], Kingman [29], Le Page [30], Guivarc'h and Raugi [22], Bougerol and Lacroix [5], Goldsheid and Guivarc'h [17], Hennion [24], Furman [13], Hennion and Hervé [26], Guivarc'h [20], Guivarc'h and Le Page [21], Benoist and Quint [2, 3] to name only a few.

In this paper we are interested in asymptotic behaviour of large deviation probabilities for $\log|G_nx|$ where $x\in\mathbb{S}^{d-1}$. Set $I_\mu=\{s\geqslant 0:\mathbb{E}(\|g_1\|^s)<+\infty\}$. For $s\in I_\mu$, let $\kappa(s)=\lim_{n\to\infty}(\mathbb{E}\|G_n\|^s)^{\frac{1}{n}}$. Define the convex function $\Lambda(s)=\log\kappa(s)$, $s\in I_\mu$, and consider its Fenchel-Legendre transform $\Lambda^*(q)=\sup_{s\in I_\mu}\{sq-\Lambda(s)\}$, $q\in\Lambda'(I_\mu)$. Our first objective is to establish the following Bahadur-Rao type precise large deviation asymptotic:

$$\mathbb{P}(\log|G_nx|\geqslant nq)\sim\bar{r}_s(x)\frac{\exp\left(-n\Lambda^*(q)\right)}{s\sigma_s\sqrt{2\pi n}},\ n\to\infty, \tag{1.2}$$

where $\sigma_s>0$, $\bar{r}_s=\frac{r_s}{\nu_s(r_s)}>0$, r_s and ν_s are, respectively, the unique up to a constant eigenfunction and unique probability eigenmeasure of the transfer operator P_s corresponding to the eigenvalue $\kappa(s)$ (see Section 2.2 for precise statements). In fact, to enlarge the area of applications in (1.2) it is useful to add a vanishing perturbation for q. In this line we obtain the following Petrov type large deviation expansion: under appropriate conditions,

uniformly in $|l| \leqslant l_n \to 0$ as $n \to \infty$,

$$\mathbb{P}(\log|G_n x| \geqslant n(q+l)) \sim \bar{r}_s(x) \frac{\exp\left(-n\Lambda^*(q+l)\right)}{s\sigma_s\sqrt{2\pi n}}, \quad n \to \infty. \tag{1.3}$$

As an consequence of (1.3) we are able to infer new results, such as large deviation principles for $\log\|G_n\|$, see Theorem 2.5. From (1.3) we also deduce a local large deviation asymptotic: there exists a sequence $\Delta_n > 0$ converging to 0 such that, uniformly in $\Delta \in [\Delta_n, o(n)]$,

$$\mathbb{P}(\log|G_n x| \in [nq, nq+\Delta)) \sim \Delta \frac{\bar{r}_s(x)}{s\sigma_s\sqrt{2\pi n}} e^{-n\Lambda^*(q)}, \ n \to \infty. \tag{1.4}$$

Our results are established for both invertible matrices and positive matrices. For invertible matrices, Le Page [30] has obtained (1.2) for $s > 0$ small enough under more restrictive conditions, such as the existence of exponential moments of $\|g_1\|$ and $\|g_1^{-1}\|$. The asymptotic (1.2) clearly implies a large deviation result due to Buraczewski and Mentemeier [8] which holds for invertible matrices and positive matrices: for $q = \Lambda'(s)$ and $s \in I_\mu^\circ$, there exist two constants $0 < c_s < C_s < +\infty$ such that

$$c_s \leqslant \liminf_{n\to\infty} \frac{\mathbb{P}(\log|G_n x| \geqslant nq)}{\frac{1}{\sqrt{n}}\, e^{-n\Lambda^*(q)}} \leqslant \limsup_{n\to\infty} \frac{\mathbb{P}(\log|G_n x| \geqslant nq)}{\frac{1}{\sqrt{n}}\, e^{-n\Lambda^*(q)}} \leqslant C_s. \tag{1.5}$$

Consider the Markov chain $X_n^x := G_n x/|G_n x|$. Our second objective is to give precise large deviations for the couple $(X_n^x, \log|G_n x|)$ with target functions. We prove that for any Hölder continuous target function φ on X_n^x, and any target function ψ on $\log|G_n x|$ such that $y \mapsto e^{-sy}\psi(y)$ is directly Riemann integrable, it holds that

$$\begin{aligned}&\mathbb{E}\Big[\varphi(X_n^x)\psi(\log|G_n x| - n(q+l))\Big] \\ &\qquad \sim \bar{r}_s(x)\nu_s(\varphi) \int_{\mathbb{R}} e^{-sy}\psi(y)dy \ \frac{\exp\left(-n\Lambda^*(q+l)\right)}{\sigma_s\sqrt{2\pi n}}, \quad n \to \infty.\end{aligned} \tag{1.6}$$

As a special case of (1.6) with $l = 0$ and ψ compactly supported we obtain Theorem 3.3 of Guivarc'h [20]. With $l = 0$, ψ the indicator function of the interval $[0, \infty)$ and $\varphi = r_s$, we get the main result in [8].

Our third objective is to establish asymptotics for lower large deviation probabilities: we prove that for $q = \Lambda'(s)$ with $s < 0$ sufficiently close to 0, it holds, uniformly in $|l| \leqslant l_n$,

$$\mathbb{P}(\log|G_n x| \leqslant n(q+l)) = \bar{r}_s(x) \frac{\exp\left(-n\Lambda^*(q+l)\right)}{-s\sigma_s\sqrt{2\pi n}}(1+o(1)). \tag{1.7}$$

This sharpens the large deviation principle established in [5, Theorem 6.1] for invertible matrices. Moreover, we extend the large deviation asymptotic (1.7) to the couple $(X_n^x, \log|G_n x|)$ with target functions.

1.2. **Proof outline.** Our proof is different from the standard approach of Dembo and Zeitouni [11] based on the Edgeworth expansion, which has been employed for instance in [8]. In contrast to [8], we start with the identity

$$\frac{e^{n\Lambda^*(q+l)}}{r_s(x)}\mathbb{P}\big(\log|G_nx|\geqslant n(q+l)\big)$$
$$= e^{nh_s(l)}\mathbb{E}_{\mathbb{Q}_s^x}\Big(\frac{\psi_s(\log|G_nx|-n(q+l))}{r_s(X_n^x)}\Big), \tag{1.8}$$

where $\mathbb{Q}_s^x$ is the change of measure defined in Section 3 for the norm cocycle $\log|G_nx|$, $\psi_s(y)=e^{-sy}\mathbb{1}_{\{y\geqslant 0\}}$ and $h_s(l)=\Lambda^*(q+l)-\Lambda^*(q)-sl$. Usually the expectation in the right-hand side of (1.8) is handled via the Edgeworth expansion for the distribution function $\mathbb{Q}_s^x\big(\frac{\log|G_nx|-nq}{\sqrt{n}\sigma_s}\leqslant t\big)$; however, the presence of the multiplier $r_s(X_n^x)^{-1}$ makes this impossible. Our idea is to replace the function ψ_s with some upper and lower smoothed bounds using a technique from Grama, Lauvergnat and Le Page [18]. For simplicity we deal only with the upper bound $\psi_s\leqslant\psi_{s,\varepsilon}^+*\rho_{\varepsilon^2}$, where $\psi_{s,\varepsilon}^+(y)=\sup_{y':|y'-y|\leqslant\varepsilon}\psi_s(y')$, for some $\varepsilon>0$, and ρ_{ε^2} is a density function on the real line satisfying the following properties: the Fourier transform $\widehat{\rho}_{\varepsilon^2}$ is supported on $[-\varepsilon^{-2},\varepsilon^{-2}]$, has a continuous extension in the complex plane and is analytic in the domain $\{z\in\mathbb{C}:|z|<\varepsilon^{-2},\Im z\neq 0\}$, see Lemma 4.2. Let $R_{s,it}$ be the perturbed operator defined by $R_{s,it}(\varphi)(x)=\mathbb{E}_{\mathbb{Q}_s^x}[\varphi(X_1)e^{it(\log|g_1x|-q)}]$, for any Hölder continuous function φ on the unit sphere $\mathbb{S}^{d-1}$. Using the inversion formula we obtain the following upper bound:

$$\mathbb{E}_{\mathbb{Q}_s^x}\Big(\frac{\psi_s(\log|G_nx|-n(q+l))}{r_s(X_n^x)}\Big)$$
$$\leqslant\frac{1}{2\pi}\int_{\mathbb{R}}e^{-itln}R_{s,it}^n(r_s^{-1})(x)\widehat{\psi}_{s,\varepsilon}^+(t)\widehat{\rho}_{\varepsilon^2}(t)dt, \tag{1.9}$$

where $R_{s,it}^n$ is the n-th iteration of $R_{s,it}$. The integral in the right-hand side of (1.9) is decomposed into two parts:

$$e^{nh_s(l)}\Big\{\int_{|t|<\delta}+\int_{|t|\geqslant\delta}\Big\}e^{-itln}R_{s,it}^n(r_s^{-1})(x)\widehat{\psi}_{s,\varepsilon}^+(t)\widehat{\rho}_{\varepsilon^2}(t)dt. \tag{1.10}$$

Since $\widehat{\rho}_{\varepsilon^2}$ is compactly supported on $\mathbb{R}$ and μ is non-arithmetic, the second integral in (1.10) decays exponentially fast to 0. To deal with the first integral in (1.10), we make use of spectral gap decomposition for the perturbed operator $R_{s,it}$: $R_{s,it}^n=\lambda_{s,it}^n\Pi_{s,it}+N_{s,it}^n$. Taking into account the fact that the remainder term $N_{s,it}^n$ decays exponentially fast to 0, the main difficulty is to investigate the integral:

$$e^{nh_s(l)}\int_{-\delta}^{\delta}e^{-itln}\lambda_{s,it}^n\Pi_{s,it}(r_s^{-1})(x)\widehat{\psi}_{s,\varepsilon}^+(t)\widehat{\rho}_{\varepsilon^2}(t)dt.$$

To find the exact asymptotic of this integral, we can apply the saddle point method (see Fedoryuk [12]). This is possible, since by the analyticity of the functions $\widehat{\psi}^+_{s,\varepsilon}$ and $\widehat{\rho}_{\varepsilon^2}$, one can apply Cauchy's integral theorem to change the integration path so that it passes through the saddle point $z_0 = z_0(l)$, which is the unique solution of the saddle point equation $\log \lambda_{s,z} = zl$.

The lower bound of the integral in (1.8) is a little more delicate, but can be treated in a similar way. The passage to the targeted version is done by using approximation techniques.

We end this section by fixing some notation, which will be used throughout the paper. We denote by c, C, eventually supplied with indices, absolute constants whose values may change from line to line. By c_α, C_α we mean constants depending only on the index α. The interior of a set A is denoted by A°. Let $\mathbb{N} = \{1, 2, \ldots\}$. For any integrable function $\psi : \mathbb{R} \to \mathbb{C}$, define its Fourier transform by $\widehat{\psi}(t) = \int_{\mathbb{R}} e^{-ity}\psi(y)dy$, $t \in \mathbb{R}$. For a matrix g, its transpose is denoted by g^{T}. For a measure ν and a function φ we write $\nu(\varphi) = \int \varphi d\nu$.

2. Main results

2.1. **Notation and conditions.** The space $\mathbb{R}^d$ is equipped with the standard scalar product $\langle \cdot, \cdot \rangle$ and the Euclidean norm $|\cdot|$. For $d \geqslant 1$, let $M(d, \mathbb{R})$ be the set of $d \times d$ matrices with entries in $\mathbb{R}$ equipped with the operator norm $\|g\| = \sup_{x \in \mathbb{S}^{d-1}} |gx|$, for $g \in M(d, \mathbb{R})$, where $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d, |x| = 1\}$ is the unit sphere.

We shall work with products of invertible or positive matrices (all over the paper we use the term positive in the wide sense, i.e. each entry is non-negative). Denote by $\mathscr{G} = GL(d, \mathbb{R})$ the general linear group of invertible matrices of $M(d, \mathbb{R})$. A positive matrix $g \in M(d, \mathbb{R})$ is said to be *allowable*, if every row and every column of g has a strictly positive entry. Denote by $\mathscr{G}_+$ the multiplicative semigroup of allowable positive matrices of $M(d, \mathbb{R})$. We write $\mathscr{G}_+^\circ$ for the subsemigroup of $\mathscr{G}_+$ with strictly positive entries.

Denote by $\mathbb{S}_+^{d-1} = \{x \geqslant 0 : |x| = 1\}$ the intersection of the unit sphere with the positive quadrant. To unify the exposition, we use the symbol $\mathcal{S}$ to denote $\mathbb{S}^{d-1}$ in the case of invertible matrices, and $\mathbb{S}_+^{d-1}$ in the case of positive matrices. The space $\mathcal{S}$ is equipped with the metric $\mathbf{d}$ which we proceed to introduce. For invertible matrices, the distance $\mathbf{d}$ is defined as the angular distance (see [21]), i.e., for any $x, y \in \mathbb{S}^{d-1}$, $\mathbf{d}(x, y) = |\sin \theta(x, y)|$, where $\theta(x, y)$ is the angle between x and y. For positive matrices, the distance $\mathbf{d}$ is the Hilbert cross-ratio metric (see [24]) defined by $\mathbf{d}(x, y) = \frac{1 - m(x,y)m(y,x)}{1 + m(x,y)m(y,x)}$, where $m(x, y) = \sup\{\lambda > 0 : \ \lambda y_i \leqslant x_i, \ \forall i = 1, \ldots, d\}$, for any two vectors $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$ in $\mathbb{S}_+^{d-1}$.

Let $\mathcal{C}(\mathcal{S})$ be the space of continuous functions on $\mathcal{S}$. We write $\mathbf{1}$ for the identity function $1(x)$, $x \in \mathcal{S}$. Throughout this paper, let $\gamma > 0$ be a fixed small constant. For any $\varphi \in \mathcal{C}(\mathcal{S})$, set

$$\|\varphi\|_\infty := \sup_{x\in\mathcal{S}} |\varphi(x)| \quad \text{and} \quad \|\varphi\|_\gamma := \|\varphi\|_\infty + \sup_{x,y\in\mathcal{S}} \frac{|\varphi(x)-\varphi(y)|}{\mathbf{d}(x,y)^\gamma},$$

and introduce the Banach space $\mathcal{B}_\gamma := \{\varphi \in \mathcal{C}(\mathcal{S}) : \|\varphi\|_\gamma < +\infty\}$.

For $g \in M(d,\mathbb{R})$ and $x \in \mathcal{S}$, write $g \cdot x = \frac{gx}{|gx|}$ for the projective action of g on $\mathcal{S}$. For any $g \in M(d,\mathbb{R})$, set $\iota(g) := \inf_{x\in\mathcal{S}} |gx|$. For both invertible matrices and allowable positive matrices, it holds that $\iota(g) > 0$. Note that for any invertible matrix g, we have $\iota(g) = \|g^{-1}\|^{-1}$.

Let $(g_n)_{n\geqslant 1}$ be a sequence of i.i.d. random matrices of the same probability law μ on $M(d,\mathbb{R})$. Set $G_n = g_n \dots g_1$, for $n \geqslant 1$. Our goal is to establish, under suitable conditions, a large deviation equivalence similar to (1.1) for the norm cocycle $\log|G_n x|$ for invertible matrices and positive matrices. In both cases, we denote by $\Gamma_\mu := [\operatorname{supp}\mu]$ the smallest closed semigroup of $M(d,\mathbb{R})$ generated by $\operatorname{supp}\mu$ (the support of μ), that is, $\Gamma_\mu = \overline{\cup_{n=1}^\infty \{\operatorname{supp}\mu\}^n}$.

