

# LIFTING IN COMPACT COVERING SPACES FOR FRACTIONAL SOBOLEV MAPPINGS

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ABSTRACT. Let  $\pi : \tilde{\mathcal{N}} \rightarrow \mathcal{N}$  be a Riemannian covering, with  $\mathcal{N}, \tilde{\mathcal{N}}$  smooth compact connected Riemannian manifolds. If  $\mathcal{M}$  is an  $m$ -dimensional compact simply-connected Riemannian manifold,  $0 < s < 1$  and  $2 \leq sp < m$ , we prove that every mapping  $u \in W^{s,p}(\mathcal{M}, \mathcal{N})$  has a lifting in  $W^{s,p}$ , i.e., we have  $u = \pi \circ \tilde{u}$  for some mapping  $\tilde{u} \in W^{s,p}(\mathcal{M}, \tilde{\mathcal{N}})$ . Combined with previous contributions of Bourgain, Brezis and Mironescu and Bethuel and Chiron, our result *settles completely* the question of the lifting in Sobolev spaces over covering spaces.

The proof relies on an a priori estimate of the oscillations of  $W^{s,p}$  maps with  $0 < s < 1$  and  $sp > 1$ , *in dimension 1*. Our argument also leads to the existence of a lifting when  $0 < s < 1$  and  $1 < sp < 2 \leq m$ , provided there is no topological obstruction on  $u$ , i.e.,  $u = \pi \circ \tilde{u}$  holds in this range provided  $u$  is in the strong closure of  $C^\infty(\mathcal{M}, \mathcal{N})$ .

However, when  $0 < s < 1$ ,  $sp = 1$  and  $m \geq 2$ , we show that an (analytical) obstruction still arises, even in absence of topological obstructions. More specifically, we construct some map  $u \in W^{s,p}(\mathcal{M}, \mathcal{N})$  in the strong closure of  $C^\infty(\mathcal{M}, \mathcal{N})$ , such that  $u = \pi \circ \tilde{u}$  does not hold for any  $\tilde{u} \in W^{s,p}(\mathcal{M}, \tilde{\mathcal{N}})$ .

## 1. INTRODUCTION

Let  $\pi \in C^\infty(\tilde{\mathcal{N}}, \mathcal{N})$  be a Riemannian covering. In most of the results we present, we make the following assumptions on the Riemannian manifolds  $\tilde{\mathcal{N}}, \mathcal{N}$  and on the cover  $\pi$ :

(1.1)  $\mathcal{N}$  is compact and connected,

(1.2)  $\tilde{\mathcal{N}}$  is connected

and

(1.3)  $\pi$  is non-trivial.

In what follows, the compactness of  $\tilde{\mathcal{N}}$  will play a crucial role. We distinguish between the *compact case* (when  $\tilde{\mathcal{N}}$  is compact) and the *non-compact case* (when  $\tilde{\mathcal{N}}$  is non-compact).

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We also consider some  $\mathcal{M}$  satisfying

- (1.4)  $\mathcal{M}$  is an  $m$ -dimensional compact simply-connected Riemannian manifold,  
possibly with boundary.

In particular, we cover the case where  $\mathcal{M}$  is a smooth bounded simply-connected domain in  $\mathbb{R}^m$ . (With a slight abuse, in this case we identify  $\mathcal{M}$  and  $\overline{\mathcal{M}}$ .)

A classical result in homotopy theory states that every map  $u \in C^k(\mathcal{M}, \mathcal{N})$  can be lifted in  $C^k$ , i.e., there exists some map  $\tilde{u} \in C^k(\mathcal{M}, \tilde{\mathcal{N}})$  such that  $u = \pi \circ \tilde{u}$  in  $\mathcal{M}$ . The *lifting problem for Sobolev mappings* consists in determining whether every map  $u \in W^{s,p}(\mathcal{M}, \mathcal{N})$  can be lifted in  $W^{s,p}$ , i.e., whether there exists some map  $\tilde{u} \in W^{s,p}(\mathcal{M}, \tilde{\mathcal{N}})$  such that  $u = \pi \circ \tilde{u}$ .

We pause here to describe the Sobolev semi-norm we consider. Although we briefly consider the case where  $s \geq 1$ , in the new results we present we always assume that  $0 < s < 1$  and  $1 \leq p < \infty$ . For such  $s$  and  $p$ , the adapted semi-norm is defined as follows. We let  $d_{\mathcal{M}}$  and  $d_{\mathcal{N}}$  denote respectively the geodesic distances on  $\mathcal{M}$  and  $\mathcal{N}$ . We embed  $\mathcal{M}$  into some Euclidean space  $\mathbb{R}^\mu$  and consider the  $m$ -dimensional Hausdorff measure on  $\mathcal{M}$ , denoted  $dx$ . We set

$$W^{s,p}(\mathcal{M}, \mathcal{N}) := \left\{ u : \mathcal{M} \rightarrow \mathcal{N}; u \text{ is measurable and } |u|_{W^{s,p}} < \infty \right\},$$

where the Gagliardo semi-norm is defined as

$$|u|_{W^{s,p}}^p := \int_{\mathcal{M}} \int_{\mathcal{M}} \frac{d_{\mathcal{N}}(u(x), u(y))^p}{d_{\mathcal{M}}(x, y)^{m+sp}} dx dy.$$

Different embeddings of  $\mathcal{M}$  lead to the same space  $W^{s,p}(\mathcal{M}, \mathcal{N})$ , with equivalent semi-norms.

In the case where the target manifold  $\mathcal{N}$  is compact, we can as well embed it into some Euclidean space  $\mathbb{R}^\nu$ , and then we may replace the geodesic distance by the Euclidean one. This leads to the same space, with equivalent semi-norm. The space  $W^{s,p}(\mathcal{M}, \tilde{\mathcal{N}})$  can be defined similarly; even when  $\mathcal{N}$  is compact, the covering space  $\tilde{\mathcal{N}}$  need not be compact.

We next present some previous results on lifting. When  $\pi : \mathbb{R} \rightarrow \mathbb{S}^1$  is the universal covering of the circle by the real line, i.e., in complex notation, we have  $\pi : \mathbb{R} \ni \tilde{x} \mapsto e^{i\tilde{x}} \in \mathbb{S}^1 \subset \mathbb{C}$ , Bourgain, Brezis and Mironescu [6] have showed that every map  $u \in W^{s,p}(\mathcal{M}, \mathbb{S}^1)$  has a lifting unless either  $1 \leq sp < 2 \leq m$  or  $[0 < s < 1 \text{ and } 1 \leq sp < m]$ . Bethuel and Chiron [4] have proved that the *same conclusion* holds, more generally, in the *non-compact case*, under the assumptions (1.1)–(1.4)<sup>1</sup>. The proof in [4] relies, among other ingredients, on the existence of a ray (i.e., an isometrically embedded real half-line) in any non-compact connected Riemannian manifold. The *compact case* was only partially settled in [4], one of the difficulties in [4] arising from the non-existence of rays in this case. More specifically, the case where  $0 < s < 1$  and  $2 \leq sp < m$  was left open in [4].

Our main result, Theorem 1 below, completes their analysis<sup>2</sup>.

**Theorem 1.** *Assume (1.1)–(1.4), with  $\tilde{\mathcal{N}}$  compact and  $m = \dim \mathcal{M} \geq 2$ . Then exactly one of the following holds.*

- (a) *Every map  $u \in W^{s,p}(\mathcal{M}, \mathcal{N})$  can be lifted into a map  $\tilde{u} \in W^{s,p}(\mathcal{M}, \tilde{\mathcal{N}})$ .*
- (b)  *$1 \leq sp < 2$ .*

<sup>1</sup>In [4],  $\pi : \tilde{\mathcal{N}} \rightarrow \mathcal{N}$  is assumed to be the universal covering of  $\mathcal{N}$ , but the proofs there use only the assumptions (1.1)–(1.4).

