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# *A non-partitionable Cohen–Macaulay simplicial complex*

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**Abstract.** A long-standing conjecture of Stanley states that every Cohen–Macaulay simplicial complex is partitionable. We disprove the conjecture by constructing an explicit counterexample. Due to a result of Herzog, Jahan and Yassemi, our construction also disproves the conjecture that the Stanley depth of a monomial ideal is always at least its depth.

**Résumé.** Une conjecture de longue date de Stanley affirme que chaque complexe simplicial Cohen–Macaulay est partitionnable. Nous réfutons la conjecture en construisant un contre-exemple explicite. En raison d’un résultat de Herzog, Jahan et Yassemi, nos construction réfute également la conjecture que la profondeur Stanley d’un idéal monôme est toujours supérieure ou égale à sa profondeur.

**Keywords.** simplicial complex, h-vector, Cohen–Macaulay, constructibility, partitionability, Stanley depth

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## 1 Introduction

Cohen–Macaulay simplicial complexes are ubiquitous in algebraic and topological combinatorics. They were introduced in 1975 by Stanley in his celebrated proof of the Upper Bound Conjecture for spheres [Sta75b]. (See [Sta14] for his engaging and personal account of how the proof came to be.) The theory of Cohen–Macaulay *rings* has long been of great importance in algebra and algebraic geometry; see, e.g., [Ree57, ZS60, Gro64, Hoc72, Hoc80, BH93]. The connection to combinatorics via what is known as Stanley–Reisner theory was established by Hochster [Hoc72], Reisner [Rei76], and Stanley [Sta75a]; standard references for this subject are [Sta96] and [BH93].

The focus of this article is the following conjecture, described by Stanley as “a central combinatorial conjecture on Cohen–Macaulay complexes” [Sta96, p. 85]. It was originally proposed by Stanley [Sta79, p. 149] in 1979 and independently by Garsia [Gar80, Remark 5.2] in 1980 for order complexes of Cohen–Macaulay posets.

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**Conjecture 1.1** (Partitionability Conjecture). *Every Cohen–Macaulay simplicial complex is partitionable.*

We explicitly construct a Cohen–Macaulay complex that is not partitionable, thus disproving the Partitionability Conjecture. In fact, we give a general method for constructing counterexamples and an explicit infinite family of non-partitionable Cohen–Macaulay complexes. We begin by giving some background for the conjecture, which will also be directly relevant in our construction.

Two basic invariants of a simplicial complex  $\Delta$  are its  $f$ - and  $h$ -vectors. The number  $f_i = f_i(\Delta)$  is the number of  $i$ -dimensional faces (simplices) in  $\Delta$ . The  $h$ -vector is more subtle. It carries the same information as the  $f$ -vector (the two are related by an invertible linear transformation), and arises naturally in algebra: the Hilbert series of the Stanley–Reisner ring of  $\Delta$  is  $(1-t)^{-d} \sum_j h_j(\Delta)t^j$ , where  $d = \dim \Delta$ . (See Section 2 for precise definitions.) It is not at all apparent if the numbers  $h_j(\Delta)$  have a combinatorial interpretation; for instance, they need not be positive in general.

A *partitioning* of a pure simplicial complex  $\Delta$  is a decomposition into pairwise-disjoint Boolean intervals whose maximal elements are exactly the facets (maximal faces) of  $\Delta$ . Partitionability was introduced by Provan [Pro77] and Ball [Bal77] in the context of reliability analysis. For a partitionable complex, the  $h$ -numbers enumerate the minimum elements of the intervals by size. In particular, shellable complexes are easily seen to be partitionable, and hence their  $h$ -vectors have this interpretation. The strict inclusions

$$\{\text{shellable complexes}\} \subsetneq \{\text{constructible complexes}\} \subsetneq \{\text{Cohen–Macaulay complexes}\}$$

are also well known. For example, the nonshellable balls constructed by Rudin [Rud58] and Ziegler [Zie98] are constructible, and any triangulation of the dunce hat is Cohen–Macaulay but not constructible [Hac08, §2]. On the other hand, the  $h$ -vectors of Cohen–Macaulay, constructible, and shellable complexes are all the same [Sta77, Theorem 6], suggesting that their entries ought to count something explicit. The Partitionability Conjecture would have provided a combinatorial interpretation of the  $h$ -vectors of Cohen–Macaulay complexes.

