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A MODEL FOR SUSPENSION OF CLUSTERS OF PARTICLE PAIRS

AMINA MECHERBET

ABSTRACT. In this paper, we consider N clusters of pairs of particles sedimenting in a viscous fluid. The particles are assumed to be rigid spheres and inertia of both particles and fluid are neglected. The distance between each two particles forming the cluster is comparable to their radii $\frac{1}{N}$ while the minimal distance between the pairs is of order $\frac{1}{N^{1/3}}$. We show that, at the mesoscopic level, the dynamics are modelled using a transport-Stokes equation describing the time evolution of the position and orientation of the clusters. We also investigate the case where the orientation of the cluster is initially correlated to its position. A local existence and uniqueness result for the limit model is provided.

INTRODUCTION

We consider the problem of N rigid particles sedimenting in a viscous fluid under gravitational force. The inertia of both fluid and particles is neglected. At the microscopic level, the fluid velocity and the pressure satisfy a Stokes equation on a perforated domain. In the analysis of the associated homogenization problem, it has been proved that the interaction between particles leads to the appearance of a Brinkman force in the fluid equation. This Brinkman force depends on the dilution of the cloud but also the geometry of the particles (see [1, 3, 8, 9]). In the dynamic case, the justification of a mesoscopic model using a coupled transport-Stokes equation has been proved in [12] where authors show that the interaction between particles is negligible in the dilute case *i.e.* when the minimal distance between particles is larger than $\frac{1}{N^{1/3}}$. In [10, 17] the justification has been extended to regimes that are not so dilute but where the minimal distance between particles is still large compared to the particles radii. The coupled equations derived are:

$$(1) \quad \begin{cases} \partial_t \rho + \operatorname{div}((\kappa g + u)\rho) & = 0 \\ -\Delta u + \nabla p & = 6\pi r_0 \kappa g \rho, \\ \operatorname{div}(u) & = 0. \end{cases}$$

Here u is the fluid velocity, p its associated pressure, ρ is the density of the cloud. $r_0 = RN$, where R is the particles radii, g the gravity vector. The velocity $\kappa g = \frac{m}{6\pi R}g$ represents the fall speed of a sedimenting single particle under gravitational force. The derivation of this model is a consequence of the method of reflections which consists in approaching the flow around several particles as the superposition of the flows associated to one particle at time, see [18], [14, Chapter 8], [16], [5, Section 4], [15], [11] for more details. This approximation

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is possible in the case where the minimal distance between particles is larger than the particles radii. Consequently, the velocity of each particle corresponds to the fall speed of one sedimenting particle κg to which we add the velocity contribution of all the other particles which is smaller but of order one.

In this paper, we are interested in the case where the cloud is made up of clusters. The main motivation is to show the influence of the clusters configuration on the mean velocity fall. A first investigation in this direction is to consider clusters of pairs of particles where the minimal distance between the particles forming the pair is comparable to their radii. The cluster configuration is determined by the center x and the orientation ξ of the pair. Starting from a microscopic model, the first result of this paper is the derivation of a mesoscopic fluid-kinetic model describing the fluid velocity and pressure (u, p) and the function $f(t, x, \xi)$ representing the density of clusters centered in x and having orientation ξ at time t . The mean velocity fall of clusters is formulated through the Stokes resistance matrices. The second result of this paper corresponds to the case where the orientation of the cluster is correlated to its center *i.e.* $\xi = F(t, x)$. We obtain a system of coupled equations on ρ the first marginal of f , the fluid velocity and pressure (u, p) and the function F describing the evolution of the cluster orientation. A local existence and uniqueness result for the former system is also presented.

The starting point is a microscopic model representing suspension of $N \in \mathbb{N}^*$ identical particle pairs in a uniform gravitational field. The pairs are defined as

$$B^i := B(x_1^i, R) \cup B(x_2^i, R), \quad 1 \leq i \leq N,$$

where x_1^i, x_2^i are the centers of the i^{th} pair and R the radius. We define (u^N, p^N) as the unique solution to the following Stokes problem :

$$(2) \quad \begin{cases} -\Delta u^N + \nabla p^N = 0, \\ \operatorname{div} u^N = 0, \end{cases} \text{ on } \mathbb{R}^3 \setminus \bigcup_{i=1}^N \overline{B^i},$$

completed with the no-slip boundary conditions :

$$(3) \quad \begin{cases} u^N = U_1^i \text{ on } \partial B(x_1^i, R), \\ u^N = U_2^i \text{ on } \partial B(x_2^i, R), \\ \lim_{|x| \rightarrow \infty} |u^N(x)| = 0, \end{cases}$$

where $(U_1^i, U_2^i) \in \mathbb{R}^3 \times \mathbb{R}^3$, $1 \leq i \leq N$ are the linear velocities. In this model, the angular velocity is neglected and we complete the PDE with the motion equation for each couple of particles :

$$(4) \quad \begin{cases} \dot{x}_1^i = U_1^i, \\ \dot{x}_2^i = U_2^i. \end{cases}$$

Newton law yields the following equations where inertia is neglected :

$$(5) \quad \begin{pmatrix} F_1^i \\ F_2^i \end{pmatrix} = - \begin{pmatrix} mg \\ mg \end{pmatrix},$$

where m is the mass of the identical particle adjusted for buoyancy, g the gravitational acceleration, F_1^i, F_2^i are the drag forces applied by the fluid on the i^{th} particle :

$$F_1^i = \int_{\partial B(x_1^i, R)} \sigma(u^N, p^N) n \quad , \quad F_2^i = \int_{\partial B(x_2^i, R)} \sigma(u^N, p^N) n,$$

with n the unit outer normal and $\sigma(u^N, p^N) = (\nabla u^N + (\nabla u^N)^\top) - p^N \mathbb{I}$ the stress tensor. In order to formulate our results we introduce the main assumptions on the cloud.

0.1. Assumptions and main results. We assume that the radius is given by $R = \frac{r_0}{2N}$. In this paper we use the following notations, given a pair of particles $B(x_1, R)$ and $B(x_2, R)$:

$$x_+ := \frac{1}{2}(x_1 + x_2), \quad x_- := \frac{1}{2}(x_1 - x_2), \quad \xi := \frac{x_-}{R}.$$

Let $T > 0$ be fixed. We introduce the empirical density $\mu^N \in \mathcal{P}([0, T] \times \mathbb{R}^3 \times \mathbb{R}^3)$:

$$\mu^N(t, x, \xi) = \frac{1}{N} \sum_1^N \delta_{(x_+^i(t), \xi_i(t))}(x, \xi),$$

and set ρ^N its first marginal:

$$(6) \quad \rho^N(t, x) := \frac{1}{N} \sum_i \delta_{x_+^i(t)}(x).$$

We denote by d_{\min} the minimal distance between the centers x_+^i :

$$d_{\min}(t) := \min \{d_{ij}(t) := |x_+^i(t) - x_+^j(t)|, i \neq j\}.$$

We assume that there exists two constants $M_1 > M_2 > 1$ independent of N such that:

$$(7) \quad M_2 \leq |\xi_i| \leq M_1, \quad i = 1, \dots, N \quad \forall t \in [0, T].$$

We assume that μ^N converges weakly to a measure μ in the sense that for all test function $\psi \in \mathcal{C}_b([0, T] \times \mathbb{R}^3 \times \mathbb{R}^3)$ we have:

$$(8) \quad \int_0^T \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \psi(t, x, \xi) \mu^N(t, dx, d\xi) dt \xrightarrow{N \rightarrow \infty} \int_0^T \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \psi(t, x, \xi) \mu(t, x, \xi) dx d\xi dt.$$

We assume that the first marginal of μ denoted by ρ is a probability measure such that $\rho \in W^{1, \infty} \cap W^{1, 1}$. We use the shortcut $W_\infty(t) := W_\infty(\rho^N(t, \cdot), \rho(t, \cdot))$ to define the infinite-Wasserstein distance between ρ^N and ρ , see [2] for a definition.

We assume that there exists a positive constant $\mathcal{E}_1 > 0$ such that for all $N \in \mathbb{N}^*$ and $t \in [0, T]$:

$$(9) \quad \sup_{N \in \mathbb{N}^*} \frac{W_\infty^3}{d_{\min}^2} \leq \mathcal{E}_1.$$

Finally, we assume that there exists a positive constant $\mathcal{E}_2 > 0$ such that for all $N \in \mathbb{N}^*$ and $t \in [0, T]$:

$$(10) \quad \sup_{N \in \mathbb{N}^*} \frac{W_\infty^3}{d_{\min}^3} \leq \mathcal{E}_2.$$

Remark 0.1. *Note that, formula (8) ensures that:*

$$(11) \quad \sup_{t \in [0, T]} W_\infty(t) \xrightarrow{N \rightarrow \infty} 0.$$

Since $\rho \in L^\infty$, this yields a lower bound for the infinite Wasserstein distance:

$$(12) \quad \frac{1}{NW_\infty^3} \lesssim \sup_{x \in \mathbb{R}^3} \frac{\rho^N(B(x, W_\infty))}{|B(x, W_\infty)|^3} \lesssim \|\rho\|_\infty.$$

The definition of the infinite Wasserstein distance ensures that

$$(13) \quad W_\infty \geq d_{\min}/2,$$

which yields according to (11)

$$(14) \quad \sup_{t \in [0, T]} d_{\min}(t) \xrightarrow{N \rightarrow \infty} 0.$$

Assumption (10) is only needed for the second Theorem 0.2.

Our main results read:

Theorem 0.1. *Assume that (7), (8) and (9) are satisfied. If $r_0 \|\rho_0\|_{L^1 \cap L^\infty}$ is small enough, μ satisfies the following transport equation :*

$$\begin{cases} \partial_t \mu + \operatorname{div}_x[(\mathbb{A}(\xi))^{-1} \kappa g + u] \mu + \operatorname{div}_\xi[\nabla u \cdot \xi \mu] = 0, & \text{on } [0, T] \times \mathbb{R}^3 \times \mathbb{R}^3, \\ -\Delta u + \nabla p = 6\pi r_0 \kappa \rho g, & \text{on } \mathbb{R}^3, \\ \operatorname{div}(u) = 0, & \text{on } \mathbb{R}^3. \end{cases}$$

Remark 0.2. *Analogously to the model (1), global existence a uniqueness result can be shown for the former model following the result of [10].*

The second result concerns the case where the vectors along the line of centers ξ_i are correlated to the positions of centers x_+^i .

Theorem 0.2. *We consider the additional assumption (10). Assume that there exists a function $F_0 \in W^{1, \infty}$ such that $\xi_i(0) = F_0(x_+^i(0))$ for all $1 \leq i \leq N$. There exists $T > 0$ independent of N and unique $F^N \in L^\infty(0, T; W^{1, \infty})$ such that for all $t \in [0, T]$ we have:*

$$\mu^N = \rho^N \otimes \delta_{F^N} \quad \text{and} \quad F^N(0, \cdot) = F_0.$$

Moreover, the sequence $(F^N)_N$ admits a limit denoted $F \in L^\infty(0, T; W^{1, \infty})$. The limiting measure μ is of the form $\mu = \rho \otimes \delta_F$ and the triplet (ρ, F, u) satisfies the following system

$$(15) \quad \left\{ \begin{array}{ll} \partial_t F + \nabla F \cdot (\mathbb{A}(F)^{-1} \kappa g + u) = \nabla u \cdot F, & \text{on } [0, T] \times \mathbb{R}^3, \\ \partial_t \rho + \operatorname{div}((\mathbb{A}(F)^{-1} \kappa g + u) \rho) = 0, & \text{on } [0, T] \times \mathbb{R}^3, \\ -\Delta u + \nabla p = 6\pi r_0 \kappa g \rho, & \text{on } \mathbb{R}^3, \\ \operatorname{div} u = 0, & \text{on } \mathbb{R}^3, \\ \rho(0, \cdot) = \rho_0, & \text{on } \mathbb{R}^3, \\ F(0, \cdot) = F_0 & \text{on } \mathbb{R}^3. \end{array} \right.$$

Remark 0.3. The matrix \mathbb{A} is defined as $\mathbb{A} := A_1 + A_2$ where A_1 and A_2 are the resistance matrices associated to the sedimentation of a couple of identical spheres, see Section 1.1 for the definition. The term $(\mathbb{A})^{-1} \kappa g$ represents the mean velocity of a couple of identical particles sedimenting under gravitational field. We assume herein that $\mathbb{A}^{-1} \in W^{2, \infty}$.

We finish with a local existence and uniqueness result for the limit model.

Theorem 0.3. Let $p > 3$, $F_0 \in W^{2, p}$ and $\rho_0 \in W^{1, p}$ compactly supported. There exists $T > 0$ and unique triplet $(\rho, F, u) \in L^\infty(0, T; W^{1, p}) \times L^\infty(0, T; W^{2, p}) \times L^\infty(0, T; W^{3, p})$ satisfying (15).

As in [17], the idea of proof of Theorem 0.1 and 0.2 is to provide a derivation of the kinetic equation satisfied weakly by μ^N . This is done by computing the first order terms of the velocities of each pair:

$$(16) \quad \left\{ \begin{array}{l} \dot{x}_+^i \sim (\mathbb{A}(\xi_i))^{-1} \kappa g + \frac{6\pi r_0}{N} \sum_{j \neq i} \Phi(x_+^i - x_+^j) \kappa g, \\ \dot{\xi}_i \sim \left(\frac{6\pi r_0}{N} \sum_{j \neq i} \nabla \Phi(x_+^i - x_+^j) \kappa g \right) \cdot \xi_i. \end{array} \right.$$

The interaction force Φ is the Oseen tensor, see formula (17). This development is a corollary of the method of reflections which consists in approaching the solution u^N of $2N$ separated particles by the superposition of fields produced by the isolated $2N$ particle solutions. We refer to [18], [16], [14, Chapter 8] and [5, Section 4], [15] for an introduction to the topic. We also refer to [11] where a converging method of reflections is developed and is used in [10]. In this paper we reproduce the same method of reflections developed in [17, Section 3]. However this method is no longer valid in the case where the minimal distance is comparable to the particle radii. The idea is then to approach the velocity field u^N by the superposition of fields produced by the isolated N couple of particles $B^i = B(x_1^i, R) \sqcup B(x_2^i, R)$. This requires an analysis of the solution of the Stokes equation past a pair of particles. In particular, we need to show that these special solutions have the same decay rate as the Stokeslets, see [17, Section 2.1].

The convergence of the method of reflections is ensured under the condition that the minimal distance d_{\min} between the centers x_+^i satisfies

$$\frac{W_\infty^3}{d_{\min}} + \frac{W_\infty^3}{d_{\min}^2} < +\infty,$$

and that the distance $|x_1^i - x_2^i|$ for each pair satisfies formula (7).

In this paper, we focus only on the derivation of the mesoscopic model. Precisely, we do not tackle the propagation in time of the dilution regime and the mean field approximation. We provide in Propositions B.3 and B.1 some estimates showing that the control on the minimal distance d_{\min} depends on the control on the infinite Wasserstein distance W_∞ . However, the gradient of the Oseen tensor appearing in equation (16) leads to a log term in the estimates involving the control of W_∞ , see Proposition B.2. This prevents us from performing a Gronwall argument in order to prove the mean field approximation in the spirit of [6, 7].

0.2. Outline of the paper. The remaining sections of this paper are organized as follows. In section 2 we present an analysis of the particular solution of two translating spheres in a Stokes flow. In section 3 we present and prove the convergence of the method of reflections. In section 4 we compute the particle velocities $(\dot{x}_+^i, \dot{\xi}_i)_{1 \leq i \leq N}$. Sections 5 and 6 are devoted to the proofs of Theorem 0.1, 0.2 and 0.3. Finally, we gather all the preliminary estimates in the appendix.

0.3. Notations. In this paper, n always refers to the unit outer normal to a surface. We recall that the Green's function for the Stokes problem also called the Oseen tensor is defined as:

$$(17) \quad \Phi(x) = \frac{1}{8\pi} \left(\frac{\mathbb{I}}{|x|} + \frac{x \otimes x}{|x|^3} \right),$$

its associated pressure P reads:

$$P(x) = \frac{1}{4\pi} \frac{x}{|x|^3}.$$

See [4, Formula (IV.2.1)] or [14, Section 2.4.1].

Given a couple of velocities $(U_1, U_2) \in \mathbb{R}^3 \times \mathbb{R}^3$ we use the following notations

$$U_+ := \frac{U_1 + U_2}{2}, U_- := \frac{U_1 - U_2}{2}.$$

Finally, in the whole paper we use the symbol \lesssim to express an inequality with a multiplicative constant independent of N and depending only on $r_0, \|\rho_0\|_{L^1 \cap L^\infty}, \mathcal{E}_1, \mathcal{E}_2$ and eventually on $\kappa|g|$ which is uniformly bounded, see [17].