<sup>2</sup>We exclude from our analysis the case where  $m = 1$ , and thus  $\mathcal{M}$  is a bounded interval. In this case, the lifting property holds for any  $s$  and  $p$  [4, 6].

The compact case covers as important examples the real projective spaces  $\mathbb{RP}^m$ , with universal covering space  $\mathbb{S}^m$ , which is relevant in the theory liquid crystals [2, 24] and the  $d$ -fold covering of the circle, with  $d \geq 2$ , corresponding to  $\mathcal{N} = \tilde{\mathcal{N}} = \mathbb{S}^1$  and, in complex notation<sup>3</sup>,  $\pi(\tilde{x}) = \tilde{x}^d$ . In this latter case, the lifting problem is also known as the  $d$ th root problem. The solution of this problem is positive unless  $1 \leq sp < 2 \leq m$  [4, 19]; the original proof of this fact is based on the existence of liftings over the universal covering of  $\mathbb{R}$  by  $\mathbb{S}^1$  in the sum  $(W^{s,p} + W^{1,sp})(\mathcal{M}, \mathbb{R})$  [17, 18] and on the fractional Gagliardo–Nirenberg interpolation inequality [10]. Our above result provides an alternative argument to the  $d$ th root problem.

As noted by Bethuel [3], Theorem 1 has as a consequence that, under the assumptions that  $p \geq 3$ , the fundamental group  $\pi_1(\mathcal{N})$  is finite and the homotopy groups  $\pi_2(\mathcal{N}), \dots, \pi_{\lfloor p-1 \rfloor}(\mathcal{N})$  are trivial, then the trace operator

$$W^{1,p}(\mathcal{M} \times (0, 1), \mathcal{N}) \ni f \mapsto \text{tr } f \in W^{1-1/p,p}(\mathcal{M}, \mathcal{N})$$

is surjective. We will come back to this in a subsequent work [21].

Returning to the lifting question, it is instructive to compare the above picture with the one in the non-compact case, already completed in [4].

**Theorem 2** (Bethuel and Chiron [4]). *Assume (1.1)–(1.4), with  $\tilde{\mathcal{N}}$  non-compact and  $m = \dim \mathcal{M} \geq 2$ .*

*Then exactly one of the following holds.*

- (a) *Every map  $u \in W^{s,p}(\mathcal{M}, \mathcal{N})$  can be lifted into a map  $\tilde{u} \in W^{s,p}(\mathcal{M}, \tilde{\mathcal{N}})$ .*
- (b)  *$1 \leq sp < 2$  or  $[0 < s < 1$  and  $1 \leq sp < \dim \mathcal{M}]$ .*

Theorem 2 contains as a special case the result established in [6] for  $\pi : \mathbb{R} \rightarrow \mathbb{S}^1$  the universal covering of the unit circle.

The proof of Theorem 1 relies on a new one-dimensional estimate, (1.6) below, that may be of independent interest. For the sake of simplicity, we state it for real-valued continuous functions  $f \in C^0(\mathbb{R}, \mathbb{R})$ . For such  $f$  and  $x, y \in \mathbb{R}$ , we define the oscillation of  $f$  on the interval  $[x, y]$  as

$$(1.5) \quad \text{osc}_{[x,y]} f := \max\{|f(z) - f(t)|; z, t \in [x, y]\}.$$

We prove that, for  $0 < s < 1$  and  $1 < p < \infty$  such that  $sp > 1$ , we have the *reverse oscillation inequality*

$$(1.6) \quad \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{[\text{osc}_{[x,y]} f]^p}{|y - x|^{1+sp}} dx dy \leq C_{s,p} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|f(y) - f(x)|^p}{|y - x|^{1+sp}} dx dy.$$

The terminology “reverse inequality” refers to the fact that, since  $\text{osc}_{[x,y]} f \geq |f(y) - f(x)|$ , we have, for any  $0 < s < 1$  and  $1 \leq p < \infty$ ,

$$(1.7) \quad \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|f(y) - f(x)|^p}{|y - x|^{1+sp}} dx dy \leq \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{[\text{osc}_{[x,y]} f]^p}{|y - x|^{1+sp}} dx dy.$$

Our result (1.6) is that the inequality (1.7) can be reversed when  $sp > 1$ .

We next turn to the nature of obstructions to the existence of lifting. They are of two types, topological and analytical ones. Topological obstructions arise when  $1 \leq sp < 2 \leq m$ , and are induced by maps which are locally of the form  $u(y, z) = f(y/|y|)$ , where  $(y, z) \in \mathbb{B}^2 \times \mathbb{B}^{m-2}$  and the map  $f \in C^0(\mathbb{S}^1, \mathcal{N})$  admits no lifting. (Here and in the sequel,  $\mathbb{B}^k$  denotes the unit ball of  $\mathbb{R}^k$ .) The existence of such  $f$  follows from our assumption (1.3). Analytical obstructions arise when  $0 < s < 1$  and  $1 \leq sp < m$ ; they are related to the existence of maps  $\tilde{u} : \mathbb{B}^m \rightarrow \tilde{\mathcal{N}}$  that

<sup>3</sup>Strictly speaking, the metrics should be adapted by a constant conformal factor so that the mapping is a local Riemannian isometry.

are smooth except at the origin, such that roughly speaking  $\pi \circ \tilde{u}$  oscillates much less than  $\tilde{u}$ , i.e.,  $\tilde{u} \in W_{loc}^{s,p}(\mathbb{B}^m \setminus \{0\}, \tilde{\mathcal{N}}) \setminus W^{s,p}(\mathbb{B}^m, \tilde{\mathcal{N}})$ , while  $\pi \circ \tilde{u} \in W^{s,p}(\mathbb{B}^m, \mathcal{N})$ .

Theorem 1 has a variant which is valid when  $1 < sp < 2$ . Indeed, the maps that include topological obstructions are not in the strong closure of  $C^\infty(\mathcal{M}, \mathcal{N})$  for the  $W^{s,p}$  norm (this can be seen by a simple topological argument [4, Lemma 1 and Appendix A.2]). With this in mind, Theorem 3 below asserts that, in absence of topological obstructions, there are no analytical obstructions.

**Theorem 3.** *Assume (1.1)–(1.4), with  $\tilde{\mathcal{N}}$  compact. Assume that  $0 < s < 1$  and  $1 < sp < 2 \leq m = \dim \mathcal{M}$ . Consider, for a map  $u \in W^{s,p}(\mathcal{M}, \mathcal{N})$ , the following properties:*

- (a)  *$u$  can be strongly approximated by maps in  $C^\infty(\mathcal{M}, \mathcal{N})$ ,*
- (b)  *$u$  can be weakly approximated by maps in  $C^\infty(\mathcal{M}, \mathcal{N})$ ,*
- (c)  *$u$  has a lifting in  $W^{s,p}(\mathcal{M}, \tilde{\mathcal{N}})$ .*

Then

- (i) *We have (a)  $\implies$  (b)  $\implies$  (c).*
- (ii) *If  $\mathcal{M}$  is diffeomorphic to a ball and  $\pi$  is the universal covering, then the properties (a), (b) and (c) are all equivalent.*

We specify the notion of strong convergence in Theorem 3, since there is no natural distance on  $W^{s,p}(\mathcal{M}, \mathcal{N})$ . We embed the manifold  $\mathcal{N}$  into some Euclidean space  $\mathbb{R}^\nu$ , and thus identify  $W^{s,p}(\mathcal{M}, \mathcal{N})$  with  $W := \{v \in W^{s,p}(\mathcal{M}, \mathbb{R}^\nu); v(x) \in \mathcal{N} \text{ for a.e. } x \in \mathcal{M}\}$ . With this identification,  $u_j \rightarrow u$  in  $W^{s,p}(\mathcal{M}, \mathcal{N})$  amounts to  $u_j, u \in W$  and  $u_j \rightarrow u$  in  $W^{s,p}(\mathcal{M}, \mathbb{R}^\nu)$  as  $j \rightarrow \infty$ . When  $\mathcal{N}$  is compact or, more generally, when the sequence  $(u_j)_{j \geq 0}$  takes its values into a fixed compact subset of  $\mathcal{N}$ , this notion of convergence does not depend on the embedding.