The idea of our construction is to work with *relative* simplicial complexes. Suppose  $Q = (X, A)$  is a relative simplicial complex that is not partitionable, but with  $X$  and  $A$  Cohen–Macaulay. Theorem 3.1 gives a general method of gluing together sufficiently many copies of  $X$  along  $A$  to obtain a counterexample to the Partitionability Conjecture, provided that  $A$  is an induced subcomplex of  $X$ . This reduces the problem to finding an appropriate pair  $(X, A)$ . Our starting point is the nonshellable simplicial 3-ball  $Z$  constructed by Ziegler [Zie98], in which we find a suitable subcomplex  $A$  and in turn the desired relative complex  $Q$  (Theorem 3.3). By refining the construction, we are able to obtain, in Theorem 3.5, a Cohen–Macaulay non-partitionable complex that is much smaller than predicted by Theorem 3.1, with  $f$ -vector  $(1, 16, 71, 98, 42)$  and  $h$ -vector  $(1, 12, 29)$ .

The existence of a Cohen–Macaulay nonpartitionable complex has an important consequence in commutative algebra. For a polynomial ring  $S = \mathbb{k}[x_1, \dots, x_n]$  and a  $\mathbb{Z}^n$ -graded  $S$ -module  $M$ , many fundamental algebraic invariants of  $M$ , such as its dimension and multigraded Hilbert series, can be profitably studied using combinatorics. On the other hand, the combinatorial properties of the *depth* of  $M$  are less well understood. In [Sta82], Stanley proposed a purely combinatorial analogue of depth, defined in terms of certain vector space decompositions of  $M$ . This invariant, now known as the *Stanley depth* and written  $\text{sdepth } M$ , has attracted considerable recent attention (see [PSFTY09] for an accessible introduction to the subject, and [Her13] for a comprehensive survey), centering around the following conjecture of Stanley [Sta82, Conjecture 5.1]:

**Conjecture 1.2** (Depth Conjecture). *For all  $\mathbb{Z}^n$ -graded  $S$ -modules  $M$ ,*

$$\text{sdepth } M \geq \text{depth } M.$$

Herzog, Jahan, and Yassemi proved [HJY08, Corollary 4.5] that when  $I$  is the Stanley–Reisner ideal of a Cohen–Macaulay complex  $\Delta$ , the inequality  $\text{sdepth } S/I \geq \text{depth } S/I$  is equivalent to the partitionability of  $\Delta$ . Therefore, our counterexample to the Partitionability Conjecture disproves the Depth Conjecture as well. We exhibit a smaller relative counterexample to the Depth Conjecture in Remark 3.6; see Section 3.2.

It was also previously not known whether all constructible complexes were partitionable; see, e.g., [Hac00, §4]. The counterexample we obtain is not only Cohen–Macaulay, but in fact constructible. Therefore, even constructibility does not imply partitionability. The counterexample also disproves a conjecture due to Kalai relating partitions and algebraic shifting [Kal02, Conjecture 22]. We discuss these ramifications and related open questions in the final section of the paper.

## 2 Preliminaries

### 2.1 Simplicial and relative simplicial complexes

Throughout the paper, all complexes will be finite. Let  $V$  be a finite set. A *simplicial complex* on  $V$  is a collection  $\Delta$  of subsets of  $V$  such that whenever  $\sigma \in \Delta$  and  $\tau \subseteq \sigma$ , then  $\tau \in \Delta$ . Equivalently,  $\Delta$  is an order ideal in the Boolean poset  $2^V$ . The symbol  $|\Delta|$  denotes the standard geometric realization of  $\Delta$ . The elements of  $\Delta$  are called the *faces* of  $\Delta$ , and the elements of  $V$  are *vertices*. Maximal faces are called *facets*. The *dimension* of a face  $\sigma$  is  $\dim \sigma = |\sigma| - 1$ , and the dimension of  $\Delta$  is  $\dim \Delta = \max\{\dim \sigma \mid \sigma \in \Delta\}$ . We often write  $\Delta^d$  to indicate that  $\dim \Delta = d$ . A complex is *pure* if all maximal faces have the same dimension. A *subcomplex* of  $\Delta$  is a simplicial complex  $\Gamma$  with  $\Gamma \subseteq \Delta$ . A subcomplex is an *induced subcomplex* if it is of the form  $\Delta|_W := \{\sigma \in \Delta \mid \sigma \subseteq W\}$  for some  $W \subseteq V$ .