1. TWO TRANSLATING SPHERES IN A STOKES FLOW

In this section, we focus on the analysis of the Stokes problem in \mathbb{R}^3 past a pair of particles. Given $x_1, x_2 \in \mathbb{R}^3$ and $R_1, R_2 > 0$, such that $|x_1 - x_2| > R_1 + R_2$, we consider two spheres $B_\alpha := B(x_\alpha, R_\alpha)$ $\alpha = 1, 2$ and focus on the following Stokes problem:

$$(18) \quad \begin{cases} -\Delta u + \nabla p = 0, \\ \operatorname{div} u = 0, \end{cases} \text{ on } \mathbb{R}^3 \setminus \bar{B}_1 \cup \bar{B}_2,$$

completed with the no-slip boundary conditions:

$$(19) \quad \begin{cases} u = U_\alpha, & \text{on } \partial B_\alpha, \alpha = 1, 2, \\ \lim_{|x| \rightarrow \infty} |u(x)| = 0, \end{cases}$$

where $U_\alpha \in \mathbb{R}^3$ for $\alpha = 1, 2$. Classical results on the Steady Stokes equations for exterior domains (see [4, Chapter V] for more details) ensures the existence and uniqueness of equations (18) – (19). In this section, we aim to describe the velocity field u in terms of the force applied by the fluid on the particles defined as:

$$F_\alpha := \int_{\partial B_\alpha} \sigma(u, p)n, \quad \alpha = 1, 2.$$

We refer to the paper [13] for the following statements. Neglecting angular velocities and torque we emphasize that there exists a linear mapping called resistance matrix satisfying:

$$(20) \quad \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = -3\pi(R_1 + R_2) \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix},$$

where $A_{\alpha\beta}$, $1 \leq \alpha, \beta \leq 2$, are 3×3 matrices depending only on the non-dimensionalized centre-to-centre separation:

$$s := 2 \frac{x_1 - x_2}{R_1 + R_2},$$

and the ratio of the spheres' radii:

$$\lambda = \frac{R_1}{R_2},$$

each of these matrices is of the form:

$$(21) \quad A_{\alpha\beta} := g_{\alpha,\beta}(|s|, \lambda)\mathbb{I} + h_{\alpha,\beta}(|s|, \lambda) \frac{s \otimes s}{|s|^2},$$

where \mathbb{I} is the 3×3 identity matrix and $g_{\alpha,\beta}$, $h_{\alpha,\beta}$ are scalar functions. We refer to the paper of Jeffrey and Onishi [13] where the authors provide a development formulas for $g_{\alpha,\beta}$ and $h_{\alpha,\beta}$ given by a convergent power series of $|s|^{-1}$. Note that the matrices satisfy

$$(22) \quad \begin{aligned} A_{22}(s, \lambda) &= A_{11}(s, \lambda^{-1}), \\ A_{12}(s, \lambda) &= A_{21}(s, \lambda), \\ A_{12}(s, \lambda) &= A_{12}(s, \lambda^{-1}). \end{aligned}$$

Inversly, there exists also a linear mapping called mobility matrix such that

$$(23) \quad \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = -\frac{1}{3\pi(R_1 + R_2)} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}.$$

The matrices $a_{\alpha,\beta}$ depend on the same parameters as matrices $A_{\alpha,\beta}$ and satisfy a formula analogous to (21). They are also symmetric in the sense of formula (22).

The resistance and mobility matrices satisfy the following formula:

$$(24) \quad \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix},$$

Again, we refer to [13] for more details.

1.1. Restriction to the case of two identical spheres. We simplify the study by assuming that $R_1 = R_2 = R$ *i.e.* $\lambda = 1$. This means that the resistance matrix depends only on the parameter s which becomes:

$$s = \frac{x_1 - x_2}{R} = 2\xi,$$

and we have:

$$A_{22}(s, 1) = A_{11}(s, 1).$$

Hence we reformulate the resistance matrix as follows:

$$(25) \quad \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = -6\pi R \begin{pmatrix} A_1(\xi) & A_2(\xi) \\ A_2(\xi) & A_1(\xi) \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix},$$

and the mobility matrix:

$$(26) \quad \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = -(6\pi R)^{-1} \begin{pmatrix} a_1(\xi) & a_2(\xi) \\ a_2(\xi) & a_1(\xi) \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}.$$

Formula (24) yields the following relations

$$(27) \quad \begin{cases} A_1 a_1 + A_2 a_2 = \mathbb{I}, \\ A_1 a_2 + A_2 a_1 = 0. \end{cases}$$

We are interested in providing a formula for the velocity u and showing some decay properties. In this paper we use the notation $(U[U_1, U_2], P([U_1, U_2]))$ for the unique solution to

$$\begin{cases} -\Delta U[U_1, U_2] + \nabla P[U_1, U_2] = 0, \\ \operatorname{div} U[U_1, U_2] = 0, \end{cases} \text{ on } \mathbb{R}^3 \setminus \bar{B}_1 \cup \bar{B}_2,$$

completed with the no-slip boundary conditions:

$$\begin{cases} U[U_1, U_2] = U_\alpha, \text{ on } \partial B_\alpha, \alpha = 1, 2, \\ \lim_{|x| \rightarrow \infty} |U[U_1, U_2](x)| = 0, \end{cases}$$

We have the following preliminary result:

Proposition 1.1. *For all $x \notin \bar{B}_1 \cup \bar{B}_2$ the following formula holds true:*

$$(28) \quad u(x) = - \int_{\partial B_1} \Phi(\xi - x) [\sigma(u, p)n](\xi) d\xi - \int_{\partial B_2} \Phi(\xi - x) [\sigma(u, p)n](\xi) d\xi.$$

Proof. Without loss of generality we assume that the pair of particles is centered in the origin *i.e.* $x_+ = 0$. In what follows we use the following shortcut

$$(u, p) = (U[U_1, U_2], P[U_1, U_2]).$$

In order to prove the main property we need some preliminary decay rates. We keep the notation u for the extension of the velocity field on \mathbb{R}^3 by U_α on B_α , $\alpha = 1, 2$. Since $\nabla u \in L^2(\mathbb{R}^3)$, the classical steady Stokes regularity results (see [4, Theorem IV.4.1]) combined with some Sobolev embeddings ensures that $(u, p) \in \mathcal{C}^2(B(0, 3d/2) \setminus B(0, d)) \times \mathcal{C}^1(B(0, 3d/2) \setminus B(0, d))$, where d is large enough to have $\bar{B}_1 \cup \bar{B}_2 \subset B(0, 2|x_-|) \subset B(0, d)$. Hence, the idea is to consider $v = \chi_d u$, $\pi = \chi_d p$ where χ_d a regular truncation function

such that $\chi_d = 0$ on $B(0, d)$ and $\chi_d = 1$ on $B(0, 3d/2)$. The couple (v, π) satisfies a Stokes equation with a source term f and a compressible condition g depending on the data and supported in $B(0, 3d/2) \setminus B(0, d)$. Hence, the convolution formula with the Oseen tensor (see [4, Formula (IV.2.1)]) holds true and we emphasize that for $|x|$ large enough we have:

$$(29) \quad |v(x)| = |u(x)| \lesssim \frac{1}{|x|}, \quad |\pi(x)| = |p(x)| \lesssim \frac{1}{|x|^2}.$$

The proof of formula (28) relies on the Lorentz reciprocal theorem, see [14, Section 2.3], which stands that for a given domain $\Omega \subset \mathbb{R}^3$ and two divergence free vector fields v, v' on Ω , there holds

$$(30) \quad \int_{\partial\Omega} v \cdot (\sigma' n) + \int_{\Omega} v \cdot \operatorname{div} \sigma' + \int_{\partial\Omega} v' \cdot (\sigma n) + \int_{\Omega} v' \cdot \operatorname{div} \sigma = 0,$$

where σ, σ' the respective stress tensor of v and v' . On the other hand, we recall the definition of the Oseen tensor Φ and its associated pressure \mathcal{P} :

$$\Phi(x) = \frac{1}{8\pi} \left(\frac{\mathbb{I}}{|x|} + \frac{x \otimes x}{|x|^3} \right), \quad \mathcal{P}(x) = \frac{1}{4\pi} \frac{x}{|x|^3}.$$

We denote by Σ its (triadic) associated stress tensor :

$$\Sigma_{ijk} = (\Phi_{ij,k} + \Phi_{kj,i}) - \delta_{ik} \mathcal{P}_j = -6 \frac{x_i x_j x_k}{|x|^5},$$

where $\Phi_{ij,k} = \partial_{x_k} \Phi_{ij}$. Since Φ is the Green's function, its stress tensor satisfies

$$(\operatorname{div} \Sigma)_{ij} = \sum_k \partial_{x_k} \Sigma_{ijk} = \delta_0(x) \delta_{ij},$$

in the sense that for all regular divergence-free vector field v

$$(31) \quad \int_{\Omega} \operatorname{div} \Sigma v = \begin{cases} v(0) & \text{if } 0 \in \Omega, \\ 0 & \text{if } 0 \notin \Omega. \end{cases}$$

We apply the reciprocal theorem, formula (30), for $v = u$ and $v' = \Phi(x - \cdot)$, we obtain for all domain Ω and all $x \in \Omega$:

$$u(x) = - \int_{\partial\Omega} \Phi(\xi - x) [\sigma(u, p)n](\xi) d\sigma(\xi) - \int_{\partial\Omega} \Sigma(\xi - x)n(\xi) u(\xi) d\sigma(\xi).$$

We may then apply this formula by choosing $\Omega = B(0, \bar{R}) \setminus \overline{B_1 \cup B_2}$ with \bar{R} large enough to satisfy $\overline{B_1 \cup B_2} \subset B(0, \bar{R})$. We obtain then for all $x \in \Omega$:

$$\begin{aligned} u(x) &= - \int_{\partial B_1 \cup \partial B_2} \Phi(\xi - x) [\sigma(u, p)n](\xi) d\sigma(\xi) - \int_{\partial B_1 \cup \partial B_2} \Sigma(\xi - x)n(\xi) u(\xi) d\sigma(\xi) \\ &\quad - \int_{\partial B(0, \bar{R})} \Phi(\xi - x) [\sigma(u, p)n](\xi) d\sigma(\xi) - \int_{\partial B(0, \bar{R})} \Sigma(\xi - x)n(\xi) u(\xi) d\sigma(\xi) \end{aligned}$$

The two last terms on the right hand side vanish when $\bar{R} \rightarrow \infty$. This is due to the fact that Φ (resp. Σ) scales like $O(\frac{1}{\bar{R}})$ (resp. $O(\frac{1}{\bar{R}^2})$) and, according to the decay rate (29),

u , $R\sigma(u, p) \rightarrow 0$ for large \bar{R} .

For the term involving the stress tensor Σ and the velocity field u on $\partial B_1 \cup \partial B_2$ we recall that $u(x) = U_\alpha$ on ∂B_α $\alpha = 1, 2$. And as $x \notin \overline{B_1 \cup B_2}$ we have then:

$$\int_{\partial B_\alpha} \Sigma(\xi - x)n(\xi) = 0.$$

Finally for all $x \notin \overline{B_1 \cup B_2}$ the following formula holds true:

$$u(x) = - \int_{\partial B_1} \Phi(\xi - x) [\sigma(u, p)n](\xi) d\xi - \int_{\partial B_2} \Phi(\xi - x) [\sigma(u, p)n](\xi) d\xi.$$

□

Corollary 1.2. *The following development holds true up to order 2:*

$$(32) \quad U[U_1, U_2](x) = -\Phi(x - x_1)F_1 - \Phi(x_2 - x)F_2.$$

There exists a function f independent of the data such that the unique solution $(U[U_1, U_2], P[U_1, U_2])$ satisfies the following decay property for all $x \notin B(x_+, 2|x_-|)$:

$$(33) \quad \frac{|U[U_1, U_2](x)|}{|x_+ - x|} + |\nabla U[U_1, U_2](x)| + |P[U_1, U_2](x)| \lesssim 6\pi R |f(|\xi|)| \frac{\max(|U_1|, |U_2|)}{|x_+ - x|^2}.$$

Precisely, we have for all $x \notin B(x_+, 2|x_-|)$:

$$(34) \quad U[U_1, U_2](x) \lesssim 6\pi R |f(|\xi|)| \left(\frac{|U_+|}{|x_+ - x|} + \frac{|U_+| + |U_-|}{|x_+ - x|^2} |x_-| \right),$$

where $\xi = \frac{|x_-|}{R}$ and $U_+ = \frac{U_1 + U_2}{2}$, $U_- = \frac{U_1 - U_2}{2}$.

Proof. As in [14, Section 2.5], we take the Taylor series of $\Phi(x - \xi)$ in ξ to obtain an approximation of the velocity field u that holds true up to the order 3. We recall that if we neglect the torque we have:

$$(35) \quad \int_{\partial B_\alpha} [\sigma(u, p)n](\xi) d\xi = F_\alpha,$$

$$(36) \quad \int_{\partial B_\alpha} [\sigma(u, p)n](\xi) \times (\xi - x_\alpha) d\xi = 0.$$

Replacing $\Phi(x - \xi)$ by its development:

$$\Phi(\xi - x) = \Phi(x_\alpha - x) + \nabla \Phi(x_\alpha - x) (\xi - x_\alpha) + \dots$$

in formula (28), we thus obtain the following formula which is exact up to second order:

$$u(x) \sim - \sum_{\alpha=1}^2 \Phi(x - x_\alpha) F_\alpha,$$

recall that the forces are given by the following formulas:

$$\begin{aligned} F_1 &= 6\pi R (A_1 U_1 + A_2 U_2), \\ F_2 &= 6\pi R (A_2 U_1 + A_1 U_2). \end{aligned}$$

We have then the existence of a scalar function f independent of the data such that:

$$|F_\alpha| \leq 6\pi R |f(|\xi|)| \max(|U_1|, |U_2|).$$

This yields the following decay rate for all $x \notin \overline{B(x_1, R) \cup B(x_2, R)}$:

$$(37) \quad |u(x)| \lesssim 6\pi R \left(\frac{1}{|x - x_1|} + \frac{1}{|x - x_2|} \right) |f(|\xi|)| \max(|U_1|, |U_2|).$$

The remaining estimates are obtained using direct computations and the following formulas:

$$(38) \quad F_1 = F_+ + F_-, \quad F_2 = F_+ - F_-.$$

□

2. THE METHOD OF REFLECTIONS

In this section, we aim to show that the method of reflections holds true in the special case where the minimal distance and the radius R are of the same order. The idea is to approach the velocity field u^N by the particular solutions developed in the section above. We recall that u^N is the unique solution to the following Stokes problem :

$$\begin{cases} -\Delta u^N + \nabla p^N = 0, \\ \operatorname{div} u^N = 0, \end{cases} \text{ on } \mathbb{R}^3 \setminus \bigcup_{i=1}^N \bar{B}^i,$$

completed with the no-slip boundary conditions :

$$\begin{cases} u^N = U_1^i, & \text{on } \partial B(x_1^i, R), \\ u^N = U_2^i, & \text{on } \partial B(x_2^i, R), \\ \lim_{|x| \rightarrow \infty} |u^N(x)| = 0, \end{cases}$$

where $(U_1^i, U_2^i) \in \mathbb{R}^3 \times \mathbb{R}^3$, $1 \leq i \leq N$ are such that:

$$\begin{pmatrix} F_1^i \\ F_2^i \end{pmatrix} = - \begin{pmatrix} mg \\ mg \end{pmatrix}, \quad \forall 1 \leq i \leq N.$$

Thanks to the superposition principle, the sum of the N solutions $\sum_{i=1}^N U[U_1^i, U_2^i]$ satisfy a Stokes equation on $\mathbb{R}^3 \setminus \bigcup_{i=1}^N B^i$, but do not match the boundary conditions. Hence, we define the error term:

$$U[u_*^{(1)}] = u - \sum_{i=1}^N U[U_1^i, U_2^i],$$

which satisfies a Stokes equation on $\mathbb{R}^3 \setminus \overline{\bigcup_{i=1}^N B^i}$ completed with the following boundary conditions for all $1 \leq i \leq N$, $\alpha = 1, 2$ and $x \in B(x_\alpha^i, R)$:

$$u_*^{(1)}(x) = - \sum_{j \neq i} U[U_1^j, U_2^j](x).$$

We set then for $\alpha = 1, 2$ and $1 \leq i \leq N$:

$$U_\alpha^{i,(1)} := u_*^{(1)}(x_\alpha^i),$$

and reproduce the same approximation to obtain:

$$U[u_*^{(2)}] := u - \sum_{i=1}^N \left(U[U_1^i, U_2^i] + U[U_1^{i,(1)}, U_2^{i,(1)}] \right),$$

which satisfies a Stokes equation with the following boundary conditions for all $1 \leq i \leq N$, $\alpha = 1, 2$ and $x \in B(x_\alpha^i, R)$:

$$u_*^{(2)}(x) = u_*^{(1)}(x) - u_*^{(1)}(x_\alpha^i) - \sum_{j \neq i} U[U_1^{j,(1)}, U_2^{j,(1)}](x).$$

By iterating the process, one can show that for all $k \geq 1$ we have:

$$u = \sum_{p=0}^k \sum_{i=1}^N U[U_1^{i,(p)}, U_2^{i,(p)}] + U[u_*^{(k+1)}],$$

where for all $\alpha = 1, 2$, $1 \leq i \leq N$ and $p \geq 0$:

$$u_*^{(p+1)}(x) = u_*^{(p)}(x) - u_*^{(p)}(x_\alpha^i) - \sum_{j \neq i} U[U_1^{j,(p)}, U_2^{j,(p)}](x),$$

$$u_*^{(0)} = \sum_{i=1}^N U_1^i 1_{B(x_1^i, R)} + U_2^i 1_{B(x_2^i, R)},$$

$$U_\alpha^{i,(p)} = u_*^{(p)}(x_\alpha^i),$$

$$(39) \quad U_\alpha^{i,(0)} = U_\alpha^i.$$

The convergence is analogous to the convergence proof in [17, Section 3.1]. We begin by the following estimates that are needed in the computations.

Lemma 2.1. *Under assumptions (7), (9) we have for all $1 \leq i \neq j \leq N$, $1 \leq \alpha, \beta \leq 2$:*

$$(40) \quad |x_\alpha^i - x_\beta^j| \geq \frac{1}{2} |x_+^i - x_+^j|.$$

The first step is to show that the sequence $\max_i(\max(|U_1^{i,(p)}|, |U_2^{i,(p)}|))$ converges when p goes to infinity.