We also specify the notion of weak convergence, since  $W^{s,p}(\mathcal{M}, \mathcal{N})$  is not a linear space. When  $0 < s < 1$  and  $1 < p < \infty$ , we adopt the following convention:  $u_j \rightarrow u$  weakly in  $W^{s,p}(\mathcal{M}, \mathcal{N})$  if  $u_j \rightarrow u$  a.e. as  $j \rightarrow \infty$  and  $|u_j|_{W^{s,p}(\mathcal{M})} \leq C, \forall j$ .

It will be clear from its proof that Theorem 3 is still valid when  $s = 1$  and  $p \geq 1$ . In the case of the universal covering of  $\mathbb{S}^1$ , the conclusion of the theorem still holds when  $s \geq 1$  [12, Chapters 9 and 11]. When  $s > 1$  and for a general covering, the definition of  $W^{s,p}(\mathcal{M}, \tilde{\mathcal{N}})$  is less obvious. Adopting the definition of  $W^{s,p}(\mathcal{M}, \tilde{\mathcal{N}})$  in [4], Theorem 3 with  $s > 1$  can possibly be obtained by combining [4, Appendix A.1] with the composition result in [10]; this is not investigated here.

Theorem 3 leaves open the question of existence of analytical obstructions when  $0 < s < 1$  and  $sp = 1$ . Such obstructions *do exist*, as shows our next result.

**Theorem 4.** *Assume (1.2)–(1.4),  $\mathcal{N}$  connected and  $m = \dim \mathcal{M} \geq 2$ . For  $0 < s < 1$  and  $p$  such that  $sp = 1$  and for every point  $a \in \mathcal{M}$ , there exists a mapping  $u : \mathcal{M} \rightarrow \mathcal{N}$  such that*

- (i)  *$u \in C^\infty(\mathcal{M} \setminus \{a\}, \mathcal{N}) \cap W^{s,p}(\mathcal{M}, \mathcal{N})$ ,*
- (ii)  *$u$  can be strongly approximated by maps in  $C^\infty(\mathcal{M}, \mathcal{N})$ ,*
- (iii)  *$u$  has no lifting  $\tilde{u} \in W^{s,p}(\mathcal{M}, \tilde{\mathcal{N}})$ .*

Theorem 4 answers negatively [19, open problem 7].

Our paper is organized as follows. In Section 2 we recall some basic facts about coverings. In Section 3, which is the main contribution of this work, we prove the reverse oscillation inequality (1.6) and its consequences, Theorems 1 and 3. In Section 4 we discuss uniqueness, in a framework more general than the one of the universal covering of the circle [6] or of universal coverings [4]. This will be needed in the proof of the existence of the analytic obstruction. In Section 5, we prove Theorem 4.

## 2. ABOUT COVERINGS

Let us start by recalling some basic fact concerning the coverings. The mapping  $\pi : \widetilde{\mathcal{N}} \rightarrow \mathcal{N}$  (with  $\mathcal{N}, \widetilde{\mathcal{N}}$  topological spaces) is a *cover* (or *covering map*) whenever  $\pi$  is continuous and every point  $y \in \mathcal{N}$  belongs to an open set  $U \subset \mathcal{N}$  *evenly covered* by  $\pi$ , i.e., the inverse image  $\pi^{-1}(U)$  is a disjoint union of open sets  $V_i, i \in I$ , with  $\pi : V_i \rightarrow U$  a homeomorphism,  $\forall i \in I$ .

If  $\mathcal{N}$  is a connected topological manifold and if the covering space  $\widetilde{\mathcal{N}}$  is connected, then the cardinality of the inverse image  $\pi^{-1}(\{y\})$  of a point does not depend on the point  $y \in \mathcal{N}$  and is at most countable; this follows from the fact that  $\pi^{-1}(\{y\})$  is isomorphic to  $\pi_1(\mathcal{N}, y)$  [14, Proposition 1.32] combined with the fact that  $\pi_1(\mathcal{N}, y)$  is at most countable,  $\forall y \in \mathcal{N}$  [16, Theorem 7.21].

If  $\mathcal{N}$  is a connected Riemannian manifold, then the cover  $\pi$  induces on  $\widetilde{\mathcal{N}}$  a unique *Riemannian structure* such that the mapping  $\pi$  is a local isometry. Conversely, if the Riemannian manifold  $\widetilde{\mathcal{N}}$  is complete and if the mapping  $\pi : \widetilde{\mathcal{N}} \rightarrow \mathcal{N}$  is a local isometry (that is, the pullback  $\pi^*g$  of the metric  $g$  of  $\mathcal{N}$  coincides with the metric  $\tilde{g}$  of  $\widetilde{\mathcal{N}}$ ), then  $\pi$  is a cover [15, Lemma 11.6]. The local isometry property implies in particular that  $\pi$  is globally a non-expansive map: for every  $\tilde{x}, \tilde{y} \in \widetilde{\mathcal{N}}$ , we have

$$d_{\mathcal{N}}(\pi(\tilde{x}), \pi(\tilde{y})) \leq d_{\widetilde{\mathcal{N}}}(\tilde{x}, \tilde{y}),$$

with equality everywhere if and only if the map  $\pi$  is a global homeomorphism.

The next lemma shows that a Riemannian covering map is always an isometry on scales smaller than the injectivity radius  $\text{inj}(\mathcal{N})$  (which is defined as the least upper bound of the radii  $\rho > 0$  such that the exponential mapping at any point  $y \in \mathcal{N}$ , restricted to a ball of radius  $\rho$  of the tangent space  $T_y\mathcal{N}$ , is a diffeomorphism).

**Lemma 2.1.** *Let  $\pi : \widetilde{\mathcal{N}} \rightarrow \mathcal{N}$  be a Riemannian covering map. Assume that  $\mathcal{N}$  has positive injectivity radius  $\text{inj}(\mathcal{N}) > 0$ .*

*Then for every  $\tilde{x}, \tilde{y} \in \widetilde{\mathcal{N}}$  such that  $d_{\widetilde{\mathcal{N}}}(\tilde{x}, \tilde{y}) \leq \text{inj}(\mathcal{N})$ , one has  $d_{\widetilde{\mathcal{N}}}(\tilde{x}, \tilde{y}) = d_{\mathcal{N}}(\pi(\tilde{x}), \pi(\tilde{y}))$ .*

The positivity assumption on the injectivity radius in Lemma 2.1 is satisfied in particular when the manifold  $\mathcal{N}$  is compact.

The proof of Lemma 2.1 follows the strategy to prove that local isometries of complete manifolds yield covering maps [15, proof of Lemma 11.6].