In the construction of our counterexample, we will work with the more general class of *relative simplicial complexes*. A relative complex  $\Phi$  on  $V$  is a subset of  $2^V$  that is *convex*: if  $\rho, \tau \in \Phi$  and  $\rho \subseteq \sigma \subseteq \tau$ , then  $\sigma \in \Phi$ . We sometimes refer to simplicial complexes as “absolute” to distinguish them from relative complexes.

Every relative complex can be expressed as a pair  $\Phi = (\Delta, \Gamma) := \Delta \setminus \Gamma$ , where  $\Delta$  is a simplicial complex and  $\Gamma \subseteq \Delta$  is a subcomplex. Topologically,  $\Phi$  corresponds to the quotient space  $|\Delta|/|\Gamma|$ . Note that there are infinitely many possibilities for the pair  $\Delta, \Gamma$ . The unique minimal expression is obtained by letting  $\Delta = \bar{\Phi}$  be the *combinatorial closure* of  $\Phi$ , i.e., the smallest simplicial complex containing  $\Phi$  as a subset, and setting  $\Gamma = \Delta \setminus \Phi$ . Note that in this case  $\dim \Gamma < \dim \Delta$ , because the maximal faces of  $\Delta$  are precisely those of  $\Phi$ .

The notation  $\tilde{H}_i(\Delta)$  denotes the  $i^{\text{th}}$  reduced simplicial homology group with coefficients in  $\mathbb{Z}$ . (The underlying ring does not matter for our purposes.) The simplicial homology groups  $\tilde{H}_i(\Phi)$  of a relative complex  $\Phi = (\Delta, \Gamma)$  are just the relative homology groups  $\tilde{H}_i(\Delta, \Gamma)$  in the usual topological sense (see, e.g., [Hat02]); in particular, the homology groups of  $\Delta$ ,  $\Gamma$ , and  $\Phi$  fit into a long exact sequence.

The *f-vector* of an (absolute or relative) complex  $\Delta^d$  is  $f(\Delta) = (f_{-1}, f_0, \dots, f_d)$ , where  $f_i = f_i(\Delta)$  is the number of  $i$ -dimensional faces of  $\Delta$ . Note that  $f_{-1}(\Delta) = 1$  for every absolute complex other than

the void complex  $\Delta = \emptyset$ . The  $h$ -vector  $h(\Delta) = (h_0, h_1, \dots, h_{d+1})$  is defined by

$$h_k = \sum_{i=0}^k (-1)^{k-i} \binom{d+1-i}{k-i} f_{i-1}, \quad 0 \leq k \leq d+1.$$

In particular, the  $f$ - and  $h$ -vectors determine each other.

The *link* of a face  $\sigma \in \Delta$  is defined as  $\text{link}_\Delta(\sigma) := \{\tau \in \Delta \mid \tau \cap \sigma = \emptyset, \tau \cup \sigma \in \Delta\}$ . Observe that if  $\Delta^d$  is pure and  $\dim \sigma = k$ , then  $\dim \text{link}_\Delta(\sigma) = d - k - 1$ . If  $\sigma$  is a facet of  $\Delta$  then  $\text{link}_\Delta(\sigma) = \{\emptyset\}$ , the trivial complex with only the empty face, and if  $\sigma \notin \Delta$  then we set  $\text{link}_\Delta(\sigma)$  to be the void complex with no faces. If  $\Phi = (\Delta, \Gamma)$  is a relative complex and  $\sigma \in \Delta$ , we can define the relative link by  $\text{link}_\Phi(\sigma) = (\text{link}_\Delta(\sigma), \text{link}_\Gamma(\sigma))$ . It is easy to check that this construction is intrinsic to  $\Phi$ , i.e., it does not depend on the choice of the pair  $\Delta, \Gamma$ . Note that  $\text{link}_\Phi(\sigma)$  is not necessarily a subset of  $\Phi$ .