Lemma 2.2. *Under assumptions (7), (8), (9) and the assumption that $r_0\|\rho_0\|_{L^1\cap L^\infty}$ is small enough, there exists a positive constant $K < 1/2$ satisfying for all $1 \leq i \leq N$, $p \geq 0$*

$$\max_i(\max(|U_1^{i,(p+1)}|, |U_2^{i,(p+1)}|)) \leq K \max_i(\max(|U_1^{i,(p)}|, |U_2^{i,(p)}|)),$$

for N large enough.

Proof. According to formulas (33) and Lemma 2.1, we have for all $\alpha = 1, 2$ and $1 \leq i \leq N$:

$$\begin{aligned} |U_\alpha^{i,(p+1)}| &\leq \left| \sum_{j \neq i} U[U_1^{j,(p)}, U_2^{j,(p)}](x_\alpha^i) \right| \\ &\lesssim \frac{6\pi r_0}{N} \left(\sum_{j \neq i} \frac{|f(|\xi_j|)|}{d_{ij}} \right) \max_j (|U_1^{j,(p)}|, |U_2^{j,(p)}|) \\ &\leq Cr_0\|\rho\|_{L^1\cap L^\infty} \left(\frac{W_\infty^3}{d_{\min}} + 1 \right), \end{aligned}$$

where we used Lemma A.1 and the fact that for all $1 \leq j \leq N$ and $N \in \mathbb{N}^*$

$$|f(|\xi_j|)| \leq \sup_{2 < |s| \leq M_1} |f(|s|)|,$$

according to assumption (7). Hence, the first term in the right-hand side vanishes according to (9) and (14).

Finally, if we assume that $r_0\|\rho\|_{L^1\cap L^\infty}$ is small enough, we obtain the existence of a positive constant $K < 1/2$ such that:

$$\max_i(\max(|U_1^{i,(p+1)}|, |U_2^{i,(p+1)}|)) \leq K \max_i(\max(|U_1^{i,(p)}|, |U_2^{i,(p)}|)).$$

□

We have the following result.

Proposition 2.3. *Under the same assumptions as Lemma 2.2, we have for N large enough:*

$$\lim_{k \rightarrow \infty} \|\nabla U[u_*^{(k+1)}]\|_2 \lesssim R \max_{\substack{1 \leq i \leq N \\ \alpha=1,2}} |U_\alpha^i|.$$

Proof. The proof is analogous to the convergence proof of [17, Proposition 3.4]. This is due to the fact that the particular solutions have the same decay rate as the Oseen-tensor. □

2.1. Two particular cases.

2.1.1. *First case.* Given $W \in \mathbb{R}^3$ we consider in this part w the unique solution to the Stokes equation (2) completed with the following boundary conditions :

$$(41) \quad w = \begin{cases} W & \text{on } B(x_1^1, R), \\ -W & \text{on } B(x_2^1, R), \\ 0 & \text{on } B(x_1^i, R) \cup B(x_2^i, R), i \neq 1. \end{cases}$$

We denote by $W_\alpha^{i,(p)}$, $\alpha = 1, 2$, $1 \leq i \leq N$, $p \in \mathbb{N}$ the velocities obtained from the method of reflections applied to the velocity field w . In other words :

$$w = \sum_{p=0}^k \sum_i U[W_1^{i,(p)}, W_2^{i,(p)}] + U[w_*^{(k+1)}].$$

We aim to show that, in this special case, the sequence of velocities $W_\alpha^{i,(p)}$ and the error term $U[w_*^{(k)}]$ are much smaller than before. This is due to the initial vanishing boundary conditions for $i \neq 1$. Indeed we have :

Proposition 2.4. *There exists two positive constants $C, L > 0$ such that for N large enough:*

$$\begin{aligned} \max_{\alpha=1,2} |W_\alpha^{i,(p+1)}| &\leq C(2Cr_0L\|\rho_0\|_{L^\infty \cap L^1})^p \frac{R|x_-^1|}{|x_+^1 - x_+^i|^2} |W|, \quad i \neq 1, \quad p \geq 0, \\ \max_\alpha |W_\alpha^{1,(p+1)}| &\leq C2^{p-1}(r_0CL\|\rho_0\|_{L^\infty \cap L^1})^p |x_-^1| \frac{R}{d_{\min}} |W|, \quad p \geq 1, \\ \max_\alpha |W_\alpha^{i,(0)}| + \max_\alpha |W_\alpha^{1,(1)}| &= 0, \quad i \neq 1. \end{aligned}$$

Proof. We show that the statement holds true for $p = 0$ then we prove it for all $p \geq 1$ by induction. According to formula (39) we have for $p = 0$:

$$W_\alpha^{1,(0)} = W\delta_{\alpha 1} - W\delta_{\alpha 2},$$

and for $i \neq 1$, $\alpha = 1, 2$, $U_\alpha^{i,(0)} = 0$. This yields for $i \neq 1$, $\alpha = 1, 2$:

$$\begin{aligned} W_\alpha^{i,(1)} &= U[W_1^{1,(0)}, W_2^{1,(0)}](x_\alpha^i), \\ &= \Phi(x_1^1 - x_\alpha^i)F_1^1 + \Phi(x_2^1 - x_\alpha^i)F_2^1, \end{aligned}$$

where:

$$F_1^1 = -6\pi R(A_1(s^1) - A_2(s^1))W, \quad F_2^1 = -6\pi R(A_2(s^1) - A_1(s^1))W.$$

Hence, $F_2^1 = -F_1^1$ we have then using Lemma 2.1:

$$\begin{aligned} |W_\alpha^{i,(1)}| &\leq |\Phi(x_1^1 - x_\alpha^i) - \Phi(x_2^1 - x_\alpha^i)| |F_1^1|, \\ &\lesssim 6\pi R \frac{|x_-^1|}{d_{i1}^2} |A_1(s^1) - A_2(s^1)| |W|, \end{aligned}$$

thus, we denote by $C > 0$ the global positive constant appearing in the estimate above. This shows that the first statement holds true for $p = 0$. For the second estimate we have

$|W_\alpha^{1,(1)}| = 0$ and for $p = 1$ we have:

$$\begin{aligned}
|W_\alpha^{1,(2)}| &= \left| \sum_{j \neq 1} U[W_1^{j,(1)}, W_2^{j,(1)}](x_\alpha^1) \right|, \\
&\leq C \sum_{j \neq 1} \frac{R}{d_{1j}} \max(|W_1^{j,(1)}|, |W_2^{j,(1)}|), \\
&\leq C \sum_{j \neq 1} \left(\frac{CR^2|x_-^1|}{d_{1j}^3} \right) |W|, \\
&\leq C \frac{|x_-^1|R}{d_{\min}} (CKr_0) |W|,
\end{aligned}$$

where we used Lemma A.1 for $k = 2$ and assumption (9). In what follows we define the constant $L > 0$ as the constant satisfying:

$$\max_i \left(\frac{1}{N} \sum_{j \neq i} \left(\frac{1}{d_{ij}^2} \right) + \frac{1}{N} \sum_{j \neq 1, i} \left(\frac{1}{d_{ij}} + \frac{1}{d_{1j}} \right) \right) \leq L \|\rho_0\|_{L^\infty \cap L^1}.$$

Now for all $p \geq 1$, $i \neq 1$ we have:

$$\begin{aligned}
|W_\alpha^{i,(p+1)}| &= \left| \sum_{j \neq i} U[W_1^{j,(p)}, W_2^{j,(p)}](x_\alpha^i) \right| \\
&\leq C \sum_{j \neq i} \frac{R}{d_{ij}} \max(|W_1^{j,(p)}|, |W_2^{j,(p)}|), \\
&\leq C \left(\sum_{j \neq i, 1} \frac{R}{d_{ij}} C(2Cr_0L\|\rho_0\|_{L^\infty \cap L^1})^{p-1} \frac{R|x_-^1|}{d_{1j}^2} \right. \\
&\quad \left. + \frac{R}{d_{i1}} \frac{R|x_-^1|}{d_{\min}} C2^{p-2}(r_0CL\|\rho_0\|_{L^\infty \cap L^1})^{p-1} \right) |W|,
\end{aligned}$$

using the fact that $\frac{1}{d_{ij}d_{kj}} \leq \frac{1}{d_{ik}} \left(\frac{1}{d_{ij}} + \frac{1}{d_{kj}} \right)$ we obtain

$$\begin{aligned}
|W_\alpha^{i,(p+1)}| &\leq C \left(\frac{R|x_-^1|}{d_{i1}} C(2Cr_0L\|\rho_0\|_{L^\infty \cap L^1})^{p-1} \left(\frac{1}{d_{1i}} \sum_{j \neq i,1} \left(\frac{R}{d_{ij}} + \frac{R}{d_{1j}} \right) + \sum_{j \neq i,1} \frac{R}{d_{1j}^2} \right) \right. \\
&\quad \left. + \frac{R}{d_{i1}} \frac{R|x_-^1|}{d_{\min}} C2^{p-2}(r_0CL\|\rho_0\|_{L^\infty \cap L^1})^{p-1} \right) |W|, \\
&\leq C \frac{R|x_-^1|}{d_{i1}} \left(C(2Cr_0L\|\rho_0\|_{L^\infty \cap L^1})^{p-1} \left(\frac{r_0L\|\rho_0\|_{L^\infty \cap L^1}}{d_{1i}} \right) \right. \\
&\quad \left. + \frac{R}{d_{\min}} C2^{p-2}(r_0CL\|\rho_0\|_{L^\infty \cap L^1})^{p-1} \right) |W|, \\
&\leq C \frac{R|x_-^1|}{d_{i1}^2} \left((Cr_0L\|\rho_0\|_{L^\infty \cap L^1})^p 2^{p-1} + \frac{Rd_{i1}}{d_{\min}} C2^{p-2}(r_0CL\|\rho_0\|_{L^\infty \cap L^1})^{p-1} \right) |W|.
\end{aligned}$$

Since $\frac{Rd_{i1}}{d_{\min}} \ll r_0L\|\rho_0\|_{L^\infty \cap L^1}$, the second term can be bounded by $(Cr_0L\|\rho_0\|_{L^\infty \cap L^1})^p 2^{p-2}$ which yields the expected result because $2^{p-1} + 2^{p-2} \leq 2^p$.

We prove now the second estimate. Let $p \geq 1$:

$$\begin{aligned}
|W_\alpha^{1,(p+1)}| &= \left| \sum_{j \neq 1} U[W_1^{j,(p)}, W_2^{j,(p)}](x_\alpha^1) \right|, \\
&\leq C \sum_{j \neq 1} \frac{R}{d_{j1}} \max(|W_1^{j,(p)}|, |W_2^{j,(p)}|), \\
&\leq C \left(\sum_{j \neq 1} \frac{R}{d_{1j}} C(2Cr_0L\|\rho_0\|_{L^\infty \cap L^1})^{p-1} \frac{R|x_-^1|}{d_{1j}^2} \right) |W|, \\
&\leq C(2Cr_0L\|\rho_0\|_{L^\infty \cap L^1})^{p-1} C \frac{R}{d_{\min}} |x_-^1| \left(\sum_{j \neq 1} \frac{R}{d_{1j}^2} \right) |W|, \\
&\leq C2^{p-1} (Cr_0L\|\rho_0\|_{L^\infty \cap L^1})^p \frac{R}{d_{\min}} |x_-^1| |W|.
\end{aligned}$$

□

According to these estimates, if we assume that $r_0\|\rho_0\|_{L^\infty \cap L^1}$ is small enough to have $2L\|\rho_0\|_{L^\infty \cap L^1}Cr_0 < 1$ then the following result holds true :

Corollary 2.5. *Under the assumption that $r_0\|\rho_0\|_{L^\infty \cap L^1}$ is small enough we have :*

$$\begin{aligned}
\sum_{p=0}^{\infty} \max_{\alpha=1,2} |W_\alpha^{i,(p)}| &\lesssim \frac{R|x_-^1|}{|x_+^1 - x_+^i|^2} |W|, \quad i \neq 1, \\
\sum_{p=1}^{\infty} \max_{\alpha=1,2} |W_\alpha^{1,(p)}| &\lesssim \frac{R|x_-^1|}{d_{\min}} |W|,
\end{aligned}$$

for N large enough.

This result shows that we can obtain a better estimate for the error term of the method of reflections in this particular case:

Proposition 2.6. *We set $\eta := 2C\|\rho_0\|_{L^\infty \cap L^1} Lr_0 < 1$ the constant introduced in Proposition 2.4. For all $i \neq 1$ we have*

$$\begin{aligned} \|\nabla w_*^{(k)}\|_{L^\infty(B_i)} &\lesssim \frac{R|x_-^1|}{d_{i1}^3}|W|, \\ \|w_*^{(k+1)}\|_{L^\infty(B_i)} &\lesssim R\|\nabla w_*^{(k)}\|_{L^\infty(B_i)} + \frac{R}{d_{1i}^2}|x_-^1|\eta^{k-1}|W|. \end{aligned}$$

And for $i = 1$ we have :

$$\begin{aligned} \|\nabla w_*^{(k)}\|_{L^\infty(B_1)} &\lesssim \frac{R}{d_{\min}}|x_-^1| \left(\frac{W_\infty^3}{d_{\min}^3} + |\log W_\infty| \right) |W|, \\ \|w_*^{(k+1)}\|_{L^\infty(B_1)} &\lesssim R\|\nabla w_*^{(k)}\|_{L^\infty(B_1)} + \frac{R}{d_{\min}}|x_-^1|\eta^{k-1}|W|, \end{aligned}$$

Proof. Estimate for $\|\nabla w_^{(k)}\|_\infty$.*

Let $x \in B(x_\alpha^i, R)$, with $\alpha = 1, 2$ and $i \neq 1$, formula (39) yields:

$$\begin{aligned} |\nabla w_*^{(k+1)}(x)| &\leq |\nabla w_*^{(k)}(x)| + \sum_{j \neq i} |\nabla U[W_1^{j,(k)}, W_2^{j,(k)}](x)|, \\ &\leq \sum_{p=0}^k \sum_{j \neq i} |\nabla U[W_1^{j,(p)}, W_2^{j,(p)}](x)|, \\ &\leq \sum_{p=0}^k \sum_{j \neq i, 1} |\nabla U[W_1^{j,(p)}, W_2^{j,(p)}](x)| + \sum_{p=1}^k |\nabla U[W_1^{1,(p)}, W_2^{1,(p)}](x)| \\ &\quad + |\nabla U[W_1^{1,(0)}, W_2^{1,(0)}](x)|. \end{aligned}$$

We estimate the first term applying Corollary 2.5:

$$\begin{aligned}
\sum_{p=0}^k \sum_{j \neq i, 1} |\nabla U[W_1^{j,(p)}, W_2^{j,(p)}](x)| &\leq C \sum_{p=0}^k \sum_{j \neq i, 1} \left(\frac{R}{|x_1^j - x_\alpha^i|^2} + \frac{R}{|x_2^j - x_\alpha^i|^2} \right) \max_{\alpha=1,2} |W_\alpha^{j,(p)}|, \\
&\leq 2C \sum_{p=0}^k \sum_{j \neq i, 1} \left(\frac{R}{d_{ij}^2} \right) \max_{\alpha=1,2} |W_\alpha^{j,(p)}|, \\
&\lesssim \sum_{j \neq i, 1} \left(\frac{R}{d_{ij}^2} \frac{R|x_-^1|}{d_{1j}^2} \right) |W|, \\
&\lesssim \frac{R|x_-^1|}{d_{1i}^2} \sum_{j \neq i, 1} \left(\frac{R}{d_{ij}^2} + \frac{R}{d_{1j}^2} \right) |W|, \\
&\lesssim \frac{R|x_-^1|}{d_{1i}^2} |W|.
\end{aligned}$$

We reproduce the same for the second term applying Corollary 2.5:

$$\begin{aligned}
\sum_{p=1}^k |\nabla U[W_1^{1,(p)}, W_2^{1,(p)}](x)| &\leq 2C \sum_{p=1}^k \left(\frac{R}{|x_+^1 - x_+^i|^2} \right) \max(|W_1^{1,(p)}|, |W_2^{1,(p)}|), \\
&\lesssim \frac{R}{|x_+^1 - x_+^i|^2} \frac{R}{d_{\min}} |x_-^1| |W|.
\end{aligned}$$

For the last term we recall that:

$$\nabla U[W_1^{1,(0)}, W_2^{1,(0)}](x) = \nabla \Phi(x_1^1 - x) F_1^1 + \nabla \Phi(x_2^1 - x) F_2^1,$$

as $(W_1^{1,(0)}, W_2^{1,(0)}) = (W, -W)$ we have:

$$\begin{cases} F_1^1 &= -6\pi R(A_1(\xi_1)W - A_2(\xi_1)W), \\ F_2^1 &= -6\pi R(A_2(\xi_1)W - A_1(\xi_1)W). \end{cases}$$

Thus $F_2^1 = -F_1^1$ and we obtain since $x \in B(x_\alpha^i, R)$, $i \neq 1$:

$$\begin{aligned}
\left| \nabla U[W_1^{1,(0)}, W_2^{1,(0)}](x) \right| &= |(\nabla \Phi(x_1^1 - x) - \nabla \Phi(x_2^1 - x)) F_1^1|, \\
&\lesssim \frac{R|x_-^1|}{|x_+^1 - x_+^i|^3} |W|.
\end{aligned}$$

Gathering all the inequalities we have for $i \neq 1$:

$$\|\nabla w_*^{(k)}\|_{L^\infty(B_i)} \lesssim \frac{R|x_-^1|}{|x_+^1 - x_+^i|^3} |W|$$