*Proof of Lemma 2.1.* Let  $\tilde{x}, \tilde{y} \in \widetilde{\mathcal{N}}$  satisfy  $d_{\widetilde{\mathcal{N}}}(\tilde{x}, \tilde{y}) \leq \text{inj}(\mathcal{N})$ . Let  $\tilde{\gamma} : [0, 1] \rightarrow \widetilde{\mathcal{N}}$  be the natural parametrization of a minimizing geodesic  $\tilde{\Gamma}$  in  $\widetilde{\mathcal{N}}$  joining the point  $\tilde{x}$  to  $\tilde{y}$ . Since  $\pi$  is a local isometry,  $\gamma := \pi \circ \tilde{\gamma} : [0, 1] \rightarrow \mathcal{N}$  is the natural parametrization of a geodesic  $\Gamma$  in  $\mathcal{N}$  joining the point  $x := \pi(\tilde{x})$  to  $y := \pi(\tilde{y})$ . Moreover, the length of  $\Gamma$  is  $d_{\widetilde{\mathcal{N}}}(\tilde{x}, \tilde{y}) \leq \text{inj}(\mathcal{N})$ . By definition of the injectivity radius, this geodesic is minimal, and thus  $d_{\widetilde{\mathcal{N}}}(\tilde{x}, \tilde{y}) = d_{\mathcal{N}}(\pi(\tilde{x}), \pi(\tilde{y}))$ .  $\square$

If  $\pi : \widetilde{\mathcal{N}} \rightarrow \mathcal{N}$  is a cover, its *group of deck transformations* is the set

$$\text{Aut}(\pi) = \left\{ \tau : \widetilde{\mathcal{N}} \rightarrow \widetilde{\mathcal{N}}; \tau \text{ is a homeomorphism and } \pi \circ \tau = \pi \right\}.$$

The set  $\text{Aut}(\pi)$  is a group under the composition operation and is also known as the *Galois group* of the cover  $\pi$ . Assuming  $\widetilde{\mathcal{N}}$  to be connected and  $\tilde{x}_0 \in \widetilde{\mathcal{N}}$ , an element  $\tau \in \text{Aut}(\pi)$  is uniquely determined by  $\tau(\tilde{x}_0)$ . Therefore, if  $\mathcal{N}$  is a connected topological manifold and if  $\widetilde{\mathcal{N}}$  is connected, then  $\text{Aut}(\pi)$  is at most countable. If  $\pi$  is a Riemannian covering, then the elements of the group  $\text{Aut}(\pi)$  are *global* isometries of the manifold  $\widetilde{\mathcal{N}}$ .

As examples of groups of deck transformations, if  $\pi : \mathbb{R} \rightarrow \mathbb{S}^1$  is the universal covering of  $\mathbb{S}^1$ , then  $\text{Aut}(\pi)$  is the group of translations of  $\mathbb{R}$  by integer multiples of  $2\pi$  and is isomorphic to  $\mathbb{Z}$ , and if  $\pi : \mathbb{S}^m \rightarrow \mathbb{RP}^m$  is the universal covering of the projective space  $\mathbb{RP}^m$ , then  $\text{Aut}(\pi) = \{\text{id}, -\text{id}\}$ , which is isomorphic to  $\mathbb{Z}_2$ .

A covering  $\pi$  is *normal* whenever the action of  $\text{Aut}(\pi)$  is transitive on the fibers of  $\pi$ , that is, whenever, given  $\tilde{x}, \tilde{y} \in \tilde{\mathcal{N}}$  such that  $\pi(\tilde{x}) = \pi(\tilde{y})$ , there exists an automorphism  $\tau \in \text{Aut}(\pi)$  such that  $\tilde{y} = \tau(\tilde{x})$ . Normal coverings are also known as *regular coverings* or as *Galois coverings*. An important case of normal covering is the universal covering of a connected Riemannian manifold [14, Proposition 1.39].

### 3. LIFTING

**3.1. Proof of the reverse oscillation inequality (1.6).** We consider some continuous function  $f \in C^0(\mathcal{I}, \mathbb{R})$ , with  $\mathcal{I} = (a, b) \subseteq \mathbb{R}$  some interval. Then (1.6) holds on  $\mathcal{I}$ , for some constant independent of  $\mathcal{I}$  and  $f$ . In order to prove (1.6), we start from the Morrey embedding  $W^{\sigma,p}(\mathcal{J}) \hookrightarrow C^{0,\sigma-1/p}(\mathcal{J})$ , valid for any interval  $\mathcal{J} = (z, t) \subseteq \mathbb{R}$  and for  $1/p < \sigma < 1$ . In a quantitative form, this embedding implies that, with a constant  $C$  depending only on  $\sigma$  and  $p$ , we have

$$(3.1) \quad |g(t) - g(z)| \leq C (t - z)^{\sigma-1/p} |g|_{W^{\sigma,p}((z,t))}, \quad \forall g \in C^0([z, t]), \quad \forall -\infty < z < t < \infty.$$

(For an elementary proof of this well-known property, see e.g. [20, Lemma 3].) In turn, (3.1) implies that

$$(3.2) \quad \text{osc}_{[x,y]} f \leq C (y - x)^{\sigma-1/p} |f|_{W^{\sigma,p}((x,y))}, \quad \forall f \in C^0(\mathcal{I}), \quad \forall a < x < y < b.$$

We next choose some  $\sigma$  such that  $1/p < \sigma < s$  (this is possible, since  $sp > 1$ ) and find, via (3.2), that<sup>4</sup>

$$\begin{aligned} \int_{\mathcal{I}} \int_{\mathcal{I}} \frac{[\text{osc}_{[x,y]} f]^p}{|y - x|^{1+sp}} dx dy &\lesssim \iint_{a < x < y < b} \frac{|f|_{W^{\sigma,p}((x,y))}^p}{(y - x)^{2+(s-\sigma)p}} dx dy \\ &\lesssim \iiint_{a < x < t < z < y < b} \frac{|f(z) - f(t)|^p}{(z - t)^{1+\sigma p}} \frac{1}{(y - x)^{2+(s-\sigma)p}} dx dy dz dt \\ &\leq \iint_{a < t < z < b} \frac{|f(z) - f(t)|^p}{(z - t)^{1+\sigma p}} \\ &\quad \times \left( \iint_{-\infty < x < t < z < y < \infty} \frac{1}{(y - x)^{2+(s-\sigma)p}} dx dy \right) dz dt \\ &\lesssim \iint_{a < z < t < b} \frac{|f(z) - f(t)|^p}{(z - t)^{1+\sigma p}} \frac{1}{(z - t)^{(s-\sigma)p}} dt dz = \frac{1}{2} |f|_{W^{s,p}(\mathcal{I})}^p, \end{aligned}$$

whence (1.6). □

In the same spirit, we have the following estimate for maps with values into manifolds. Let  $\mathcal{I} = (a, b) \subseteq \mathbb{R}$  and  $u \in C^0(\mathcal{I}, \mathcal{N})$ , where  $\mathcal{N}$  is a connected Riemannian manifold. By analogy with (1.5), we define the oscillation

$$(3.3) \quad \text{osc}_{[x,y]} u := \max\{d_{\mathcal{N}}(u(z), u(t)); z, t \in [x, y]\}.$$

**Lemma 3.1.** *Let  $0 < s < 1$  and  $1 < p < \infty$  be such that  $sp > 1$ . Let  $\mathcal{N}$  be a connected Riemannian manifold.*

<sup>4</sup>In what follows,  $A \lesssim B$  stands for  $A \leq CB$ , with  $C$  an absolute constant.

Let  $\mathcal{I} = (a, b) \subseteq \mathbb{R}$  and  $u \in C^0(\mathcal{I}, \mathcal{N})$ .

Then

$$(3.4) \quad \int_{\mathcal{I}} \int_{\mathcal{I}} \frac{[\text{osc}_{[x,y]} u]^p}{|y-x|^{1+sp}} dx dy \leq C_{s,p} |u|_{W^{s,p}(\mathcal{I})}^p = C_{s,p} \int_{\mathcal{I}} \int_{\mathcal{I}} \frac{d_{\mathcal{N}}(u(x), u(y))^p}{|y-x|^{1+sp}} dx dy.$$

*Proof.* Write  $\mathcal{I} = (a, b)$  and let  $a < z < t < b$ . Applying (3.1) with  $g(\alpha) := d_{\mathcal{N}}(u(\alpha), u(z))$ ,  $\forall \alpha \in [z, t]$ , and using the inequality  $|g(\alpha) - g(\beta)| \leq d_{\mathcal{N}}(u(\alpha), u(\beta))$ ,  $\forall \alpha, \beta \in [z, t]$ , we find that

$$(3.5) \quad |d_{\mathcal{N}}(u(t), u(z))| \leq C (t-z)^{\sigma-1/p} |u|_{W^{\sigma,p}((z,t))}, \quad \forall a < z < t < b,$$

and thus

$$(3.6) \quad \text{osc}_{[x,y]} u \leq C (y-x)^{\sigma-1/p} |u|_{W^{\sigma,p}((x,y))}, \quad \forall a < x < y < b.$$

We then continue as in the proof of (1.6). □

**3.2. The one-dimensional estimate for lifting.** We assume here that

$$(3.7) \quad 0 < s < 1 \text{ and } 1 < p < \infty \text{ are such that } sp > 1,$$

$$(3.8) \quad \pi \in C^\infty(\widetilde{\mathcal{N}}, \mathcal{N}) \text{ is a Riemannian covering and } \widetilde{\mathcal{N}} \text{ is compact.}$$

Let us note that (3.8) implies that  $\mathcal{N}$  is compact and thus  $0 < \text{inj}(\mathcal{N}) < \infty$ , and that  $\text{diam}(\widetilde{\mathcal{N}}) < \infty$ .