Set

$$I_\mu = \{s \geqslant 0 : \mathbb{E}(\|g_1\|^s) < +\infty\}.$$

Applying Hölder's inequality to $\mathbb{E}(\|g_1\|^s)$, it is easily seen that I_μ is an interval. We make use of the following exponential moment condition:

A1. *There exist $s \in I_\mu^\circ$ and $\alpha \in (0,1)$ such that $\mathbb{E}\|g_1\|^{s+\alpha}\iota(g_1)^{-\alpha} < +\infty$.*

For invertible matrices, we introduce the following strong irreducibility and proximality conditions, where we recall that a matrix g is said to be *proximal* if it has an algebraic simple dominant eigenvalue.

A2. (i)(Strong irreducibility) *No finite union of proper subspaces of $\mathbb{R}^d$ is Γ_μ-invariant.*

(ii)(Proximality) *Γ_μ contains at least one proximal matrix.*

The conditions of strong irreducibility and proximality are always satisfied for $d = 1$. If g is proximal, denote by λ_g its dominant eigenvalue and by v_g the associated normalized eigenvector ($|v_g| = 1$). In fact, g is proximal iff the space $\mathbb{R}^d$ can be decomposed as $\mathbb{R}^d = \mathbb{R}\lambda_g \oplus V'$ such that $gV' \subset V'$ and the spectral radius of g on the invariant subspace V' is strictly less than $|\lambda_g|$. For invertible matrices, condition **A2** implies that the Markov chain X_n^x has a unique μ-stationary measure, which is supported on

$$V(\Gamma_\mu) = \overline{\{\pm v_g \in \mathbb{S}^{d-1} : g \in \Gamma_\mu,\ g \text{ is proximal}\}}.$$

For positive matrices, introduce the following condition:

A3. (i) (Allowability) *Every $g \in \Gamma_\mu$ is allowable.*

(ii) (Positivity) *Γ_μ contains at least one matrix belonging to $\mathscr{G}_+^\circ$.*

It can be shown (see [7, Lemma 4.3]) that for positive matrices, condition **A3** ensures the existence and uniqueness of the invariant measure for the Markov chain X_n^x supported on

$$V(\Gamma_\mu) = \overline{\{v_g \in \mathbb{S}_+^{d-1} : g \in \Gamma_\mu,\ g \in \mathscr{G}_+^\circ\}}.$$

In addition, $V(\Gamma_\mu)$ is the unique minimal Γ_μ-invariant subset (see [7, Lemma 4.2]). According to the Perron-Frobenius theorem, a strictly positive matrix always has a unique dominant eigenvalue, so condition **A3**(ii) implies condition **A2**(ii) for $d > 1$.

For any $s \in I_\mu$, for invertible matrices and for positive matrices, the following limit exists (see [21] and [8]):

$$\kappa(s) = \lim_{n\to\infty} (\mathbb{E}\|G_n\|^s)^{\frac{1}{n}}.$$

The function $\Lambda = \log\kappa : I_\mu \to \mathbb{R}$ is convex and analytic on I_μ° (it plays the same role as the log-Laplace transform of X_1 in the real i.i.d. case). Introduce the Fenchel-Legendre transform of Λ by $\Lambda^*(q) = \sup_{s\in I_\mu}\{sq - \Lambda(s)\}$, $q \in \Lambda'(I_\mu)$. We have that $\Lambda^*(q) = sq - \Lambda(s)$ if $q = \Lambda'(s)$ for some $s \in I_\mu$, which implies $\Lambda^*(q) \geqslant 0$ on $\Lambda'(I_\mu)$ since $\Lambda(0) = 0$ and $\Lambda(s)$ is convex on I_μ.

We say that the measure μ is *arithmetic*, if there exist $t > 0$, $\beta \in [0, 2\pi)$ and a function $\vartheta : \mathcal{S} \to \mathbb{R}$ such that for any $g \in \Gamma_\mu$ and any $x \in V(\Gamma_\mu)$, we have $\exp[it\log|gx| - i\beta + i\vartheta(g\cdot x) - i\vartheta(x)] = 1$. For positive matrices, we need the following condition:

A4. (Non-arithmeticity) *The measure μ is non-arithmetic.*

A simple sufficient condition established in [28] for the measure μ to be non-arithmetic is that the additive subgroup of $\mathbb{R}$ generated by the set $\{\log\lambda_g : g \in \Gamma_\mu,\ g \in \mathscr{G}_+^\circ\}$ is dense in $\mathbb{R}$ (see [8, Lemma 2.7]).

Note that for positive matrices, condition **A4** is used to ensure that $\sigma_s^2 = \Lambda''(s) > 0$. For invertible matrices, condition **A2** implies the non-arithmeticity of the measure μ, hence, σ_s is also strictly positive (for a proof see Guivarc'h and Urban [23, Proposition 4.6]).

For any $s \in I_\mu$, the transfer operator P_s and the conjugate transfer operator P_s^* are defined, for any $\varphi \in \mathcal{C}(\mathcal{S})$ and $x \in \mathcal{S}$, by

$$P_s\varphi(x) = \int_{\Gamma_\mu} |g_1 x|^s \varphi(g_1\cdot x)\mu(dg_1),\ \ P_s^*\varphi(x) = \int_{\Gamma_\mu} |g_1^{\mathrm{T}} x|^s \varphi(g_1^{\mathrm{T}}\cdot x)\mu(dg_1), \quad (2.1)$$

which are bounded linear on $\mathcal{C}(\mathcal{S})$. Under condition **A2** for invertible matrices, or condition **A3** for positive matrices, the operator P_s has a unique probability eigenmeasure ν_s on $\mathcal{S}$ corresponding to the eigenvalue $\kappa(s)$: $P_s\nu_s = \kappa(s)\nu_s$. Similarly, the operator P_s^* has a unique probability eigenmeasure ν_s^* corresponding to the eigenvalue $\kappa(s)$: $P_s^*\nu_s^* = \kappa(s)\nu_s^*$. Set, for

$x \in \mathcal{S}$,

$$r_s(x) = \int_{\mathcal{S}} |\langle x, y\rangle|^s \nu_s^*(dy), \quad r_s^*(x) = \int_{\mathcal{S}} |\langle x, y\rangle|^s \nu_s(dy).$$

Then, r_s is the unique, up to a scaling constant, strictly positive eigenfunction of P_s: $P_s r_s = \kappa(s) r_s$; similarly r_s^* is the unique, up to a scaling constant, strictly positive eigenfunction of P_s^*: $P_s^* r_s^* = \kappa(s) r_s^*$. We refer for details to Section 3.

Below we shall also make use of normalized eigenfunction $\bar{r}_s$ defined by $\bar{r}_s(x) = \frac{r_s(x)}{\nu_s(r_s)}$, $x \in \mathcal{S}$, which is strictly positive and Hölder continuous on the projective space $\mathcal{S}$, see Proposition 3.1.

2.2. **Large deviations for the norm cocycle.** The following theorem gives the exact asymptotic behavior of the large deviation probabilities for the norm cocycle.

Theorem 2.1. *Assume that μ satisfies either conditions **A1**, **A2** for invertible matrices, or conditions **A1**, **A3**, **A4** for positive matrices. Let $q = \Lambda'(s)$, where $s \in I_\mu^\circ$. Then for any positive sequence $(l_n)_{n\geqslant 1}$ satisfying $\lim_{n\to\infty} l_n = 0$, we have, as $n \to \infty$, uniformly in $x \in \mathcal{S}$ and $|l| \leqslant l_n$,*

$$\mathbb{P}\big(\log|G_n x| \geqslant n(q+l)\big) = \bar{r}_s(x) \frac{\exp\big(-n\Lambda^*(q+l)\big)}{s\sigma_s\sqrt{2\pi n}}(1+o(1)). \tag{2.2}$$

In particular, with $l = 0$, as $n \to \infty$, uniformly in $x \in \mathcal{S}$,

$$\mathbb{P}\big(\log|G_n x| \geqslant nq\big) = \bar{r}_s(x) \frac{\exp\big(-n\Lambda^*(q)\big)}{s\sigma_s\sqrt{2\pi n}}(1+o(1)). \tag{2.3}$$

The rate function $\Lambda^*(q+l)$ admits the following expansion: for $q = \Lambda'(s)$ and l in a small neighborhood of 0, we have

$$\Lambda^*(q+l) = \Lambda^*(q) + sl + \frac{l^2}{2\sigma_s^2} - \frac{l^3}{\sigma_s^3}\zeta_s\Big(\frac{l}{\sigma_s}\Big), \tag{2.4}$$

where $\zeta_s(t)$ is the Cramér series, $\zeta_s(t) = \sum_{k=3}^{\infty} c_{s,k} t^{k-3} = \frac{\Lambda'''(s)}{6\sigma_s^3} + O(t)$, with $\Lambda'''(s)$ and σ_s defined in Proposition 3.3. We refer for details to Lemma 4.1, where the coefficients $c_{s,k}$ are given in terms of the cumulant generating function $\Lambda = \log\kappa$.

For invertible matrices, a point-wise version of (2.3), without $\sup_{x\in\mathcal{S}}$ and with $l = 0$, namely the asymptotic (1.2), has been first established by Le Page [30, Theorem 8] for small enough $s > 0$ under a stronger exponential moment condition. For positive matrices, the asymptotic (2.3) is new and implies the large deviation bounds (1.5) established in Buraczewski and Mentemeier [8, Corollary 3.2]. We note that there is a misprint in [8], where e^{nsq} should be replaced by $e^{\Lambda^*(q)}$.

Now we consider the precise large deviations for the couple $(X_n^x, \log|G_n x|)$ with target functions φ and ψ on $X_n^x := G_n \cdot x$ and $\log|G_n x|$, respectively.

Theorem 2.2. *Assume the conditions of Theorem 2.1 and let $q = \Lambda'(s)$ for $s \in I_\mu^\circ$. Then, for any $\varphi \in \mathcal{B}_\gamma$, any measurable function ψ on $\mathbb{R}$ such that $y \mapsto e^{-sy}\psi(y)$ is directly Riemann integrable, and any positive sequence $(l_n)_{n\geqslant 1}$ satisfying $\lim_{n\to\infty} l_n = 0$, we have, as $n \to \infty$, uniformly in $x \in \mathcal{S}$ and $|l| \leqslant l_n$,*

$$\mathbb{E}\Big[\varphi(X_n^x)\psi(\log|G_n x| - n(q+l))\Big] = \bar{r}_s(x)\frac{\exp\left(-n\Lambda^*(q+l)\right)}{\sigma_s\sqrt{2\pi n}}\Big[\nu_s(\varphi)\int_{\mathbb{R}} e^{-sy}\psi(y)dy + o(1)\Big]. \quad (2.5)$$

With $\varphi = \mathbf{1}$ and $\psi(y) = \mathbb{1}_{\{y\geqslant 0\}}$ for $y \in \mathbb{R}$, we obtain Theorem 2.1. For invertible matrices and with $l = 0$, Theorem 2.2 strengthens the point-wise large deviation result stated in Theorem 3.3 of Guivarc'h [20], since we do not assume the function ψ to be compactly supported and our result is uniform in $x \in \mathcal{S}$. By the way we would like to remark that in Theorem 3.3 of [20] $\kappa^n(s)$ should be replaced by $\kappa^{-n}(s)$, and $\nu_s(\varphi r_s^{-1})$ should be replaced by $\frac{\nu_s(\varphi)}{\nu_s(r_s)}$. For positive matrices, Theorem 2.2 is new. Since r_s is a strictly positive and Hölder continuous function on $\mathcal{S}$ (see Proposition 3.1), taking $\varphi = r_s$ and $\psi(y) = \mathbb{1}_{\{y\geqslant 0\}}$, $y \in \mathbb{R}$ in Theorem 2.2, we get the main result of [8] (Theorem 3.1).

Unlike the case of i.i.d. real-valued random variables, Theorems 2.1 and 2.2 do not imply the similar asymptotic for lower large deviation probabilities $\mathbb{P}(\log|G_n x| \leqslant n(q+l))$, where $q < \Lambda'(0)$. To formulate our results, we need an exponential moment condition, as in Le Page [30]. For $g \in \Gamma_\mu$, set $N(g) = \max\{\|g\|, \iota(g)^{-1}\}$, which reduces to $N(g) = \max\{\|g\|, \|g^{-1}\|\}$ for invertible matrices.

A5. *There exists a constant $\eta \in (0,1)$ such that $\mathbb{E}[N(g_1)^\eta] < +\infty$.*

Under condition **A5**, the functions $s \mapsto \kappa(s)$ and $s \mapsto \Lambda(s) = \log\kappa(s)$ can be extended analytically in a small neighborhood of 0 of the complex plane; in this case the expansion (2.4) still holds and we have $\sigma_s = \Lambda''(s) > 0$ for $s < 0$ small enough. We also need to extend the function $\bar{r}_s$ for small $s < 0$, which is positive and Hölder continuous on the projective space $\mathcal{S}$, as in the case of $s > 0$: we refer to Proposition 3.2 for details.

Theorem 2.3. *Assume that μ satisfies either conditions **A2**, **A5** for invertible matrices or conditions **A3**, **A4**, **A5** for positive matrices. Then, there exists $\eta_0 < \eta$ such that for any $s \in (-\eta_0, 0)$ and $q = \Lambda'(s)$, for any positive sequence $(l_n)_{n\geqslant 1}$ satisfying $\lim_{n\to\infty} l_n = 0$, we have, as $n \to \infty$, uniformly in $x \in \mathcal{S}$ and $|l| \leqslant l_n$,*

$$\mathbb{P}\big(\log|G_n x| \leqslant n(q+l)\big) = \bar{r}_s(x)\frac{\exp\left(-n\Lambda^*(q+l)\right)}{-s\sigma_s\sqrt{2\pi n}}(1+o(1)).$$

In particular, with $l = 0$, as $n \to \infty$, uniformly in $x \in \mathcal{S}$,

$$\mathbb{P}\big(\log|G_n x| \leqslant nq\big) = \bar{r}_s(x)\frac{\exp\left(-n\Lambda^*(q)\right)}{-s\sigma_s\sqrt{2\pi n}}(1+o(1)).$$

For invertible matrices, this result sharpens the large deviation principle established in [5]. For positive matrices, our result is new, even for the large deviation principle.

More generally, we also have the precise large deviations result for the couple $(X_n^x, \log|G_n x|)$ with target functions.

Theorem 2.4. *Assume the conditions of Theorem 2.3. Then, there exists $\eta_0 < \eta$ such that for any $s \in (-\eta_0, 0)$ and $q = \Lambda'(s)$, for any $\varphi \in \mathcal{B}_\gamma$, any measurable function ψ on $\mathbb{R}$ such that $y \mapsto e^{-sy}\psi(y)$ is directly Riemann integrable, and any positive sequence $(l_n)_{n\geqslant 1}$ satisfying $\lim_{n\to\infty} l_n = 0$, we have, as $n \to \infty$, uniformly in $x \in \mathcal{S}$ and $|l| \leqslant l_n$,*

$$\mathbb{E}\Big[\varphi(X_n^x)\psi(\log|G_n x| - n(q+l))\Big] = \bar{r}_s(x)\frac{\exp\left(-n\Lambda^*(q+l)\right)}{\sigma_s\sqrt{2\pi n}}\Big[\nu_s(\varphi)\int_{\mathbb{R}} e^{-sy}\psi(y)dy + o(1)\Big].$$

With $\varphi = \mathbf{1}$ and $\psi(y) = \mathbb{1}_{\{y\leqslant 0\}}$ for $y \in \mathbb{R}$, we obtain Theorem 2.3.

2.3. **Applications to large deviation principle for the matrix norm.** We use Theorems 2.1 and 2.3 to deduce large deviation principles for the matrix norm $\|G_n\|$. Our first result concerns the upper tail and the second one deals with lower tail.