Analogously for $i = 1$ we obtain:

$$\begin{aligned}
|\nabla w_*^{(k+1)}(x)| &\leq |\nabla w_*^{(k)}(x)| + \sum_{j \neq 1} |\nabla U[W_1^{j,(k)}, W_2^{j,(k)}](x)|, \\
&\leq \sum_{p=0}^k \sum_{j \neq 1} |\nabla U[W_1^{j,(p)}, W_2^{j,(p)}](x)|, \\
&\leq 2C \sum_{p=0}^k \sum_{j \neq 1} \left(\frac{R}{d_{1j}^2} \right) \max(|W_1^{j,(p)}|, |W_2^{j,(p)}|), \\
&\lesssim \sum_{j \neq 1} \left(\frac{R}{d_{1j}^2} \frac{R|x_-^1|}{d_{1j}^2} \right) |W|, \\
&\lesssim \frac{R|x_-^1|}{d_{\min}} \left(\frac{W_\infty^3}{d_{\min}^3} + |\log W_\infty| \right) |W|.
\end{aligned}$$

Estimate for $\|w_*^{(k)}\|_\infty$. Let $x \in B(x_\alpha^i, R)$, $\alpha = 1, 2$, $i \neq 1$. We have according to formula (39) :

$$\begin{aligned}
|w_*^{(k+1)}(x)| &= \left| w_*^{(k)}(x) - w_*^{(k)}(x_\alpha^i) - \sum_{j \neq i} U[W_1^{j,(k)}, W_2^{j,(k)}](x) \right|, \\
&\leq R \|\nabla w_*^{(k)}\|_\infty + \sum_{j \neq i} \left| U[W_1^{j,(k)}, W_2^{j,(k)}](x) \right|, \\
&\leq R \|\nabla w_*^{(k)}\|_\infty + C \sum_{j \neq i} \frac{R}{d_{ij}} \max(|W_1^{j,(k)}|, |W_2^{j,(k)}|), \\
&\lesssim R \|\nabla w_*^{(k)}\|_\infty + \left(\sum_{j \neq i, 1} \frac{R}{d_{ij}} \eta^{k-1} \frac{R}{d_{1j}^2} + \frac{R}{d_{1i}} \eta^{k-1} \frac{R}{d_{\min}} \right) |x_-^1| |W|.
\end{aligned}$$

where $\eta = 2Cr_0L < 1$ is the constant appearing in Proposition 2.4. Reproducing the same computations as before yields:

$$\|w_*^{(k+1)}\|_{L^\infty(B_i)} \lesssim R \|\nabla w_*^{(k)}\|_\infty + \frac{R}{d_{1i}^2} |x_-^1| \eta^{k-1} |W|.$$

In the case $i = 1$ we have:

$$\begin{aligned}
|w_*^{(k+1)}(x)| &= \left| w_*^{(k)}(x) - w_*^{(k)}(x_\alpha^i) - \sum_{j \neq i} U[W_1^{j,(k)}, W_2^{j,(k)}](x) \right|, \\
&\leq R \|\nabla w_*^{(k)}\|_\infty + C \sum_{j \neq 1} \frac{R}{d_{1j}} \max(|W_1^{j,(k)}|, |W_2^{j,(k)}|), \\
&\lesssim R \|\nabla w_*^{(k)}\|_\infty + \sum_{j \neq 1} \frac{R}{d_{1j}} \eta^{k-1} \frac{R}{d_{1j}^2} |x_-^1| |W|, \\
&\lesssim R \|\nabla w_*^{(k)}\|_\infty + \frac{R}{d_{\min}} |x_-^1| \eta^{k-1} |W|.
\end{aligned}$$

□

Thanks to these estimates we have the following convergence rate:

Proposition 2.7.

$$\lim_{k \rightarrow \infty} \|\nabla U[w_*^{(k+1)}]\|_2 \lesssim R |x_-^1| |W|.$$

Proof. Reproducing exactly the same proof as in [17, Proposition 3.4], the main difference appears in the last estimate where we apply Proposition 2.6:

$$\begin{aligned}
\|\nabla U[w_*^{(k+1)}]\|_2^2 &\lesssim R^3 \sum_i \left(\|\nabla w_*^{(k+1)}\|_{L^\infty(B_i)} + \frac{1}{R} \|w_*^{(k+1)}\|_{L^\infty(B_i)} \right)^2, \\
&\lesssim R^3 \left[\sum_{i \neq 1} \left(\frac{R^2}{d_{1i}^6} + \frac{1}{d_{1i}^4} \eta^{2(k-1)} \right) \right. \\
&\quad \left. + \frac{R^2}{d_{\min}^2} \left(\frac{W_\infty^3}{d_{\min}^3} + |\log W_\infty| \right)^2 + \frac{1}{d_{\min}^2} \eta^{2(k-1)} \right] |x_-^1|^2 |W|^2, \\
&\lesssim \left(\frac{R^4}{d_{\min}^3} + \frac{R^2}{d_{\min}} \eta^{2(k-1)} \right) \left(\frac{W_\infty^3}{d_{\min}^3} + |\log W_\infty| \right) |x_-^1|^2 |W|^2 \\
&\quad + |x_-^1|^2 |W|^2 \frac{R^5}{d_{\min}^2} \left(\frac{W_\infty^3}{d_{\min}^3} + |\log W_\infty| \right)^2 + \frac{R^3}{d_{\min}^2} \eta^{2(k-1)} |x_-^1|^2 |W|^2.
\end{aligned}$$

Taking the limit when k goes to infinity we get:

$$\|\nabla U[w_*^{(k+1)}]\|_2^2 \lesssim R^2 |x_-^1|^2 |W|^2 \left\{ \frac{R^2}{d_{\min}^3} \left(\frac{W_\infty^3}{d_{\min}^3} + |\log W_\infty| \right) + \frac{R^3}{d_{\min}^2} \left(\frac{W_\infty^3}{d_{\min}^3} + |\log W_\infty| \right)^2 \right\}.$$

The term inside brackets is bounded as follows:

$$\begin{aligned} \frac{R^2}{d_{\min}^3} \left(\frac{W_\infty^3}{d_{\min}^3} + |\log W_\infty| \right) + \frac{R^3}{d_{\min}^2} \left(\frac{W_\infty^3}{d_{\min}^3} + |\log W_\infty| \right)^2 \\ \leq \frac{R^2}{d_{\min}^2} \frac{W_\infty^3}{d_{\min}^2} + R |\log W_\infty| + \frac{R}{d_{\min}^2} \left(\frac{R}{d_{\min}} \frac{W_\infty^3}{d_{\min}^2} + R |\log W_\infty| \right)^2, \end{aligned}$$

we recall that $\frac{R}{d_{\min}} < +\infty$ and $\frac{R}{d_{\min}^2} \leq \frac{r_0}{2} \|\rho_0\|_{L^\infty \cap L^1} \frac{W_\infty^3}{d_{\min}^2}$ according to (12). \square

2.1.2. *Second case.* Given $W \in \mathbb{R}^3$ we consider in this part w the unique solution to the Stokes equation (2) completed with the following boundary conditions :

$$(42) \quad w = \begin{cases} W & \text{on } B(x_1^1, R), \\ W & \text{on } B(x_2^1, R), \\ 0 & \text{on } B(x_1^i, R) \cup B(x_2^i, R), i \neq 1. \end{cases}$$

Denote by $U_\alpha^{i,(p)}$, $\alpha = 1, 2$, $1 \leq i \leq N$, $p \in \mathbb{N}$ the velocities obtained from the method of reflections applied to the velocity field w . In other words :

$$w = \sum_{p=0}^{\infty} \sum_i U[W_1^{i,(p)}, W_2^{i,(p)}] + O(R).$$

We aim to show that, in this special case, the sequence of velocities $W_\alpha^{i,(p)}$ are also smaller than the general case. This is due to the initial boundary conditions which vanish for $i \neq 1$. Indeed we have :

Proposition 2.8. *There exists two positive constants $C, L > 0$ such that :*

$$\begin{aligned} \max_{\alpha=1,2} |W_\alpha^{i,(p+1)}| &\leq C(2Cr_0L\|\rho_0\|_{L^\infty \cap L^1})^p \frac{R}{|x_+^1 - x_+^i|} |W|, i \neq 1, p \geq 0, \\ \max_\alpha |W_\alpha^{1,(p+1)}| &\leq C2^{p-1}(r_0CL\|\rho_0\|_{L^\infty \cap L^1})^p R |W|, p \geq 1, \\ \max_\alpha |W_\alpha^{1,(1)}| &= 0, \end{aligned}$$

for N large enough.

Proof. The proof is analogous to the one of Proposition 2.4. \square

According to these estimates, if we assume that $r_0\|\rho_0\|_{L^\infty \cap L^1}$ is small enough to have $2LCr_0\|\rho_0\|_{L^\infty \cap L^1} < 1$ then the following result holds true:

Corollary 2.9. *Under the assumption that $r_0\|\rho_0\|_{L^\infty \cap L^1}$ is small enough we have for N large enough:*

$$\begin{aligned} \sum_{k=0}^{\infty} \max_{\alpha=1,2} |W_\alpha^{i,(p+1)}| &\lesssim \frac{R}{|x_+^1 - x_+^i|} |W|, i \neq 1, \\ \sum_{k=0}^{\infty} \max_\alpha |W_\alpha^{1,(p+1)}| &\lesssim R |W|. \end{aligned}$$

3. EXTRACTION OF THE FIRST ORDER TERMS FOR THE VELOCITIES

In this section, we apply the method of reflections to the velocity field u^N as presented above and we set :

$$\sum_{p=0}^{\infty} U_{\alpha}^{i,(p)} = U_{\alpha}^{i,\infty}, \quad 1 \leq \alpha \leq 2, \quad 1 \leq i \leq N,$$

we also use the following notations for the forces associated to the solutions $U[U_1^{i,\infty}, U_2^{i,\infty}]$:

$$(43) \quad \begin{aligned} F_1^{i,\infty} &= -6\pi R(A_1(\xi_i)U_1^{i,\infty} + A_2(\xi_i)U_2^{i,\infty}), \\ F_2^{i,\infty} &= -6\pi R(A_2(\xi_i)U_1^{i,\infty} + A_1(\xi_i)U_2^{i,\infty}). \end{aligned}$$

3.1. Preliminary estimates.

Proposition 3.1. *If assumptions (7), (8) (9) hold true and $r_0 \|\rho_0\|_{L^\infty \cap L^1}$ is small enough we have for N large enough and for all $1 \leq i \leq N$*

$$\begin{aligned} \frac{U_1^i + U_2^i}{2} &= (A_1(\xi_i) + A_2(\xi_i))^{-1} \frac{m}{6\pi R} g \\ &\quad + \frac{1}{2} \sum_{j \neq i} (U[U_1^{j,\infty}, U_2^{j,\infty}](x_1^j) + U[U_1^{j,\infty}, U_2^{j,\infty}](x_2^j)) + O(R) \max_{\substack{1 \leq i \leq N \\ \alpha=1,2}} |U_{\alpha}^i|. \\ \frac{U_1^{i,\infty} + U_2^{i,\infty}}{2} &= (A_1(\xi_i) + A_2(\xi_i))^{-1} \frac{m}{6\pi R} g + O(R) \max_{\substack{1 \leq i \leq N \\ \alpha=1,2}} |U_{\alpha}^i|. \end{aligned}$$

Proof. We prove the formula for $i = 1$ and the same holds true for all $1 \leq i \leq N$. We set w the unique solution to the Stokes equation (2) completed with the following boundary conditions :

$$(44) \quad w = \begin{cases} W & \text{on } B(x_1^1, R), \\ W & \text{on } B(x_2^1, R), \\ 0 & \text{on } B(x_1^i, R) \cup B(x_2^i, R), \quad i \neq 1, \end{cases}$$

with W an arbitrary vector of \mathbb{R}^3 . We use the method of reflections to obtain :

$$\begin{aligned} 2mg \cdot W &= 2 \int D(u^N) : \nabla w \\ &= -(F_1^{1,\infty} + F_2^{1,\infty}) \cdot W + \lim_{k \rightarrow \infty} 2 \int D(U[u_*^{(k+1)}]) : \nabla w. \end{aligned}$$

For the last term we apply again the method of reflections to the velocity field w , see Section 2.1.2. We set:

$$w_1 = \sum_{p=0}^k \sum_{i=1}^N U[W_1^{i,(p)}, W_2^{i,(p)}],$$

with

$$\|\nabla w - \nabla w_1\|_{L^2(\mathbb{R}^3 \setminus \cup \bar{B}_i)} \leq R|W|.$$

We obtain :

$$2 \int D(U[u_*^{(k+1)}]) : \nabla w = 2 \int \nabla U[u_*^{(k+1)}] : D(w_1) + 2 \int D(U[u_*^{(k+1)}]) : \nabla(w - w_1).$$

Thanks to the method of reflections, the second term on the right hand side can be bounded by $R^2|W| \max_{\substack{1 \leq i \leq N \\ \alpha=1,2}} |U_\alpha^i|$ (see Proposition 2.3). For the first term we write :

$$\begin{aligned} \lim_{k \rightarrow \infty} 2 \int D(U[u_*^{(k+1)}]) : \nabla w_1 = \\ - \sum_{p=0}^{\infty} \sum_j \sum_i \int_{\partial B(x_1^i, R) \cup \partial B(x_2^i, R)} \sigma(U[W_1^{j,(p)}, W_2^{j,(p)}]) n U[u_*^{(k+1)}]. \end{aligned}$$

We have

$$\begin{aligned} \|\sigma(U[W_1^{j,(p)}, W_2^{j,(p)}])\|_{L^\infty(\partial B(x_1^i, R) \cup \partial B(x_2^i, R))} &\lesssim \frac{R}{d_{ij}^2} \max(|W_1^{j,(p)}|, |W_2^{j,(p)}|), \quad \forall i \neq j, \\ \|\sigma(U[W_1^{i,(p)}, W_2^{i,(p)}])\|_{L^\infty(\partial B(x_1^i, R) \cup \partial B(x_2^i, R))} &\lesssim \frac{\max(|W_1^{i,(p)}|, |W_2^{i,(p)}|)}{R}. \end{aligned}$$

for the sake of clarity we set

$$\Gamma_{i,j} := \sum_{p=0}^{\infty} \|\sigma(U[W_1^{j,(p)}, W_2^{j,(p)}])\|_{L^\infty(\partial B(x_1^i, R) \cup \partial B(x_2^i, R))}.$$

We have then

$$\lim_{k \rightarrow \infty} \left| \int \nabla U[u_*^{(k+1)}] : D(w_1) \right| \lesssim 4\pi R^2 \left(\sum_i \sum_j \Gamma_{i,j} \right) \overline{\lim}_{k \rightarrow \infty} \|u_*^{(k+1)}\|_\infty.$$

Recall that $\|u_*^{(k+1)}\|_\infty = O(R) \max_{\substack{1 \leq i \leq N \\ \alpha=1,2}} |U_\alpha^i|$ when k goes to infinity. Thus, we focus only on

the remaining terms by splitting the sum as follow :

$$\sum_i \sum_j \Gamma_{i,j} = \sum_{i \neq 1} \left(\sum_{j \neq 1, i} \Gamma_{i,j} + \Gamma_{i,i} + \Gamma_{i,1} \right) + \sum_{j \neq 1} \Gamma_{1,j} + \Gamma_{1,1}.$$

For the first term, we have thanks to Corollary 2.9 and the estimates for $\Gamma_{i,j}$:

$$\begin{aligned} \sum_{i \neq 1} \left(\sum_{j \neq 1, i} \Gamma_{i,j} + \Gamma_{i,i} + \Gamma_{i,1} \right) &\lesssim \sum_{i \neq 1} \left(\sum_{j \neq 1, i} \frac{R}{d_{ij}^2} \frac{R}{d_{j1}} + \frac{1}{R} \frac{R}{d_{i1}} + \frac{R}{d_{i1}^2} R \right) |W|, \\ &\lesssim \sum_{i \neq 1} \frac{1}{d_{i1}} |W|. \end{aligned}$$

For the second term we have:

$$\sum_{j \neq 1} \Gamma_{1,j} \lesssim \sum_{j \neq 1} \frac{R}{d_{1j}^2} \frac{R}{d_{1j}} |W| \lesssim |W|.$$

The third term gives finally:

$$\Gamma_{1,1} \lesssim \frac{1}{R} R |W| \lesssim |W|.$$

Gathering all the inequalities we obtain:

$$\lim_{k \rightarrow \infty} \left| \int U[u_*^{(k+1)}] : \nabla w_1 \right| \lesssim R^3 \left(\sum_{i \neq 1} \frac{1}{d_{i1}} \right) |W| \max_{\substack{1 \leq i \leq N \\ \alpha=1,2}} |U_\alpha^i| \lesssim R^2 |W| \max_{\substack{1 \leq i \leq N \\ \alpha=1,2}} |U_\alpha^i|.$$

Finally, we have:

$$2mg \cdot W = -(F_1^{1,\infty} + F_2^{1,\infty}) \cdot W + O(R^2) |W| \max_{\substack{1 \leq i \leq N \\ \alpha=1,2}} |U_\alpha^i|.$$

This being true for all $W \in \mathbb{R}^3$ it yields:

$$2mg = -(F_1^{1,\infty} + F_2^{1,\infty}) + O(R^2) \max_{\substack{1 \leq i \leq N \\ \alpha=1,2}} |U_\alpha^i|.$$

Using the definitions of $F_1^{1,\infty}$ and $F_2^{1,\infty}$, see (43), this becomes:

$$2mg = 6\pi R (A_1(\xi_1) + A_2(\xi_1)) (U_1^{1,\infty} + U_2^{1,\infty}) + O(R^2) \max_{\substack{1 \leq i \leq N \\ \alpha=1,2}} |U_\alpha^i|.$$