Let  $\mathcal{I} = (a, b) \subseteq \mathbb{R}$  and  $u \in C^0(\mathcal{I}, \mathcal{N})$ . Then we may lift  $u$  as  $u = \pi \circ \tilde{u}$ , for some  $\tilde{u} \in C^0(\mathcal{I}, \widetilde{\mathcal{N}})$ , uniquely determined by its value at some point of  $\mathcal{I}$ .

**Lemma 3.2.** Assume (3.7)–(3.8).

Let  $\mathcal{I} \subseteq \mathbb{R}$  be an interval and  $u \in C^0(\mathcal{I}, \mathcal{N})$ .

Then every continuous lifting  $\tilde{u} \in C^0(\mathcal{I}, \widetilde{\mathcal{N}})$  of  $u$  satisfies

$$(3.9) \quad |\tilde{u}|_{W^{s,p}(\mathcal{I})}^p \leq C_{s,p} \left( \frac{\text{diam}(\widetilde{\mathcal{N}})}{\text{inj}(\mathcal{N})} \right)^p |u|_{W^{s,p}(\mathcal{I})}^p,$$

for some absolute constant  $C_{s,p}$ .

*Proof.* Let  $\mathcal{I} = (a, b)$ . We have the obvious estimate

$$(3.10) \quad d_{\widetilde{\mathcal{N}}}(\tilde{u}(x), \tilde{u}(y)) \leq \text{diam}(\widetilde{\mathcal{N}}), \quad \forall x, y \in \mathcal{I}.$$

On the other hand, if  $x, y \in \mathcal{I}$  and  $\text{osc}_{[x,y]} u \leq \text{inj}(\mathcal{N})$ , then  $d_{\widetilde{\mathcal{N}}}(\tilde{u}(x), \tilde{u}(y)) \leq \text{inj}(\mathcal{N})$  and thus, by Lemma 2.1,

$$(3.11) \quad d_{\widetilde{\mathcal{N}}}(\tilde{u}(x), \tilde{u}(y)) \leq \text{osc}_{[x,y]} u.$$

Combining (3.10) with the conditional inequality (3.11), and noting that  $\text{diam}(\widetilde{\mathcal{N}}) \geq \text{inj}(\mathcal{N})$ , we find that

$$(3.12) \quad d_{\widetilde{\mathcal{N}}}(\tilde{u}(x), \tilde{u}(y)) \leq \frac{\text{diam}(\widetilde{\mathcal{N}})}{\text{inj}(\mathcal{N})} \text{osc}_{[x,y]} u, \quad \forall x, y \in \mathcal{I}.$$

We obtain (3.9) from Lemma 3.1 and (3.12). □

*Remark 3.3.* The estimate (3.9) has to depend on  $\text{diam}(\widetilde{\mathcal{N}})/\text{inj}(\mathcal{N})$ . Indeed, consider the  $d$ -fold covering  $\pi_d$  of  $\mathbb{S}^1$ , with  $d \geq 1$ . In this case, we have  $\widetilde{\mathcal{N}} = d\mathbb{S}^1$ ,  $\pi_d(d \cos t, d \sin t) = (\cos(dt), \sin(dt))$ ,  $\forall t \in \mathbb{R}$ ,  $\text{inj}(\mathcal{N}) = \pi$ ,  $\text{diam}(\widetilde{\mathcal{N}}) = \pi d$ . Let  $\xi \in \mathbb{R}$ . If we set  $u_{d,\xi}(x) :=$

$d(\cos(\xi x), \sin(\xi x)) \in \widetilde{\mathcal{N}}$ ,  $\forall x \in (0, 1)$ , then we have  $\pi_d(u_{d,\xi}) = u_{1,d\xi}$ . On the other hand, we have, with  $0 < C < \infty$  some absolute constant,

$$\lim_{|\xi| \rightarrow \infty} \frac{|u_{d,\xi}|_{W^{s,p}((0,1))}^p}{d^p |\xi|^{sp-1}} = C, \quad \lim_{|\xi| \rightarrow \infty} \frac{|\pi_d \circ u_{d,\xi}|_{W^{s,p}((0,1))}^p}{(d|\xi|)^{sp-1}} = C,$$

and thus

$$(3.13) \quad \lim_{|\xi| \rightarrow \infty} \frac{|u_{d,\xi}|_{W^{s,p}((0,1))}^p}{|\pi_d \circ u_{d,\xi}|_{W^{s,p}((0,1))}^p} = d^{p-sp+1} = \left( \frac{\text{diam}(\widetilde{\mathcal{N}})}{\text{inj}(\mathcal{N})} \right)^{p-sp+1}.$$

Note, however, that the estimates (3.9) and (3.13) do not yield the same power of  $\text{diam}(\widetilde{\mathcal{N}})/\text{inj}(\mathcal{N})$ . The question about the optimal power in (3.9) is open.

**3.3. The dimensional reduction argument.** In this section and the next one, we explain how to derive  $m$ -dimensional estimates from the one-dimensional estimate provided by Lemma 3.2. To start with, we consider the case of a cube, which is very simple. The case of a general domain requires slightly more work and is presented in the next section.

**Lemma 3.4.** *Assume (3.7)–(3.8).*

*Let  $\mathcal{C} := a + (0, \ell)^m$ , with  $\ell \in (0, \infty)$  and  $a \in \mathbb{R}^m$ . Let  $\mathcal{Q} \subset \mathcal{C}$  be an open set such that*

- (i)  *$\mathcal{Q}$  is simply-connected,*
- (ii) *for every  $i = 1, \dots, m$  and for a.e.  $\widehat{x}_i := (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_m) \in (0, \ell)^{m-1}$ , we have  $a + (x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_m) \in \mathcal{Q}$ ,  $\forall t \in (0, \ell)$ .*

*Let  $u : \mathcal{C} \rightarrow \mathcal{N}$  be such that  $u \in C^0(\mathcal{Q}, \mathcal{N})$ .*

*Then every continuous lifting  $\widetilde{u} \in C^0(\mathcal{Q}, \widetilde{\mathcal{N}})$  of  $u$  satisfies*

$$(3.14) \quad |\widetilde{u}|_{W^{s,p}(\mathcal{C})}^p \leq C_{s,p,m} \left( \frac{\text{diam}(\widetilde{\mathcal{N}})}{\text{inj}(\mathcal{N})} \right)^p |u|_{W^{s,p}(\mathcal{C})}^p,$$

*for some absolute constant  $C_{s,p,m}$ .*

The existence of the lifting  $\widetilde{u}$  follows from assumption (i) on  $\mathcal{Q}$ . By assumption (ii) on  $\mathcal{Q}$ ,  $\mathcal{C} \setminus \mathcal{Q}$  is a null set, and thus  $\widetilde{u}$  is defined a.e. on  $\mathcal{C}$ .