Theorem 2.5. *Assume the conditions of Theorem 2.1. Let $q = \Lambda'(s)$, where $s \in I_\mu^\circ$. Then, for any positive sequence $(l_n)_{n\geqslant 1}$ with $l_n \to 0$ as $n \to \infty$, we have, uniformly in $|l| \leqslant l_n$,*

$$\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}\big(\log\|G_n\| \geqslant n(q+l)\big) = -\Lambda^*(q).$$

For invertible matrices, with $l = 0$, Theorem 2.5 improves the large deviation bounds in Benoist and Quint [3, Theorem 14.19], where the authors consider general groups, but without giving the rate function. For positive matrices, the result is new for $l = 0$ and $l = O(l_n)$.

Theorem 2.6. *Assume the conditions of Theorem 2.3. Then, there exists $\eta_0 < \eta$ such that for any $s \in (-\eta_0, 0)$ and $q = \Lambda'(s)$, for any positive sequence $(l_n)_{n\geqslant 1}$ with $l_n \to 0$ as $n \to \infty$, we have, uniformly in $|l| \leqslant l_n$,*

$$\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}\big(\log\|G_n\| \leqslant n(q+l)\big) = -\Lambda^*(q).$$

This result is new for both invertible matrices and positive matrices.

2.4. **Local limit theorems with large deviations.** Local limit theorems and large and moderate deviations for sums of i.i.d. random variables have been studied by Gnedenko [16], Sheep [34], Stone [35], Breuillard [6], Borovkov and Borovkov [4]. Moderate deviation results in the local limit theorem for products of invertible random matrices have been obtained in [3, Theorems 17.9 and 17.10].

Taking $\varphi = \mathbf{1}$ and $\psi = \mathbb{1}_{[a,a+\Delta]}$, where $a \in \mathbb{R}$ and $\Delta > 0$ do not depend on n, it is easy to understand that Theorem 2.2 becomes, in fact, a statement on large deviations in the local limit theorem. It turns out that with the Petrov type extension (2.5) we can derive the following more general statement where Δ can increase with n.

Theorem 2.7. *Assume conditions of Theorem 2.1 and let $q = \Lambda'(s)$. Then there exists a sequence $\Delta_n > 0$ converging to 0 as $n \to \infty$ such that, for any $\varphi \in \mathcal{B}_\gamma$, for any positive sequence $(l_n)_{n\geqslant 1}$ with $l_n \to 0$ as $n \to \infty$ and any fixed $a \in \mathbb{R}$, we have, as $n \to \infty$, uniformly in $\Delta \in [\Delta_n, o(n)]$, $x \in \mathcal{S}$ and $|l| \leqslant l_n$,*

$$\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log|G_nx|\in n(q+l)+[a,a+\Delta)\}}\Big] = \bar{r}_s(x)e^{-sa}\big(1-e^{-s\Delta}\big)\frac{\exp(-n\Lambda^*(q+l))}{s\sigma_s\sqrt{2\pi n}}\Big[\nu_s(\varphi)+o(1)\Big].$$

Taking $\varphi = \mathbf{1}$, as $n \to \infty$, uniformly in $\Delta \in [\Delta_n, o(n)]$, $x \in \mathcal{S}$ and $|l| \leqslant l_n$,

$$\mathbb{P}\big(\log|G_nx| \in n(q+l)+[a,a+\Delta)\big) = \bar{r}_s(x)e^{-sa}\big(1-e^{-s\Delta}\big)\frac{\exp(-n\Lambda^*(q+l))}{s\sigma_s\sqrt{2\pi n}}\Big[1+o(1)\Big].$$

We can compare this result with Theorem 3.3 in [20], from which the above equivalence can be deduced for $l = 0$ and Δ fixed.

It is easy to see that, under additional assumption **A5**, the assertion of Theorem 2.7 remains true for $s < 0$ small enough. This can be deduced from Theorem 2.4: the details are left to the reader.

3. Spectral gap theory for the norm

3.1. **Properties of the transfer operator.** Recall that the transfer operator P_s and the conjugate operator P_s^* are defined by (2.1). Below $P_s\nu_s$ stands for the measure on $\mathcal{S}$ such that $P_s\nu_s(\varphi) = \nu_s(P_s\varphi)$, for continuous functions φ on $\mathcal{S}$, and $P_s^*\nu_s^*$ is defined similarly. The following result was proved in [7, 8] for positive matrices, and in [21] for invertible matrices.

Proposition 3.1. *Assume that μ satisfies either conditions **A1**, **A2** for invertible matrices, or conditions **A1**, **A3** for positive matrices. Let $s \in I_\mu$. Then the spectral radii $\varrho(P_s)$ and $\varrho(P_s^*)$ are both equal to $\kappa(s)$, and there*

exist a unique, up to a scaling constant, strictly positive Hölder continuous function r_s and a unique probability measure ν_s on $\mathcal{S}$ such that

$$P_s r_s = \kappa(s) r_s, \quad P_s \nu_s = \kappa(s) \nu_s.$$

Similarly, there exist a unique strictly positive Hölder continuous function r_s^* and a unique probability measure ν_s^* on $\mathcal{S}$ such that

$$P_s^* r_s^* = \kappa(s) r_s^*, \quad P_s^* \nu_s^* = \kappa(s) \nu_s^*.$$

Moreover, the functions r_s and r_s^* are given by

$$r_s(x) = \int_{\mathcal{S}} |\langle x, y\rangle|^s \nu_s^*(dy), \quad r_s^*(x) = \int_{\mathcal{S}} |\langle x, y\rangle|^s \nu_s(dy), \quad x \in \mathcal{S}.$$

It is easy to see that the family of kernels $q_n^s(x, g) = \frac{|gx|^s}{\kappa^n(s)} \frac{r_s(g\cdot x)}{r_s(x)}$, $n \geqslant 1$ satisfies the following cocycle property:

$$q_n^s(x, g_1) q_m^s(g_1\cdot x, g_2) = q_{n+m}^s(x, g_2 g_1). \tag{3.1}$$

The equation $P_s r_s = \kappa(s) r_s$ implies that, for any $x \in \mathcal{S}$ and $s \in I_\mu$, the probability measures $\mathbb{Q}_{s,n}^x(dg_1, \ldots, dg_n) = q_n^s(x, g_n \ldots g_1)\mu(dg_1)\ldots\mu(dg_n)$, $n \geqslant 1$, form a projective system on $M(d, \mathbb{R})^{\mathbb{N}}$. By the Kolmogorov extension theorem, there is a unique probability measure $\mathbb{Q}_s^x$ on $M(d, \mathbb{R})^{\mathbb{N}}$, with marginals $\mathbb{Q}_{s,n}^x$; denote by $\mathbb{E}_{\mathbb{Q}_s^x}$ the corresponding expectation.

If $(g_n)_{n\in\mathbb{N}}$ denotes the coordinate process on the space of trajectories $M(d, \mathbb{R})^{\mathbb{N}}$, then the sequence $(g_n)_{n\geqslant 1}$ is i.i.d. with the common law μ under $\mathbb{Q}_0^x$. However, for any $s \in I_\mu^\circ$ and $x \in \mathcal{S}$, the sequence $(g_n)_{n\geqslant 1}$ is Markov-dependent under the measure $\mathbb{Q}_s^x$. Let

$$X_0^x = x, \quad X_n^x = G_n\cdot x, \quad n \geqslant 1.$$

By the definition of $\mathbb{Q}_s^x$, for any bounded measurable function f on $(\mathcal{S}\times\mathbb{R})^n$, it holds that

$$\frac{1}{\kappa^n(s) r_s(x)} \mathbb{E}\Big[r_s(X_n^x)|G_n x|^s f\big(X_1^x, \log|G_1 x|, \ldots, X_n^x, \log|G_n x|\big)\Big] = \mathbb{E}_{\mathbb{Q}_s^x}\Big[f\big(X_1^x, \log|G_1 x|, \ldots, X_n^x, \log|G_n x|\big)\Big]. \tag{3.2}$$

Under the measure $\mathbb{Q}_s^x$, the process $(X_n^x)_{n\in\mathbb{N}}$ is a Markov chain with the transition operator given by

$$Q_s\varphi(x) = \frac{1}{\kappa(s) r_s(x)} P_s(\varphi r_s)(x) = \frac{1}{\kappa(s) r_s(x)} \int_{\Gamma_\mu} |gx|^s \varphi(g\cdot x) r_s(g\cdot x)\mu(dg).$$

It has been proved in [7] for positive matrices, and in [21] for invertible matrices, that Q_s has a unique invariant probability measure π_s supported on $V(\Gamma_\mu)$ and that, for any $\varphi \in \mathcal{C}(\mathcal{S})$,

$$\lim_{n\to\infty} Q_s^n \varphi = \pi_s(\varphi), \quad \text{where } \pi_s(\varphi) = \frac{\nu_s(\varphi r_s)}{\nu_s(r_s)}. \tag{3.3}$$

Moreover, letting $\mathbb{Q}_s = \int \mathbb{Q}_s^x \pi_s(dx)$, from the results of [7, 21], it follows that, under the assumptions of Theorem 2.1, for any $s \in I_\mu$, we have $\lim_{n\to\infty} \frac{\log|G_n x|}{n} = \Lambda'(s)$, $\mathbb{Q}_s$-a.s. and $\mathbb{Q}_s^x$-a.s., where $\Lambda'(s) = \frac{\kappa'(s)}{\kappa(s)}$.

When $s \in (-\eta_0, 0)$ for small enough $\eta_0 > 0$, define the transfer operator P_s as follows: for any $\varphi \in \mathcal{C}(\mathcal{S})$,

$$P_s\varphi(x) = \int_{\Gamma_\mu} |g_1 x|^s \varphi(g_1 \cdot x)\mu(dg_1), \quad x \in \mathcal{S},$$

which is well-defined under condition **A5**. The following proposition is proved in [36].

Proposition 3.2. *Assume that μ satisfies either conditions **A2**, **A5** for invertible matrices, or conditions **A3**, **A5** for positive matrices. Then there exists $\eta_0 < \eta$ such that for any $s \in (-\eta_0, 0)$, the spectral radius $\varrho(P_s)$ of the operator P_s is equal to $\kappa(s)$. Moreover there exist a unique, up to a scaling constant, strictly positive Hölder continuous function r_s and a unique probability measure ν_s on $\mathcal{S}$ such that*

$$P_s r_s = \kappa(s) r_s, \quad P_s \nu_s = \kappa(s)\nu_s.$$

Based on Proposition 3.2, in the same way as for $s > 0$, one can define the measure $\mathbb{Q}_s^x$ for negative values $s < 0$ sufficiently close to 0, and one can extend the change of measure formula (3.2) to $s < 0$. Under the measure $\mathbb{Q}_s^x$, the process $(X_n^x)_{n\in\mathbb{N}}$ is a Markov chain with the transition operator Q_s and the assertion (3.3) holds true. We refer to [36] for details.

3.2. **Spectral gap of the perturbed operator.** Recall that the Banach space B_γ consists of all γ-Hölder continuous function on $\mathcal{S}$, where $\gamma > 0$ is a fixed small constant. Denote by $\mathcal{L}(\mathcal{B}_\gamma, \mathcal{B}_\gamma)$ the set of all bounded linear operators from $\mathcal{B}_\gamma$ to $\mathcal{B}_\gamma$ equipped with the operator norm $\|\cdot\|_{\mathcal{B}_\gamma \to \mathcal{B}_\gamma}$. For $s \in I_\mu^\circ$ and $z \in \mathbb{C}$ with $s + \Re z \in I_\mu$, define a family of perturbed operators $R_{s,z}$ as follows: for any $\varphi \in \mathcal{B}_\gamma$,

$$R_{s,z}\varphi(x) = \mathbb{E}_{\mathbb{Q}_s^x}\left[e^{z(\log|g_1 x| - q)}\varphi(X_1^x)\right], \quad x \in \mathcal{S}. \tag{3.4}$$

It follows from the cocycle property (3.1) that

$$R_{s,z}^n\varphi(x) = \mathbb{E}_{\mathbb{Q}_s^x}\left[e^{z(\log|G_n x| - nq)}\varphi(X_n^x)\right], \quad x \in \mathcal{S}.$$

The following proposition collects useful assertions that we will use in the proofs of our results. Denote $B_\delta(0) := \{z \in \mathbb{C} : |z| \leqslant \delta\}$.

Proposition 3.3. *Assume that μ satisfies either conditions **A1**, **A2** for invertible matrices, or conditions **A1**, **A3** for positive matrices. Then, there exists $\delta > 0$ such that for any $z \in B_\delta(0)$,*

$$R_{s,z}^n = \lambda_{s,z}^n \Pi_{s,z} + N_{s,z}^n, \ n \geqslant 1. \tag{3.5}$$

Moreover, for any $s \in I_\mu^\circ$, the following assertions hold:

(i) *$\Pi_{s,z}$ is a rank-one projection for $|z| \leqslant \delta$, with $\Pi_{s,0}(\varphi)(x) = \pi_s(\varphi)$ for any $\varphi \in \mathcal{B}_\gamma$ and $x \in \mathcal{S}$, $\Pi_{s,z}N_{s,z} = N_{s,z}\Pi_{s,z} = 0$ and*

$$\lambda_{s,z} = e^{-qz}\frac{\kappa(s+z)}{\kappa(s)}, \quad \text{for } z \in B_\delta(0). \tag{3.6}$$

For any fixed $k \geqslant 1$, there exist $\varkappa_s \in (0,1)$ and c_s such that

$$\sup_{|z|<\delta} \|\frac{d^k}{dz^k}N_{s,z}^n\|_{\mathcal{B}_\gamma \to \mathcal{B}_\gamma} \leqslant c_s \varkappa_s^n, \ n \geqslant 1.$$

In addition, the mappings $z \mapsto \Pi_{s,z} : B_\delta(0) \to \mathcal{L}(\mathcal{B}_\gamma, \mathcal{B}_\gamma)$ and $z \mapsto N_{s,z} : B_\delta(0) \to \mathcal{L}(\mathcal{B}_\gamma, \mathcal{B}_\gamma)$ are analytic in the strong operator sense.

(ii) *For any compact set $K \subseteq \mathbb{R}\backslash\{0\}$, there exists a constant $C_K > 0$ such that for any $n \geqslant 1$ and $\varphi \in \mathcal{B}_\gamma$, we have*

$$\sup_{t\in K}\sup_{x\in\mathcal{S}} |R_{s,it}^n \varphi(x)| \leqslant e^{-nC_K} \sup_{x\in\mathcal{S}} |\varphi(x)|.$$

(iii) *The mapping $z \mapsto \lambda_{s,z} : B_\delta(0) \to \mathbb{C}$ is analytic, and*

$$\lambda_{s,z} = 1 + \frac{\sigma_s^2}{2}z^2 + \frac{\Lambda'''(s)}{6}z^3 + o(z^3) \quad \text{as } z \to 0,$$

where

$$\sigma_s^2 = \Lambda''(s) = \lim_{n\to\infty}\frac{1}{n}\mathbb{E}_{\mathbb{Q}_s}(\log|G_n x| - nq)^2$$

and

$$\Lambda'''(s) = \lim_{n\to\infty}\frac{1}{n}\mathbb{E}_{\mathbb{Q}_s}(\log|G_n x| - nq)^3.$$

In addition, if the measure μ is non-arithmetic, then the asymptotic variance σ_s^2 is strictly positive.

The assertions (i), (ii), (iii) of Proposition 3.3, except (3.6), have been proved in [8] for imaginary-valued $z \in (-i\delta, i\delta)$, based on the perturbation theory (see [25]). The assertions (i), (iii) can be extended to the complex-valued $z \in B_\delta(0)$ without changes in the proof in [8].