Recall that $A_1(\xi)$ and $A_2(\xi)$ are of the form $h_1(|\xi|)\mathbb{I} + h_2(|\xi|) \frac{\xi \otimes \xi}{|\xi|^2}$. Moreover, according to formulas (27) $A_1 + A_2$ (resp. $A_1 - A_2$) is invertible and its inverse is $(a_1 + a_2)$ (resp. $a_1 - a_2$). Thus :

$$(45) \quad U_1^{1,\infty} + U_2^{1,\infty} = 2(A_1(\xi_1) + A_2(\xi_1))^{-1} \frac{m}{6\pi R} g + \frac{1}{6\pi} (A_1(\xi_1) + A_2(\xi_1))^{-1} O(R) \max_{\substack{1 \leq i \leq N \\ \alpha=1,2}} |U_\alpha^i|.$$

We use the fact that $\|(A_1(\xi_1) + A_2(\xi_1))^{-1}\|$ is uniformly bounded independently of the particles and N to get

$$U_1^{1,\infty} + U_2^{1,\infty} = 2(A_1(\xi_1) + A_2(\xi_1))^{-1} \frac{m}{6\pi R} g + O(R) \max_{\substack{1 \leq i \leq N \\ \alpha=1,2}} |U_\alpha^i|.$$

On the other hand, as $(U_1^{1,(0)}, U_2^{1,(0)}) = (U_1^1, U_2^1)$ we rewrite formula (45) as :

$$\begin{aligned} U_1^1 + U_2^1 &= - \sum_{p=1}^{\infty} (U_1^{1,(p)} + U_2^{1,(p)}) + (A_1(\xi_1) + A_2(\xi_1))^{-1} \frac{m}{6\pi R} g \\ &\quad + 2 \frac{1}{6\pi} (A_1(\xi_1) + A_2(\xi_1))^{-1} O(R) \max_{\substack{1 \leq i \leq N \\ \alpha=1,2}} |U_\alpha^i|. \end{aligned}$$

Using again formula (39) this yields :

$$\begin{aligned}
 U_1^1 + U_2^1 &= \sum_{p=1}^{\infty} \sum_{j \neq 1} U[U_1^{j,(p-1)}, U_2^{j,(p-1)}](x_1^1) + U[U_1^{j,(p-1)}, U_2^{j,(p-1)}](x_2^1), \\
 &+ 2(A_1(\xi_1) + A_2(\xi_1))^{-1} \frac{m}{6\pi R} g + \frac{1}{6\pi} (A_1(\xi_1) + A_2(\xi_1))^{-1} O(R) \max_{\substack{1 \leq i \leq N \\ \alpha=1,2}} |U_\alpha^i|, \\
 &= \sum_{j \neq 1} (U[U_1^{j,\infty}, U_2^{j,\infty}](x_1^1) + U[U_1^{j,\infty}, U_2^{j,\infty}](x_2^1)) + 2(A_1(\xi_1) + A_2(\xi_1))^{-1} \frac{m}{6\pi R} g, \\
 &+ \frac{1}{6\pi} (A_1(\xi_1) + A_2(\xi_1))^{-1} O(R) \max_{\substack{1 \leq i \leq N \\ \alpha=1,2}} |U_\alpha^i|.
 \end{aligned}$$

We conclude by emphasizing that $\|(A_1 + A_2)^{-1}\|$ can be uniformly bounded. \square

Applying the same ideas we obtain the following result:

Proposition 3.2. *for all $1 \leq i \leq N$ we have :*

$$\begin{aligned}
 U_1^i - U_2^i &= \sum_{j \neq i} (U[U_1^{j,\infty}, U_2^{j,\infty}](x_1^i) - U[U_1^{j,\infty}, U_2^{j,\infty}](x_2^i)) + O(R|x_-^i|) \max_{\substack{1 \leq i \leq N \\ \alpha=1,2}} |U_\alpha^i|. \\
 U_1^{i,\infty} - U_2^{i,\infty} &= O(R|x_-^i|) \max_{\substack{1 \leq i \leq N \\ \alpha=1,2}} |U_\alpha^i|.
 \end{aligned}$$

Proof. The proof is analogous to the one of Proposition 3.1. The idea is to consider this time w the unique solution to the Stokes equation (2) completed with the following boundary conditions :

$$(46) \quad w = \begin{cases} W & \text{on } B(x_1^1, R), \\ -W & \text{on } B(x_2^1, R), \\ 0 & \text{on } B(x_1^i, R) \cup B(x_2^i, R), i \neq 1, \end{cases}$$

with W an arbitrary vector of \mathbb{R}^3 . Using the method of reflections, Propositions 2.7 and 2.3 we obtain the desired result. \square

3.2. Estimates for x_+^i . Propositions 3.1 and 3.2 yields the following result:

Corollary 3.3. *For all $1 \leq i \leq N$ we have :*

$$U_+^i := (\mathbb{A}(\xi_i))^{-1} \frac{m}{6\pi R} g + \frac{6\pi r_0}{N} \sum_{j \neq i} \Phi(x_+^i - x_+^j) \kappa g + O(R),$$

where $\mathbb{A} = A_1 + A_2$.

Proof. First of all, from Propositions 3.1 and 3.2 we can show that the velocities U_α^i are uniformly bounded with respect to N for all $1 \leq i \leq N$ and $\alpha = 1, 2$. Indeed, using formula

(39) together with the decay properties (33) and Propositions 3.1 and 3.2 we have :

$$\begin{aligned}
\max_{\substack{\alpha=1,2 \\ 1 \leq i \leq N}} |U_\alpha^i| &\leq \max_{1 \leq i \leq N} (|U_+^i| + |U_-^i|), \\
&\lesssim 1 + \max_{1 \leq i \leq N} (|U_+^{i,\infty}| + |U_-^{i,\infty}|) + O(R) \max_{\substack{\alpha=1,2 \\ 1 \leq i \leq N}} |U_\alpha^i|, \\
&\lesssim 1 + O(R) \max_{\substack{\alpha=1,2 \\ 1 \leq i \leq N}} |U_\alpha^i|.
\end{aligned}$$

This allows us to bound the terms $\max_{\substack{\alpha=1,2 \\ 1 \leq i \leq N}} |U_\alpha^i|$ by a constant independent of N in the estimates of Propositions 3.1 and 3.2. From Proposition 3.2 we have

$$U_+^i = (A_1(\xi_i) + A_2(\xi_i))^{-1} \frac{m}{6\pi R} g + \frac{1}{2} \sum_{j \neq i} (U[U_1^{j,\infty}, U_2^{j,\infty}](x_1^1) + U[U_1^{j,\infty} + U_2^{j,\infty}](x_2^1)) + O(R),$$

with:

$$\begin{aligned}
U[U_1^{j,\infty}, U_2^{j,\infty}](x_1^1) + U[U_1^{j,\infty}, U_2^{j,\infty}](x_2^1) &= -\Phi(x_1^1 - x_1^j) F_1^{j,\infty} - \Phi(x_1^1 - x_2^j) F_2^{j,\infty} \\
&\quad - \Phi(x_2^1 - x_1^j) F_1^{j,\infty} - \Phi(x_2^1 - x_2^j) F_2^{j,\infty}, \\
&= -(\Phi(x_1^1 - x_1^j) + \Phi(x_2^1 - x_1^j)) F_1^{j,\infty} \\
&\quad - (\Phi(x_1^1 - x_2^j) + \Phi(x_2^1 - x_2^j)) F_2^{j,\infty},
\end{aligned}$$

recall that:

$$\begin{cases} F_1^{j,\infty} = F_+^{j,\infty} + F_-^{j,\infty} \\ F_2^{j,\infty} = F_+^{j,\infty} - F_-^{j,\infty} \end{cases}, \quad \begin{cases} F_+^{j,\infty} = -mg + O(R^2) \\ F_-^{j,\infty} = O(R^2) \end{cases}$$

see proof of Propositions 3.2 and 3.1. Hence, we replace F_1^j and F_2^j by their formula and bound the sum of terms involving the error term $O(R^2)$ by $O(R)$. We get

$$\begin{aligned}
(47) \quad \sum_{j \neq i} U[U_1^{j,\infty}, U_2^{j,\infty}](x_1^1) + U[U_1^{j,\infty}, U_2^{j,\infty}](x_2^1) &= \\
6\pi R \sum_{j \neq i} (\Phi(x_1^1 - x_1^j) + \Phi(x_2^1 - x_1^j) + \Phi(x_1^1 - x_2^j) + \Phi(x_2^1 - x_2^j)) \kappa g + O(R),
\end{aligned}$$

where $mg = 6\pi R \kappa g$. Now the idea is to replace each of the four terms by $\Phi(x_+^1 - x_+^j)$. Direct computations shows that for all $1 \leq \alpha, \beta \leq 2$ we have:

$$|x_\alpha^1 - x_\beta^j - x_+^1 + x_+^j| \leq |x_-^1| + |x_-^j|$$

which yields for all $1 \leq \alpha, \beta \leq 2$:

$$|\Phi(x_\alpha^1 - x_\beta^j) - \Phi(x_+^1 - x_+^j)| \lesssim \frac{|x_-^1| + |x_-^j|}{|x_+^1 - x_+^j|^2}.$$

Hence the error term can be bounded by $(|x_-^1| + |x_-^j|)$ which is of order R . \square

3.3. **Estimates for \dot{x}_- .** Analogously, Propositions 3.1 and 3.2 yields the following result:

Corollary 3.4. *For all $1 \leq i \neq N$ we have:*

$$\frac{U_1^i - U_2^i}{2} = \left(\frac{6\pi r_0}{N} \sum_{j \neq i} \nabla \Phi(x_+^i - x_+^j) \kappa g \right) \cdot x_-^i + O(|x_-^i| d_{\min}).$$

Proof. The first formula of Proposition 3.2 together with the uniform bound on the velocities (U_+^i, U_-^i) , see proof of Corollary 3.3, yields:

$$U_1^i - U_2^i = \sum_{j \neq i} U[U_1^{j,\infty}, U_2^{j,\infty}](x_1^i) - U[U_1^{j,\infty}, U_2^{j,\infty}](x_2^i) + O(R|x_-^i|).$$

We want to estimate the first term, we have:

$$\begin{aligned} U[U_1^{j,\infty}, U_2^{j,\infty}](x_1^i) - U[U_1^{j,\infty}, U_2^{j,\infty}](x_2^i) &= -\Phi(x_1^i - x_1^j) F_1^{j,\infty} - \Phi(x_1^i - x_2^j) F_2^{j,\infty} \\ &\quad + \Phi(x_2^i - x_1^j) F_1^{j,\infty} + \Phi(x_2^i - x_2^j) F_2^{j,\infty}, \\ &= -(\Phi(x_1^i - x_1^j) - \Phi(x_2^i - x_1^j)) F_1^{j,\infty} \\ &\quad - (\Phi(x_1^i - x_2^j) - \Phi(x_2^i - x_2^j)) F_2^{j,\infty}, \\ &= -2[\nabla \Phi(x_2^i - x_1^j) \cdot x_-^i] F_1^{j,\infty} \\ &\quad + 2[\nabla \Phi(x_2^i - x_2^j) \cdot x_-^i] F_2^{j,\infty} + \mathcal{E}_{i,j}^1, \\ &= -2[\nabla \Phi(x_+^i - x_+^j) \cdot x_-^i] (F_1^{j,\infty} + F_2^{j,\infty}) + \mathcal{E}_{i,j}^1 + \mathcal{E}_{i,j}^2. \end{aligned}$$

Now recall that, from the proof of Proposition 3.1 we have:

$$F_1^{j,\infty} + F_2^{j,\infty} = -2mg + O(R^2).$$

Thus, we get the following formula:

$$U[U_1^{j,\infty}, U_2^{j,\infty}](x_1^i) - U[U_1^{j,\infty}, U_2^{j,\infty}](x_2^i) = 2[\nabla \Phi(x_+^i - x_+^j) \cdot x_-^i] mg + \mathcal{E}_{i,j}^1 + \mathcal{E}_{i,j}^2 + \mathcal{E}_j^3,$$

with

$$\mathcal{E}_j^3 = -2[\nabla \Phi(x_+^i - x_+^j) \cdot x_-^i] (F_1^{j,\infty} + F_2^{j,\infty} + 2mg).$$

Finally we obtain:

$$\frac{U_1^i - U_2^i}{2} = \sum_{j \neq i} [\nabla \Phi(x_+^i - x_+^j) \cdot x_-^i] mg + \frac{1}{2} \sum_{j \neq i} \mathcal{E}_{i,j}^1 + \mathcal{E}_{i,j}^2 + \mathcal{E}_j^3 + O(R|x_-^i|).$$

It remains to bound the error terms. We begin by the first one:

$$|\mathcal{E}_{i,j}^1| \leq 2 \left(\sup_{y \in [x_1^i, x_2^i]} (|\nabla^2 \Phi(x_1^j - y)| + |\nabla^2 \Phi(x_2^j - y)|) \right) |x_-^i|^2 (|F_1^{j,\infty}| + |F_2^{j,\infty}|).$$

We emphasize that for all $y \in [x_1^i, x_2^i]$:

$$|y - x_1^j| \geq |x_1^i - x_1^j| - |x_1^i - y| \geq |x_1^i - x_1^j| - 2|x_-^i| \geq \frac{1}{4}|x_+^i - x_+^j|,$$

where we used the fact that

$$|x_-^i| \leq \frac{C}{R} \leq \frac{1}{8}d_{\min} \leq \frac{1}{8}|x_+^i - x_+^j|,$$

and

$$|x_1^i - x_1^j| \geq \frac{1}{2}|x_+^i - x_+^j|,$$

This yields :

$$\sum_{j \neq i} |\mathcal{E}_{i,j}^1| \leq C \sum_{j \neq i} \frac{1}{d_{ij}^3} |x_-^i|^2 R \kappa |g| \leq C |x_-^i| \frac{R}{d_{\min}} \left(\sum_{j \neq i} \frac{R}{d_{ij}^2} \right) \leq C |x_-^i| \frac{R}{d_{\min}} \leq C d_{\min} |x_-^i|.$$

For the second error term we have:

$$\begin{aligned} \mathcal{E}_{i,j}^2 = & -2[\nabla\Phi(x_2^i - x_1^j) - \nabla\Phi(x_+^i - x_+^j)] \cdot x_-^i \cdot F_1^{j,\infty} \\ & - [\nabla\Phi(x_2^i - x_2^j) - \nabla\Phi(x_+^i - x_+^j)] \cdot x_-^i \cdot F_2^{j,\infty}, \end{aligned}$$

where

$$|\nabla\Phi(x_2^i - x_1^j) - \nabla\Phi(x_+^i - x_+^j)| \leq C \left(\frac{1}{|x_2^i - x_1^j|^3} + \frac{1}{|x_+^i - x_+^j|^3} \right) |x_-^i + x_-^j|$$

As $|x_-^j| \sim R \sim |x_-^i|$ the second error term is bounded by:

$$\sum_{j \neq i} |\mathcal{E}_{i,j}^2| \leq C \sum_{j \neq i} \frac{1}{d_{ij}^3} |x_-^i|^2 R \kappa |g|,$$

which yields the same estimate as for the first error term. Finally, the last error term gives:

$$\begin{aligned} \sum_{j \neq i} |\mathcal{E}_{i,j}^3| & \leq 2|\nabla\Phi(x_+^i - x_+^j)| |x_-^i| |F_1^{j,\infty} + F_2^{j,\infty} + 2mg|, \\ & \leq CR^2. \end{aligned}$$

where we used the fact that $F_1^{j,\infty} + F_2^{j,\infty} = -2mg + O(R^2)$ and $|x_-^i| \sim R$. \square

4. PROOF OF THEOREM 0.1

In order to derive the transport-Stokes equation satisfied at the limit, the idea is to show that the discrete density μ^N satisfies weakly a transport equation. We introduce the following notations. Given a density ρ , we define the operator $\mathcal{K}\rho$ as:

$$\mathcal{K}\rho(x) := 6\pi r_0 \int_{\mathbb{R}^3} \Phi(x-y) \kappa g \rho(dy).$$

The operator is well defined and is Lipschitz in the case where $\rho \in L^1 \cap L^\infty$. Moreover, note that $\mathcal{K}\rho$ satisfies the Stokes equation

$$-\Delta \mathcal{K}(\rho) + \nabla p = 6\pi r_0 \kappa g \rho,$$

on \mathbb{R}^3 . Analogously, we define $\mathcal{K}^N \rho^N$ as:

$$\mathcal{K}^N \rho^N(x) := 6\pi r_0 \int_{\mathbb{R}^3} \chi \Phi(x-y) \kappa g \rho^N(dy),$$

where $\chi \Phi(\cdot) = \chi\left(\frac{\cdot}{d_{\min}}\right) \Phi(\cdot)$, χ is a truncation function such that $\chi = 0$ on $B(0, 1/4)$ and $\chi = 1$ on ${}^c B(0, 1/2)$.