*Proof of Lemma 3.4.* With no loss of generality, we assume that  $a = 0$ . For  $i = 1, \dots, m$  and  $\widehat{x}_i \in (0, \ell)^{m-1}$ , set

$$u_{\widehat{x}_i}(t) := u(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_m), \quad \forall t \in (0, \ell).$$

By assumption (ii),  $u_{\widehat{x}_i}$  is well-defined on  $(0, \ell)$ , for  $\widehat{x}_i$  in the complement of a null subset of  $(0, \ell)^{m-1}$ , and for such  $\widehat{x}_i$  we define similarly  $\widetilde{u}_{\widehat{x}_i}(t)$ . By Lemma 3.2, we have

$$(3.15) \quad \sum_{i=1}^m \int_{(0,\ell)^{m-1}} |\widetilde{u}_{\widehat{x}_i}|_{W^{s,p}((0,\ell))}^p d\widehat{x}_i \leq C_{s,p} \left( \frac{\text{diam}(\widetilde{\mathcal{N}})}{\text{inj}(\mathcal{N})} \right)^p \sum_{i=1}^m \int_{(0,\ell)^{m-1}} |u_{\widehat{x}_i}|_{W^{s,p}((0,\ell))}^p d\widehat{x}_i.$$

We conclude by combining (3.15) with the  $\ell$ -independent semi-norm equivalence

$$(3.16) \quad \sum_{i=1}^m \int_{(0,\ell)^{m-1}} |\widehat{f}_{\widehat{x}_i}|_{W^{s,p}((0,\ell))}^p d\widehat{x}_i \sim |f|_{W^{s,p}(\mathcal{C})}^p, \quad \forall f : \mathcal{C} \rightarrow \mathcal{N}$$

(and the similar equivalence for  $\widetilde{\mathcal{N}}$ -valued maps). For  $\mathbb{R}$ -valued maps defined on  $\mathbb{R}^m$ , this equivalence is well-known, see e.g. [1, Lemma 7.44]. The argument for manifold-valued maps defined on a cube is exactly the same as the one in [1, proof of Lemma 7.44]. The fact that the constant  $C_{s,p,m}$  does not depend on  $\ell$  follows by scaling.  $\square$

**3.4. From local to global estimates.** Here, we explain how to pass from *local estimates* (on cubes) to *global estimates* (on general domains). The basic ingredient is the semi-norm control provided by the next result.

**Lemma 3.5.** *Let  $0 < s < 1$  and  $1 \leq p < \infty$ .*

*Let  $\mathcal{N}$  be a compact Riemannian manifold.*

*Let  $\mathcal{M}$  be a connected compact manifold, possibly with boundary.*

*Let  $(\mathcal{C}_j)_{j \in J}$  be a finite family of open subsets of  $\mathcal{M}$ , covering  $\mathcal{M}$ .*

*Then*

$$(3.17) \quad |u|_{W^{s,p}(\mathcal{M})}^p \leq C_{s,p,\mathcal{M}} \sum_{j \in J} |u|_{W^{s,p}(\mathcal{C}_j)}^p, \quad \forall u : \mathcal{M} \rightarrow \mathcal{N}.$$

*Proof.* Let  $m$  be the dimension of  $\mathcal{M}$ . Let  $\delta > 0$  be such that

$$[x, y \in \mathcal{M}, d_{\mathcal{M}}(x, y) < \delta] \implies [x, y \in \mathcal{C}_j \text{ for some } j \in J].$$

The existence of  $\delta$  implies that

$$(3.18) \quad \begin{aligned} |u|_{W^{s,p}(\mathcal{M})}^p &\leq \sum_{j \in J} |u|_{W^{s,p}(\mathcal{C}_j)}^p + \iint_{x,y \in \mathcal{M}, d_{\mathcal{M}}(x,y) \geq \delta} \frac{d_{\mathcal{N}}(u(x), u(y))^p}{d_{\mathcal{M}}(x, y)^{m+sp}} dx dy \\ &\lesssim \sum_{j \in J} |u|_{W^{s,p}(\mathcal{C}_j)}^p + \iint_{x,y \in \mathcal{M}} d_{\mathcal{N}}(u(x), u(y))^p dx dy, \end{aligned}$$

and thus (3.17) amounts to proving, the Poincaré type estimate

$$(3.19) \quad \iint_{x,y \in \mathcal{M}} d_{\mathcal{N}}(u(x), u(y))^p dx dy \lesssim \sum_{j \in J} |u|_{W^{s,p}(\mathcal{C}_j)}^p.$$

We may assume that every  $\mathcal{C}_j$  is non-empty. Since  $\mathcal{M}$  is connected, we can relabel the sets  $(\mathcal{C}_j)_{j \in J}$  as  $(\mathcal{C}_j)_{1 \leq j \leq k}$  in such a way that  $\mathcal{C}_{i+1} \cap \bigcup_{j=1}^i \mathcal{C}_j \neq \emptyset$ ,  $\forall 1 \leq i \leq k-1$ . We then have, by the triangle inequality, for every  $x \in \bigcup_{j=1}^i \mathcal{C}_j$  and  $y \in \mathcal{C}_{i+1}$ ,

$$d_{\mathcal{M}}(u(x), u(y))^p \lesssim \int_{\mathcal{C}_{i+1} \cap \bigcup_{j=1}^i \mathcal{C}_j} [d(u(x), u(z))^p + d(u(z), u(y))^p] dz,$$

and hence, by induction, we obtain

$$\begin{aligned} \iint_{x,y \in \bigcup_{j=1}^{i+1} \mathcal{C}_j} d(u(x), u(y))^p dx dy &\lesssim \int_{x,y \in \bigcup_{j=1}^i \mathcal{C}_j} d(u(x), u(y))^p dx dy + \int_{x,y \in \mathcal{C}_{i+1}} d(u(x), u(y))^p dx dy \\ &\lesssim \sum_{j=1}^{i+1} |u|_{W^{s,p}(\mathcal{C}_j)}^p. \end{aligned} \quad \square$$

Combining Lemma 3.4 with Lemma 3.5, we obtain the following

**Corollary 3.6.** *Assume (3.7)–(3.8).*

*Let  $\mathcal{M} \subset \mathbb{R}^m$  be a smooth bounded open set. Let  $\mathcal{M}' \subset \mathbb{R}^m$  be an open set such that  $\overline{\mathcal{M}} \subset \mathcal{M}'$ .*

*Let  $\mathcal{R} \subset \mathcal{M}'$  and  $u : \mathcal{M}' \rightarrow \mathcal{N}$  be such that*

- (i) *for every cube  $\mathcal{C} \subset \mathcal{M}'$ , the set  $\mathcal{Q} := \mathcal{R} \cap \mathcal{C}$  satisfies assumption (ii) in Lemma 3.4,*
- (ii)  *$u \in C^0(\mathcal{R}, \mathcal{N})$  and  $u$  has a lifting  $\tilde{u} \in C^0(\mathcal{R}, \tilde{\mathcal{N}})$ .*

*Then*

$$(3.20) \quad |\tilde{u}|_{W^{s,p}(\mathcal{M})}^p \leq C_{s,p,\mathcal{M}} \left( \frac{\text{diam}(\tilde{\mathcal{N}})}{\text{inj}(\mathcal{N})} \right)^p |u|_{W^{s,p}(\mathcal{M}')}^p,$$

*for some absolute constant  $C_{s,p,\mathcal{M}}$ .*

**3.5. Proof of Theorem 3.** Since, clearly, (a)  $\implies$  (b), it suffices to prove that (b)  $\implies$  (c) (always) and (c)  $\implies$  (a) (in the case of the universal covering, with  $\mathcal{M}$  a ball).

*Proof of (b)  $\implies$  (c).* We work on a compact manifold  $\mathcal{M}$ . In order to obtain (c), it suffices to obtain the following *a priori estimate*. If  $u \in C^\infty(\mathcal{M}, \mathcal{N})$ , then  $u$  has a lifting  $\tilde{u} \in C^\infty(\mathcal{M}, \tilde{\mathcal{N}})$  such that

$$(3.21) \quad |\tilde{u}|_{W^{s,p}(\mathcal{M})}^p \lesssim |u|_{W^{s,p}(\mathcal{M})}^p.$$

Indeed, assuming that (3.21) holds for smooth maps, a straightforward limiting procedure shows that (3.21) still holds for weak limits of smooth maps.

