From heterogeneous microscopic traffic flow models to macroscopic models

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Abstract

The goal of this paper is to derive rigorously macroscopic traffic flow models from microscopic models. More precisely, for the microscopic models, we consider follow-the-leader type models with different types of drivers and vehicles which are distributed randomly on the road. After a rescaling, we show that the cumulative distribution function converge to the solution of a macroscopic model. We also make the link between this macroscopic model and the so-called LWR model.

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1 Introduction

The modelling and the simulation of traffic flow is a challenging task in particular in order to design infrastructure. In particular, it can allow us to understand how the traffic will react to a change in the infrastructure of the road (if it is interesting to place a traffic light, if a bridge would help the traffic flow, how would a moderator affect the traffic...). Indeed, there are some examples in which the construction of a new infrastructure did not improve the traffic. For example, in Stuttgart, Germany, after investments into the road network in 1969, the traffic situation did not improve until a section of newly built road was closed for traffic again (see [11]). This is known as the Braess' paradox. During the last years, a lot of work has been done concerning the modelling of traffic flows problems.

The goal of this paper is to obtain rigorously macroscopic trafic flow models by rescaling microscopic models. From a modelling point of view, microscopic models describe the dynamics of each vehicles individually. The most famous microscopic models are those of follow-the-leader type, also known as car-following models. In such a model, the dynamics of each vehicle depends on the vehicles in front, so that, in a cascade, the whole traffic flow can be determined by the dynamics of the very first vehicle (the leader). Two very popular models are the Bando model [4] and the Newell model [14], which describe respectively the acceleration and the velocity of each vehicle as a function (called optimal velocity function) of the inter-distance with the vehicle in

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front of it. The main advantage of these methods is that one can easily distinguish each vehicle and then associate different attributes (like maximal velocity, maximal acceleration...) to each vehicle. It is also possible to describe microscopic phenomena like red lights, slowdown or change of the maximal velocity. The main drawback is for numerical simulations where we have to treat a large number of data, which can be very expensive for example if we want to simulate the traffic at the scale of a town.

On the contrary, macroscopic models consist in describing the collective behaviour of the particles for example by giving an evolution law on the density of vehicles. Concerning the modelling of the traffic flow, the oldest macroscopic model is the LWR model (Lighthill, Whitham [13], Richards [15], see also the book [9] for a good introduction to this models), which dates back to 1955 and is inspired by the laws of fluid dynamics. More recently, some macroscopic models propose to describe the flow of vehicles in terms of the averaged spacing between the vehicles (in some sense, the inverse of the density, see the works of Leclercq, Laval and Chevallier [12]). The main advantage of these macroscopic models is that it is possible to make numerical simulations on large portion of road. On the other hand, it is more complicated to describe microscopic phenomena or attributes. Generally speaking, microscopic models are considered more justifiable because the behaviour of every single particle can be described with high precision and it is immediately clear which kind of interactions are considered. On the contrary, macroscopic models are based on assumptions that are hardly correct or at least verifiable. As a consequence, it is often desirable establishing a connection between microscopic and macroscopic models so to justify and validate the latter on the basis of the verifiable modelling assumptions of the former.

Connections between microscopic and macroscopic fluid-dynamics traffic flow models are already well understood in certain cases of vehicles moving on a single road. The goal of this paper is to generalise these results to the case of different types of drivers which are distributed randomly on the road. The existing results mainly consider the case of one type of vehicles (see for example [6] or [10] for the case of non-local intercations) or several types of vehicles/drivers but distributed in a periodic way on the road (see [8]). To the best of our knowledge, the only result in the random case is a formal derivation obtain in the paper of Chiabaut, Leclercq and Buisson [5].

1.1 Description of the model and assumptions

In this paper, we consider a stochastic first order follow-the-leader model. More precisely, we assume that each car has a type z and that the optimal velocity $V_z(p)$ of a car, which expresses the velocity of the car in function of the distance p of this car to the car in front of it, depends on this type z. The set of types is a compact metric space $(\mathcal{Z}, d_{\mathcal{Z}})$. A typical (and realistic) example is when $\mathcal{Z}$ is a finite set. We suppose that the cars are labelled by an index i, with $i \in \mathbb{Z}$, the position of the cars being an increasing function of i. Our main structure assumption is that the type of car i is a random variable Z_i with values in $\mathcal{Z}$ and that the (Z_i) are i.i.d. random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The types (Z_i) do not evolve in time and are fixed throughout the paper. *Without loss of generality we also suppose throughout the paper that the law of Z_0 (and therefore of all the Z_i) has full support,* since we can always restrict the compact metric space $(\mathcal{Z}, d_{\mathcal{Z}})$ to the support of the law of Z_0.

On the optimal velocity map $V : \mathcal{Z} \times \mathbb{R}_+ \to \mathbb{R}$, we assume the following:

(H_1) The map $(z, p) \to V_z(p)$ is uniformly continuous on $\mathcal{Z} \times \mathbb{R}_+$ and $p \to V_z(p)$ is Lipschitz continuous, uniformly with respect to $z \in \mathcal{Z}$;

(H_2) For any $z \in \mathcal{Z}$, there exists $h_0^z > 0$ (depending in a measurable way on z) such that

$V_z(p) = 0$ for all $p \in [0, h_0^z]$;

(H_3) For any $z \in \mathcal{Z}$, $p \to V_z(p)$ is increasing in $[h_0^z, +\infty)$;

(H_4) There exists $V_{\max} > 0$ and, for any $z \in \mathcal{Z}$, there exists $V_{\max}^z \leqslant V_{\max}$, such that $\lim_{p \to +\infty} V_z(p) = V_{\max}^z$.