4.1. Derivation of the transport-Stokes equation. The transport equation satisfied by μ^N is obtained directly using the ODE system derived for each couple (x_+^i, ξ_i) . We recall that:

$$\begin{aligned} U_+^i &= (\mathbb{A}(\xi_i))^{-1} \kappa g + \mathcal{K}^N \rho^N(x_+^i) + O(R), \\ \frac{U_-^i}{R} &= \nabla \mathcal{K}^N \rho^N(x_+^i) \cdot \xi_i + O(d_{\min}). \end{aligned}$$

Following the idea of [17, Section 5.2], one can show that we can construct two divergence-free velocity fields E^N and \tilde{E}^N such that :

$$(48) \quad \begin{aligned} U_+^i &= (\mathbb{A}(\xi_i))^{-1} \kappa g + \mathcal{K}^N \rho^N(x_+^i) + E^N(x_+^i), \\ \frac{U_-^i}{R} &= \nabla \mathcal{K}^N \rho^N(x_+^i) \cdot \xi_i + \tilde{E}^N(\xi_i), \end{aligned}$$

and there exists a positive constant independent of N such that

$$(49) \quad \|E^N\|_\infty = O(R), \quad \|\tilde{E}^N\|_\infty = O(d_{\min}), \quad \|\nabla E^N\|_\infty + \|\nabla \tilde{E}^N\|_\infty < C.$$

This construction yields the following result

Proposition 4.1. μ^N satisfies weakly the transport equation:

$$(50) \quad \partial_t \mu^N + \operatorname{div}_x [(\mathbb{A}(\xi))^{-1} \kappa g \mu^N + \mathcal{K}^N \rho^N(x) \mu^N + E^N \mu^N] + \operatorname{div}_\xi [\nabla \mathcal{K}^N \rho^N(x) \cdot \xi \mu^N + \tilde{E}^N \mu^N] = 0.$$

We can prove now Theorem 0.1.

4.2. proof of Theorem 0.1. The proof is a corollary of Proposition 4.1. Indeed, we want to show that for all $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^3)$ we have:

$$(51) \quad \int_0^T \int_{\mathbb{R}^3 \times \mathbb{R}^3} \left\{ \partial_t \psi(t, x, \xi) + \nabla_x \psi(t, x, \xi) \cdot [(\mathbb{A}(\xi))^{-1} \kappa g + 2\mathcal{K}\rho(x)] \right. \\ \left. + \nabla_\xi \psi(t, x, \xi) \cdot [\nabla \mathcal{K}\rho(x) \cdot \xi] \right\} \mu(t, dx, d\xi) dt.$$

which is obtained directly by passing through the limit in each term of formula (50). Indeed we recall that we have the following estimates:

$$\begin{aligned}\|\mathcal{K}^N \rho^N - \mathcal{K} \rho\|_\infty &\lesssim W_\infty, \\ \|\nabla \mathcal{K}^N \rho^N - \nabla \mathcal{K} \rho\|_\infty &\lesssim W_\infty(1 + |\log W_\infty|), \\ \|E^N\|_\infty = O(R) &\quad , \quad \|\tilde{E}^N\|_\infty = O(d_{\min}).\end{aligned}$$

5. PROOF OF THEOREM 0.2 AND 0.3

This section is devoted to the proof of Theorem 0.2 and 0.3. The Lipschitz-like estimates proved in Proposition B.3 suggests a correlation between the vectors along the line of centers ξ_i and the centers x_+^i . In this section, we show in particular that this correlation is well propagated in time.

5.1. Derivation of the transport-Stokes equation. We assume now that there exists a Lipschitz function F_0 such that

$$\xi_i(0) = F_0(x_+^i(0)), \quad 1 \leq i \leq N,$$

which means that $\mu_0^N = \rho_0^N \otimes \delta_{F_0}$. In order to propagate this correlation we search for a function $F^N(t, \cdot) \in W^{1,\infty}$ such that for all $t \in [0, T]$ we have

$$\xi_i(t) = F^N(t, x_+^i(t)), \quad 1 \leq i \leq N.$$

According to the ODE satisfied by ξ_i , see (48), F^N must satisfy the following equation

$$\begin{cases} \partial_t F^N + \nabla F^N \cdot (\mathbb{A}(F^N)^{-1} \kappa g + \mathcal{K}^N \rho^N + E^N) &= \nabla \mathcal{K}^N \rho^N \cdot F^N + \tilde{E}^N(F^N), \\ F^N(0, \cdot) &= F_0. \end{cases}$$

The following proposition shows the existence and uniqueness of F^N .

Proposition 5.1. *There exists $T > 0$ such that for all $N \in \mathbb{N}^*$, there exists a unique (local) solution $F^N \in L^\infty(0, T; W^{1,\infty}(\mathbb{R}^3))$ of the following equation*

$$(52) \quad \begin{cases} \partial_t F^N + \nabla F^N \cdot (\mathbb{A}(F^N)^{-1} \kappa g + \mathcal{K}^N \rho^N + E^N) &= \nabla \mathcal{K}^N \rho^N \cdot F^N + \tilde{E}^N(F^N), \\ F^N(0, \cdot) &= F_0. \end{cases}$$

Proof. The idea is to apply a fixed-point argument. We define the mapping \mathcal{A} which associates to any $F \in L^\infty(0, T; W^{1,\infty}(\mathbb{R}^3))$ the unique solution $\mathcal{A}(F) = \hat{F}$ to the transport equation

$$(53) \quad \begin{cases} \partial_t \hat{F} + \nabla \hat{F} \cdot (\mathbb{A}(F)^{-1} \kappa g + \mathcal{K}^N \rho^N + E^N) &= \nabla \mathcal{K}^N \rho^N \cdot F + \tilde{E}^N(F), \\ \hat{F}(0, \cdot) &= F_0. \end{cases}$$

We define X^N as the characteristic flow satisfying :

$$\begin{aligned}\partial_s X^N(s, t, x) &= \mathbb{A}(F(s, X^N(s, t, x)))^{-1} \kappa g + \mathcal{K}^N \rho^N(s, X^N(s, t, x)) + E^N(s, X^N(s, t, x)). \\ X^N(t, t, x) &= x.\end{aligned}$$

The Lipschitz property of \mathbb{A}^{-1} , F , $\mathcal{K}^N \rho^N$ and E^N ensures the existence, uniqueness and regularity of such a flow, see Proposition B.1 and formula (49). Moreover, direct estimates show that for all $0 \leq s \leq t$:

$$(54) \quad \|\nabla X^N(s, t, \cdot)\|_\infty \leq \exp([\kappa g \|\nabla \mathbb{A}^{-1}\|_\infty \|F\|_{L^\infty(0, T; W^{1, \infty})} + \|\mathcal{K}^N \rho^N + E^N\|_{L^\infty(0, T; W^{1, \infty})}] (t - s)).$$

Hence, we can write

$$\begin{aligned} \hat{F}(t, x) &= F_0(X^N(0, t, x)) \\ &\quad + \int_0^t \nabla \mathcal{K}^N \rho^N(s, X^N(s, t, x)) \cdot F(s, X^N(s, t, x)) + \tilde{E}(s, F(X^N(s, t, x))) ds. \end{aligned}$$

Direct computations yield

$$\|\mathcal{A}(F)\|_{L^\infty(0, T; L^\infty)} \leq \|F_0\|_\infty + T \|\nabla \mathcal{K}^N \rho^N\|_{L^\infty(0, T; L^\infty)} \|F\|_{L^\infty(0, T; L^\infty)} + \|\tilde{E}^N\|_{L^\infty(0, T; L^\infty)},$$

and

$$\begin{aligned} \|\nabla \mathcal{A}(F)\|_{L^\infty(0, T; L^\infty)} &\leq [\|F_0\|_{1, \infty} + T \left\{ \|\nabla \mathcal{K}^N \rho^N\|_{L^\infty(0, T; W^{1, \infty})} \right. \\ &\quad \left. + \|\tilde{E}^N\|_{L^\infty(0, T; W^{1, \infty})} \right\} \|F\|_{L^\infty(0, T; W^{1, \infty})}] \|\nabla X^N(\cdot, t, \cdot)\|_{L^\infty(0, T; L^\infty)}, \end{aligned}$$

Gathering all the estimates and using Proposition B.1 and the uniform bounds (49), there exists some constants independent of N such that:

$$(55) \quad \|\mathcal{A}(F)\|_{L^\infty(0, T; W^{1, \infty}(\mathbb{R}^3))} \leq (\|F_0\|_{W^{1, \infty}} + TC_1 \|F\|_{L^\infty(0, T; W^{1, \infty}(\mathbb{R}^3))}) e^{C_2 T}.$$

On the other hand, given $F_1, F_2 \in L^\infty(0, T; W^{1, \infty}(\mathbb{R}^3))$ we set X_i the associated characteristic flow and we have

$$\begin{aligned} \|\mathcal{A}(F_1)(t, \cdot) - \mathcal{A}(F_2)(t, \cdot)\|_\infty &\leq \\ &\left(\|\nabla F_0\|_\infty + t \|F_1\|_{L^\infty(0, T; W^{1, \infty})} \|\mathcal{K}^N \rho^N\|_{L^\infty(0, T; W^{2, \infty})} \right) \|X_1(0, t, \cdot) - X_2(0, t, \cdot)\|_\infty \\ &\quad + t \|\nabla \mathcal{K}^N \rho^N\|_{L^\infty(0, T; L^\infty)} \|F_1 - F_2\|_{L^\infty(0, T; L^\infty(\mathbb{R}^3))}. \end{aligned}$$

The characteristic flows satisfies

$$\begin{aligned} |X_1(s, t, x) - X_2(s, t, x)| &\leq \|\nabla \mathbb{A}^{-1}\|_\infty \int_s^t \|F_1(\tau, \cdot) - F_2(\tau, \cdot)\|_\infty + \\ &\quad (\|F_1\|_{L^\infty(0, T; L^\infty)} |\kappa g| + 2 \|\nabla \mathcal{K}^N \rho^N + \nabla E^N\|_{L^\infty(0, T; L^\infty)}) |X_1(\tau, t, x) - X_2(\tau, t, x)| d\tau, \end{aligned}$$

hence

$$\|X_1(s, t, \cdot) - X_2(s, t, \cdot)\|_\infty \leq \left(\int_s^t \|\nabla \mathbb{A}^{-1}\|_\infty \|F_1(\tau, \cdot) - F_2(\tau, \cdot)\|_\infty d\tau \right) e^{C(t-s)}.$$

This yields

$$(56) \quad \|\mathcal{A}(F_1) - \mathcal{A}(F_2)\|_{L^\infty(0, T; L^\infty(\mathbb{R}^3))} \leq C(\|F_1\|_{L^\infty(0, T; W^{1, \infty}(\mathbb{R}^3))}) T \|F_1 - F_2\|_{L^\infty(0, T; L^\infty)}.$$

We construct the following sequence $(F_k)_{k \in \mathbb{N}} \subset L^\infty(0, T; W^{1, \infty}(\mathbb{R}^3))$ defined as

$$\begin{cases} F^{k+1} &= \mathcal{A}(F^k), k \in \mathbb{N}, \\ F^0 &= F_0. \end{cases}$$

For T small enough and independent of N , using estimates (55) and (56), the sequence $(F^k)_k$ is bounded in $L^\infty(0, T; W^{1, \infty}(\mathbb{R}^3))$ and is a Cauchy sequence in the Banach space $L^\infty([0, T], L^\infty(\mathbb{R}^3))$. There exists a limit $F \in L^\infty(0, T; W^{1, \infty}(\mathbb{R}^3))$ such that $F^k \rightarrow F$ in $L^\infty(0, T, L^\infty)$ and $\nabla F^k \rightharpoonup \nabla F$ weakly-* in $L^\infty(0, T, L^\infty)$. It remains to show that $F = \mathcal{A}(F)$. The weak formulation of the transport equation writes

$$\int_0^T \int_{\mathbb{R}^3} (\partial_t \psi + \operatorname{div}(\psi \cdot [\mathbb{A}^{-1}(F^k) \kappa g + \mathcal{K}^N \rho^N])) F^k = \int_0^T \int_{\mathbb{R}^3} (\nabla \mathcal{K}^N \rho^N \cdot F^k + \tilde{E}^N(F^k)) \cdot \psi,$$

for all $\psi \in C_c^1((0, T) \times \mathbb{R}^3)$. Using the strong convergence of F^N to F and the weak-* convergence of its derivative, we get

$$\int_0^T \int_{\mathbb{R}^3} (\partial_t \psi + \operatorname{div}(\psi \cdot [\mathbb{A}^{-1}(F) \kappa g + \mathcal{K}^N \rho^N])) F = \int_0^T \int_{\mathbb{R}^3} (\nabla \mathcal{K}^N \rho^N \cdot F + \tilde{E}^N(F)) \cdot \psi,$$

Uniqueness of the fixed-point is ensured thanks to estimate (55) and (56). \square

Proposition 5.1 and formula (48) yield the following result

Corollary 5.2. *There exists a unique solution of (52) $F^N \in L^\infty([0, T], W^{1, \infty})$ such that $\mu^N = (id, F^N) \# \rho^N$ and ρ^N satisfies weakly*

$$(57) \quad \partial_t \rho^N + \operatorname{div}[(\mathbb{A}(F^N))^{-1} \kappa g + \mathcal{K}^N \rho^N(x) + E^N] \rho^N = 0.$$

5.2. proof of Theorem 0.2 and 0.3. In the previous part we showed the existence of a unique function F^N such that:

$$\xi_i = F^N(x_+^i).$$

In order to provide the limiting behaviour of the system, we need to extract the limiting equation satisfied by $F = \lim_{N \rightarrow \infty} F^N$ and to estimate and specify the convergence. It is straightforward that the limit function F should satisfy the following equation:

$$(58) \quad \begin{cases} \partial_t F + \nabla F \cdot (\mathbb{A}(F)^{-1} \kappa g + \mathcal{K} \rho) &= \nabla \mathcal{K} \rho \cdot F, \text{ on } [0, T] \times \mathbb{R}^3, \\ F(0, \cdot) &= F_0. \end{cases}$$

We begin with the proof of local existence and uniqueness of the solution to system (15).

Proof of Theorem 0.3. Let $p > 3$, $F_0 \in W^{2, p}$, $\rho_0 \in W^{1, p}$ having compact support. The idea is to apply a fixed-point argument. We define the operator A which associates to each $u \in L^\infty(0, T; W^{3, p})$ the following divergence free velocity

$$u \mapsto F(u) \mapsto \rho(u) \mapsto \mathcal{A}(u),$$

where $F(u) \in L^\infty(0, T; W^{2, p})$ is the unique solution, see Proposition C.1, to the following equation

$$\begin{cases} \partial_t F + \nabla F \cdot (\mathbb{A}^{-1}(F) \kappa g + u) &= \nabla u \cdot F, & \text{on } [0, T] \times \mathbb{R}^3, \\ F(0, \cdot) &= F_0, & \text{on } \mathbb{R}^3. \end{cases}$$

$\rho(u) \in L^\infty(0, T; W^{1,p})$ is the unique solution, see Proposition C.2, to the transport equation

$$\begin{cases} \partial_t \rho + \operatorname{div}((\mathbb{A}^{-1}(F(u))\kappa g + u)\rho) = 0, & \text{on } [0, T] \times \mathbb{R}^3, \\ \rho(0, \cdot) = \rho_0, & \text{on } \mathbb{R}^3. \end{cases}$$

and $\mathcal{A}(u) = \mathcal{K}\rho(u) = 6\pi r_0 \Phi * (\kappa\rho(u)g)$. The mapping is well-defined, indeed, since $\rho_0 \in W^{1,p}$ we have $\rho \in L^\infty(0, T; W^{1,p})$, see Proposition C.2. Consequently, applying [4, Theorem IV.2.1] shows that $\nabla^3 A(u), \nabla^2 A(u) \in L^p$ and we have

$$\|\nabla^3 A(u)\|_p \leq C\|\nabla\rho(u)\|_p, \quad \|\nabla^2 A(u)\|_p \leq C\|\rho(u)\|_p.$$

On the other hand, since $\rho(t, \cdot) \in L^p$ and is compactly supported, see Remark C.1, we have in particular $\rho(t, \cdot) \in L^{q_1} \cap L^{q_2}$ with

$$q_1 = \frac{3p}{3+p} \in]3/2, 3[, \quad q_2 = \frac{3p}{3+2p} \in]1, 3/2[.$$

We apply again [4, Theorem IV.2.1] for $q = q_1$ (resp. $q = q_2$) to get $\nabla A(u) \in L^p$ (resp. $A(u) \in L^p$) and we have according to [4, Formula IV.2.22] (resp. [4, Formula IV.2.23])

$$\|\nabla A(u)\|_p \leq C\|\rho(u)\|_{q_1}, \quad \|A(u)\|_p \leq C\|\rho(u)\|_{q_2},$$

Hence, since $q_1, q_2 < 3 < p$, Holder's inequality yields

$$\|\nabla A(u)\|_p + \|A(u)\|_p \lesssim \left(\sup_{[0,T]} |\operatorname{supp} \rho(u)(t, \cdot)|^{1/3} + \sup_{[0,T]} |\operatorname{supp} \rho(u)(t, \cdot)|^{2/3} \right) \|\rho(u)\|_p,$$

where $\sup_{[0,T]} |\operatorname{supp} \rho(u)(t, \cdot)|$ depends on $T, \|\mathbb{A}^{-1}\|_\infty, \|F\|_{L^\infty(0,T;W^{2,p})}$ and $\|u\|_{L^\infty(0,T;W^{2,p})}$ according to Remark C.1

$$(59) \quad \operatorname{diam}(\operatorname{supp}(\rho(u)(t, \cdot))) \leq C(\rho_0, T, \|u\|_{L^\infty(0,T;W^{2,p})}, \|F\|_{L^\infty(0,T;W^{2,p})}),$$

Finally we have

$$(60) \quad \|A(u)\|_{L^\infty(0,T;W^{3,p})} \leq C(1 + M(T))\|\rho(u)\|_{L^\infty(0,T;W^{1,p})},$$

$$(61) \quad \|A(u)\|_{L^\infty(0,T;W^{2,p})} \leq C(1 + M(T))\|\rho(u)\|_{L^\infty(0,T;L^p)},$$

$$M(T) = \sup_{[0,T]} |\operatorname{supp} \rho(u)(t, \cdot)|^{1/3} (1 + \sup_{[0,T]} |\operatorname{supp} \rho(u)(t, \cdot)|^{1/3}).$$