## 2.2 Cohen–Macaulay simplicial complexes

A ring is *Cohen–Macaulay* if its depth equals its (Krull) dimension. Reisner’s criterion [Rei76, Theorem 1] states that Cohen–Macaulayness of the Stanley–Reisner ring [Sta96, §II.1] of a simplicial complex can be expressed in terms of simplicial homology, and we will take this criterion as our definition. The relative version of Reisner’s criterion is Theorem 5.3 of [Sta87].

**Theorem 2.1.** [Rei76, Sta87] *A simplicial complex  $\Delta$  is Cohen–Macaulay if for every face  $\sigma \in \Delta$ ,*

$$\tilde{H}_i(\text{link}_\Delta(\sigma)) = 0 \quad \text{for } i < \dim \text{link}_\Delta(\sigma).$$

*Similarly, a relative complex  $\Phi = (\Delta, \Gamma)$  is Cohen–Macaulay if for every  $\sigma \in \Delta$ ,*

$$\tilde{H}_i(\text{link}_\Delta(\sigma), \text{link}_\Gamma(\sigma)) = 0 \quad \text{for } i < \dim \text{link}_\Delta(\sigma).$$

In fact, Cohen–Macaulayness is a *topological* invariant: it depends only on the homeomorphism type of the geometric realization  $|\Delta|$ . This was proved by Munkres [Mun84]. Topological invariance holds for relative complexes as well [Sta96, Corollary III.7.3]. Importantly, if  $|\Delta|$  is homeomorphic to a ball or to a sphere, then  $\Delta$  is Cohen–Macaulay [Mun84, §2]. The following technical fact will be central to our construction; the proof has been omitted in this extended abstract.

**Proposition 2.2.** *Let  $\Delta_1, \dots, \Delta_n$  be  $d$ -dimensional Cohen–Macaulay simplicial complexes on disjoint vertex sets. Let  $\Gamma$  be a Cohen–Macaulay simplicial complex of dimension  $d - 1$  or  $d$ , and suppose that each  $\Delta_i$  contains a copy of  $\Gamma$  as an induced subcomplex. Then the complex  $\Omega$  obtained from  $\Delta_1, \dots, \Delta_n$  by identifying the  $n$  copies of  $\Gamma$  is Cohen–Macaulay.*

## 2.3 Shellability, partitionability, and constructibility

**Definition 2.3.** Let  $\Delta$  be a pure simplicial complex. A *shelling* of  $\Delta$  is a total ordering  $F_1, \dots, F_n$  of its facets so that for every  $j$ , the set  $\{\sigma \subseteq F_j \mid \sigma \not\subseteq F_i \text{ for all } i < j\}$  has a unique minimal element  $R_j$ .

The  $h$ -vector of a shellable complex has a simple combinatorial interpretation:

$$h_k(\Delta) = \#\{j \mid \#R_j = k\}. \quad (1)$$

In particular  $h_k(\Delta) \geq 0$  for all  $k$ , and in fact  $h_k(\Delta) = 0$  implies  $h_\ell(\Delta) = 0$  for all  $\ell > k$  (a consequence of [BH93, Theorem 5.1.15]). Shellable complexes are Cohen–Macaulay, although the converse is not

true: well-known counterexamples include any triangulation of the dunce hat, as well as the nonshellable balls constructed by Rudin [Rud58] and Ziegler [Zie98]. On the other hand, Cohen–Macaulay complexes satisfy the same conditions on the  $h$ -vector, so it is natural to look for a combinatorial interpretation of their  $h$ -vectors.

**Definition 2.4.** Let  $\Delta$  be a pure simplicial complex with facets  $F_1, \dots, F_n$ . A *partitioning*  $\mathcal{P}$  of  $\Delta$  is a decomposition into pairwise-disjoint Boolean intervals

$$\Delta = \bigsqcup_{i=1}^n [R_i, F_i]$$

where  $[R_i, F_i] = \{\sigma \in \Delta \mid R_i \subseteq \sigma \subseteq F_i\}$ . We say that each  $F_i$  is *matched* to the corresponding  $R_i$ .