Under assumptions (H1)—(H4), the map $V_z : [h_0^z, +\infty) \to [0, V_{\max}^z)$ is increasing and continuous for any $z \in \mathcal{Z}$ and we denote by V_z^{-1} its inverse. For simplicity of notation, we set $\bar{h}_0 := \mathbb{E}[h_0^{Z_0}]$ and by $\underline{V_{\max}} := \inf_{z \in \mathcal{Z}} V_{\max}^z$. We need a last assumption, on the statistical distribution of $V_{\max}^{Z_0}$:

(H_5) $\lim_{\theta \to \underline{V_{\max}}^-} \mathbb{E}\left[V_{Z_0}^{-1}(\theta)\right] = +\infty.$

The last condition (H5) is merely technical. It is clearly satisfied if $\mathcal{Z}$ is finite because in this case $\mathbb{P}[V_{\max}^{Z_0} = \underline{V_{\max}}] > 0$. It roughly says that, statistically, $V_{\max}^{Z_0}$ is close to $\underline{V_{\max}}$ with a sufficiently large probability. The schematic representation of the optimal velocity functions is given in Fig. 1.

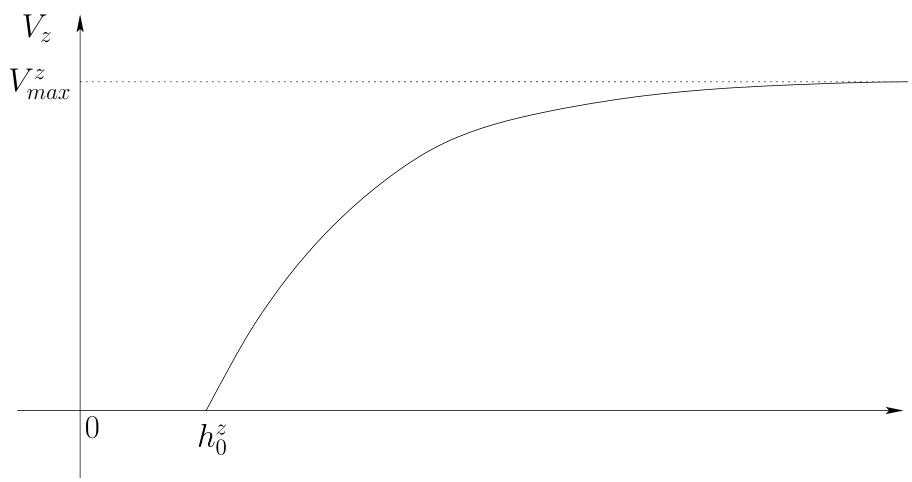

Figure 1: Schematic representation of the optimal velocity functions.

With these conditions in mind, we consider the (random) follow-the-leader model :

$$\frac{d}{dt} U_i(t) = V_{Z_i}(U_{i+1}(t) - U_i(t)), \; t \geqslant 0, \forall i \geqslant 0. \tag{1}$$

In this model, U_i denotes the position of car i and $\frac{d}{dt} U_i$ its velocity.

1.2 Main results

We now want to rescale the microscopic model. For $\epsilon > 0$, we consider an initial condition $(U_i^{\epsilon,0})$ such that there exists a Lipschitz continuous function $u_0 : \mathbb{R} \to \mathbb{R}$ with

$$\lim_{\epsilon \to 0, \; \epsilon i \to x} \epsilon U_i^{\epsilon,0} = u_0(x), \tag{2}$$

locally uniformly with respect to x. Let (U_i^ϵ) be the solution of (1) with initial condition $(U_i^{\epsilon,0})$.

Theorem 1.1 (Convergence result)**.** *Under assumptions* $(H1)-(H5)$*, the limit*

$$u(x,t) := \lim_{\epsilon\to 0,\ \epsilon(i,s)\to(x,t)} \epsilon U_i^\epsilon(s)$$

exists a.s., locally uniformly in (x,t)*, and* u *is the unique (deterministic) viscosity solution of*

$$\begin{cases} \partial_t u = \bar{F}(\partial_x u) & \text{in } \mathbb{R}\times]0,+\infty[\\ u(x,0) = u_0(x) & \text{in } \mathbb{R} \end{cases} \tag{3}$$

where the effective velocity $\bar{F} : [0,+\infty) \to [0,\bar{V}_{\max})$ *is the continuous and increasing map defined by* $\bar{F}(p) = 0$ *if* $p \leqslant \bar{h}_0$ *and* $\mathbb{E}[V_{Z_0}^{-1}(\bar{F}(p))] = p$ *if* $p > \bar{h}_0$*.*

1.3 Link with the LWR model

In this subsection, we explicit the link between the macroscopic model (3) and the classical Lighthill-Whitham-Richards (LWR) model (see [13, 15]). Let U^ϵ be the solution of (1) with an initial condition $U^{\epsilon,0}$ such that (2) holds. We consider the (rescaled) empirical density of cars:

$$\rho^\epsilon(t) = \epsilon \sum_{i\in\mathbb{Z}} \delta_{\epsilon U_i^\epsilon(t/\epsilon)}, \qquad t \geqslant 0.$$