We recall the following bounds, see Proposition C.2 and Proposition C.1

$$(62) \quad \|\rho(u)\|_{L^\infty(0,T;W^{1,p})} \leq \|\rho_0\|_{1,p} e^{CT}, \quad C = C(\|F(u)\|_{L^\infty(0,T;W^{2,p})}, \|u\|_{L^\infty(0,T;W^{3,p})}).$$

According to Proposition C.1, for a small time interval we have for a fixed $\lambda > 1$

$$(63) \quad \|F(u)\|_{2,p} \leq \lambda \|F_0\|_{2,p}.$$

On the other hand, gathering the stability estimates of Proposition C.2 and Proposition C.1 and (61) we get for $u_i \in W^{3,p}$, $i = 1, 2$

$$\begin{aligned} & \|A(u_1) - A(u_2)\|_{L^\infty(0,T;W^{2,p})} \\ & \leq C(1 + M(u_1, u_2)(T))\|\rho(u_1) - \rho(u_2)\|_{L^\infty(0,T;L^p)} \\ & \leq C(1 + M(u_1, u_2)(T))T (\|F(u_1) - F(u_2)\|_{L^\infty(0,T;W^{1,p})} + \|u_1 - u_2\|_{L^\infty(0,T;W^{1,p})}) e^{C_1 T} \\ & \leq C(1 + M(u_1, u_2)(T))T(1 + T)\|u_1 - u_2\|_{L^\infty(0,T;W^{2,p})} e^{C_1 T}, \end{aligned}$$

where C depends on $\|u_i\|_{L^\infty(0,T;W^{3,p})}$, $\|F(u_i)\|_{L^\infty(0,T;W^{2,p})}$, $\|\rho(u_i)\|_{L^\infty(0,T;W^{1,p})}$ and

$$\begin{aligned} M(u_1, u_2)(T) & := \sup_{[0,T]} |\text{supp}(\rho(u_1)) \cup \text{supp}(\rho(u_2))|^{1/3} (1 + \sup_{[0,T]} |\text{supp}(\rho(u_1)) \cup \text{supp}(\rho(u_2))|^{1/3}), \\ & \lesssim C(T, \|u_i\|_{L^\infty(0,T;W^{2,p})}, \|F_i\|_{L^\infty(0,T;W^{2,p})}, \text{supp}(\rho_0)). \end{aligned}$$

We consider the following sequence

$$\begin{cases} u^{k+1} & = \mathcal{A}(u^k), k \in \mathbb{N}, \\ u^0 & = \mathcal{K}\rho_0. \end{cases}$$

We set $F^k := A(u^k)$, $\rho^k := \rho(u^k)$. Previous estimates show that the sequences $(u_k)_{k \in \mathbb{N}}$, $(F_k)_{k \in \mathbb{N}}$, $(\rho_k)_{k \in \mathbb{N}}$ are uniformly bounded in $L^\infty(0, T; W^{3,p})$, $L^\infty(0, T; W^{2,p})$, $L^\infty(0, T; W^{1,p})$, respectively, and are Cauchy sequences in $L^\infty(0, T; W^{2,p})$, $L^\infty(0, T; W^{1,p})$, $L^\infty(0, T; L^p)$, respectively for T small enough. Consequently, there exists (u, F, ρ) such that

$$\begin{aligned} u^k & \rightarrow u & \text{in } L^\infty(0, T; W^{2,p}), \\ F^k & \rightarrow F & \text{in } L^\infty(0, T; W^{1,p}), \\ \rho^k & \rightarrow \rho & \text{in } L^\infty(0, T; L^p). \end{aligned}$$

This allows to pass through the limit in the weak formulations of u^k and ρ^k . In addition, we use the fact that ∇F_k converges weakly-* in $L^\infty(0, T; L^\infty)$ in order to pass through the limit in the weak formulation of F^k . Hence, the triplet (u, ρ, F) satisfies equation (15). We recover the regularity of each term using the a priori bounds. Uniqueness is a consequence of the previous stability estimates. \square

5.3. Proof of Theorem 0.2.

Proof of Theorem 0.2. Since $\rho^N \rightharpoonup \rho$ weakly in the sense of measure, this yields that $W_\infty(\rho^N, \rho) \rightarrow 0$. We want to show that the triplet $(\rho^N, F^N, \mathcal{K}^N \rho^N)$ converges to $(\rho, F, \mathcal{K}\rho)$ the unique solution of equation (15). From Proposition B.2 and using the same arguments as in Proposition C.1 we have

$$\|F^N(t, \cdot) - F(t, \cdot)\|_\infty \leq C \int_0^t W_\infty(s) \left(1 + |\log W_\infty(s)| + \frac{W_\infty^2(s)}{d_{\min}^2} \right) + \|E^N\|_\infty + \|\tilde{E}^N\|_\infty,$$

where $W_\infty(s) := W_\infty(\rho^N(s, \cdot), \rho(s, \cdot))$. Hence $F^N \rightarrow F$ in $L^\infty(0, T; L^\infty)$ and $\mathcal{K}^N \rho^N \rightarrow \mathcal{K}\rho$ in $L^\infty(0, T; W^{1,\infty})$ if the Wasserstein distance is preserved in finite time. This allows us to

pass through the limit in the weak formulation of ρ^N

$$\int_0^t \int_{\mathbb{R}^3} (\partial_t \psi + \nabla \psi \cdot (\mathcal{A}^{-1}(F^N) \kappa g + \mathcal{K}^N \rho^N)) \rho^N = 0.$$

□

APPENDIX A. SOME PRELIMINARY ESTIMATES

This section is devoted to the proof of the following lemma which is analogous to [12, Lemma 2.1].

Lemma A.1. *There exists a positive constant C such that for $k \in [0, 2]$*

$$\begin{aligned} \frac{1}{N} \sum_{j \neq i} \frac{1}{d_{ij}^k} &\leq C \|\rho\|_{L^1 \cap L^\infty} \left(\frac{W_\infty^3}{d_{\min}^k} + 1 \right), \\ \frac{1}{N} \sum_{j \neq i} \frac{1}{d_{ij}^3} &\leq C \|\rho\|_{L^1 \cap L^\infty} \left(\frac{W_\infty^3}{d_{\min}^3} + |\log W_\infty| \right). \end{aligned}$$

Proof. We introduce a radial truncation function χ such that $\chi = 0$ on $B(0, 1/2)$ and $\chi = 1$ on ${}^c B(0, 3/4)$. We have for all $k \geq 0$:

$$\begin{aligned} \frac{1}{N} \sum_{j \neq i} \frac{1}{d_{ij}^k} &= \int_{\mathbb{R}^3} \chi \left(\frac{x_i - y}{d_{\min}} \right) \frac{1}{|x_i - y|^k} \rho^N(t, dy), \\ &= \int_{\mathbb{R}^3} \chi \left(\frac{x_i - T(y)}{d_{\min}} \right) \frac{1}{|x_i - T(y)|^k} \rho(t, dy), \\ &= \left(\int_{B(x_i, 3W_\infty)} + \int_{{}^c B(x_i, 3W_\infty)} \right) \chi \left(\frac{x_i - T(y)}{d_{\min}} \right) \frac{1}{|x_i - T(y)|^k} \rho(t, dy). \end{aligned}$$

Recall that $W_\infty \geq d_{\min}/2$. Since $\chi \left(\frac{x_i - T(y)}{d_{\min}} \right) = 0$ if $|x_i - T(y)| \leq d_{\min}/2$, the first term yields:

$$\int_{B(x_i, 3W_\infty)} \chi \left(\frac{x_i - T(y)}{d_{\min}} \right) \frac{1}{|x_i - T(y)|^k} \rho(t, dy) \leq C \|\rho\|_\infty \frac{W_\infty^3}{d_{\min}^k}.$$

For the second term, we have $|x_i - T(y)| \geq |x_i - y| - |y - T(y)| \geq \frac{|x_i - y|}{2}$ and we get:

$$\int_{{}^c B(x_i, 3W_\infty)} \chi \left(\frac{x_i - y}{d_{\min}} \right) \frac{1}{|x_i - T(y)|^k} \rho(t, dy) \leq \|\rho\|_\infty \int_{3W_\infty}^1 \frac{1}{r^{k-2}} dr + \|\rho\|_{L^1},$$

which yields the desired result. □

APPENDIX B. ESTIMATES ON $\mathcal{K}^N \rho^N$, $\mathcal{K}\rho$ AND CONTROL OF THE MINIMAL DISTANCE

In this part we present some estimates for the convergence of the velocity field $\mathcal{K}^N \rho^N$ and its gradient towards $\mathcal{K}\rho$ and its gradient. We estimate the ∞ norm of the error using the infinite Wasserstein distance between ρ^N and ρ in the spirit of [6, 7].

We recall that, according to [2][Theorem 5.6], at fixed time $t \geq 0$, there exists a (unique) optimal transport map T satisfying :

$$W_\infty := W_\infty(\rho(t, \cdot), \rho^N(t, \cdot)) = \rho - \text{esssup } |T(x) - x|,$$

with $\rho^N(t, \cdot) = T\#\rho(t, \cdot)$. This allows us to write $\mathcal{K}^N \rho^N$ as follows

$$\mathcal{K}^N \rho^N(x) = 6\pi r_0 \int \chi\Phi(x - T(y))\rho(y)dy.$$

This important property allows us to show the following results.

Proposition B.1 (Boundedness). *Under the assumption that $\rho \in W^{1,1} \cap W^{1,\infty}$, there exists a positive constant $C > 0$ independent of N such that:*

$$\|\mathcal{K}^N \rho^N\|_{W^{2,\infty}} \leq C \left(\frac{W_\infty^3}{d_{\min}} + \frac{W_\infty^3}{d_{\min}^2} + \frac{W_\infty^3}{d_{\min}^3} \right) \|\rho\|_{W^{1,\infty} \cap W^{1,1}}.$$

Remark B.1. *The term $\frac{W_\infty^3}{d_{\min}^3}$ appears only for the second derivative of $\mathcal{K}^N \rho^N$ which is needed for the proof of Theorem 0.2.*

Proof. Let $x \in \mathbb{R}^3$, we have :

$$\begin{aligned} |\mathcal{K}^N \rho^N(x)| &\leq C \int |\chi\Phi(x - T(y))\rho(y)dy|, \\ &\leq C \|\rho\|_\infty \int_{B(x, 3W_\infty)} |\chi\Phi(x - T(y))| + \int_{{}^c B(x, W_\infty)} |\chi\Phi(x - T(y))| |\rho(y)| dy. \end{aligned}$$

Recall that for all $y \in B(x, 3W_\infty)$ such that $|x - T(y)| \leq d_{\min}/2$ we have $\chi\Phi(x - T(y)) = 0$. Hence in all cases we have the following bound for all $y \in B(x, 3W_\infty)$:

$$|\chi\Phi(x - T(y))| \leq \frac{C}{d_{\min}},$$

this yields the following bound

$$\int_{B(x, 3W_\infty)} |\chi\Phi(x - T(y))| \leq C \frac{W_\infty^3}{d_{\min}}.$$

For all $y \in {}^c B(x, W_\infty)$ we have that $|x - T(y)| \geq |x - y| - |T(y) - y| \geq 2W_\infty \geq d_{\min}$. This ensures that $\chi\Phi(x - T(y)) = \Phi(x - T(y))$ on ${}^c B(x, W_\infty)$. Moreover we have

$$|x - T(y)| \geq |x - y| - W_\infty \geq \frac{1}{2}|x - y|,$$

which yields

$$\begin{aligned} \int_{cB(x, W_\infty)} |\chi\Phi(x - T(y))| |\rho(y)| dy &\leq C \|\rho\|_\infty \int_{cB(x, W_\infty) \cap B(x, 1)} \frac{dy}{|x - y|} + \|\rho\|_{L^1}, \\ &\leq C \|\rho\|_{L^1 \cap L^\infty}. \end{aligned}$$

Analogously we obtain a similar bound for $\nabla \mathcal{K}^N$. We focus now on the bound for $\nabla^2 \mathcal{K}^N \rho^N$. We have

$$\begin{aligned} |\nabla^2 \mathcal{K}^N \rho^N(x)| &\leq C \|\rho\|_\infty \int_{B(x, 3W_\infty)} |\nabla^2 \chi\Phi(x - T(y))| dy \\ &\quad + \left| \int_{cB(x, W_\infty)} \nabla^2 \chi\Phi(x - T(y)) \rho(y) dy \right|. \end{aligned}$$

We use the same estimates as before to bound the first term by $\|\rho\|_\infty \frac{W_\infty^3}{d_{\min}^3}$. For the second term we write

$$(64) \quad \left| \int_{cB(x, W_\infty)} \nabla^2 \chi\Phi(x - T(y)) \rho(y) dy \right| \leq \left| \int_{cB(x, W_\infty)} \nabla^2 \Phi(x - y) \rho(y) dy \right| \\ + \int_{cB(x, W_\infty)} |\nabla^2 \chi\Phi(x - T(y)) - \nabla^2 \Phi(x - y)| |\rho(y)| dy.$$

Using an integration by parts for the first term in the right hand side of (64) we get

$$\begin{aligned} \left| \int_{cB(x, W_\infty)} \nabla^2 \Phi(x - y) \rho(y) dy \right| &\leq \left| \int_{cB(x, W_\infty)} \nabla \Phi(x - y) \nabla \rho(y) dy \right| \\ &\quad + \int_{\partial B(x, W_\infty)} |\nabla \Phi(x - y)| |\rho(y)| d\sigma(y), \\ &\leq C \|\nabla \rho\|_{L^1 \cap L^\infty} + \|\rho\|_\infty. \end{aligned}$$

Finally, for the second term in the right hand side of (64) we have

$$\begin{aligned} &\int_{cB(x, W_\infty)} |\nabla^2 \chi\Phi(x - T(y)) - \nabla^2 \Phi(x - y)| |\rho(y)| dy \\ &\leq \int_{cB(x, W_\infty)} \left(\frac{1}{|x - y|^4} + \frac{1}{|x - T(y)|^4} \right) |y - T(y)| |\rho(y)| dy \\ &\leq C \|\rho\|_{L^1 \cap L^\infty}. \end{aligned}$$

□

The following convergence estimates are used in the proof of Theorem 0.2.

Proposition B.2 (Convergence estimates). *The following estimates hold true:*

$$\begin{aligned} \|\mathcal{K}^N \rho^N - \mathcal{K}\rho\|_{L^\infty} &\lesssim \|\rho\|_\infty W_\infty(\rho^N, \rho) \left(1 + \frac{W_\infty(\rho^N, \rho)^2}{d_{\min}}\right), \\ \|\nabla \mathcal{K}^N \rho^N - \nabla \mathcal{K}\rho\|_{L^\infty} &\lesssim \|\rho\|_\infty W_\infty(\rho^N, \rho) \left(|\log W_\infty(\rho^N, \rho)| + \frac{W_\infty(\rho^N, \rho)^2}{d_{\min}^2} + 1\right). \end{aligned}$$

Proof. We use in the proof the shortcut $W_\infty := W_\infty(\rho^N, \rho)$. Let $x \in \mathbb{R}^3$, we have

$$|\mathcal{K}^N \rho^N(x) - \mathcal{K}\rho(x)| \leq 6\pi r_0 \int_{\text{supp } \rho} |\chi\Phi(x - T(y)) - \Phi(x - y)| \rho(y) dy.$$

We split the integral into two disjoint domains $J := \{y \in \text{supp } \rho, |x - y| \leq 3W_\infty\}$ and its complementary. Note that on J , according to the definition of the truncation function χ , we have $\chi\Phi(x - T(y)) = 0$ for all $y \in J$ such that $|x - T(y)| \leq \frac{d_{\min}}{4}$. We can then bound directly the first integral as follows

$$\begin{aligned} \int_J |\chi\Phi(x - T(y)) - \Phi(x - y)| \rho(y) dy &\leq \int_J |\chi\Phi(x - T(y))| \rho(y) dy + \int_J |\Phi(x - y)| \rho(y) dy \\ &\lesssim \|\rho\|_\infty \left(|B(x, 3W_\infty)| \frac{4}{d_{\min}} + \int_{B(x, 3W_\infty)} \frac{1}{|x - y|} dy \right). \end{aligned}$$

Direct computations yields

$$\int_J |\chi\Phi(x - T(y)) - \Phi(x - y)| \lesssim \|\rho\|_\infty \left(\frac{W_\infty^3}{d_{\min}} + W_\infty^2 \right).$$

We focus now on the remaining term, note that for all $y \in {}^c J := {}^c B(x, 3W_\infty)$ we have

$$|x - T(y)| \geq |x - y| - |T(y) - y| \geq 2W_\infty \geq d_{\min},$$

which yields that $\chi\Phi(x - T(y)) = \Phi(x - T(y))$ on ${}^c J$. Moreover, we have $|x - T(y)| \geq \frac{1}{2}|x - y|$ on ${}^c J$. We have then

$$\begin{aligned} \int_{{}^c J} |\chi\Phi(x - T(y)) - \Phi(x - y)| &= \int_{{}^c J} |\Phi(x - T(y)) - \Phi(x - y)|, \\ &\leq K \int_{{}^c J} \left(\frac{1}{|x - T(y)|^2} + \frac{1}{|x - y|^2} \right) |y - T(y)| \rho(y) dy, \\ &\lesssim W_\infty \|\rho\|_\infty \int_{{}^c J} \frac{1}{|x - y|^2} dy, \\ &\lesssim W_\infty \|\rho\|_\infty. \end{aligned}$$

In the last line we use the fact that $\frac{1}{|x - y|^2}$ is integrable on ${}^c B(x, 3W_\infty)$. The proof for the second estimate is analogous to the first one. The main difference occurs for the last estimate where the *log* term appears. This is due to the fact that we integrate $\frac{1}{|x - y|^3}$ on ${}^c B(x, 3W_\infty)$. \square

We present now an estimate for the conservation of the particle configuration. This estimate combined with Proposition B.1 shows that the dilution regime is conserved provided that we have a control on the infinite Wasserstein distance.