The identity (3.6) is not proved in [8], but can be obtained by using the arguments from [36]. By the perturbation theory, the operator P_s and its spectral radius $\kappa(s)$ can be extended to P_{s+z} and the eigenvalue $\kappa(s+z)$, respectively, with z in the small neighborhood of 0, see [21]. By the definitions of $R_{s,z}$ and P_z using the change of measure (3.2), we obtain for any $\varphi \in \mathcal{B}_\gamma$, $n \geqslant 1$, $s \in I_\mu^\circ$ and $z \in B_\delta(0)$,

$$R_{s,z}^n(\varphi) = e^{-nz\Lambda'(s)}\frac{P_{s+z}^n(\varphi r_s)}{\kappa^n(s) r_s}. \tag{3.7}$$

Since r_s is uniformly bounded, using (3.7) and the fact that $\kappa(s+z)$ is the unique eigenvalue of P_{s+z}, we deduce (3.6).

For negative values $s < 0$ sufficiently close to 0, we can define the perturbed operator $R_{s,z}$ as in (3.4). The following spectral gap property of $R_{s,z}$ is established in [36].

Proposition 3.4. *Assume that μ satisfies conditions **A2**, **A5** for invertible matrices, or conditions **A3**, **A5** for positive matrices. Then, there exist $\eta_0 < \eta$ and $\delta > 0$ such that for any $s \in (-\eta_0, 0)$ and $z \in B_\delta(0)$,*

$$R_{s,z}^n = \lambda_{s,z}^n \Pi_{s,z} + N_{s,z}^n, \ n \geqslant 1.$$

Moreover, for any $s \in (-\eta_0, 0)$, the assertions (i), (ii), (iii) of Proposition 3.3 hold true.

4. Proof of Theorems 2.1 and 2.3

4.1. **Auxiliary results.** We need some preliminary statements. Following Petrov [32], under the changed measure $\mathbb{Q}_s^x$, define the Cramér series ζ_s by

$$\zeta_s(t) = \frac{\gamma_{s,3}}{6\gamma_{s,2}^{3/2}} + \frac{\gamma_{s,4}\gamma_{s,2} - 3\gamma_{s,3}^2}{24\gamma_{s,2}^3} t + \frac{\gamma_{s,5}\gamma_{s,2}^2 - 10\gamma_{s,4}\gamma_{s,3}\gamma_{s,2} + 15\gamma_{s,3}^3}{120\gamma_{s,2}^{9/2}} t^2 + \ldots,$$

where $\gamma_{s,k} = \Lambda^{(k)}(s)$ and $\Lambda(s) = \log \kappa(s)$. The following lemma gives a full expansion of $\Lambda^*(q+l)$ in terms of power series in l in a neighborhood of 0, for $q = \Lambda'(s)$ and $s \in I_\mu^\circ \cup (\eta_0, 0)$, where η_0 is from Proposition 3.4.

Lemma 4.1. *Assume conditions of Theorem 2.1 or Theorem 2.3. Let $q = \Lambda'(s)$. Then, there exists $\delta > 0$ such that, for any $|l| \leqslant \delta$,*

$$\Lambda^*(q+l) = \Lambda^*(q) + sl + h_s(l),$$

where h_s is linked to the Cramér series ζ_s by the identity

$$h_s(l) = \frac{l^2}{2\sigma_s^2} - \frac{l^3}{\sigma_s^3} \zeta_s\Big(\frac{l}{\sigma_s}\Big). \tag{4.1}$$

Proof. Let $(\Lambda')^{-1}$ be the inverse function of Λ'. With the notation $l_s = (\Lambda')^{-1}(q+l) - s$, we have $\Lambda'(s+l_s) = q + l$. By the definition of Λ^*, it follows that $\Lambda^*(q+l) = (s+l_s)(q+l) - \Lambda(s+l_s)$. This, together with $\Lambda^*(q) = sq - \Lambda(s)$ and Taylor's formula, gives

$$h_s(l) := \Lambda^*(q+l) - \Lambda^*(q) - sl = l_s l - \sum_{k=2}^{\infty} \frac{\Lambda^{(k)}(s)}{k!} l_s^k. \tag{4.2}$$

From $\Lambda'(s+l_s) = q + l$ and $\Lambda'(s) = q$, we deduce that $l = \Lambda'(s+l_s) - \Lambda'(s)$, so that, by Taylor's formula,

$$l = \sum_{k=1}^{\infty} \frac{\Lambda^{(k+1)}(s)}{k!} l_s^k. \tag{4.3}$$

The rest of the proof is similar to that in Petrov [32] (chapter VIII, section 2). For $|l|$ small enough, the equation (4.3) has a unique solution l_s given by

$$l_s = \frac{l}{\sigma_s^2} - \frac{\Lambda^{(3)}(s)}{2\sigma_s^6} l^2 - \frac{\Lambda^{(4)}(s)\sigma_s^2 - 3(\Lambda^{(3)}(s))^2}{6\sigma_s^{10}} l^3 + \cdots .$$

Together with (4.2) and (4.3), this implies

$$h_s(l) = \sum_{k=2}^{\infty} \Lambda^{(k)}(s) \frac{k-1}{k!} l_s^k = \frac{l^2}{2\sigma_s^2} - \frac{l^3}{\sigma_s^3} \zeta_s(\frac{l}{\sigma_s}).$$

□

Let us fix a non-negative Schwartz function ρ on $\mathbb{R}$ with $\int_{\mathbb{R}} \rho(y)dy = 1$, whose Fourier transform $\widehat{\rho}$ is supported on $[-1,1]$ and has a continuous extension in the complex plane. Moreover, $\widehat{\rho}$ is analytic in the domain $D := \{z \in \mathbb{C} : |z| < 1, \Im z \neq 0\}$. Such a function can be constructed as follows. On the real line define $\widehat{\varsigma}(t) = e^{-\frac{1}{1-t^2}}$ if $t \in [-1,1]$, and $\widehat{\varsigma} = 0$ elsewhere. The function $\widehat{\varsigma}$ is compactly supported and has finite derivatives of all orders. Its inverse Fourier transform ς, however, is not non-negative. Let $\widehat{\rho}_0 = \widehat{\varsigma} * \widehat{\varsigma}$ be the convolution of $\widehat{\varsigma}$ with itself. It is supported by $[-2,2]$ and its inverse Fourier transform ρ_0 satisfies $\rho_0 = 2\pi\varsigma^2 \geqslant 0$. We show below that $\widehat{\rho}_0$ has a continuous extension in the complex plane, and $\widehat{\rho}_0$ is analytic in the domain D. Finally we rescale and renormalize ρ_0 by setting $\rho(y) = \rho_0(y/2)/[2\widehat{\rho}_0(0)]$ for $y \in \mathbb{R}$.

Lemma 4.2. *$\widehat{\rho}_0$ has a continuous extension in the complex plane, and $\widehat{\rho}_0$ is analytic in the domain D.*

Proof. The function $\widehat{\varsigma}$ can be extended to the complex plane as follows:

$$\widehat{\varsigma}(z) = \begin{cases} e^{-\frac{1}{1-z^2}} & |z| < 1,\ z \in \mathbb{C} \\ 0 & |z| \geqslant 1,\ z \in \mathbb{C}. \end{cases}$$

It is easily verified that $\widehat{\varsigma}$ is continuous in the interior of the unit disc and outside it, but is not continuous at any point on the unit circle $|z| = 1$. Note also that $\widehat{\varsigma}$ is uniformly bounded on $\mathbb{C}$. Recall that the function $\widehat{\rho}_0 = \widehat{\varsigma} * \widehat{\varsigma}$ is defined on the real line. We extend it to the complex plane by setting $\widehat{\rho}_0(z) = \int_{-1}^{1} \widehat{\varsigma}(t)\widehat{\varsigma}(z-t)\mathbb{1}_{\{|z-t|<1\}} dt$. The latter integral is well defined for any $z \in \mathbb{C}$, since $\widehat{\varsigma}$ is bounded. We are going to show that $\widehat{\varsigma}$ is continuous in $\mathbb{C}$. For any fixed $z \in \mathbb{C}$ and $h \in \mathbb{C}$ with $|h|$ small, we write

$$|\widehat{\rho}_0(z+h) - \widehat{\rho}_0(z)| \leqslant \int_{-1}^{1} \widehat{\varsigma}(t)|\widehat{\varsigma}(z-t+h) - \widehat{\varsigma}(z-t)|dt. \tag{4.4}$$

The set $T_z = \{t : |z-t| = 1\}$ of points of discontinuity of the function $t \mapsto \widehat{\varsigma}(z-t)$ consists of at most two points. For any $t \in [-1,1]$, $t \notin T_z$, by the

definition of $\widehat{\varsigma}$, we have that $|\widehat{\varsigma}(z-t+h)-\widehat{\varsigma}(z-t)| \to 0$ as $|h| \to 0$. Since the Lebesgue measure of T_z is 0, applying the Lebesgue dominated convergence theorem and taking into account the boundedness of the function $\widehat{\varsigma}$ on $\mathbb{C}$, we see that $\widehat{\rho}_0$ is continuous in the complex plane.

We next show that $\widehat{\rho}_0$ is analytic in the domain $D = \{z' \in \mathbb{C} : |z'| < 1, \Im z' \neq 0\}$. Fix $z \in D$. Let $\varepsilon = \Im z/2 \in (0, \frac{1}{2})$. Denote $D(\varepsilon) =: \{z' \in D : |\Im z'| > \varepsilon\}$. One can verify that the derivative $\widehat{\varsigma}'(z)$ exists and is uniformly bounded by $\frac{c}{\varepsilon^4}$ on the domain $D(\varepsilon)$. For any $h \in \mathbb{C}$ with $|h|$ small enough, we have

$$\begin{aligned}\frac{\widehat{\rho}_0(z+h)-\widehat{\rho}_0(z)}{h} &= \int_{[-1,1]\setminus T_z} \widehat{\varsigma}(t)\frac{\widehat{\varsigma}(z-t+h)-\widehat{\varsigma}(z-t)}{h}dt \\ &= \int_{[-1,1]\setminus T_z} \widehat{\varsigma}(t)\left(\int_0^1 \widehat{\varsigma}'(z-t+\theta h)\mathbb{1}_{\{|z-t+\theta h|<1\}}d\theta\right)dt.\end{aligned}$$

Since for any $t \in [-1,1]$ and $\theta \in [0,1]$, we have $|\Im(z-t+\theta h)| \geqslant \varepsilon$ uniformly in $|h| < \varepsilon$. This implies that $z-t+\theta h \in D(\varepsilon)$ and thus $\widehat{\varsigma}'(z-t+\theta h)$ is bounded, uniformly in $|h| < \varepsilon$ and $t \in [-1,1]$. Applying twice the Lebesgue dominated convergence theorem, we obtain that $\widehat{\rho}_0'(z)$ exists and is given by $\widehat{\rho}_0'(z) = \int_{[-1,1]\setminus T_z} \widehat{\varsigma}(t)\widehat{\varsigma}'(z-t)dt$. Hence $\widehat{\rho}_0$ is analytic in the domain D. □

For any $\varepsilon > 0$, define the density $\rho_\varepsilon(y) = \frac{1}{\varepsilon}\rho(\frac{y}{\varepsilon})$, $y \in \mathbb{R}$, whose Fourier transform has a compact support in $[-\varepsilon^{-1}, \varepsilon^{-1}]$ and is analytically extendable in a neighborhood of 0. For any non-negative integrable function ψ, following the paper [19], we introduce two modified functions related to ψ and establish some two-sided bounds. For any $\varepsilon > 0$ and $y \in \mathbb{R}$, set $\mathbb{B}_\varepsilon(y) = \{y' \in \mathbb{R} : |y'-y| \leqslant \varepsilon\}$ and

$$\psi_\varepsilon^+(y) = \sup_{y' \in \mathbb{B}_\varepsilon(y)} \psi(y') \quad \text{and} \quad \psi_\varepsilon^-(y) = \inf_{y' \in \mathbb{B}_\varepsilon(y)} \psi(y'). \tag{4.5}$$

Lemma 4.3. *Suppose that ψ is a non-negative integrable function and that ψ_ε^+ and ψ_ε^- are measurable for any $\varepsilon > 0$, then for sufficiently small ε, there exists a positive constant $C_\rho(\varepsilon)$ with $C_\rho(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that, for any $x \in \mathbb{R}$,*

$$\psi_\varepsilon^- * \rho_{\varepsilon^2}(x) - \int_{|y| \geqslant \varepsilon} \psi_\varepsilon^-(x-y)\rho_{\varepsilon^2}(y)dy \leqslant \psi(x) \leqslant (1+C_\rho(\varepsilon))\psi_\varepsilon^+ * \rho_{\varepsilon^2}(x).$$

The proof of the above lemma, being similar to that of Lemma 5.2 in [18], will not be detailed here.

The next assertion is the key point in establishing Theorem 2.1. Its proof is based on the spectral gap properties of the perturbed operator $R_{s,z}$ (see Proposition 3.3) and on the saddle point method, see Daniels [10], Richter [33], Ibragimov and Linnik [27] and Fedoryuk [12]. Let us introduce the necessary notation. In the following, let φ be a γ-Hölder continuous function

on $\mathcal{S}$. Assume that $\psi : \mathbb{R} \mapsto \mathbb{C}$ is a continuous function with compact support in $\mathbb{R}$, and moreover, ψ has a continuous extension in some neighborhood of 0 in the complex plane and can be extended analytically to the domain $D_\delta := \{z \in \mathbb{C} : |z| < \delta, \Im z \neq 0\}$ for some small $\delta > 0$. Recall that π_s is the invariant measure of the Markov chain X_n^x under the changed measure $\mathbb{Q}_s^x$, see (3.3).