Corollary 1.2. *As* $\epsilon \to 0$*,* $\rho^\epsilon(t)$ *converges, a.s., in distribution and locally uniformly in time, to*

$$\rho(t) := u(\cdot,t)\sharp dx,$$

where u *is the solution of* (3) *and the notation* $u(\cdot,t)\sharp dx$ *denotes the push-forward measure of* dx *by* u*. If, in addition, there exists* $C > 0$ *such that*

$$C^{-1} \leqslant \partial_x u_0(x) \leqslant C, \tag{4}$$

then ρ *has an absolutely continuous density which is locally bounded and is the entropy solution of the LWR model*

$$\partial_t \rho + \partial_x(\rho \bar{v}(\rho)) = 0 \qquad \text{in } \mathbb{R}\times\mathbb{R}_+, \tag{5}$$

with initial condition $u_0(\cdot)\sharp dx$ *and where* $\bar{v}(\rho) = \bar{F}(1/\rho)$*.*

Proof. Let $\varphi \in \mathcal{C}_c^0(\mathbb{R})$. Then, for any $t' \geqslant 0$,

$$\int_{\mathbb{R}} \varphi(x)\rho^\epsilon(dx,t') = \epsilon\sum_{i\in\mathbb{Z}} \varphi(\epsilon U_i^\epsilon(t'/\epsilon)) = \int_{\mathbb{R}} \varphi(\epsilon U_{[x/\epsilon]}^\epsilon(t'/\epsilon))dx.$$

As $\epsilon([x/\epsilon], t'/\epsilon) \to (x,t)$ as $\epsilon \to 0$ and $t' \to t$, Theorem 1.1 implies:

$$\lim_{\epsilon\to 0,\ t'\to t} \int_{\mathbb{R}} \varphi(x)\rho^\epsilon(dx,t) = \int_{\mathbb{R}} \varphi(u(x,t))dx = \int_{\mathbb{R}} \varphi(x)d(u(\cdot,t)\sharp dx) = \int_{\mathbb{R}} \varphi(x)\rho(dx,t).$$

This proves that $\rho^\epsilon(t)$ converges locally uniformly in time and in the sense of measures to $\rho(t) := u(\cdot,t)\sharp dx$.

Let us now assume that (4) holds. From standard arguments (as $\bar{F}$ does not depend on space), one easily checks that the bounds (4) are preserved in time:

$$C^{-1} \leqslant \partial_x u(x,t) \leqslant C.$$

In particular $\rho(t)$ is absolutely continuous with a density satisfying the same estimate:

$$C^{-1} \leqslant \rho(x,t) \leqslant C.$$

In addition, one easily checks that $w(x,t) := u^{-1}(x,t)$ (where the inverse is taken in space) is a viscosity solution to

$$\partial_t w + \partial_x w \bar{F}(1/\partial_x w) = 0,$$

which in turn implies that $\rho(x,t) = \partial_x w(x,t)$ is an entropy solution to (5). □

2 Some properties of the microscopic and macroscopic models

2.1 The microscopic model

The following lemma is a direct consequence of the fact that V_z is non-negative joint to assumption (H_4).

Lemma 2.1. *Let U_i be a solution of* (1)*. Then, for all $t \geqslant 0$,*

$$0 \leqslant U_i(t) - U_i(0) \leqslant V_{\max} t.$$

Proposition 2.2. *Let U_i and $\tilde{U}_i$ be two solutions of* (1) *such that there exists $i_0 \geqslant 0$ such that*

$$U_i(0) \leqslant \tilde{U}_i(0) \quad \forall i \geqslant i_0.$$

Then

$$U_i(t) \leqslant \tilde{U}_i(t) \quad \forall t \geqslant 0 \text{ and } i \geqslant i_0.$$

Proof. Let $T \geqslant 0$. We want to prove that

$$M = \sup_{t\in[0,T]} \max_{i\geqslant i_0} (U_i(t) - \tilde{U}_i(t)) \leqslant 0.$$

By contradiction, assume that $M > 0$ and consider for $\varepsilon, \eta, \beta > 0$

$$M_\varepsilon = \sup_{t,s\in[0,T]} \max_{i\geqslant i_0} \left\{ U_i(t) - \tilde{U}_i(s) - \frac{(t-s)^2}{2\varepsilon} - \frac{\eta}{T-t} - \beta(i-i_0)^2 \right\} > 0$$

for η and β small enough. We assume that the maximum is reached at some point $(t,s) \in [0,T]$ and for an index $i \geqslant i_0$. We first prove that $t > 0$ and $s > 0$. Indeed, if $t = 0$, then we get

$$\frac{s^2}{2\varepsilon} + \frac{\eta}{T} < U_i(0) - \tilde{U}_i(s) \leqslant U_i(0) - \tilde{U}_i(0) \leqslant 0$$

which is a contradiction. In the same way, if $s = 0$, then

$$\frac{t^2}{2\varepsilon} + \frac{\eta}{T} \leqslant U_i(t) - \tilde{U}_i(0) \leqslant V_{\max} t$$

which is a contradiction for ε small enough. We then deduce that $t, s > 0$. This implies that

$$\frac{\eta}{T^2} \leqslant V_{Z_i}(U_{i+1}(t) - U_i(t)) - V_{Z_i}(\tilde{U}_{i+1}(s) - \tilde{U}_i(s)).$$

Using that

$$U_{i+1}(t) - \tilde{U}_{i+1}(s) - \beta(i+1-i_0)^2 \leqslant U_i(t) - \tilde{U}_i(s) - \beta(i-i_0)^2,$$

and the fact that V_Z are uniformly Lipschitz continuous, we deduce that

$$\frac{\eta}{T^2} \leqslant C\beta(1 + 2(i-i_0)).$$

Sending $\beta \to 0$ (and using the classical fact that $\beta(i-i_0) \to 0$), we get a contradiction. □

2.2 The macroscopic model

In this subsection, we recall the definition of viscosity solution for the macroscopic model (3) and give a comparison principle.