In order to prove (3.21), we consider a finite covering of  $\mathcal{M}$  with open sets  $\mathcal{C}_j$ , each one bi-Lipschitz homeomorphic to a cube in  $\mathbb{R}^m$ . On each  $\mathcal{C}_j$ , we have

$$(3.22) \quad |\tilde{u}|_{W^{s,p}(\mathcal{C}_j)}^p \lesssim |u|_{W^{s,p}(\mathcal{C}_j)}^p;$$

this follows (after composition with a suitable homeomorphism) from Lemma 3.4.

We conclude using (3.22) and Lemma 3.5 (applied to  $\tilde{u}$ ).  $\square$

*Proof of (c)  $\implies$  (a).* We work on an open ball. Write  $u = \pi \circ \tilde{u}$ , with  $\tilde{u} \in W^{s,p}(\mathcal{M}, \tilde{\mathcal{N}})$ . Since  $1 < sp < 2$  and  $\tilde{\mathcal{N}}$  is compact and simply-connected (by definition of the universal covering),  $C^\infty(\overline{\mathcal{M}}, \tilde{\mathcal{N}})$  is dense in  $W^{s,p}(\mathcal{M}, \tilde{\mathcal{N}})$  [11, Theorem 4] (see also [8, Theorem 1.3; 23, Theorem 2]). Consider a sequence  $(\tilde{u}_n)_{n \geq 0}$  in  $C^\infty(\overline{\mathcal{M}}, \tilde{\mathcal{N}})$  such that  $\tilde{u}_n \rightarrow \tilde{u}$  in  $W^{s,p}(\mathcal{M})$  as  $n \rightarrow \infty$ . Set  $u_n := \pi \circ \tilde{u}_n \in C^\infty(\overline{\mathcal{M}}, \mathcal{N})$ . Using the fact that  $\pi$  is Lipschitz-continuous, we find that  $u_n \rightarrow u$  in  $W^{s,p}(\mathcal{M})$  as  $n \rightarrow \infty$ .  $\square$

*Remark 3.7.* We have proved the following quantitative version of (c). If  $u \in W^{s,p}(\mathcal{M}, \mathcal{N})$  has a lifting  $\tilde{u} \in W^{s,p}(\mathcal{M}, \tilde{\mathcal{N}})$ , then

$$|\tilde{u}|_{W^{s,p}(\mathcal{M})}^p \leq C_{s,p,\mathcal{M}} \left( \frac{\text{diam}(\tilde{\mathcal{N}})}{\text{inj}(\mathcal{N})} \right)^p |u|_{W^{s,p}(\mathcal{M})}^p.$$

**3.6. Proof of Theorem 1.** In view of the partial results of Bethuel and Chiron [4], it suffices to consider the case where  $0 < s < 1$ ,  $2 \leq sp < m = \dim \mathcal{M}$ .

*Proof of Theorem 1 when  $\mathcal{M}$  is a smooth bounded domain of  $\mathbb{R}^m$ .* As in the previous section, it suffices to prove the *a priori estimate*

$$(3.23) \quad |\tilde{u}|_{W^{s,p}(\mathcal{M})}^p \leq C_{s,p,\mathcal{M}} |u|_{W^{s,p}(\mathcal{M})}^p,$$

for a lifting  $\tilde{u}$  of  $u$ , where  $u$  belongs to a dense subset of  $W^{s,p}(\mathcal{M}, \mathcal{N})$ . Weak density would suffice, but it turns out that we have at our disposal a convenient strongly dense class. Such a class is obtained as follows [11, Theorem 6]. Extend first every  $u \in W^{s,p}(\mathcal{M}, \mathcal{N})$  by reflection across  $\partial\mathcal{M}$  to a larger set  $\mathcal{M}'$ . The extension, still denoted  $u$ , satisfies  $u : \mathcal{M}' \rightarrow \mathcal{N}$  and

$$(3.24) \quad |u|_{W^{s,p}(\mathcal{M}')}^p \leq C_{s,p,\mathcal{M}} |u|_{W^{s,p}(\mathcal{M})}^p.$$

Since  $\mathcal{M}$  is smooth, bounded and simply-connected, we can assume without loss of generality that  $\mathcal{M}'$  is also smooth, bounded and simply-connected.

Let  $j := \lfloor sp \rfloor$  denote the integer part of  $sp$ , so that  $2 \leq j < m$ . Consider the  $\varepsilon$ -grids  $\mathcal{T}_{a,\varepsilon}$ ,  $\forall \varepsilon > 0$ ,  $\forall a \in \mathbb{R}^m$ , defined by the cubes  $\mathcal{C}_{a,\varepsilon,k} := a + \varepsilon k + [0, \varepsilon]^m$ ,  $k \in \mathbb{Z}^m$ . Let  $\mathcal{T}_{a,\varepsilon}^j$  denote the  $j$ th skeleton of  $\mathcal{T}_{a,\varepsilon}$  and  $\mathcal{U}_{a,\varepsilon}^{m-j-1}$  denote the  $((m-j-1)$ -dimensional) dual skeleton of  $\mathcal{T}_{a,\varepsilon}^j$ .

We use the following approximation result [11, Theorem 6]: given  $u \in W^{s,p}(\mathcal{M}', \mathcal{N})$ , there exist sequences  $\varepsilon_n \searrow 0$ ,  $(a_n)_{n \geq 0} \subset \mathbb{R}^m$ ,  $(u_n)_{n \geq 0} \subset W^{s,p}(\mathcal{M}', \mathcal{N})$  such that

$$(a) \quad u_n \rightarrow u \text{ as } n \rightarrow \infty, \text{ strongly in } W^{s,p}(\mathcal{M}').$$

(b)  $u_n$  is continuous in  $\overline{\mathcal{M}} \setminus \mathcal{U}_{a_n, \varepsilon_n}^{m-j-1}$ ,  $\forall n \geq 0$ .

In view of item (a) above and of Corollary 3.6, in order to obtain (3.23) (and thus to complete the proof of Theorem 1) it suffices to prove that  $u_n$  and the set  $\mathcal{R} = \mathcal{R}_n := \mathcal{M}' \setminus \mathcal{U}_{a_n, \varepsilon_n}^{m-j-1}$  satisfy the assumptions (i) and (ii) in Corollary 3.6.

Clearly, assumption (i) is satisfied, since  $\mathcal{U}_{a_n, \varepsilon_n}^{m-j-1}$  is a finite union of  $(m-j-1)$ -dimensional affine subspaces and since  $j \geq 1$ . Moreover, by a straightforward induction argument relying on the next lemma (which is a particular case of general position arguments), the set  $\mathcal{R}_n$  is simply-connected, and thus  $u_n|_{\mathcal{R}_n}$  has a lifting  $\tilde{u}_n \in C^0(\mathcal{R}_n, \tilde{\mathcal{N}})$ .  $\square$

**Lemma 3.8.** *Let  $m \geq 3$ . Let  $\mathcal{V} \subset \mathbb{R}^m$  be open and let  $\Sigma$  be an affine subspace of dimension  $n \leq m-3$ . If  $\mathcal{V}$  is simply-connected, then  $\mathcal{V} \setminus \Sigma$  is simply-connected.*

*Proof of Lemma 3.8.* Without loss of generality, we assume that  $0 \in \Sigma$ . Let  $\gamma \in C^1(\mathbb{S}^1, \mathcal{V} \setminus \Sigma)$ . Our aim is to prove that  $\gamma$  is null homotopic in  $\mathcal{V} \setminus \Sigma$ .