Clearly, shellable complexes are partitionable. If  $\Delta$  is partitionable, then its  $h$ -vector automatically carries the combinatorial interpretation (1) [Sta96, Proposition III.2.3]. Moreover, Definitions 2.3 and 2.4, and the interpretation of the  $h$ -vector, carry over precisely from absolute to relative complexes. A partitionable complex need not be Cohen–Macaulay, much less shellable; see the example due to Björner in [Sta96, p. 85].

Constructibility, introduced by Hochster [Hoc72], is a combinatorial condition intermediate between shellability and Cohen–Macaulayness.

**Definition 2.5.** A complex  $\Delta^d$  is *constructible* if it is a simplex, or if it can be written as  $\Delta = \Delta_1 \cup \Delta_2$ , where  $\Delta_1$ ,  $\Delta_2$ , and  $\Delta_1 \cap \Delta_2$  are constructible of dimensions  $d$ ,  $d$ , and  $d - 1$  respectively.

Hachimori [Hac00] investigated the question of whether constructibility implies partitionability. Our counterexample to the Partitionability Conjecture is in fact constructible, resolving this question as well.

### 3 The counterexample

We first give a general construction that reduces the problem of finding a counterexample to the problem of constructing a certain kind of non-partitionable Cohen–Macaulay relative complex.

**Theorem 3.1.** *Let  $Q = (X, A)$  be a relative complex such that*

- (i)  *$X$  and  $A$  are Cohen–Macaulay;*
- (ii)  *$A$  is an induced subcomplex of  $X$  of codimension at most 1; and*
- (iii)  *$Q$  is not partitionable.*

*Let  $k$  be the total number of faces of  $A$ , let  $N > k$ , and let  $C = C_N$  be the simplicial complex constructed from  $N$  disjoint copies of  $X$  identified along the subcomplex  $A$ . Then  $C$  is Cohen–Macaulay and not partitionable.*

*Proof.* First,  $C$  is Cohen–Macaulay by Proposition 2.2. Second, suppose that  $C$  has a partitioning  $\mathcal{P}$ . Let  $X_1, X_2, \dots, X_N$  be the  $N$  copies of  $X$ . By the pigeonhole principle, since  $N > k$ , there is some copy of  $X$ , say  $X_N$ , none of whose facets is matched to a face in  $A$ . Let  $[R_1, F_1], \dots, [R_\ell, F_\ell]$  be the intervals in  $\mathcal{P}$  for which  $F_i \in X_N$ ; then

$$\bigcup_{i=1}^{\ell} [R_i, F_i] \subseteq X_N \setminus A. \tag{2}$$

No other interval in  $\mathcal{P}$  can intersect  $X_N \setminus A$  nontrivially, so in fact equality must hold in (2). But then (2) is in fact a partitioning of  $X_N \setminus A = Q$ , which was assumed to be non-partitionable.  $\square$

**Remark 3.2.** It is easy to see that a subcomplex  $A \subset X$  is an induced subcomplex if and only if every minimal face of  $X \setminus A$  has dimension 0. Therefore, this condition may be viewed as a restriction on the relative complex  $(X, A)$ .

### 3.1 The construction

Throughout, we abbreviate the simplex on vertices  $\{v_1, \dots, v_k\}$  by  $v_1 \cdots v_k$ . Our construction begins with Ziegler's nonshellable 3-ball  $Z$ , which is a nonshellable triangulation of the 3-ball with 10 vertices labeled  $0, 1, \dots, 9$ , and the following 21 facets [Zie98, §4]:

$$\begin{array}{cccccccc} 0123, & 0125, & 0237, & 0256, & 0267, & 1234, & 1249, \\ 1256, & 1269, & 1347, & 1457, & 1458, & 1489, & 1569, \\ 1589, & 2348, & 2367, & 2368, & 3478, & 3678, & 4578. \end{array}$$

The complex  $Z$  is not shellable, but it is partitionable, Cohen–Macaulay and, in fact, constructible [Hac01].