Proposition 4.4. *Assume conditions of Theorem 2.1. Let $q = \Lambda'(s)$, where $s \in I_\mu^\circ$. Then, for any positive sequence $(l_n)_{n\geqslant 1}$ satisfying $l_n \to 0$ as $n \to \infty$, we have, uniformly in $x \in \mathcal{S}$, $|l| \leqslant l_n$ and $\varphi \in \mathcal{B}_\gamma$,*

$$\Big|\sqrt{n}\ \sigma_s e^{nh_s(l)} \int_{\mathbb{R}} e^{-itln} R_{s,it}^n(\varphi)(x)\psi(t)dt - \sqrt{2\pi}\psi(0)\pi_s(\varphi)\Big| \leqslant C\|\varphi\|_\gamma\Big(\frac{\log n}{\sqrt{n}} + l_n\Big).$$

Proof. Denote $c_s(\psi) = \frac{\sqrt{2\pi}}{\sigma_s}\psi(0)\pi_s(\varphi)$. Taking sufficiently small $\delta > 0$, we write

$$\begin{aligned}
&\Big|\sqrt{n}\ e^{nh_s(l)} \int_{\mathbb{R}} e^{-itln} R_{s,it}^n(\varphi)(x)\psi(t)dt - c_s(\psi)\Big| \\
&\leqslant \Big|\sqrt{n}\ e^{nh_s(l)} \int_{|t|\geqslant\delta} e^{-itln} R_{s,it}^n(\varphi)(x)\psi(t)dt\Big| \\
&\quad + \Big|\sqrt{n}\ e^{nh_s(l)} \int_{|t|<\delta} e^{-itln} R_{s,it}^n(\varphi)(x)\psi(t)dt - c_s(\psi)\Big| \\
&= I(n) + J(n). \qquad (4.6)
\end{aligned}$$

For $I(n)$, since ψ is bounded and compactly supported on the real line, taking into account Proposition 3.3 (ii), the fact $|e^{-itln}| = 1$ and equality (4.1), we get

$$\sup_{x\in\mathcal{S}} \sup_{|l|\leqslant l_n} |I(n)| \leqslant C_\delta e^{-c_\delta n}\|\varphi\|_\gamma. \qquad (4.7)$$

For $J(n)$, by Proposition 3.3 (i), we have

$$R_{s,it}^n(\varphi)(x) = \lambda_{s,it}^n \Pi_{s,it}(\varphi)(x) + N_{s,it}^n(\varphi)(x).$$

Set for brevity $\psi_{s,x}(t) = \Pi_{s,it}(\varphi)(x)\psi(t)$. It follows that

$$\begin{aligned}
J(n) \leqslant\ & \Big|\sqrt{n}\ e^{nh_s(l)} \int_{|t|<\delta} e^{-itln} \lambda_{s,it}^n \psi_{s,x}(t)dt - c_s(\psi)\Big| \\
&+ \Big|\sqrt{n}\ e^{nh_s(l)} \int_{|t|<\delta} e^{-itln} N_{s,it}^n(\varphi)(x)\psi(t)dt\Big| \\
=\ & J_1(n) + J_2(n). \qquad (4.8)
\end{aligned}$$

For the second term $J_2(n)$, applying Proposition 3.3 (i), we get that there exist constants $c_\delta > 0$ and $\varkappa \in (0,1)$ such that

$$\sup_{x\in\mathcal{S}} \sup_{|t|<\delta} |N^n_{s,it}(\varphi)(x)| \leqslant \sup_{|t|<\delta} \|N^n_{s,it}\|_{\mathcal{B}_\gamma\to\mathcal{B}_\gamma} \|\varphi\|_\gamma \leqslant c_\delta \varkappa^n \|\varphi\|_\gamma.$$

Combining this with the continuity of the function ψ at the point 0 and the fact $|e^{-itln}| = 1$, we obtain that, uniformly in $|l| \leqslant l_n$, $x \in \mathcal{S}$ and $\varphi \in \mathcal{B}_\gamma$,

$$J_2(n) \leqslant C_\delta e^{-c_\delta n} \|\varphi\|_\gamma. \tag{4.9}$$

For the first term $J_1(n)$, we shall use the method of steepest descends to derive a precise asymptotic expansion. We make a change of variable $z = it$ to rewrite $J_1(n)$ as an integral over the complex interval $L_0 = (-i\delta, i\delta)$:

$$J_1(n) = \Big| - i\sqrt{n}\; e^{nh_s(l)} \int_{-i\delta}^{i\delta} e^{n(K_s(z)-zl)} \psi_{s,x}(-iz)dz - c_s(\psi)\Big|, \tag{4.10}$$

where $K_s(z) = \log \lambda_{s,z}$ (we choose the branch where $K_s(0) = 0$), which is an analytic function for $|z| \leqslant \delta$ by Proposition 3.3 (iii). Since the function $z \mapsto e^{n(K_s(z)-zl)}$ is analytic in the neighborhood of 0, and the function $z \mapsto \psi_{s,x}(-iz)$ has an analytic extension in the domain $D_\delta := \{z \in \mathbb{C} : |z| < \delta, \Im z \neq 0\}$ and has a continuous extension in the domain $\overline{D}_\delta := \{z \in \mathbb{C} : |z| \leqslant \delta\}$, by Cauchy's integral theorem we can choose a special path of the integration which passes through the saddle point of the function $K_s(z) - zl$. From (3.6), we have

$$K_s(z) = -qz + \log \kappa(s+z) - \log \kappa(s),$$

which implies that for $|z| < \delta$,

$$K_s(z) = \sum_{k=2}^{\infty} \gamma_{s,k} \frac{z^k}{k!}, \tag{4.11}$$

where $\gamma_{s,k} = \Lambda^{(k)}(s)$ and $\Lambda(s) = \log \kappa(s)$. From this Taylor's expansion and the fact that $\Lambda^{(2)}(s) = \sigma_s^2 > 0$, it follows that the function $K_s(z) - zl$ is convex in the neighborhood of 0. Consider the saddle point equation

$$K_s'(z) - l = 0. \tag{4.12}$$

An equivalent formulation of (4.12) is $l = \sum_{k=2}^{\infty} \gamma_{s,k} \frac{z^{k-1}}{(k-1)!}$, which by simple series inversion techniques gives the following solution:

$$z_0 = z_0(l) := \frac{l}{\gamma_{s,2}} - \frac{\gamma_{s,3}}{2\gamma_{s,2}^3} l^2 - \frac{\gamma_{s,4}\gamma_{s,2} - 3\gamma_{s,3}^2}{6\gamma_{s,2}^5} l^3 + \cdots. \tag{4.13}$$

From (4.13), it follows that the solution $z_0 = z_0(l)$ is real for sufficiently small l and that $z_0 = z_0(l) \to 0$ as $l \to 0$. Moreover, $z_0 > 0$ for sufficiently

small $l > 0$, and $z_0 < 0$ for sufficiently small $l < 0$. By Cauchy's integral theorem, $J_1(n)$ can be rewritten as

$$J_1(n) = \Big| - i\sqrt{n}\ e^{nh_s(l)} \Big\{ \int_{L_1} + \int_{L_2} + \int_{L_3} \Big\} e^{n(K_s(z)-zl)} \psi_{s,x}(-iz)dz - c_s(\psi) \Big|,$$

where $L_1 = (-i\delta, z_0 - i\delta)$, $L_2 = (z_0 - i\delta, z_0 + i\delta)$ and $L_3 = (z_0 + i\delta, i\delta)$. By (4.11), we get $K_s(it) = -\frac{1}{2}\sigma_s^2 t^2 + O(t^3)$, which implies that $|e^{nK_s(it)}| \leqslant e^{-\frac{n}{3}\sigma_s^2 t^2}$, when t is sufficiently small. Combining this with (4.13) and the continuity of $K_s(z)$ in the neighborhood of 0 yields that, for sufficiently small l, $|e^{nK_s(z)}| \leqslant e^{-\frac{n}{4}\sigma_s^2\delta^2}$, for any $z \in L_1 \cup L_3$. Since, for sufficiently small l, $lz_0 > 0$, we get that, for $z \in L_1 \cup L_3$, $|e^{-nzl}| = |e^{-nlz_0}| \leqslant 1$. Moreover, using the continuity of the function $z \mapsto \psi_{s,x}(-iz)$ in a small neighborhood of 0 in the complex plane, there exists a constant $C_s > 0$ such that, on L_1 and L_3, we have $\sup_{x\in\mathcal{S}} |\psi_{s,x}(-iz)| \leqslant C_s\|\varphi\|_\gamma$. Therefore, we obtain, for n sufficiently large, uniformly in $|l| \leqslant l_n$ and $x \in \mathcal{S}$,

$$\Big| - i\sqrt{n}\ e^{nh_s(l)} \Big\{ \int_{L_1} + \int_{L_3} \Big\} e^{n(K_s(z)-zl)} \psi_{s,x}(-iz)dz \Big| \leqslant O(e^{-\frac{n}{5}\sigma_s^2\delta^2})\|\varphi\|_\gamma.$$

It follows that

$$\begin{aligned} J_1(n) \leqslant & \Big| - i\sqrt{n}\ e^{nh_s(l)} \int_{z_0-i\delta}^{z_0+i\delta} e^{n(K_s(z)-zl)} \psi_{s,x}(-iz)dz - c_s(\psi) \Big| \\ & + O(e^{-\frac{n}{5}\sigma_s^2\delta^2})\|\varphi\|_\gamma. \end{aligned}$$

Without loss of generality, assume that $n \geqslant 3$. Making a change of variable $z = z_0 + it$ gives

$$\begin{aligned} J_1(n) &\leqslant \Big| \sqrt{n}\ e^{nh_s(l)} \int_{-\delta}^{\delta} e^{n[K_s(z_0+it)-(z_0+it)l]} \psi_{s,x}(t - iz_0)dt - c_s(\psi) \Big| \\ &\qquad + O(e^{-\frac{n}{5}\sigma_s^2\delta^2})\|\varphi\|_\gamma \\ &\leqslant \Big| \sqrt{n}\, e^{nh_s(l)} \int_{n^{-\frac{1}{2}}\log n \leqslant |t| < \delta} e^{n[K_s(z_0+it)-(z_0+it)l]} \psi_{s,x}(t - iz_0)dt \Big| \\ &\quad + \Big| \sqrt{n} e^{nh_s(l)} \int_{|t|<n^{-\frac{1}{2}}\log n} e^{n[K_s(z_0+it)-(z_0+it)l]} \psi_{s,x}(t - iz_0)dt - c_s(\psi) \Big| \\ &\quad + O(e^{-\frac{n}{5}\sigma_s^2\delta^2})\|\varphi\|_\gamma. \end{aligned} \tag{4.14}$$

From (4.12) and (4.13), we have $K_s'(z_0) = l$. By Taylor's formula, we get that for $|t| < \delta$,

$$K_s(z_0 + it) - (z_0 + it)l = K_s(z_0) - z_0 l + \sum_{k=2}^{\infty} \frac{K_s^{(k)}(z_0)(it)^k}{k!}.$$

Using $K_s'(z_0) = l$ and (4.11), it follows that

$$K_s(z_0) - z_0 l = K_s(z_0) - z_0 K_s'(z_0) = -\sum_{k=2}^{\infty} \frac{k-1}{k!} \gamma_{s,k} z_0^k.$$

Combining this with (4.13) and Lemma 4.1 gives $K_s(z_0) - z_0 l = -h_s(l)$. Thus

$$K_s(z_0 + it) - (z_0 + it)l = -h_s(l) + \sum_{k=2}^{\infty} \frac{K_s^{(k)}(z_0)(it)^k}{k!}. \tag{4.15}$$

Since $K_s''(z_0) = \sigma_s^2 + O(z_0) > \frac{1}{2}\sigma_s^2$, for small enough z_0, δ and l, we obtain that $\Re(\sum_{k=2}^{\infty} \frac{K_s^{(k)}(z_0)(it)^k}{k!}) < -\frac{1}{8}\sigma_s^2 t^2$. Therefore, using (4.15) and the fact that uniformly in $x \in \mathcal{S}$, the function $z \mapsto \psi_{s,x}(z)$ is continuous in a neighborhood of 0 in the complex plane, we obtain that, uniformly in $x \in \mathcal{S}$ and $|l| \leqslant l_n$,

$$\begin{aligned} &\Big| \sqrt{n}\, e^{n h_s(l)} \int_{n^{-\frac{1}{2}} \log n \leqslant |t| < \delta} e^{n[K_s(z_0+it) - (z_0+it)l]} \psi_{s,x}(t - iz_0) dt \Big| \\ &\leqslant c_1 \sqrt{n} \int_{n^{-\frac{1}{2}} \log n \leqslant |t| < \delta} e^{-\frac{1}{8} n \sigma_s^2 t^2} dt \|\varphi\|_\gamma = O(e^{-c \log^2 n}) \|\varphi\|_\gamma. \end{aligned}$$

This, together with (4.14)-(4.15), implies

$$\begin{aligned} J_1(n) \leqslant \sup_{x \in \mathcal{S}} &\Big| \sqrt{n} \int_{|t| < n^{-\frac{1}{2}} \log n} e^{n \sum_{k=2}^{\infty} \frac{K_s^{(k)}(z_0)(it)^k}{k!}} \psi_{s,x}(t - iz_0) dt - c_s(\psi) \Big| \\ &+ O(e^{-c \log^2 n}) \|\varphi\|_\gamma. \end{aligned}$$

Noting that $\Pi_{s,0}(\varphi)(x) = \pi_s(\varphi)$ and $\psi_{s,x}(0) = \psi(0)\pi_s(\varphi)$, we write

$$\begin{aligned} J_1(n) \leqslant &\sup_{x \in \mathcal{S}} \Big| \sqrt{n} \int_{|t| < n^{-\frac{1}{2}} \log n} \Big(e^{n \sum_{k=2}^{\infty} \frac{K_s^{(k)}(z_0)(it)^k}{k!}} - e^{-\frac{n\sigma_s^2 t^2}{2}} \Big) \psi_{s,x}(t - iz_0) dt \Big| \\ &+ \sup_{x \in \mathcal{S}} \Big| \sqrt{n} \int_{|t| < n^{-\frac{1}{2}} \log n} e^{-\frac{n\sigma_s^2 t^2}{2}} \big[\psi_{s,x}(t - iz_0) - \psi_{s,x}(0) \big] dt \Big| \\ &+ \sqrt{n} \psi(0) \pi_s(\varphi) \int_{|t| \geqslant n^{-\frac{1}{2}} \log n} e^{-\frac{n\sigma_s^2 t^2}{2}} dt + O(e^{-c \log^2 n}) \|\varphi\|_\gamma \\ = &\, J_{11}(n) + J_{12}(n) + J_{13}(n) + O(e^{-c \log^2 n}) \|\varphi\|_\gamma. \end{aligned} \tag{4.16}$$

We give a control of $J_{11}(n)$. Note that $|\psi_{s,x}(t - iz_0)|$ is bounded by $C_s \|\varphi\|_\gamma$, uniformly in $|t| < n^{-\frac{1}{2}} \log n$. Note also that for $|t| < n^{-\frac{1}{2}} \log n$ and for large enough n, we have $|e^{n \Re \sum_{k=3}^{\infty} \frac{K^{(k)}(z_0)(it)^k}{k!}}| \leqslant e^{cnt^4} \leqslant C$. Hence using the inequality $|e^z - 1| \leqslant e^{\Re z} |z|$ yields

$$J_{11}(n) \leqslant C_s \|\varphi\|_\gamma \sqrt{n} \int_{|t| < n^{-\frac{1}{2}} \log n} e^{-\frac{n\sigma_s^2 t^2}{2}} n|t|^3 dt \leqslant \frac{C_s}{\sqrt{n}} \|\varphi\|_\gamma. \tag{4.17}$$

Now we control $J_{12}(n)$. Recalling that $z_0 = z_0(l) \leqslant c_s l_n$, using the fact that uniformly with respect to $x \in \mathcal{S}$, the map $z \mapsto \psi_{s,x}(z)$ is continuous in the neighborhood of 0 in the complex plane, we get that for $|t| \leqslant n^{-\frac{1}{2}} \log n$,

$$\sup_{x\in\mathcal{S}} |\psi_{s,x}(t - iz_0) - \psi_{s,x}(0)| < c_s(n^{-\frac{1}{2}} \log n + l_n)\|\varphi\|_\gamma.$$

We then obtain

$$J_{12}(n) \leqslant c_s(n^{-\frac{1}{2}} \log n + l_n)\|\varphi\|_\gamma.$$

It is easy to see that $J_{13}(n) \leqslant C\|\varphi\|_\gamma e^{-c_s \log^2 n}$. This, together with (4.16)-(4.17), proves that $J_1(n) \leqslant c_s(n^{-\frac{1}{2}} \log n + l_n)\|\varphi\|_\gamma$. The desired result follows by combining this with (4.6)-(4.9). □

Assume that the functions φ and ψ satisfy the same properties as in Proposition 4.4. The following result, for $s < 0$ small enough, will be used to prove Theorem 2.3.