Definition 2.3 (Definition of viscosity solutions for (3))**.** *An upper semi-continuous (resp. lower semi-continuous) function* $u : \mathbb{R} \times [0, +\infty) \to \mathbb{R}$ *is a viscosity sub-solution (resp. super-solution) of* (3) *if* $u(x, 0) \leqslant u_0(x)$ *(resp.* $u(x, 0) \geqslant u_0(x)$*) and for all* $(x, t) \in \mathbb{R} \times [0, +\infty)$ *and for all* $\varphi \in C^1(\mathbb{R} \times [0, +\infty)$ *such that* $u - \varphi$ *reaches a local maximum (resp. minimum) in* (x, t)*, we have*

$$\partial_t \varphi(x, t) \leqslant \bar{F}(\partial_x \varphi(x, t)) \quad \big(\text{resp. } \partial_t \varphi(x, t) \geqslant \bar{F}(\partial_x \varphi(x, t))\big) \tag{6}$$

We say that u *is a viscosity solution of* (3) *if* u^* *and* u_* *are respectively a sub-solution and a super-solution of* (3)*.*

Proposition 2.4 (Comparison principle for (3))**.** *Let* u *and* v *be respectively a sub and a super-solution of* (3) *and assume that* u_0 *is Lipschitz continuous. Then*

$$u \leqslant v \quad \text{in } \mathbb{R} \times [0, +\infty).$$

3 Construction of the effective velocity

Recall that $\bar{h}_0 := \mathbb{E}[h_0^{Z_0}]$. Given $p > \bar{h}_0$, in order to construct the effective velocity $\bar{F}(p)$, we consider the solution to

$$\frac{d}{dt} U_i(t) = V_{Z_i}(U_{i+1}(t) - U_i(t)), \ t \geqslant 0, \qquad U_i(0) = pi \qquad \forall i \geqslant 0. \tag{7}$$

We begin to prove that, as $t \to \infty$, $U_i(t)/t$ is bounded by $\underline{V_{\max}} := \inf_{z \in \mathcal{Z}} V_{\max}^z$.

Proposition 3.1. *Let* (U_i) *be the solution of* (7)*. Then, for every* $i \geqslant 0$*, we have,*

$$\limsup_{t \to \infty} \frac{U_i(t)}{t} \leqslant \inf_{z \in \mathcal{Z}} V_{\max}^z =: \underline{V_{\max}}.$$

Proof. Let $\varepsilon > 0$ and fix $i \geqslant 0$. Since the Z_j are i.i.d. and have a law which has a full support in $\mathcal{Z}$, there exists $i_0 \geqslant i$ such that

$$V_{\max}^{Z_{i_0}} \leqslant \underline{V_{\max}} + \varepsilon.$$

Moreover, we have

$$U_{i_0}(t) \leqslant U_{i_0}(0) + V_{\max}^{Z_{i_0}} \cdot t.$$

Since the U_j are increasing in j, we deduce that

$$\frac{U_i(t)}{t} \leqslant \frac{U_{i_0}(t)}{t} \leqslant \frac{U_{i_0}(0)}{t} + V_{\max}^{Z_{i_0}}.$$

This implies that

$$\limsup_{t \to +\infty} \frac{U_i(t)}{t} \leqslant V_{\max}^{Z_{i_0}} \leqslant \underline{V_{\max}} + \varepsilon.$$

We finally get the result by taking $\varepsilon \to 0$. □

We now construct the limit of $U_i(t)/t$.

Proposition 3.2. *There exists Ω_0 with $\mathbb{P}(\Omega_0) = 1$ such that for every $p \geqslant 0$, $i \in \mathbb{N}$ and $\omega \in \Omega_0$*

$$\lim_{t \to +\infty} \frac{U_i(t)}{t} = \bar{F}(p) \qquad \forall i \geqslant 0,$$

where (U_i) is the solution of (7) and where the continuous and non-decreasing map $\bar{F} : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by $\bar{F}(p) = 0$ if $p \leqslant \bar{h}_0$ and by the relation $\mathbb{E}[V_{Z_0}^{-1}(\bar{F}(p))] = p$ if $p > \bar{h}_0$.