Let  $B$  be the induced subcomplex  $Z|_{\{0,2,3,4,6,7,8\}}$ . That is,  $B$  is the pure 3-dimensional complex with facets

$$0237, \quad 0267, \quad 2367, \quad 2368, \quad 2348, \quad 3678, \quad 3478.$$

The given order is a shelling of  $B$ ; in particular  $B$  is Cohen–Macaulay. Define  $Q$  to be the relative complex  $Q = (Z, B)$ . Then  $Q$  is also Cohen–Macaulay by [Duv96, Corollary 3.2].

The facets of  $Q$  are

$$\begin{array}{cccccccc} 1249, & 1269, & 1569, & 1589, & 1489, & 1458, & 1457, \\ 4578, & 1256, & 0125, & 0256, & 0123, & 1234, & 1347. \end{array} \tag{3}$$

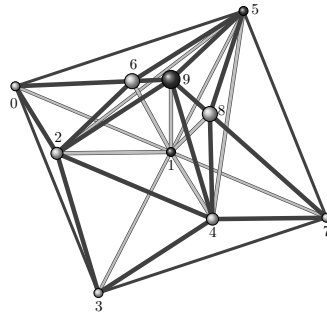
The minimal faces of  $Q$  are just the vertices  $1, 5, 9$ . We can picture  $Q$  easily by considering its combinatorial closure  $\bar{Q}$ , that is, the 3-dimensional simplicial complex generated by the facets (3). In fact  $\bar{Q}$  is a shellable ball; the ordering of facets given in (3) is a shelling. The complement  $A = \bar{Q} \setminus Q = \bar{Q}|_{\{0,2,3,4,6,7,8\}}$  is the shellable 2-ball with facets

$$026, \quad 023, \quad 234, \quad 347, \quad 478. \tag{4}$$

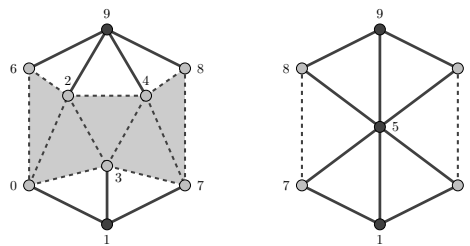
Thus  $Q = (\bar{Q}, A)$ . The  $f$ - and  $h$ -vectors of these complexes are

$$\begin{array}{ll} f(\bar{Q}) = (1, 10, 31, 36, 14), & h(\bar{Q}) = (1, 6, 7, 0, 0), \\ f(A) = (1, 7, 11, 5, 0), & h(A) = (1, 4, 0, 0, 0), \\ f(Q) = (0, 3, 20, 31, 14), & h(Q) = (0, 3, 11, 0, 0). \end{array}$$

The 1-skeleton of  $\bar{Q}$  is shown in Figure 1. The triangles on the boundary of  $Q$ , i.e., those contained in exactly one facet, are illustrated in Figure 2, which shows the boundary of  $Q$  as seen from the front (left) and back (right). The five shaded triangles are the facets of  $A$ , and hence are missing from  $Q$ . Note that the triple transposition  $\tau = (0\ 7)(2\ 4)(6\ 8)$  is a simplicial automorphism of  $\bar{Q}$ ; this symmetry is apparent as a reflection through the plane containing vertices  $1, 3, 5$ , and  $9$  in Figure 1, and as a vertical reflection in each part of Figure 2.



**Fig. 1:** A perspective view of the 1-skeleton of  $\bar{Q}$  from the top. Dark edges are exterior edges visible from the top; light edges are interior edges, or exterior edges at the bottom. Light vertices are in  $A$ .



**Fig. 2:** Left: A front view of  $Q$ . Right: A back view of  $Q$ . The shaded and dashed faces are in  $A$ .

**Theorem 3.3.** *The relative complex  $Q$  is not partitionable.*

The proofs of Theorem 3.3 and Theorem 3.5 have been omitted here; see [DGKMar] for the full article.

We can now give an explicit description of our counterexample to the Partitionability Conjecture. Since  $X = \bar{Q}$  and  $A$  are both shellable balls, they are Cohen–Macaulay. We may therefore apply Theorem 3.1, with  $N = 25$  (since  $A$  has 24 faces total).