Proposition 4.5. *Assume conditions of Theorem 2.3. Then, there exists $\eta_0 < \eta$ such that for any $s \in (-\eta_0, 0)$, $q = \Lambda'(s)$ and for any positive sequence $(l_n)_{n\geqslant 1}$ satisfying $l_n \to 0$ as $n \to \infty$, we have, uniformly in $x \in \mathcal{S}$, $|l| \leqslant l_n$ and $\varphi \in \mathcal{B}_\gamma$,*

$$\begin{aligned}&\Big|\sqrt{n}\ \sigma_s e^{nh_s(l)} \int_{\mathbb{R}} e^{-itln} R^n_{s,it}(\varphi)(x)\psi(t)dt - \sqrt{2\pi}\psi(0)\pi_s(\varphi)\Big| \\ &\leqslant C\|\varphi\|_\gamma\Big(\frac{\log n}{\sqrt{n}} + l_n\Big).\end{aligned}$$

Proof. Using Propositions 3.2 and 3.4, the proof of Proposition 4.5 can be carried out as the proof of Proposition 4.4. We omit the details. □

4.2. **Proof of Theorem 2.1.** Recall that $q = \Lambda'(s)$, $\Lambda^*(q + l) = \Lambda^*(q) + sl + h_s(l)$, $x \in \mathcal{S}$, and $|l| \leqslant l_n \to 0$, as $n \to \infty$. Taking into account that $e^{n\Lambda^*(q)} = e^{sqn}/\kappa^n(s)$ and using the change of measure (3.2), we write

$$\begin{aligned}A_n(x, l) &:= \sqrt{2\pi n}\ s\sigma_s e^{n\Lambda^*(q+l)} \frac{1}{r_s(x)} \mathbb{P}(\log|G_n x| \geqslant n(q + l)) \\ &= \sqrt{2\pi n}\, s\sigma_s e^{nsl} e^{nh_s(l)} e^{sqn} \mathbb{E}_{\mathbb{Q}^x_s}\Big(\frac{1}{r_s(X^x_n)} e^{-s\log|G_n x|} \mathbb{1}_{\{\log|G_n x|\geqslant n(q+l)\}}\Big). \quad (4.18)\end{aligned}$$

Setting $T^x_n = \log|G_n x| - nq$ and $\psi_s(y) = e^{-sy}\mathbb{1}_{\{y\geqslant 0\}}$, from (4.18) we get

$$A_n(x, l) = \sqrt{2\pi n}\ s\sigma_s e^{nh_s(l)} \mathbb{E}_{\mathbb{Q}^x_s}\left(\frac{1}{r_s(X^x_n)} \psi_s(T^x_n - nl)\right). \quad (4.19)$$

Upper bound. Let $\varepsilon \in (0,1)$ and $\psi^+_{s,\varepsilon}(y) = \sup_{y' \in \mathbb{B}_\varepsilon(y)} \psi_s(y')$ be defined as in (4.5) but with ψ_s instead of ψ. Using Lemma 4.3 leads to

$$\begin{aligned} A_n(x,l) &\leqslant (1+C_\rho(\varepsilon))\sqrt{2\pi n}\; s\sigma_s e^{nh_s(l)} \mathbb{E}_{\mathbb{Q}^x_s}\left[\frac{1}{r_s(X^x_n)}(\psi^+_{s,\varepsilon} * \rho_{\varepsilon^2})(T^x_n - nl)\right] \\ &=: B^+_n(x,l). \end{aligned} \tag{4.20}$$

Denote by $\widehat{\psi}^+_{s,\varepsilon}$ the Fourier transform of $\psi^+_{s,\varepsilon}$. Elementary calculations give

$$\sup_{t\in\mathbb{R}} |\widehat{\psi}^+_{s,\varepsilon}(t)| \leqslant \widehat{\psi}^+_{s,\varepsilon}(0) = \int_{-\varepsilon}^{\varepsilon} dy + \int_{\varepsilon}^{+\infty} e^{-s(y-\varepsilon)} dy = \frac{1+2s\varepsilon}{s}. \tag{4.21}$$

By the inversion formula, for any $y \in \mathbb{R}$,

$$\psi^+_{s,\varepsilon} * \rho_{\varepsilon^2}(y) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ity} \widehat{\psi}^+_{s,\varepsilon}(t)\widehat{\rho}_{\varepsilon^2}(t) dt.$$

Substituting $y = T^x_n - nl$, taking expectation with respect to $\mathbb{E}_{\mathbb{Q}^x_s}$, and using Fubini's theorem, we get

$$\mathbb{E}_{\mathbb{Q}^x_s}\Big[\frac{1}{r_s(X^x_n)}(\psi^+_{s,\varepsilon} * \rho_{\varepsilon^2})(T^x_n - nl)\Big] = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itln} R^n_{s,it}(r_s^{-1})(x)\widehat{\psi}^+_{s,\varepsilon}(t)\widehat{\rho}_{\varepsilon^2}(t) dt, \tag{4.22}$$

where

$$R^n_{s,it}(r_s^{-1})(x) = \mathbb{E}_{\mathbb{Q}^x_s}\left[e^{itT^x_n}\frac{1}{r_s(X^x_n)}\right].$$

Note that $\widehat{\psi}^+_{s,\varepsilon}\widehat{\rho}_{\varepsilon^2}$ is compactly supported in $\mathbb{R}$ since $\widehat{\rho}_{\varepsilon^2}$ has a compact support. One can verify that $\widehat{\psi}^+_{s,\varepsilon}$ has an analytic extension in a neighborhood of 0. By Lemma 4.2, we see that the function $\widehat{\rho}_{\varepsilon^2}$ has a continuous extension in the complex plane, and has an analytic in the domain $D_{\varepsilon^2} := \{z \in \mathbb{C} : |z| < \varepsilon^2, \Im z \neq 0\}$. Using Proposition 4.4 with $\varphi = r_s^{-1}$ and $\psi = \widehat{\psi}^+_{s,\varepsilon}\widehat{\rho}_{\varepsilon^2}$, it follows that

$$\lim_{n\to\infty} \sup_{x\in\mathcal{S}} \sup_{|l|\leqslant l_n} \left| B^+_n(x,l) - (1+C_\rho(\varepsilon))\pi_s(r_s^{-1}) s\widehat{\psi}^+_{s,\varepsilon}(0)\widehat{\rho}_{\varepsilon^2}(0)\right| = 0. \tag{4.23}$$

Since $\widehat{\rho}_{\varepsilon^2}(0) = 1$, from (4.19)-(4.23), we have that for sufficiently small $\varepsilon \in (0,1)$,

$$\begin{aligned} \limsup_{n\to\infty} \sup_{x\in\mathcal{S}} \sup_{|l|\leqslant l_n} A_n(x,l) &\leqslant (1+C_\rho(\varepsilon)) s\pi_s(r_s^{-1})\widehat{\psi}^+_{s,\varepsilon}(0)\widehat{\rho}_\varepsilon(0) \\ &\leqslant (1+C_\rho(\varepsilon))(1+2s\varepsilon)\pi_s(r_s^{-1}). \end{aligned}$$

Letting $\varepsilon \to 0$ and noting that $C_\rho(\varepsilon) \to 0$, we obtain the upper bound:

$$\limsup_{n\to\infty} \sup_{x\in\mathcal{S}} \sup_{|l|\leqslant l_n} A_n(x,l) \leqslant \pi_s(r_s^{-1}) = \frac{1}{\nu_s(r_s)}. \tag{4.24}$$

Lower bound. For $\varepsilon \in (0,1)$, let $\psi^-_{s,\varepsilon}(y) = \inf_{y' \in \mathbb{B}_\varepsilon(y)} \psi_s(y')$ be defined as in (4.5) with ψ_s instead of ψ. From (4.19) and Lemma 4.3, we get

$$\begin{aligned} A_n(x,l) &\geqslant \sqrt{2\pi n}\ s\sigma_s e^{nh_s(l)} \mathbb{E}_{\mathbb{Q}_s^x}\left[\frac{1}{r_s(X_n^x)}(\psi^-_{s,\varepsilon} * \rho_{\varepsilon^2})(T_n^x - nl)\right] \\ &\quad - \sqrt{2\pi n}\ s\sigma_s e^{nh_s(l)} \int_{|y|\geqslant\varepsilon} \mathbb{E}_{\mathbb{Q}_s^x}\left[\frac{1}{r_s(X_n^x)}\psi^-_{s,\varepsilon}(T_n^x - nl - y)\right]\rho_{\varepsilon^2}(y)dy \\ &:= B_n^-(x,l) - C_n^-(x,l). \end{aligned} \tag{4.25}$$

For the first term $B_n^-(x,l)$, applying (4.22) with $\psi^+_{s,\varepsilon}\rho_{\varepsilon^2}$ replaced by $\psi^-_{s,\varepsilon}\rho_{\varepsilon^2}$, we get

$$B_n^-(x,l) = \sqrt{\frac{n}{2\pi}}\ s\sigma_s e^{nh_s(l)} \int_{\mathbb{R}} e^{-itln} R^n_{s,it}(r_s^{-1})(x)\widehat{\psi}^-_{s,\varepsilon}(t)\widehat{\rho}_{\varepsilon^2}(t)dt.$$

In the same way as for the upper bound, using $\widehat{\psi}^-_{s,\varepsilon}(0) = \frac{e^{-2s\varepsilon}}{s}$ and Proposition 4.4 with $\varphi = r_s^{-1}$ and $\psi = \widehat{\psi}^-_{s,\varepsilon}\widehat{\rho}_{\varepsilon^2}$ (one can check that the functions φ and ψ satisfy the required conditions in Proposition 4.4), we obtain the lower bound:

$$\liminf_{n\to\infty} \sup_{x\in\mathcal{S}} \sup_{|l|\leqslant l_n} B_n^-(x,l) \geqslant \pi_s(r_s^{-1}) = \frac{1}{\nu_s(r_s)}. \tag{4.26}$$

For the second term $C_n^-(x,l)$, noting that $\psi^-_{s,\varepsilon} \leqslant \psi_s$ and applying Lemma 4.3 to ψ_s, we get $\psi^-_{s,\varepsilon} \leqslant \psi_s \leqslant (1+C_\rho(\varepsilon))\psi^+_{s,\varepsilon} * \rho_{\varepsilon^2}$. We use the same argument as in (4.22) to obtain

$$\begin{aligned} C_n^-(x,l) &\leqslant (1+C_\rho(\varepsilon))\sqrt{2\pi n}\ s\sigma_s e^{nh_s(l)} \\ &\quad \times \int_{|y|\geqslant\varepsilon} \mathbb{E}_{\mathbb{Q}_s^x}\left[\frac{1}{r_s(X_n^x)}(\psi^+_{s,\varepsilon} * \rho_{\varepsilon^2})(T_n^x - nl - y)\right]\rho_{\varepsilon^2}(y)dy \\ &= (1+C_\rho(\varepsilon))\sqrt{\frac{n}{2\pi}}\ s\sigma_s e^{nh_s(l)} \\ &\quad \times \int_{|y|\geqslant\varepsilon}\left(\int_{\mathbb{R}} e^{-it(ln+y)} R^n_{s,it}(r_s^{-1})(x)\widehat{\psi}^+_{s,\varepsilon}(t)\widehat{\rho}_{\varepsilon^2}(t)dt\right)\rho_{\varepsilon^2}(y)dy. \end{aligned}$$

Notice that, from Lemma 4.1, for any fixed $y \in \mathbb{R}$, it holds, uniformly in l satisfying $|l| \leqslant l_n$, that $e^{nh_s(l) - nh_s(l+\frac{y}{n})} \to 1$ as $n \to \infty$. Applying Proposition 4.4 again with $\varphi = r_s^{-1}$, $\psi = \widehat{\psi}^+_{s,\varepsilon}\widehat{\rho}_{\varepsilon^2}$, and using the Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned} \limsup_{n\to\infty} \sup_{x\in\mathcal{S}} \sup_{|l|\leqslant l_n} C_n^-(x,l) &\leqslant (1+C_\rho(\varepsilon))s\pi_s(r_s^{-1})\widehat{\psi}^+_{s,\varepsilon}(0)\widehat{\rho}_{\varepsilon^2}(0)\int_{|y|\geqslant\varepsilon}\rho_{\varepsilon^2}(y)dy \\ &= (1+C_\rho(\varepsilon))\pi_s(r_s^{-1})(1+2s\varepsilon)\int_{|y|\geqslant\frac{1}{\varepsilon}}\rho(y)dy \to 0, \quad \text{as } \varepsilon \to 0, \end{aligned}$$

since ρ is integrable on $\mathbb{R}$. This, together with (4.25)-(4.26), implies the lower bound:

$$\liminf_{n\to\infty} \sup_{x\in\mathcal{S}} \sup_{|l|\leqslant l_n} A_n(x,l) \geqslant \pi_s(r_s^{-1}) = \frac{1}{\nu_s(r_s)}, \tag{4.27}$$

as required. We conclude the proof of Theorem 2.1 by combining (4.24) and (4.27).

4.3. **Proof of Theorem 2.3.** Since the change of measure formula can be extended for small $s < 0$, under the conditions of Theorem 2.3, we have, similar to (4.18),

$$\begin{aligned} &- s\sigma_s\sqrt{2\pi n}\, e^{n\Lambda^*(q+l)} \frac{1}{r_s(x)} \mathbb{P}(\log|G_n x| \leqslant n(q+l)) \\ &= - s\sigma_s\sqrt{2\pi n}\, e^{nsl} e^{nh_s(l)} e^{sqn} \mathbb{E}_{\mathbb{Q}_s^x}\Big(\frac{1}{r_s(X_n^x)} e^{-s\log|G_n x|} \mathbb{1}_{\{\log|G_n x|\leqslant n(q+l)\}}\Big). \end{aligned}$$

Applying Proposition 4.5, we can follow the proof of Theorem 2.1 to show Theorem 2.3. We omit the details.

5. Proof of Theorems 2.2 and 2.4

We first establish the following assertion which will be used to prove Theorem 2.2, but which is of independent interest. Let ψ be a measurable function on $\mathbb{R}$ and $\varepsilon > 0$. Denote, for brevity, $\psi_s(y) = e^{-sy}\psi(y)$ and

$$\psi_{s,\varepsilon}^+(y) = \sup_{y'\in\mathbb{B}_\varepsilon(y)} \psi_s(y'), \quad \psi_{s,\varepsilon}^-(y) = \inf_{y'\in\mathbb{B}_\varepsilon(y)} \psi_s(y').$$

Introduce the following condition: for any $s \in I_\mu^\circ$ and $\varepsilon > 0$, the functions $y \mapsto \psi_{s,\varepsilon}^+(y)$ and $y \mapsto \psi_{s,\varepsilon}^-(y)$ are measurable and

$$\lim_{\varepsilon\to 0^+} \int_{\mathbb{R}} \psi_{s,\varepsilon}^+(y)dy = \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}} \psi_{s,\varepsilon}^-(y)dy = \int_{\mathbb{R}} e^{-sy}\psi(y)dy < +\infty. \tag{5.1}$$

Theorem 5.1. *Suppose the assumptions of Theorem 2.1 hold true. Let $q = \Lambda'(s)$, where $s \in I_\mu^\circ$. Assume that φ is a Hölder continuous function on $\mathcal{S}$ and ψ is a measurable function on $\mathbb{R}$ satisfying condition (5.1). Then, for any positive sequence $(l_n)_{n\geqslant 1}$ satisfying $\lim_{n\to\infty} l_n = 0$, we have*

$$\lim_{n\to\infty} \sup_{x\in\mathcal{S}} \sup_{|l|\leqslant l_n} \Big| \sqrt{2\pi n}\, \sigma_s e^{n\Lambda^*(q+l)} \mathbb{E}\Big[\varphi(X_n^x)\psi(\log|G_n x| - n(q+l))\Big] - \bar{r}_s(x)\nu_s(\varphi) \int_{\mathbb{R}} e^{-sy}\psi(y)dy \Big| = 0. \tag{5.2}$$

Before proceeding with the proof of this theorem, let us give some examples of functions satisfying condition (5.1). It is easy to see that (5.1) holds for increasing non-negative functions ψ satisfying $\int_{\mathbb{R}} e^{-sy}\psi(y)dy < +\infty$, in particular, for the indicator function $\psi(y) = \mathbb{1}_{\{y\geqslant c\}}$, $y \in \mathbb{R}$, where $c \in \mathbb{R}$ is a fixed constant. Another example for which (5.1) holds true is when ψ is non-negative, continuous and there exists $\varepsilon > 0$ such that

$$\int_{\mathbb{R}} e^{-sy}\psi_\varepsilon^+(y)dy < +\infty, \tag{5.3}$$

where the function $\psi_\varepsilon^+(y) = \sup_{y'\in\mathbb{B}_\varepsilon(y)} \psi(y')$ is assumed to be measurable.