Proposition B.3. *For all $1 \leq i \leq N$ and $j \neq i$ we have*

$$\begin{aligned} |\dot{\xi}_i| &\lesssim \|\nabla \mathcal{K}^N \rho^N\|_\infty |\xi_i| + O(d_{\min}), \\ |\dot{x}_+^i - \dot{x}_+^j| &\lesssim \|\nabla \mathcal{K}^N \rho^N\|_\infty |x_+^i - x_+^j| + |\xi_i - \xi_j| + O(R), \\ |\dot{\xi}_i - \dot{\xi}_j| &\lesssim \|\nabla \mathcal{K}^N \rho^N\|_\infty |\xi_i - \xi_j| + \|\nabla^2 \mathcal{K}^N \rho^N\|_\infty |x_+^i - x_+^j| + O(d_{\min}). \end{aligned}$$

where

$$W_\infty := W_\infty(\rho(t, \cdot), \bar{\rho}^N(t, \cdot)) = \rho - \text{esssup} |T_t(x) - x|.$$

We remark that the conservation of the infinite Wasserstein distance, which is initially of order $\frac{1}{N^{1/3}}$, ensures the control of the particle distance. Unfortunately, due to the log term appearing in Proposition B.2 we are not able to prove the conservation in time of the infinite Wasserstein distance.

APPENDIX C. EXISTENCE, UNIQUENESS AND SOME STABILITY PROPERTIES

In this section we present some existence, uniqueness and stability estimates.

Proposition C.1. *Let $p > 3$. Given $F_0 \in W^{2,p}$ and $u \in L^\infty(0, T; W^{3,p})$, there exists a time $T > 0$ such that $F \in L^\infty(0, T; W^{2,p})$ is the unique local solution of*

$$(65) \quad \begin{cases} \partial_t F + \nabla F \cdot (\mathbb{A}^{-1}(F)\kappa g + u) = \nabla u \cdot F, & \text{on } [0, T] \times \mathbb{R}^3, \\ F(0, \cdot) = F_0, & \text{on } \mathbb{R}^3. \end{cases}$$

We have the following stability estimates

$$\|F_1 - F_2\|_{L^\infty(0, T; W^{1,p})} \leq C_1 T \|u_1 - u_2\|_{L^\infty(0, T; W^{2,p})} e^{C_2 T},$$

with C_1 and C_2 depending on $\|\mathbb{A}^{-1}\|_{2,\infty}$, $\|u_i\|_{L^\infty(0, T; W^{3,p})}$, $\|F_i\|_{L^\infty(0, T; W^{2,p})}$.

Proof. Since $p > 3$, we have $F_0 \in W^{2,p} \hookrightarrow W^{1,\infty}$ and $u \in W^{2,\infty}$. We can apply the existence proof analogous to the existence proof of Proposition 5.1 to get a unique solution $F \in L^\infty(0, T; W^{1,\infty})$ for a given $T > 0$. It remains to show that $F \in L^\infty(0, T; W^{2,p})$ for a finite time interval. We have for $\alpha = 0, 1, 2$

$$\partial_t D^\alpha F + \nabla D^\alpha F \cdot (\mathbb{A}^{-1}(F)\kappa g + u) = -\nabla F \cdot D^\alpha (\mathbb{A}^{-1}(F)\kappa g + u) + (D^\alpha \nabla u)F + (\nabla u)D^\alpha F.$$

Multiplying by $|D^\alpha F|^{p-1}$ and integrating by parts the second term using the fact that $\operatorname{div}(u) = 0$, we get

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} \int |D^\alpha F|^p &= \frac{1}{p} \int |D^\alpha F|^p \operatorname{div}(\mathbb{A}^{-1}(F)) + \nabla F \cdot |D^\alpha F|^{p-1} (D^\alpha [\mathbb{A}^{-1}(F)] \kappa g + D^\alpha u) \\ &+ (D^\alpha \nabla u) F |D^\alpha F|^{p-1} + (\nabla u) D^\alpha F |D^\alpha F|^{p-1}, \\ &\lesssim \|F\|_{2,p}^p (\|\nabla \mathbb{A}^{-1}\|_\infty \|F\|_{1,\infty} + \|\nabla u\|_\infty) \\ &+ \|D^\alpha F\|^{p-1} (\|\mathbb{A}^{-1}\|_{2,\infty} + 1) \left(\|\nabla F\|_\infty \{ \|\nabla F\|_p + \|\nabla F\|_\infty \|\nabla F\|_p + \|\nabla^2 F\|_p + \|D^\alpha u\|_p \} \right. \\ &\left. + \|F\|_\infty \|D^\alpha \nabla u\|_p \right). \end{aligned}$$

Since $\|F\|_{1,\infty} \lesssim \|F\|_{2,p}$, $\|u\|_{1,\infty} \lesssim \|F\|_{2,p}$, we get up to a constant depending on $\|\mathbb{A}^{-1}\|_{2,\infty}$

$$\frac{d}{dt} \|D^\alpha F\|_p^p \lesssim \|D^\alpha F\|_p^p (\|F\|_{2,p} + \|u\|_{3,p}) + \|D^\alpha F\|_p^{p-1} \|F\|_{2,p} (\|F\|_{2,p} + \|u\|_{3,p}).$$

Applying Young's inequality and summing over $\alpha = 0, 1, 2$ we get

$$\|F\|_{L^\infty(0,T;W^{2,p})} \lesssim \|F_0\|_{2,p} e^{C(p,\|F\|_{2,p},\|u\|_{3,p},\|\mathbb{A}^{-1}\|_{2,\infty})T},$$

which shows that $F \in L^\infty(0, T; W^{2,p})$ for a finite time $T > 0$. Now consider two divergence free velocity fields $u_1, u_2 \in L^\infty(0, T; W^{3,p})$ and denote by F_i the solution to (65). We have

$$\begin{aligned} \partial_t(F_1 - F_2) + (\nabla F_1 - \nabla F_2)(\mathbb{A}^{-1}(F_1)\kappa g + u_1) \\ = \nabla F_2 (\mathbb{A}^{-1}(F_1) - \mathbb{A}^{-1}(F_2) + u_1 - u_2) + (\nabla u_1 - \nabla u_2)F_1 + (F_1 - F_2)\nabla u_2. \end{aligned}$$

Multiplying by $|F_1 - F_2|^{p-1}$ and integrating by parts the second term in the left hand side using the divergence free property of u , we get

$$\begin{aligned} \frac{d}{dt} \|F_1 - F_2\|_p^p &\lesssim \|F_1 - F_2\|_p^p (\|\nabla \mathbb{A}^{-1}\|_\infty (\|\nabla F_1\|_\infty + \|\nabla F_2\|_\infty) + \|\nabla u_2\|_\infty) \\ &+ \|F_1 - F_2\|_p^{p-1} \|u_1 - u_2\|_{2,p} (\|\nabla F_1\|_\infty + \|\nabla F_2\|_\infty). \end{aligned}$$

For the derivative we have

$$\begin{aligned} \partial_t(\nabla F_1 - \nabla F_2) + \nabla(\nabla F_1 - \nabla F_2)(\mathbb{A}^{-1}(F)\kappa g + u_1) \\ = -(\nabla F_1 - \nabla F_2)(\nabla \mathbb{A}^{-1}(F_1)\nabla F_1\kappa g + \nabla u_1) + \nabla^2 F_2 (\mathbb{A}^{-1}(F_1) - \mathbb{A}^{-1}(F_2) + u_1 - u_2) \\ + \nabla F_2 (\{ [\nabla \mathbb{A}^{-1}(F_1) - \nabla \mathbb{A}^{-1}(F_2)] \nabla F_1 + \nabla \mathbb{A}^{-1}(F_2)(\nabla F_1 - \nabla F_2) \} \kappa g + \nabla u_1 - \nabla u_2) \\ + (\nabla^2 u_1 - \nabla^2 u_2)F_1 + (\nabla u_1 - \nabla u_2)\nabla F_1 + \nabla u_2(\nabla F_1 - \nabla F_2) + \nabla^2 u_2(F_1 - F_2). \end{aligned}$$

Using the same estimates as previously, we obtain

$$\frac{d}{dt} \|F_1 - F_2\|_{1,p}^p \leq C_1 \|F_1 - F_2\|_{1,p}^p + C_2 \|F_1 - F_2\|_{1,p}^{p-1} \|u_1 - u_2\|_{2,p},$$

where C_1, C_2 depend on $\|\mathbb{A}^{-1}\|_{2,\infty}$, $\|u_i\|_{3,p}$, $\|F_i\|_{2,p}$. We conclude by integrating with respect to time and apply Gronwall's inequality. \square

Proposition C.2. *Let $T > 0$, $p > 3$. We consider $\rho_0 \in W^{1,p}$, $u \in L^\infty(0, T; W^{3,p})$ and $F \in L^\infty(0, T; W^{2,p})$. There exists a unique solution $\rho \in L^\infty(0, T; W^{1,p})$ to the transport equation*

$$(66) \quad \begin{cases} \partial_t \rho + \operatorname{div}((\mathbb{A}^{-1}(F)\kappa g + u)\rho) &= 0, \\ \rho(0, \cdot) &= \rho_0, \end{cases}$$

for all $T > 0$. ρ satisfies

$$\|\rho(t, \cdot)\|_{L^\infty(0, T; W^{1,p})} \leq \|\rho_0\|_{1,p} e^{Ct},$$

where C depends on p , $\|\mathbb{A}^{-1}\|_{2,\infty}$, $\|F\|_{L^\infty(0, T; W^{2,p})}$, $\|u\|_{L^\infty(0, T; W^{2,p})}$. In addition, we have the following stability estimate

$$\|\rho_1 - \rho_2\|_{L^\infty(0, T; L^p)} \leq C_1 T \left(\|u_1 - u_2\|_{L^\infty(0, T; W^{1,p})} + \|F_1 - F_2\|_{L^\infty(0, T; W^{1,p})} \right) e^{C_2 T},$$

with constants depending on $\|\mathbb{A}^{-1}\|_{1,\infty}$, $\|\rho_i\|_{L^\infty(0, T; W^{1,p})}$, $\|F_i\|_{L^\infty(0, T; W^{1,p})}$.

Remark C.1. *If we assume in addition that ρ_0 is compactly supported then classical transport theory ensures that $\rho(t, \cdot)$ is compactly supported and using the characteristic flow, which is well defined since $F, u \in W^{1,\infty}$, one can show that*

$$\operatorname{diam}(\operatorname{supp}(\rho(t, \cdot))) \leq \operatorname{diam}(\operatorname{supp}(\rho_0)) e^{Ct},$$

with $C = C(\|\nabla \mathbb{A}^{-1}\|_\infty, \|\nabla F\|_{L^\infty(0, t; L^\infty)}, \|\nabla u\|_{L^\infty(0, t; L^\infty)})$.

Proof. Since $g = -|g|e_3$, we have the following formula

$$\operatorname{div}(\mathbb{A}^{-1}(F)\kappa g) = -\nabla \mathbb{A}_3^{-1}(F) \cdot \nabla F \kappa |g|,$$

where \mathbb{A}_3^{-1} is the third column of \mathbb{A}^{-1} . Note that since $p > 3$, we have the following Sobolev embedding

$$(67) \quad \|F\|_{1,\infty} \lesssim \|F\|_{2,p}, \quad \|u\|_{1,\infty} \lesssim \|u\|_{2,p}, \quad \|\rho\|_\infty \lesssim \|\rho\|_{1,p}.$$

The idea is to apply a fixed point argument. We define the operator A which maps any $\rho \in L^\infty(0, T; W^{1,p})$ to the unique density $A(\rho)$ solution of

$$(68) \quad \partial_t A(\rho) + \nabla A(\rho) \cdot (\mathbb{A}^{-1}(F)\kappa g + u) = (\nabla \mathbb{A}_3^{-1}(F) \cdot \nabla F \kappa |g|) \rho.$$

Thanks to (67), $u \in W^{1,\infty}$ and $F \in W^{1,\infty}$, hence DiPerna-Lions renormalization theory ensures the existence of $\mathcal{A}(\rho) \in L^\infty(0, T; L^p)$. Multiplying (68) by $|A(\rho)|^{p-1}$, integrating by parts and using Young's inequality we get

$$\begin{aligned} \frac{1}{p} \|A(\rho)\|_p^p &\leq \frac{1}{p} \|\rho_0\|_p^p + \frac{1}{p} \int_0^t \|A(\rho)\|_p^p \|\mathbb{A}^{-1}\|_\infty \|\nabla F\|_\infty + \int_0^t \|\mathbb{A}^{-1}\|_\infty \|\nabla F\|_\infty \|\rho\|_p \|A(\rho)\|_p^{p-1}, \\ &\leq \frac{1}{p} \|\rho_0\|_p^p + C \int_0^t \left(\frac{1}{p} \|A(\rho)\|_p^p + \frac{1}{p} \|\rho\|_p^p + \frac{p-1}{p} \|A(\rho)\|_p^p \right), \\ &\leq \frac{1}{p} \|\rho_0\|_p^p + C \int_0^t \|A(\rho)\|_p^p + \frac{C}{p} t \|\rho\|_{L^\infty(L^p)}^p \end{aligned}$$

with $C = C(\|\mathbb{A}^{-1}\|_\infty, \|\nabla F\|_{L^\infty(0, T; L^\infty)})$. Hence, Gronwall's inequality yields

$$\|A(\rho)\|_p \leq (\|\rho_0\|_p + TC\|\rho\|_p) e^{Ct}.$$

Moreover, we have

$$\begin{aligned} & \partial_t \nabla A(\rho) + \nabla(\nabla A(\rho)) \cdot (\mathbb{A}^{-1}(F)\kappa g + u) \\ &= -\nabla A(\rho) \nabla(\mathbb{A}^{-1}(F)\kappa g + u) + \nabla^2 \mathbb{A}_3^{-1}(F)\kappa|g| \nabla F \nabla F \rho \\ &+ \nabla \mathbb{A}_3^{-1}(F)\kappa|g| \nabla^2 F \rho + \nabla \mathbb{A}_3^{-1}(F) \cdot \nabla F \kappa|g| \nabla \rho. \end{aligned}$$

Multiplying by $|\nabla A(\rho)|^{p-1}$ and reproducing the same computations as before we get

$$\|\nabla A(\rho)\|_p \leq (\|\nabla \rho_0\|_p + TC_1 \|\rho\|_{1,p}) e^{C_2 t},$$

where we used (67). Hence, the constants C_1, C_2 depend on $p, \|\mathbb{A}^{-1}\|_{2,\infty}, \|u\|_{L^\infty(0,T;W^{2,p})}, \|F\|_{L^\infty(0,T;W^{2,p})}$ and $\|\rho\|_{L^\infty(0,T;W^{1,p})}$. Gathering the two estimates we obtain

$$(69) \quad \|A(\rho)\|_{L^\infty(0,T;W^{1,p})} \leq (\|\rho_0\|_{1,p} + TC_1 \|\rho\|_{1,p}) e^{C_2 T}.$$

Given ρ_1, ρ_2 , since equation (68) is linear, $A(\rho_1) - A(\rho_2)$ satisfies the same equation with $\rho_0 = 0$. Consequently, for $T > 0$ small enough, estimate (69) shows that the mapping A is a contraction and hence there exists a unique fixed point. Estimate (69) shows also global existence.

Let $u_i \in L^\infty(0, T, W^{3,p})$ and $F_i \in L^\infty(0, T, W^{2,p})$ for $i = 1, 2$. Denote by ρ_i the unique solution to equation (66). We have

$$\begin{aligned} & \partial_t(\rho_1 - \rho_2) + \nabla(\rho_1 - \rho_2) \cdot (\mathbb{A}^{-1}(F_1)\kappa g + u_1) \\ &= -\nabla \rho_2 \cdot ([\mathbb{A}^{-1}(F_1) - \mathbb{A}^{-1}(F_2)]\kappa g + u_1 - u_2) \\ &+ (\rho_1 - \rho_2) \nabla \mathbb{A}_3^{-1}(F_1)\kappa|g| \\ &+ \rho_1 ([(\nabla \mathbb{A}_3^{-1}(F_1) - \nabla \mathbb{A}_3^{-1}(F_2))] \nabla F_1 + \nabla \mathbb{A}_3^{-1}(F_2)(\nabla F_1 - \nabla F_2)) \kappa|g|. \end{aligned}$$

Multiplying by $|\rho_1 - \rho_2|^{p-1}$ and integrating we get

$$\frac{d}{dt} \|\rho_1 - \rho_2\|_p^p \lesssim C_1 \|\rho_1 - \rho_2\|_p^p + C_2 (\|u_1 - u_2\|_\infty + \|F_1 - F_2\|_\infty + \|\nabla F_1 - \nabla F_2\|_p) \|\rho_1 - \rho_2\|_p^{p-1},$$

with constants depending on $\|\mathbb{A}^{-1}\|_{1,\infty}, \|\rho_i\|_{1,p}, \|F_i\|_{1,p}$. We conclude using again the embedding $\|F_1 - F_2\|_\infty \leq C \|F_1 - F_2\|_{1,p}$ and analogously for $\|u_1 - u_2\|_\infty$. \square

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