**Theorem 3.4.** *Let  $X = \bar{Q}$  be the combinatorial closure of  $Q$ , and let  $A = X \setminus Q$ . That is,  $X$  and  $A$  are the absolute simplicial complexes whose facets are listed in (3) and (4), respectively. Then the simplicial complex  $C_{25}$  constructed in Theorem 3.1 is Cohen–Macaulay and non-partitionable.*

The  $f$ -vector is  $f(C_{25}) = f(A) + 25f(Q) = (1, 82, 511, 780, 350)$ .

For this particular construction, the full power of Theorem 3.1 is not necessary; there is a much smaller counterexample.

**Theorem 3.5.** *Let  $Q$ ,  $A$ , and  $X = \bar{Q}$  be as described above. Then the simplicial complex  $C_3$  obtained by gluing together three copies of  $X$  along  $A$  is Cohen–Macaulay and non-partitionable.*

The  $f$ -vector is  $f(C_3) = f(A) + 3f(Q) = (1, 16, 71, 98, 42)$ . We do not know if there exists a smaller counterexample (for example, the complex  $C_2$  obtained by gluing two copies of  $X$  together along  $A$  is



partitionable). In particular, it is still open whether every *two-dimensional* Cohen–Macaulay simplicial complex is partitionable; see Hachimori [Hac08].

We have previously observed that  $X$  and  $A$  are shellable. We note that  $X$  and  $A$  are contractible, and it is easily seen that  $X$  deformation-retracts onto  $A$ , so  $C_3$  is contractible as well, although it is not homeomorphic to a ball.

**Remark 3.6.** There is a much smaller *relative* simplicial complex that is Cohen–Macaulay but not partitionable, with  $f$ -vector  $(0, 0, 5, 10, 5)$ . This complex can be written as  $Q' = (X', A')$ , where  $X' = \overline{Q'} = Z|_{\{1,4,5,7,8,9\}}$  is the complex with facets  $\{1589, 1489, 1458, 1457, 4578\}$ , and  $A'$  is the (non-induced) subcomplex of  $X'$  with facets  $\{489, 589, 578, 157\}$ . These complexes are shellable balls of dimensions 3 and 2 respectively (the given orders of facets are shelling orders), and  $A'$  is contained in the boundary of  $X'$  (note that each facet in  $A'$  is contained in only one facet of  $X'$ ), so  $Q'$  is Cohen–Macaulay by [Sta87, Corollary 5.4]. On the other hand, one can check directly that there is no partitioning of  $Q'$ . Because  $A'$  is not an induced subcomplex, it is not possible to obtain a counterexample to the Partitionability Conjecture by applying Theorem 3.1.

**Remark 3.7.** It is easily seen that  $C_3$  is constructible. Therefore, it is also a counterexample to the conjecture that every constructible simplicial complex is partitionable [Hac00, §4]. Furthermore, since all constructible complexes are Cohen–Macaulay [BH93, p. 219], the constructibility and non-partitionability of  $C_3$  are sufficient to disprove the Partitionability Conjecture.

### 3.2 Stanley depth

Let  $\mathbb{k}$  be a field and  $S = \mathbb{k}[x_1, \dots, x_n]$ , and let  $M$  be a  $\mathbb{Z}^n$ -graded  $S$ -module. A *Stanley decomposition*  $\mathcal{D}$  of  $M$  is a vector space decomposition

$$M = \bigoplus_{i=1}^r \mathbb{k}[X_i] \cdot m_i$$

where each  $X_i$  is a subset of  $\{x_1, \dots, x_n\}$  and each  $m_i$  is a homogeneous element of  $M$ . The *Stanley depth* of  $M$  is defined as

$$\text{sdepth } M = \max_{\mathcal{D}} \{ \min(|X_1|, \dots, |X_r|) \},$$

where  $\mathcal{D}$  ranges over all Stanley decompositions of  $M$ . If  $\Phi$  is an (absolute or relative) simplicial complex, then we define its Stanley depth to be the Stanley depth of its associated Stanley–Reisner ring or module. This invariant has received substantial recent attention [PSFTY09, Her13], centering on the Depth Conjecture of Stanley [Sta82, Conjecture 5.1], which we now restate.