Proof of Theorem 5.1. Without loss of generality, we assume that both φ and ψ are non-negative (otherwise, we decompose the functions $\varphi = \varphi^+ - \varphi^-$ and $\psi = \psi^+ - \psi^-$). Let $T_n^x = \log|G_n x| - nq$. Since $e^{n\Lambda^*(q)} = e^{sqn}/\kappa^n(s)$, using the change of measure (3.2), we have

$$\begin{aligned} A_n(x,l) &:= \sqrt{2\pi n}\ \sigma_s e^{n\Lambda^*(q+l)} \frac{1}{r_s(x)} \mathbb{E}\Big[\varphi(X_n^x)\psi(\log|G_n x| - n(q+l))\Big] \\ &= \sqrt{2\pi n}\ \sigma_s e^{nsl} e^{nh_s(l)} e^{sqn} \mathbb{E}_{\mathbb{Q}_s^x}\left[(\varphi r_s^{-1})(X_n^x) e^{-s\log|G_n x|}\psi(T_n^x - nl)\right] \\ &= \sqrt{2\pi n}\ \sigma_s e^{nh_s(l)} \mathbb{E}_{\mathbb{Q}_s^x}\left[(\varphi r_s^{-1})(X_n^x) e^{-s(T_n^x - nl)}\psi(T_n^x - nl)\right]. \end{aligned}$$

For brevity, set $\Phi_s(x) = (\varphi r_s^{-1})(x)$, $x \in \mathcal{S}$, and $\Psi_s(y) = e^{-sy}\psi(y)$, $y \in \mathbb{R}$. Then,

$$A_n(x,l) = \sqrt{2\pi n}\ \sigma_s e^{nh_s(l)} \mathbb{E}_{\mathbb{Q}_s^x}\left[\Phi_s(X_n^x)\Psi_s(T_n^x - nl)\right]. \tag{5.4}$$

Upper bound. We wish to write the expectation in (5.4) as an integral of the Fourier transform of Ψ_s, which, however, may not belong to the space $L^1(\mathbb{R})$. As in the proof of Theorem 2.1 (see Section 4.2), we make use of the convolution technique to overcome this difficulty. Applying Lemma 4.3 to Ψ_s, one has, for sufficiently small $\varepsilon > 0$,

$$\begin{aligned} A_n(x,l) &\leqslant (1 + C_\rho(\varepsilon))\sqrt{2\pi n}\ \sigma_s e^{nh_s(l)} \mathbb{E}_{\mathbb{Q}_s^x}\left[\Phi_s(X_n^x)(\Psi_{s,\varepsilon}^+ * \rho_{\varepsilon^2})(T_n^x - nl)\right] \\ &:= B_n(x,l), \end{aligned} \tag{5.5}$$

where $\Psi_{s,\varepsilon}^+(y) = \sup_{y'\in\mathbb{B}_\varepsilon(y)} \Psi_s(y')$, $y \in \mathbb{R}$. Using the same arguments as for deducing (4.22), we have

$$B_n(x,l) = (1 + C_\rho(\varepsilon))\frac{\sigma_s}{\sqrt{2\pi}}\sqrt{n}\ e^{nh_s(l)} \int_{\mathbb{R}} e^{-itln} R_{s,it}^n \Phi_s(x) \widehat{\Psi}_{s,\varepsilon}^+(t)\widehat{\rho}_{\varepsilon^2}(t)dt, \tag{5.6}$$

where $R_{s,it}^n \Phi_s(x) = \mathbb{E}_{\mathbb{Q}_s^x}\left[e^{itT_n^x}\Phi_s(X_n^x)\right]$ and $\widehat{\Psi}_{s,\varepsilon}^+$ is the Fourier transform of $\Psi_{s,\varepsilon}^+$. Note that Φ_s is strictly positive and γ-Hölder continuous function on $\mathcal{S}$, and $\widehat{\Psi}_{s,\varepsilon}^+\widehat{\rho}_{\varepsilon^2}$ has a compact support in $\mathbb{R}$. Applying Proposition 4.4 with

$\varphi = \Phi_s$ and $\psi = \widehat{\Psi}^+_{s,\varepsilon}\widehat{\rho}_{\varepsilon^2}$ (one can verify that the functions φ and ψ satisfy the required conditions in Proposition 4.4), we obtain

$$\lim_{n\to\infty} \sup_{x\in\mathcal{S}} \sup_{|l|\leqslant l_n} B_n(x,l) = (1+C_\rho(\varepsilon))\pi_s(\Phi_s)\widehat{\Psi}^+_{s,\varepsilon}(0)\widehat{\rho}_{\varepsilon^2}(0).$$

Since $\widehat{\Psi}^+_{s,\varepsilon}(0) = \int_{\mathbb{R}} \sup_{y'\in\mathbb{B}_\varepsilon(y)} e^{-sy'}\psi(y')dy$ and $\widehat{\rho}_{\varepsilon^2}(0) = 1$, letting ε go to 0, using the condition (5.1) and the fact that $C_\rho(\varepsilon) \to 0$ as $\varepsilon \to 0$, we get the upper bound:

$$\limsup_{n\to\infty} \sup_{x\in\mathcal{S}} \sup_{|l|\leqslant l_n} A_n(x,l) \leqslant \pi_s(\Phi_s) \int_{\mathbb{R}} e^{-sy}\psi(y)dy. \tag{5.7}$$

Lower bound. Denote $\Psi^-_{s,\varepsilon}(y) = \inf_{y'\in\mathbb{B}_\varepsilon(y)} \Psi_s(y')$. From (5.4), using Lemma 4.3, we get

$$\begin{aligned} A_n(x,l) &\geqslant \sqrt{2\pi n}\ \sigma_s e^{nh_s(l)}\mathbb{E}_{\mathbb{Q}^x_s}\left[\Phi_s(X^x_n)(\Psi^-_{s,\varepsilon} * \rho_{\varepsilon^2})(T^x_n - nl)\right] \\ &\quad - \sqrt{2\pi n}\ \sigma_s e^{nh_s(l)} \int_{|y|\geqslant\varepsilon} \mathbb{E}_{\mathbb{Q}^x_s}\left[\Phi_s(X^x_n)\Psi^-_{s,\varepsilon}(T^x_n - nl - y)\right]\rho_{\varepsilon^2}(y)dy \\ &:= B^-_n(x,l) - C^-_n(x,l). \end{aligned} \tag{5.8}$$

For $B^-_n(x,l)$, we proceed as for (5.5) and (5.6), with $\Psi^+_{s,\varepsilon}$ replaced by $\Psi^-_{s,\varepsilon}$. Using Proposition 4.4, with $\varphi = \Phi_s$ and $\psi = \widehat{\Psi}^-_{s,\varepsilon}\widehat{\rho}_{\varepsilon^2}$, and the fact that $\widehat{\rho}_{\varepsilon^2}(0) = 1$ and $\widehat{\Psi}^-_{s,\varepsilon}(0) = \int_{\mathbb{R}} \inf_{y'\in\mathbb{B}_\varepsilon(y)} e^{-sy'}\psi(y')dy$, in an analogous way as in (5.7), we obtain that

$$\begin{aligned} &\lim_{n\to\infty} \sup_{x\in\mathcal{S}} \sup_{|l|\leqslant l_n} B^-_n(x,l) \\ &= \pi_s(r_s^{-1}) \int_{\mathbb{R}} \inf_{y\in\mathbb{B}_\varepsilon(z)} e^{-sy}\psi(y)dz \to \pi_s(r_s^{-1}) \int_{\mathbb{R}} e^{-sy}\psi(y)dy, \text{ as } \varepsilon\to 0, \end{aligned} \tag{5.9}$$

where the last convergence is due to the condition (5.1). For $C^-_n(x,l)$, noting that $\Psi^-_{s,\varepsilon} \leqslant \Psi_s$, applying Lemma 4.3 to Ψ_s we get $\Psi^-_{s,\varepsilon} \leqslant (1+C_\rho(\varepsilon))\widehat{\Psi}^+_{s,\varepsilon}\widehat{\rho}_{\varepsilon^2}$. Similarly to (5.6), we show that

$$\begin{aligned} C^-_n(x,l) &\leqslant (1+C_\rho(\varepsilon))\sqrt{\frac{n}{2\pi}}\ \sigma_s e^{nh_s(l)} \\ &\quad \times \int_{|y|\geqslant\varepsilon} \left(\int_{\mathbb{R}} e^{-it(ln+y)} R^n_{s,it}(\Phi_s)(x)\widehat{\Psi}^+_{s,\varepsilon}(t)\widehat{\rho}_{\varepsilon^2}(t)dt\right)\rho_{\varepsilon^2}(y)dy. \end{aligned}$$

From Lemma 4.1, for any fixed $y \in \mathbb{R}$, it holds that $e^{nh_s(l)-nh_s(l+\frac{y}{n})} \to 1$, uniformly in $|l| \leqslant l_n$ as $n \to \infty$. Applying Proposition 4.4 with $\varphi = \Phi_s$ and $\psi = \widehat{\Psi}^+_{s,\varepsilon}\widehat{\rho}_{\varepsilon^2}$, it follows, from the Lebesgue dominated convergence theorem,

that

$$\limsup_{n\to\infty}\sup_{x\in\mathcal{S}}\sup_{|l|\leqslant l_n} C_n^-(x,l)$$
$$\leqslant (1+C_\rho(\varepsilon))\pi_s(\Phi_s)\widehat{\Psi}^+_{s,\varepsilon}(0)\widehat{\rho}_{\varepsilon^2}(0)\int_{|y|\geqslant\varepsilon}\rho_{\varepsilon^2}(y)dy\to 0$$

as $\varepsilon\to 0$. Combining this with (5.8)-(5.9), we get the lower bound

$$\liminf_{n\to\infty}\sup_{x\in\mathcal{S}}\sup_{|l|\leqslant l_n} A_n(x,l)\geqslant \pi_s(\Phi_s)\int_{\mathbb{R}} e^{-sy}\psi(y)dy. \tag{5.10}$$

Putting together (5.7) and (5.10), and noting that $\pi_s(\Phi_s)=\pi_s(\varphi r_s^{-1})=\frac{\nu_s(\varphi)}{\nu_s(r_s)}$, the result follows. □

In the sequel, we deduce Theorem 2.2 from Theorem 5.1 using approximation techniques.

Proof of Theorem 2.2. Without loss of generality, we assume that $\varphi\geqslant 0$ and $\psi\geqslant 0$. Let $\Psi_s(y)=e^{-sy}\psi(y)$, $y\in\mathbb{R}$. We construct two step functions as follows: for any $\eta\in(0,1)$, $m\in\mathbb{Z}$ and $y\in[m\eta,(m+1)\eta)$, set

$$\Psi^+_{s,\eta}(y)=\sup_{y\in[m\eta,(m+1)\eta)}\Psi_s(y)\quad\text{and}\quad \Psi^-_{s,\eta}(y)=\inf_{y\in[m\eta,(m+1)\eta)}\Psi_s(y).$$

By the definition of the direct Riemann integrability, the following two limits exist and are equal:

$$\lim_{\eta\to 0^+}\int_{\mathbb{R}}\Psi^+_{s,\eta}(y)dy=\lim_{\eta\to 0^+}\int_{\mathbb{R}}\Psi^-_{s,\eta}(y)dy. \tag{5.11}$$

Since Ψ_s is directly Riemann integrable, we have $M:=\sup_{y\in\mathbb{R}}\Psi_s(y)<+\infty$. Let $\varepsilon\in(0,M\eta)$ be fixed. Denote $I_m=[(m-1)\eta,m\eta)$, $I_m^-=(m\eta-\frac{\varepsilon}{M4^{|m|}},m\eta)$, and $I_m^+=[m\eta,m\eta+\frac{\varepsilon}{M4^{|m|}})$, $m\in\mathbb{Z}$. Set $k_m^+:=M4^{|m|}\frac{\Psi^+_{s,\eta}(m\eta)-\Psi^+_{s,\eta}((m-1)\eta)}{\varepsilon}$, $m\in\mathbb{Z}$. For the step function $\Psi^+_{s,\eta}$, in the neighborhood of every possible discontinuous point $m\eta$, $m\in\mathbb{Z}$, if $\Psi^+_{s,\eta}(m\eta)\geqslant\Psi^+_{s,\eta}((m-1)\eta)$, then for any $y\in I_m\cup I_{m+1}$, $m\in\mathbb{Z}$, we define

$$\Psi^+_{s,\eta,\varepsilon}(y)=\begin{cases}\Psi^+_{s,\eta}((m-1)\eta), & y\in I_m\setminus I_m^-\\ \Psi^+_{s,\eta}((m-1)\eta)+k_m^+\left(y-m\eta+\frac{\varepsilon}{M4^{|m|}}\right), & y\in I_m^-\\ \Psi^+_{s,\eta}(m\eta), & y\in I_{m+1}.\end{cases}$$

If $\Psi^+_{s,\eta}(m\eta)<\Psi^+_{s,\eta}((m-1)\eta)$, then we define

$$\Psi^+_{s,\eta,\varepsilon}(y)=\begin{cases}\Psi^+_{s,\eta}((m-1)\eta), & y\in I_m\\ \Psi^+_{s,\eta}((m-1)\eta)+k_m^+(y-m\eta), & y\in I_m^+\\ \Psi^+_{s,\eta}(m\eta), & y\in I_{m+1}\setminus I_m^+.\end{cases}$$

From this construction, the non-negative continuous function $\Psi^+_{s,\eta,\varepsilon}$ satisfies $\Psi^+_{s,\eta} \leqslant \Psi^+_{s,\eta,\varepsilon}$ and $\int_{\mathbb{R}}[\Psi^+_{s,\eta,\varepsilon}(y) - \Psi^+_{s,\eta}(y)]dy < \varepsilon$. Similarly, for the step function $\Psi^-_{s,\eta}$, one can construct a non-negative continuous function $\Psi^-_{s,\eta,\varepsilon}$ which satisfies $\Psi^-_{s,\eta,\varepsilon} \leqslant \Psi^-_{s,\eta}$ and $\int_{\mathbb{R}}[\Psi^-_{s,\eta}(y) - \Psi^-_{s,\eta,\varepsilon}(y)]dy < \varepsilon$. Consequently, in view of (5.11), we obtain that, for η small enough,

$$\int_{\mathbb{R}} |\Psi^+_{s,\eta,\varepsilon}(y) - \Psi^-_{s,\eta,\varepsilon}(y)|dy < 3\varepsilon. \tag{5.12}$$

For brevity, set $c_{s,l,n} = \sqrt{2\pi n}\ \sigma_s e^{n\Lambda^*(q+l)}$ and $T^x_{n,l} = \log|G_n x| - n(q+l)$. Recalling that $\Psi_s(y) = e^{-sy}\psi(y)$, we write

$$\begin{aligned}
&\left|c_{s,l,n}\mathbb{E}\left[\varphi(X^x_n)\psi(T^x_{n,l})\right] - \bar{r}_s(x)\nu_s(\varphi)\int_{\mathbb{R}}\Psi_s(y)dy\right| \\
\leqslant\ &\left|c_{s,l,n}\mathbb{E}\left\{\varphi(X^x_n)e^{sT^x_{n,l}}\left[\Psi_s(T^x_{n,l}) - \Psi^+_{s,\eta,\varepsilon}(T^x_{n,l})\right]\right\}\right| \\
&+ \left|c_{s,l,n}\mathbb{E}\left[\varphi(X^x_n)e^{sT^x_{n,l}}\Psi^+_{s,\eta,\varepsilon}(T^x_{n,l})\right] - \bar{r}_s(x)\nu_s(\varphi)\int_{\mathbb{R}}\Psi^+_{s,\eta,\varepsilon}(y)dy\right| \\
&+ \left|r_s(x)\pi_s(\varphi r_s^{-1})\int_{\mathbb{R}}\Psi^+_{s,\eta,\varepsilon}(y)dy - \bar{r}_s(x)\nu_s(\varphi)\int_{\mathbb{R}}\Psi_s(y)dy\right| \\
=\ &J_1 + J_2 + J_3.
\end{aligned} \tag{5.13}$$