Proof. Let us first check that the map $\bar{F}$ is well defined and continuous. For this we first recall that $V_z^{-1} : [0, V_{\max}^z) \to [h_0^z, +\infty)$ is increasing and continuous. We claim that, for any $\theta \in [0, \underline{V_{\max}})$, $V_z^{-1}(\theta)$ is bounded with respect to $z \in \mathcal{Z}$. Indeed, assume that, contrary to our claim, there exists $z_n \in \mathcal{Z}$ such that $p_n := V_{z_n}^{-1}(\theta) \to +\infty$. As $\mathcal{Z}$ is compact, there exists z a limit of a subsequence of the (z_n), still denoted z_n. Then, by uniform continuity of the map V,

$$\theta = \lim_n V_{z_n}(p_n) = V_{\max}^z \geqslant \underline{V_{\max}} > \theta,$$

which is a contradiction. Therefore the map $\theta \to \mathbb{E}\left[V_{Z_0}^{-1}(\theta)\right]$ is well-defined, increasing and continuous on $[0, \underline{V_{\max}})$. Moreover, by monotone convergence, $\mathbb{E}[V_{Z_0}^{-1}(\theta)] \to \mathbb{E}[h_0^{Z_0}] =: \bar{h}_0$ as $\theta \to 0^+$ while, by assumption (H5), we have

$$\lim_{\theta \to \underline{V_{\max}}} \mathbb{E}[V_{Z_0}^{-1}(\theta)] = +\infty.$$

So we can define, for $p \in (\bar{h}_0, +\infty)$, $\bar{F}(p)$ as the unique real number in $(0, \underline{V_{\max}})$ such that

$$\mathbb{E}\left[V_{Z_0}^{-1}(\bar{F}(p))\right] = p.$$

We extend $\bar{F}$ by setting $\bar{F}(p) = 0$ for $p \in [0, \bar{h}_0]$. Then $\bar{F}$ is increasing (in $[\bar{h}_0, +\infty)$), continuous and satisfies

$$\lim_{p \to \bar{h}_0^+} \bar{F}(p) = 0, \qquad \lim_{p \to +\infty} \bar{F}(p) = \underline{V_{\max}}.$$

In order to study the behavior of the solution to (7), we introduce next the correctors of the problem. Given $\theta \in (0, \underline{V_{\max}})$, we consider the random sequence (c_i^θ) defined by $c_0^\theta = 0$ and $c_{i+1}^\theta = c_i^\theta + V_{Z_i}^{-1}(\theta)$. In other words,

$$V_{Z_i}(c_{i+1}^\theta - c_i^\theta) = \theta \qquad \forall i \geqslant 0. \tag{8}$$

The reals c_i^θ represent the optimal position of the vehicle i in the sense that all the vehicles drive at velocity θ. In this sense, the sequence (c_i^θ) is related to the so-called hull functions (see the pionnering work of Aubry [1, 2], and Aubry, Le Daeron [3] as well as [7, 8]). We note that, by the law of large numbers, there exists Ω_0 with $\mathbb{P}(\Omega_0) = 1$ such that for every $\omega \in \Omega_0$, we have

$$\frac{c_n^\theta}{n} = \frac{1}{n} \sum_{i=0}^{n-1} V_{Z_i}^{-1}(\theta) \to \mathbb{E}\left[V_{Z_0}^{-1}(\theta)\right] \qquad \text{as } n \to +\infty. \tag{9}$$

We now study the limit of $\frac{U_i(t)}{t}$ as $t \to +\infty$. Let us fix $p > h_0$ and $\epsilon > 0$ small. We first define

$$\tilde{U}_i(t) = c_i^{p+\epsilon} + t\bar{F}(p + \epsilon),$$

(where by a "small" abuse of notation, $c_i^{p+\epsilon} := c_i^{\bar{F}(p+\epsilon)}$, i.e., $\theta = \bar{F}(p+\epsilon)$). Note that, by (8), $\tilde{U}_i$ solves

$$V_{Z_i}(\tilde{U}_{i+1}(t) - \tilde{U}_i(t)) = V_{Z_i}(c_{i+1}^{p+\epsilon} - c_i^{p+\epsilon}) = \bar{F}(p+\epsilon) = \frac{d}{dt}\tilde{U}_i(t), \qquad \forall i \geqslant 0.$$

Moreover, by (9), we have a.s.

$$\lim_n \frac{c_n^{p+\epsilon}}{n} = \mathbb{E}\left[V_{Z_0}^{-1}(\bar{F}(p+\epsilon)))\right] = p + \epsilon.$$

Thus (since $U_i(0) = pi$), there exists a random integer i_0 such that

$$U_i(0) \leqslant \tilde{U}_i(0) \qquad \forall i \geqslant i_0.$$

By comparison (Proposition 2.2) we obtain

$$U_i(t) \leqslant \tilde{U}_i(t) = \tilde{U}_i(0) + t\bar{F}(p+\epsilon) \qquad \forall i \geqslant i_0,\ t \geqslant 0.$$

Hence, for any $i \geqslant i_0$,

$$\limsup_{t\to+\infty} \frac{U_i(t)}{t} \leqslant \bar{F}(p+\epsilon).$$

This actually holds for any i (since $i \to U_i(t)$ is increasing) and for any $\epsilon > 0$. So

$$\limsup_{t\to+\infty} \frac{U_i(t)}{t} \leqslant \bar{F}(p) \qquad \forall i \geqslant 0.$$

Next we want to prove the bound below, which is slightly more difficult. As above we set

$$\tilde{U}_i(t) = c_i^{p-\epsilon} + t\bar{F}(p-\epsilon),$$

(where by again a "small" abuse of notation, $c_i^{p-\epsilon} := c_i^{\bar{F}(p-\epsilon)}$). Note that $\tilde{U}_i$ solves

$$V_{Z_i}(\tilde{U}_{i+1}(t) - \tilde{U}_i(t)) = V_{Z_i}(c_{i+1}^{p-\epsilon} - c_i^{p-\epsilon}) = \bar{F}(p-\epsilon) = \frac{d}{dt}\tilde{U}_i(t), \qquad \forall i \geqslant 0$$

and, by (9), we have a.s.