**Conjecture 1.2** (Depth Conjecture). *For all  $\mathbb{Z}^n$ -graded  $S$ -modules  $M$ ,*

$$\text{sdepth } M \geq \text{depth } M.$$

Herzog, Jahan and Yassemi [HJY08, Corollary 4.5] proved that if  $\Delta$  is a Cohen–Macaulay simplicial complex whose Stanley–Reisner ring [Sta96, §II.1] is  $\mathbb{k}[\Delta] := S/I_\Delta$  (so that  $\text{depth } \mathbb{k}[\Delta] = \dim \mathbb{k}[\Delta] = \dim \Delta + 1$ ), then Conjecture 1.2 holds for  $\mathbb{k}[\Delta]$  if and only if  $\Delta$  is partitionable. Therefore, our construction provides a counterexample to the Depth Conjecture. Katthän has conjectured that the inequality  $\text{sdepth } S/I \geq \text{depth } S/I - 1$  holds for every monomial ideal  $I$ ; for a detailed exposition and the evidence for this conjecture, see [Kat16].

A smaller counterexample to Conjecture 1.2 is provided by the relative complex  $Q'$  in Remark 3.6. The depth of each of  $C_3$  and  $Q'$  is easily seen to be 4, but the Stanley depth of each of  $C_3$  and  $Q'$  is 3. The Stanley depth computations were made by Katthän [Kat], using the algorithm developed by Ichim and Zarojanu [IZ14].

## 4 Open questions

Now that we know that Cohen–Macaulayness and even constructibility are not sufficient to guarantee partitionability, it is natural to ask what other conditions do suffice. Hachimori defined a related but more restricted class of *strongly constructible* complexes and showed that they are always partitionable [Hac00, Corollary 4.7]. Here are two additional possibilities, inspired by what our counterexample  $C_3$  is *not*. First,  $C_3$  is not homeomorphic to a ball, because the triangles in  $A$  are each contained in three facets. On the other hand, balls are Cohen–Macaulay, motivating the following question:

**Question 4.1.** *Is every simplicial ball partitionable?*

This conjecture is true if we further assume the ball is convexly realizable, by [Sta96, Proposition III.2.8]; see also [KS91]. On the other hand, there exist non-convex simplicial balls in dimensions as small as 3; see, e.g., [Lut04, Lut08].

Garsia [Gar80, Remark 5.2] proposed the Partitionability Conjecture for the special class of order complexes of *Cohen–Macaulay posets* (see also [Bac76, Bac80, BGS82]), which give rise to balanced Cohen–Macaulay simplicial complexes. Recall that a  $d$ -dimensional simplicial complex is *balanced* if its vertices can be colored with  $d + 1$  colors so that every facet has one vertex of each color. For instance, if  $P$  is a ranked poset, then its order complex is easily seen to be balanced by associating colors with ranks. The complex  $\bar{Q}$  with facets listed in (3) is not balanced (because its 1-skeleton is not 4-colorable), hence neither is  $C_3$  or  $C_{25}$ , nor indeed  $C_N$  for any  $N$ .

**Question 4.2.** *Is every balanced Cohen–Macaulay simplicial complex partitionable?*

Although Cohen–Macaulay complexes are not necessarily partitionable, their  $h$ -vectors are still nice; they are always non-negative and in fact coincide with the  $h$ -vectors of shellable complexes. Without the Partitionability Conjecture, the question remains:

**Question 4.3.** *What does the  $h$ -vector of a Cohen–Macaulay simplicial complex count?*

One answer is given by [DZ01], where it is shown that every simplicial complex can be decomposed into *Boolean trees* indexed by iterated Betti numbers; see [DZ01, Corollary 3.5]. The starting point of that paper is a conjecture of Kalai’s [Kal02, Conjecture 22] that any simplicial complex can be partitioned into intervals in a way related to algebraic shifting. Kalai’s conjecture would have implied that simplicial complexes could be decomposed into Boolean intervals. Such a decomposition into intervals, however, would have implied the Partitionability Conjecture. Hence our result provides a counterexample to Kalai’s conjecture. Moreover, the decomposition in [DZ01] may be best possible at this level of generality.

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