To control J_2, we shall verify the conditions of Theorem 5.1. Noting that the function $y \mapsto e^{sy}\Psi^+_{s,\eta,\varepsilon}(y)$ is non-negative and continuous, it remains to check the condition (5.3). By the construction of $\Psi^+_{s,\eta,\varepsilon}$ one can verify that there exists a constant $\varepsilon_1 \in (0, \min\{M\eta, \eta/3\})$ such that

$$\begin{aligned}
\int_{\mathbb{R}} \sup_{y'\in\mathbb{B}_{\varepsilon_1}(y)} \Psi^+_{s,\eta,\varepsilon}(y')dy &\leqslant 2\eta \sum_{m\in\mathbb{Z}} \sup_{y\in[m\eta,(m+1)\eta)} \Psi^+_{s,\eta}(y) \\
&= 2\eta \sum_{m\in\mathbb{Z}} \sup_{y\in[m\eta,(m+1)\eta)} \Psi_s(y) < +\infty,
\end{aligned} \tag{5.14}$$

where the series is finite since the function Ψ_s is directly Riemann integrable. Hence, applying Theorem 5.1 to $y \mapsto e^{sy}\Psi^+_{s,\eta,\varepsilon}(y)$, we get

$$\lim_{n\to\infty} \sup_{x\in\mathcal{S}} \sup_{|l|\leqslant l_n} J_2 = 0. \tag{5.15}$$

For $J_3(x)$, recall that $\Psi^-_{s,\eta,\varepsilon} \leqslant \Psi_s \leqslant \Psi^+_{s,\eta,\varepsilon}$. Using (5.12) and the fact that r_s is uniformly bounded on $\mathcal{S}$, we get that there exists a constant $C_s > 0$ such that

$$\sup_{x\in\mathcal{S}} J_3 \leqslant C_s\varepsilon. \tag{5.16}$$

For J_1, note that $e^{sy}\Psi^-_{s,\eta,\varepsilon}(y) \leqslant e^{sy}\Psi_s(y) \leqslant e^{sy}\Psi^+_{s,\eta,\varepsilon}(y)$, $y \in \mathbb{R}$. Combining this with the positivity of φ, it holds that

$$\begin{aligned}
|J_1| &\leqslant \left|c_{s,l,n}\mathbb{E}\left\{\varphi(X^x_n)e^{sT^x_{n,l}}\left[\Psi^+_{s,\eta,\varepsilon}(T^x_{n,l}) - \Psi^-_{s,\eta,\varepsilon}(T^x_{n,l})\right]\right\}\right| \\
&\leqslant \left|c_{s,l,n}\mathbb{E}\left[\varphi(X^x_n)e^{sT^x_{n,l}}\Psi^+_{s,\eta,\varepsilon}(T^x_{n,l})\right] - \bar{r}_s(x)\nu_s(\varphi)\int_{\mathbb{R}}\Psi^+_{s,\eta,\varepsilon}(y)dy\right| \\
&\quad + \left|c_{s,l,n}\mathbb{E}\left[\varphi(X^x_n)e^{sT^x_{n,l}}\Psi^-_{s,\eta,\varepsilon}(T^x_{n,l})\right] - \bar{r}_s(x)\nu_s(\varphi)\int_{\mathbb{R}}\Psi^-_{s,\eta,\varepsilon}(y)dy\right| \\
&\quad + \left|\bar{r}_s(x)\nu_s(\varphi)\int_{\mathbb{R}}\Psi^+_{s,\eta,\varepsilon}(y)dy - \bar{r}_s(x)\nu_s(\varphi)\int_{\mathbb{R}}\Psi^-_{s,\eta,\varepsilon}(y)dy\right| \\
&= J_{11} + J_{12} + J_{13}.
\end{aligned}$$

Using (5.15), it holds that, as $n \to \infty$, $J_{11} \to 0$, uniformly in $x \in \mathcal{S}$ and $|l| \leqslant l_n$. For J_{12}, note that the function $y \mapsto e^{sy}\Psi^-_{s,\eta,\varepsilon}(y)$ is non-negative and continuous. By the construction of $\Psi^-_{s,\eta,\varepsilon}$, similarly to (5.14), one can verify that there exists $\varepsilon_2 > 0$ such that $\int_{\mathbb{R}}\sup_{y'\in\mathbb{B}_{\varepsilon_2}(y)}\Psi^-_{s,\eta,\varepsilon}(y')dy < +\infty$. We deduce from Theorem 5.1 that $J_{12} \to 0$ as $n \to \infty$, uniformly in $x \in \mathcal{S}$ and $|l| \leqslant l_n$. For J_{13}, we use (5.12) to get that $J_{13} \leqslant C_s\varepsilon$. Consequently, we obtain that, as $n \to \infty$, $J_1 \leqslant C_s\varepsilon$, uniformly in $x \in \mathcal{S}$ and $|l| \leqslant l_n$. This, together with (5.13), (5.15)-(5.16), implies that

$$\lim_{n\to\infty}\sup_{x\in\mathcal{S}}\sup_{|l|\leqslant l_n}\left|c_{s,l,n}\mathbb{E}\left[\varphi(X^x_n)\psi(T^x_{n,l})\right] - \bar{r}_s(x)\nu_s(\varphi)\int_{\mathbb{R}}\Psi_s(y)dy\right| \leqslant C_s\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude the proof of Theorem 2.2. □

Proof of Theorem 2.4. Following the proof of Theorem 5.1, one can verify that the asymptotic (5.2) holds true for $s < 0$ small enough and for ψ satisfying condition (5.1). The passage to a directly Riemann integrable function ψ can be done by using the same approximation techniques as in the proof of Theorem 2.2. □

6. Proof of Theorems 2.5, 2.6 and 2.7

Proof of Theorems 2.5 and 2.6. We first give a proof of Theorem 2.5. Since $\log|G_nx| \leqslant \log\|G_n\|$ and the function $\bar{r}_s$ is strictly positive and uniformly bounded on $\mathcal{S}$, applying Theorem 2.1 we get the lower bound:

$$\liminf_{n\to\infty}\inf_{|l|\leqslant l_n}\frac{1}{n}\log\mathbb{P}(\log\|G_n\| \geqslant n(q+l)) \geqslant -\Lambda^*(q). \tag{6.1}$$

For the upper bound, since all matrix norms are equivalent, there exists a positive constant C which does not depend on the product G_n such that

$\log\|G_n\| \leqslant \max_{1\leqslant i\leqslant d}\log|G_n e_i| + C$, where $(e_i)_{1\leqslant i\leqslant d}$ is the canonical orthonormal basis in $\mathbb{R}^d$. From this inequality, we deduce that

$$\mathbb{P}(\log\|G_n\| \geqslant n(q+l)) \leqslant \sum_{i=1}^{d}\mathbb{P}\Big(\log|G_n e_i| \geqslant n\big(q+l-C/n\big)\Big).$$

Using Lemma 4.1, we see that there exists a constant $C_s > 0$ such that $e^{n[\Lambda^*(q+l-C/n)-\Lambda^*(q+l)]} \leqslant C_s$, uniformly in $|l| \leqslant l_n$ and $n \geqslant 1$. Again by Theorem 2.1, we obtain the upper bound:

$$\limsup_{n\to\infty}\sup_{|l|\leqslant l_n}\frac{1}{n}\log\mathbb{P}(\log\|G_n\| \geqslant n(q+l)) \leqslant -\Lambda^*(q).$$

This, together with (6.1), proves Theorem 2.5. Using Theorem 2.3, the proof of Theorem 2.6 can be carried out in the same way. □

Proof of Theorem 2.7. Without loss of generality, we assume that the function φ is non-negative. From Theorem 2.2, we deduce that there exists a sequence $(r_n)_{n\geqslant 1}$, determined by the matrix law μ such that $r_n \to 0$ as $n\to\infty$ and, uniformly in $x\in\mathcal{S}$, $|l|\leqslant l_n$ and $0\leqslant\Delta\leqslant o(n)$, it holds that

$$\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log|G_n x|\geqslant n(q+l)+a+\Delta\}}\Big] = \frac{\bar{r}_s(x)}{s\sigma_s\sqrt{2\pi n}}e^{-n\Lambda^*(q+l+\frac{a+\Delta}{n})}\Big[\nu_s(\varphi)+r_n\Big]. \tag{6.2}$$

Taking the difference of (6.2) with $\Delta=0$ and with $\Delta>0$, we get, as $n\to\infty$,

$$\mathbb{E}\Big[\varphi(X_n^x)\mathbb{1}_{\{\log|G_n x|\in n(q+l)+[a,a+\Delta)\}}\Big] = I_\Delta(n)\frac{\bar{r}_s(x)}{s\sigma_s\sqrt{2\pi n}}e^{-n\Lambda^*(q+l)}\Big[\nu_s(\varphi)+r_n\Big],$$

where

$$I_\Delta(n) := e^{n\Lambda^*(q+l)-n\Lambda^*(q+l+\frac{a}{n})} - e^{n\Lambda^*(q+l)-n\Lambda^*(q+l+\frac{a+\Delta}{n})}.$$

An elementary analysis using Lemma 4.1 shows that

$$I_\Delta(n) \sim e^{-sa}(1-e^{-s\Delta}),$$

uniformly in $|l|\leqslant l_n$ and $\Delta_n\leqslant\Delta\leqslant o(n)$, for any $(\Delta_n)_{n\geqslant 1}$ converging to 0 slowly enough ($\Delta_n^{-1} = o(r_n^{-1})$). This concludes the proof of Theorem 2.7. □

References

[1] Bahadur R. R., Rao R. R.: On deviations of the sample mean. *The Annals of Mathematical Statistics*, 31(4), 1015-1027, 1960.

[2] Benoist Y., Quint J. F.: Central limit theorem for linear groups. *The Annals of Probability*, 44(2), 1308-1340, 2016.

[3] Benoist Y., Quint J. F.: Random walks on reductive groups. *Springer International Publishing*, 2016.
[4] Borovkov A. A., Borovkov K. A.: Asymptotic analysis of random walks. *Cambridge University Press*, 2008.
[5] Bougerol P., Lacroix J.: Products of random matrices with applications to Schrödinger operators. Birkhäuser Boston, 1985.
[6] Breuillard E.: Distributions diophantiennes et théorème limite local sur $\mathbb{R}^d$. *Probability Theory and Related Fields*, 132(1): 13-38, 2005.
[7] Buraczewski D., Damek E., Guivarc'h Y., Mentemeier S.: On multidimensional Mandelbrot cascades. *Journal of Difference Equations and Applications*, 20(11), 1523-1567, 2014.
[8] Buraczewski D., Mentemeier S.: Precise large deviation results for products of random matrices. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*. Vol. 52, No. 3, 1474-1513, 2016.
[9] Buraczewski D., Collamore J., Damek E., Zienkiewicz J.: Large deviation estimates for exceedance times of perpetuity sequences and their dual processes. *The Annals of Probability* 44(6), 3688-3739, 2016.
[10] Daniels H. E.: Saddlepoint approximations in statistics. *The Annals of Mathematical Statistics*, 631-650, 1954.
[11] Dembo A., Zeitouni O.: Large deviations techniques and applications. *Springer Science and Business Media*, 2009.
[12] Fedoryuk M. V.: Asymptotic, Integrals and Series, Nauka, 1987 (in Russian).
[13] Furman A.: Random walks on groups and random transformations. *Handbook of dynamical systems*, 1, 931-1014, 2002.
[14] Furstenberg H., Noncommuting random products. *Transactions of the American Mathematical Society*, 108(3), 377-428, 1963.
[15] Furstenberg H., Kesten H.: Products of random matrices. *The Annals of Mathematical Statistics*, 31(2), 457-469, 1960.
[16] Gnedenko B. V.: On a local limit theorem of the theory of probability. *Uspekhi Matematicheskikh Nauk*, 3(3):187-194, 1948.
[17] Goldsheid I. Y., Guivarc'h Y.: Zariski closure and the dimension of the Gaussian law of the product of random matrices. *Probability Theory and Related Fields*, 105(1), 109-142, 1996.
[18] Grama I., Lauvergnat R., Le Page É.: Conditioned local limit theorems for random walks defined on finite Markov chains. *arXiv preprint* arXiv:1707.06129, 2017.
[19] Grama I., Le Page É.: Bounds in the local limit theorem for a random walk conditioned to stay positive. *In International Conference on Modern Problems of Stochastic Analysis and Statistics.* 103-127, Springer 2017.
[20] Guivarc'h Y.: Spectral gap properties and limit theorems for some random walks and dynamical systems. *Proc. Sympos. Pure Math.* 89, 279-310, 2015.
[21] Guivarc'h Y., Le Page É.: Spectral gap properties for linear random walks and Pareto's asymptotics for affine stochastic recursions. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*. Vol. 52. No. 2, 503-574, 2016.
[22] Guivarc'h Y., Raugi A.: Frontiere de Furstenberg, propriétés de contraction et théorèmes de convergence. *Probability Theory and Related Fields*, 69(2): 187-242, 1985.
[23] Guivarc'h Y., Urban R.: Semigroup actions on tori and stationary measures on projective spaces. *Studia Math.* 171, no. 1, 33-66, 2005.
[24] Hennion H.: Limit theorems for products of positive random matrices. *The Annals of Probability*, 25(4): 1545-1587, 1997.

[25] Hennion H., Hervé L.: Limit theorems for Markov chains and stochastic properties of dynamical systems by quasi-compactness. Vol. 1766, *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2001.
[26] Hennion H., Hervé L.: Central limit theorems for iterated random Lipschitz mappings. *The Annals of Probability*, 32: 1934-1984, 2004.
[27] Ibragimov I.A., Linnik Yu.V.: Independent and stationary sequences of random variables. Wolters, Noordhoff Pub., 1975.
[28] Kesten H.: Random difference equations and renewal theory for products of random matrices. *Acta Mathematica*, vol. 131(1): 207-248, 1973.
[29] Kingman J. F. C.: Subadditive ergodic theory. *The Annals of Probability*, 883-899, 1973.
[30] Le Page É.: Théorèmes limites pour les produits de matrices aléatoires. *In Probability measures on groups*. Springer Berlin Heidelberg, 258-303, 1982.
[31] Petrov V. V.: On the probabilities of large deviations for sums of independent random variables. *Theory of Probability and its Applications*, 10(2): 287-298, 1965.
[32] Petrov V. V.: Sums of independent random variables. *Springer*, 1975.
[33] Richter W.: Local limit theorems for large deviations. *Theory of Probability and its Applications*, 2(2): 206-220, 1957.
[34] Sheep L. A.: A local limit theorem. *The Annals of Mathematical Statistics*, 35: 419-423, 1964.
[35] Stone C.: A local limit theorem for nonlattice multi-dimensional distribution functions. *The Annals of Mathematical Statistics*, 36(2): 546-551, 1965.
[36] Xiao H., Grama I., Liu Q.: Berry-Esseen bound and precise moderate deviations for products of random matrices, *Submitted*, 2019.