$$\lim_n \frac{c_n^{p-\epsilon}}{n} = \mathbb{E}\left[V_{Z_0}^{-1}(\bar{F}(p-\epsilon)))\right] = p - \epsilon.$$

Thus there exists a random integer i_0 such that

$$U_i(0) \geqslant \tilde{U}_i(0) \qquad \forall i \geqslant i_0.$$

By comparison (Proposition 2.2) we obtain

$$U_i(t) \geqslant \tilde{U}_i(t) = \tilde{U}_i(0) + t\bar{F}(p-\epsilon) \qquad \forall i \geqslant i_0,\ t \geqslant 0.$$

Hence, for any $i \geqslant i_0$,

$$\liminf_{t\to+\infty} \frac{U_i(t)}{t} \geqslant \bar{F}(p-\epsilon).$$

It remains to check that this inequality still holds for $i = 0$ (and thus for all $i \geqslant 0$).

For this we set
$$W_i(t) = U_i(t) - \tilde{U}_i(t),$$
and note that
$$\begin{aligned}\frac{d}{dt}W_i(t) &= V_{Z_i}(U_{i+1}(t) - U_i(t)) - V_{Z_i}(\tilde{U}_{i+1}(t) - \tilde{U}_i(t))\\ &= V'_{Z_i}(\sigma_i(t))\ (W_{i+1}(t) - W_i(t))\end{aligned}$$
for some $\sigma_i(t) \geqslant h_0$ that can be chosen measurable with respect to t and ω. We integrate this equality to get
$$W_i(t) = W_i(0)\exp\Big\{-\int_0^t V'_{Z_i}(\sigma_i(\tau))d\tau\Big\} + \int_0^t V'_{Z_i}(\sigma_i(s))\exp\Big\{-\int_s^t V'_{Z_i}(\sigma_i(\tau))d\tau\Big\} W_{i+1}(s)ds.$$
Now we consider the indices $i \in \{0,\dots i_0\}$ and set $C = \sup_{j\in\{0,\dots,i_0\}}|W_i(0)|$ (this is a random variable). As $V'_k \geqslant 0$, we have (for $(w)_- = \max\{0,-w\}$)
$$\begin{aligned}W_i(t) &\geqslant -C - \int_0^t V'_{Z_i}(\sigma_i(s))\exp\Big\{-\int_s^t V'_{Z_i}(\sigma_i(\tau))d\tau\Big\} ds\|(W_{i+1})_-\|_{L^\infty(0,t)}\\ &\geqslant -C - \Big(1-\exp\Big\{-\int_0^t V'_{Z_i}(\sigma_i(\tau))d\tau\Big\}\Big)\|(W_{i+1})_-\|_{L^\infty(0,t)}\\ &\geqslant -C - \|(W_{i+1})_-\|_{L^\infty(0,t)}.\end{aligned}$$
We infer by induction that
$$\|(W_0)_-\|_{L^\infty(0,t)} \leqslant Ci_0 + \|(W_{i_0})_-\|_{L^\infty(0,t)} = Ci_0,$$
since $W_{i_0} \geqslant 0$. This shows that
$$U_0(t) \geqslant \tilde{U}_0(t) - Ci_0 = \tilde{U}_0(0) + \bar{F}(p-\epsilon)t - Ci_0,$$
and therefore that
$$\liminf_{t\to+\infty}\frac{U_0(t)}{t} \geqslant \bar{F}(p-\epsilon).$$
Since $i \to U_i(t)$ is nondecreasing, this holds for any $i \geqslant 0$. And, finally, as ϵ is arbitrary, this shows that
$$\liminf_{t\to+\infty}\frac{U_i(t)}{t} \geqslant \bar{F}(p) \qquad \forall i \geqslant 0.$$

It just remains to treat the case $p \in [0,\bar{h}_0]$. We begin to show that $\lim\limits_{p\to\bar{h}_0^+}\bar{F}(p) = 0$. Taking the limit $\theta \to 0^+$ in the equality $\mathbb{E}[V_{Z_0}^{-1}(\theta)] = \bar{F}^{-1}(\theta)$, we deduce that $\lim\limits_{\theta\to 0^+}\bar{F}^{-1}(\theta) = \bar{h}_0$, which implies that
$$\lim_{p\to\bar{h}_0^+}\bar{F}(p) = 0.$$

Now, let $p \in [0,\bar{h}_0]$. Using that $U_i(t) \geqslant 0$, we just have to show that $\limsup_{t\to+\infty}\frac{U_i(t)}{t} \leqslant 0$. Let $\varepsilon > 0$ and consider $\tilde{U}_i$ solution of
$$\frac{d}{dt}\tilde{U}_i(t) = V_{Z_i}(\tilde{U}_{i+1}(t) - \tilde{U}_i(t)),\ t \geqslant 0, \qquad \tilde{U}_i(0) = (\bar{h}_0 + \varepsilon)i \qquad \forall i \geqslant 0.$$

Using the previous result (i.e., the case $p > \bar{h}_0$), we get that

$$\lim_{t\to+\infty} \frac{\tilde{U}_i(t)}{t} = \bar{F}(\bar{h}_0 + \varepsilon).$$

Using that, by the comparison principle, we have $\tilde{U}_i(t) \geqslant U_i(t)$ for all $i \geqslant 0$ and $t \geqslant 0$, we deduce that

$$\limsup_{t\to+\infty} \frac{U_i(t)}{t} \leqslant \lim_{t\to+\infty} \frac{\tilde{U}_i(t)}{t} = \bar{F}(\bar{h}_0 + \varepsilon).$$

Letting $\epsilon \to 0$ completes the proof of the proposition. □

4 Proof of convergence

We recall that the initial condition $(U_i^{\epsilon,0})_{i\in\mathbb{Z}}$ is fixed and we consider the solution of

$$\frac{d}{dt}U_i^\epsilon(t) = V_{Z_i}(U_{i+1}^\epsilon(t) - U_i^\epsilon(t)),\ t \geqslant 0,\ i \in \mathbb{Z} \qquad U_i^\epsilon(0) = U_i^{\epsilon,0},\ i \in \mathbb{Z}.$$

We recall that we assumed the existence of a Lipschitz continuous function $u_0 : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{\epsilon\to 0,\ \epsilon i\to x} \epsilon U_i^{\epsilon,0} = u_0(x).$$

Proof of Theorem 1.1. Let

$$u^*(x,t) = \limsup_{\epsilon\to 0,\ \epsilon(i,s)\to(x,t)} \epsilon U_i^\epsilon(s) \quad \text{and} \quad u_*(x,t) = \liminf_{\epsilon\to 0,\ \epsilon(i,s)\to(x,t)} \epsilon U_i^\epsilon(s).$$

We want to prove that u^* and u_* are respectively sub- and super-solution of (3). Indeed, if we show these statements, then by the comparison principle for (3) (see Proposition 2.4), we get $u^* \leqslant u_*$. The reverse inequality being obvious, we get the convergence result.

We begin with the initial condition. Since the velocity V_Z is uniformly bounded and non-negative, we have that

$$\epsilon U_i^\epsilon(0) \leqslant \epsilon U_i^\epsilon(s) \leqslant \epsilon U_i^\epsilon(0) + C\epsilon s.$$

Taking the lim sup as $\epsilon \to 0$, with $\epsilon(i,s) \to (x,0)$, we infer that

$$u^*(x,0) = u_0(x).$$

We now turn to the equation. We only prove that u^* is a sub-solution since the proof for u_* is similar. Let ϕ be a smooth test function such that $u^* - \phi$ has a strict maximum at some point $(\bar{x},\bar{t})$. Then there exists a subsequence of ϵ, still denoted in the same way, and (i_ϵ, s_ϵ), such that the map

$$(i,s) \to \epsilon U_i^\epsilon(s) - \phi(\epsilon i, \epsilon s)$$

has a maximum at (i_ϵ, s_ϵ). Moreover we have $(\epsilon i_\epsilon, \epsilon s_\epsilon) \to (\bar{x},\bar{t})$. The optimality of (i_ϵ, s_ϵ) can be rewritten as

$$\frac{1}{\epsilon}\left(\phi(\epsilon(i_\epsilon + i), \epsilon(s_\epsilon + s)) - \phi(\epsilon i_\epsilon, \epsilon s_\epsilon)\right) \geqslant U_{i_\epsilon+i}^\epsilon(s) - U_{i_\epsilon}^\epsilon(s_\epsilon) \qquad \forall (i,s). \tag{10}$$

Let us fix $\delta_\epsilon > 0$, with $\delta_\epsilon \to 0$ as $\epsilon \to 0$, to be choose later. Then by (10), we have, for any $i \geqslant 0$,

$$
\begin{aligned}
U^\epsilon_{i_\epsilon+i}(s_\epsilon - \delta_\epsilon/\epsilon) &\leqslant U^\epsilon_{i_\epsilon}(s_\epsilon) + \frac{1}{\epsilon}\left(\phi(\epsilon(i_\epsilon+i), \epsilon s_\epsilon - \delta_\epsilon)) - \phi(\epsilon i_\epsilon, \epsilon s_\epsilon)\right) \\
&\leqslant U^\epsilon_{i_\epsilon}(s_\epsilon) + \partial_x\phi(\epsilon i_\epsilon, \epsilon s_\epsilon - \delta_\epsilon)i + C\epsilon i^2 + \frac{1}{\epsilon}\left(\phi(\epsilon i_\epsilon, \epsilon s_\epsilon - \delta_\epsilon)) - \phi(\epsilon i_\epsilon, \epsilon s_\epsilon)\right).
\end{aligned}
$$

Let us set $p = \partial_x\phi(\bar{x}, \bar{t})$ and fix $\theta > 0$ small. The RHS of the above inequality can be bounded from above as follows, for ϵ small enough:

$$
\begin{aligned}
U^\epsilon_{i_\epsilon+i}(s_\epsilon - \delta_\epsilon/\epsilon) \leqslant U^\epsilon_{i_\epsilon}(s_\epsilon) + (p+\theta)i + \frac{1}{\epsilon}\left(\phi(\epsilon i_\epsilon, \epsilon s_\epsilon - \delta_\epsilon)) - \phi(\epsilon i_\epsilon, \epsilon s_\epsilon)\right) \\
\forall i \in \{0, \ldots, \theta/(C\epsilon)\}. \qquad (11)
\end{aligned}
$$

We now need the following localization lemma:

Lemma 4.1. *Let (U_i) and $(\tilde{U}_i)$ be two solutions to* (1) *such that $U_i(0) \leqslant \tilde{U}_i(0)$ for $i \in \{0, \ldots, K\}$ (where $K \geqslant 1$). Then*

$$
U_0(t) \leqslant \tilde{U}_0(t) + V_{\max} t\left(1 - \exp(-\alpha t)\right)^K \qquad \forall t \geqslant 0,
$$

where $\alpha = \sup_k \|\partial_x V_k\|_\infty$.

We postpone the proof of the lemma and proceed with the ongoing argument. Let $(\tilde{U}_i)$ be the solution to

$$
\frac{d}{dt}\tilde{U}_i(t) = V_{Z_i}(\tilde{U}_{i+1}(t) - \tilde{U}_i(t)),\ t \geqslant 0,\ i \in \mathbb{N} \qquad \tilde{U}_i(0) = (p+\theta)i,\ i \in \mathbb{N}.
$$

We know from Proposition 3.2 that, a.s. in Ω_0,

$$
\lim_{t\to+\infty} \frac{1}{t}\tilde{U}_0(t) = \bar{F}(p+\theta).
$$

We consider the flow generated by the left-hand side of (11) (i.e., (U^ϵ_i)) and by its right-hand side (i.e., up to a constant, $(\tilde{U}_i)$) at time δ_ϵ/ϵ. We have, by Lemma 4.1 (with $t = \delta_\epsilon/\epsilon$ and $K = \theta/(C\epsilon)$):

$$
\begin{aligned}
U^\epsilon_{i_\epsilon}(s_\epsilon) \leqslant U^\epsilon_{i_\epsilon}(s_\epsilon) + \tilde{U}_0(\delta_\epsilon/\epsilon) + V_{\max}\frac{\delta_\epsilon}{\epsilon}\left(1 - \exp(-\alpha\delta_\epsilon/\epsilon)\right)^{\theta/(C\epsilon)} \\
+ \frac{1}{\epsilon}\left(\phi(\epsilon i_\epsilon, \epsilon s_\epsilon - \delta_\epsilon)) - \phi(\epsilon i_\epsilon, \epsilon s_\epsilon)\right). \qquad (12)
\end{aligned}
$$

Choosing $\delta_\epsilon := -\gamma\epsilon\ln(\epsilon)$ where $\gamma > 0$ is small enough (such that $\alpha\gamma < 1$), we have

$$
\delta_\epsilon/\epsilon \to +\infty \qquad \text{and} \qquad \left(1 - \exp(-\alpha\delta_\epsilon/\epsilon)\right)^{\theta/(C\epsilon)} \to 0.
$$

Dividing (12) by δ_ϵ/ϵ and letting $\epsilon \to 0$ gives

$$
0 \leqslant \bar{F}(p+\theta) - \partial_t\phi(\bar{x}, \bar{t}).
$$

In view of the definition of p and the fact that $\theta > 0$ is arbitrary, we conclude that u^ϵ is a subsolution to the equation. □

Proof of Lemma 4.1. Let $W_i(t) := \tilde{U}_i(t) - U_i(t)$. Then (W_i) solves

$$\begin{aligned}\frac{d}{dt}W_i(t) &= V_{Z_i}(\tilde{U}_{i+1}(t) - \tilde{U}_i(t)) - V_{Z_i}(U_{i+1}(t) - U_i(t)) \\ &= V'_{Z_i}(\sigma_i(t))(W_{i+1}(t) - W_i(t)),\end{aligned}$$

for some σ_i between $U_{i+1}(t) - U_i(t)$ and $\tilde{U}_{i+1}(t) - \tilde{U}_i(t)$. We integrate this equation and find

$$W_i(t) = W_i(0)\exp\Big\{-\int_0^t V'_{Z_i}(\sigma_i(\tau))d\tau\Big\} + \int_0^t V'_{Z_i}(\sigma_i(s))\exp\Big\{-\int_s^t V'_{Z_i}(\sigma_i(\tau))d\tau\Big\} W_{i+1}(s)ds.$$

For $i \in \{0, \dots, K-1\}$, we have, since $W_i(0) \geqslant 0$ and $V'_{Z_i} \geqslant 0$,

$$\begin{aligned}W_i(t) &\geqslant -\|(W_{i+1})_-\|_{[0,t]} \int_0^t V'_{Z_i}(\sigma_i(s))\exp\{-\int_s^t V'_{Z_i}(\sigma_i(\tau))d\tau\}ds \\ &\geqslant -\|(W_{i+1})_-\|_{[0,t]} \left(1 - \exp\{-\int_0^t V'_{Z_i}(\sigma_i(\tau))d\tau\}\right) \\ &\geqslant -\|(W_{i+1})_-\|_{[0,t]} \left(1 - \exp\{-\alpha t\}\right).\end{aligned}$$

By induction, we infer that

$$\|(W_0)_-\|_{[0,t]} \leqslant \|(W_K)_-\|_{[0,t]} \left(1 - \exp\{-\alpha t\}\right)^K,$$

where, because V_Z is uniformly bounded by $V_{\max}$ and $W_K(0) \geqslant 0$, $\|(W_K)_-\|_{[0,t]} \leqslant V_{\max}t$. □

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