Since the set  $\mathcal{V}$  is simply-connected, there exists  $\sigma \in C^1(\overline{\mathbb{B}^2}, \mathcal{V})$  such that  $\sigma|_{\mathbb{S}^1} = \gamma$ . Since the set  $\mathcal{V}$  is open, there exists  $\delta > 0$ , such that, for every  $x \in \mathbb{S}^1$ ,  $B_\delta(\gamma(x)) \subset \mathcal{V} \setminus \Sigma$  and, for every  $x \in \overline{\mathbb{B}^2}$ ,  $B_\delta(\sigma(x)) \subset \mathcal{V}$ .<sup>5</sup> Let  $P : \mathbb{R}^m \rightarrow \Sigma^\perp$  be the orthogonal projection on  $\Sigma^\perp$ . Since  $\dim \Sigma^\perp \geq 3$ ,  $P(\sigma(\overline{\mathbb{B}^2}))$  is a negligible subset of  $\Sigma^\perp$ . Hence, for almost every  $\xi \in B_\delta \cap \Sigma^\perp$ , we have  $-\xi \notin P(\sigma(\overline{\mathbb{B}^2}))$ . For any such  $\xi$ , we have  $(\sigma + \xi)(\overline{\mathbb{B}^2}) \subset \mathcal{V} \setminus \Sigma$ , and thus  $\gamma + \xi : \mathbb{S}^1 \rightarrow \mathcal{V} \setminus \Sigma$  is null homotopic in  $\mathcal{V} \setminus \Sigma$ . We conclude by noting that, by the choice of  $\delta$ , the maps  $\gamma : \mathbb{S}^1 \rightarrow \mathcal{V} \setminus \Sigma$  and  $\gamma + \xi : \mathbb{S}^1 \rightarrow \mathcal{V} \setminus \Sigma$  are homotopic in  $\mathcal{V} \setminus \Sigma$ .

By a similar argument, if  $\mathcal{V}$  is connected, then  $\mathcal{V} \setminus \Sigma$  is connected.  $\square$

*Proof of Theorem 1 when  $\mathcal{M}$  is a compact manifold without boundary.* We embed  $\mathcal{M}$  isometrically into some Euclidean space  $\mathbb{R}^\mu$ . Then there exists  $\delta > 0$  such that:

- (a) the nearest point projection  $\Pi : \mathcal{O} \rightarrow \mathcal{M}$  is well-defined and smooth on the set  $\mathcal{O} := \{x \in \mathbb{R}^\mu; \text{dist}(x, \mathcal{M}) < \delta\}$ ;
- (b)  $\mathcal{O}$  is smooth;
- (c) for every  $x \in \mathcal{M}$ ,  $\Pi^{-1}(\{x\})$  is diffeomorphic to  $\mathbb{B}^{\mu-m}$ ;
- (d) if  $u : \mathcal{M} \rightarrow \mathcal{N}$  and we set  $U := u \circ \Pi : \mathcal{O} \rightarrow \mathcal{N}$ , then

$$(3.25) \quad C'|u|_{W^{s,p}(\mathcal{M})}^p \leq |U|_{W^{s,p}(\mathcal{O})}^p \leq C|u|_{W^{s,p}(\mathcal{M})}^p$$

for some  $C', C \in (0, \infty)$  depending on  $0 < s < 1$ ,  $1 \leq p < \infty$ , the embedding,  $\delta$ , but independent of  $u$ .

Let  $u \in W^{s,p}(\mathcal{M}, \mathcal{N})$ , and let  $\mathcal{O}, U$  as above. Then  $\mathcal{O}$  is simply-connected, since  $\Pi : \mathcal{O} \rightarrow \mathcal{M}$  is a retraction and  $\mathcal{M}$  is simply-connected.

By the first part of the proof of the theorem, there exists a map  $\tilde{U} \in W^{s,p}(\mathcal{O}, \tilde{\mathcal{N}})$  such that  $\pi \circ \tilde{U} = U$  in  $\mathcal{O}$ . Moreover, for a.e.  $x \in \mathcal{M}$ ,  $U|_{\Pi^{-1}(\{x\})}$  is constant on  $\Pi^{-1}(\{x\})$  (that we identify with a ball, see (c) above) and  $\tilde{U}|_{\Pi^{-1}(\{x\})} \in W^{s,p}(\Pi^{-1}(\{x\}), \tilde{\mathcal{N}})$ . Set  $b := u(x)$  and let  $\pi^{-1}(b) = \{\tilde{b}_i; i \in I\}$ , so that  $U(y) \in \{\tilde{b}_i; i \in I\}$ , for a.e.  $y \in \Pi^{-1}(\{x\})$ . Consider some  $i \in I$  such that the set  $\{y \in \Pi^{-1}(\{x\}); U(y) = \tilde{b}_i\}$  is non-negligible (such an  $i$  does exist, since  $I$  is at most countable). Since  $\pi \circ U = \pi \circ \tilde{b}_i$  on  $\Pi^{-1}(\{x\})$ , Proposition 4.4 below implies that  $U = \tilde{b}_i$  a.e. on  $\Pi^{-1}(\{x\})$ . For any  $x$  as above, set  $\tilde{u}(x) := \tilde{b}_i$ , so that  $\tilde{u}$  is defined a.e. on  $\mathcal{M}$  and  $\tilde{U} \circ \Pi = \tilde{u}$ . By (3.25), we have  $\tilde{u} \in W^{s,p}(\mathcal{M}, \tilde{\mathcal{N}})$  and, clearly,  $\pi \circ \tilde{u} = u$ .  $\square$

<sup>5</sup> $B_\varepsilon(x)$  is the Euclidean ball of centre  $x$  and radius  $\varepsilon$ , with  $x \in \mathbb{R}^m$ . When  $x = 0$ , we write  $B_\varepsilon$  instead of  $B_\varepsilon(0)$ .

*Proof of Theorem 1 when  $\mathcal{M}$  is a compact manifold with boundary.* This is a slightly more subtle case. We consider two larger smooth compact manifolds with boundary,  $\mathcal{M}'$  and  $\mathcal{M}''$ , such that  $\mathcal{M} \subset \text{int}(\mathcal{M}')$ ,  $\mathcal{M}' \subset \text{int}(\mathcal{M}'')$  (where  $\text{int}$  stands for the interior), and we can extend maps from  $\mathcal{M}$  to  $\mathcal{M}'$  by reflection across the boundary such that (3.24) holds.

We next embed  $\mathcal{M}''$  isometrically into some  $\mathbb{R}^\mu$ . Let  $\Pi$  denote the nearest point projection on  $\mathcal{M}''$ . Then, for small  $\delta > 0$ , if we set  $\mathcal{O} := \{x \in \mathbb{R}^\mu; \text{dist}(x, \mathcal{M}) < \delta \text{ and } \Pi(x) \in \mathcal{M}\}$ , then  $\mathcal{O}$  satisfies (a), (c) and (d), above, but not (b). Thus we cannot directly apply directly [11, Theorem 6] to the map  $U$  in  $\mathcal{O}$  as above. However, we note that in order to invoke this result, we do not need a smooth domain. It suffice to know that there exists an open set  $\mathcal{O}'$  such that  $\overline{\mathcal{O}} \subset \mathcal{O}'$  and an extension  $V \in W^{s,p}(\mathcal{O}', \mathcal{N})$  of  $U$ . In our case, we let (again, for sufficiently small  $\delta > 0$ )  $\mathcal{O}' := \{x \in \mathbb{R}^\mu; \text{dist}(x, \mathcal{M}') < 2\delta \text{ and } \Pi(x) \in \mathcal{M}'\}$ . The extension  $V$  of  $U$  to  $\mathcal{O}'$  is defined as follows. Let  $\bar{u}$  be the extension of  $u$  to  $\mathcal{M}'$  by reflection across  $\partial\mathcal{M}$ . Then we set, in  $\mathcal{O}'$ ,  $V := \bar{u} \circ \Pi$ . Clearly,  $V$  has the required properties. We continue the proof as in the case of compact manifolds without boundary.

The proof of Theorem 1 is complete.  $\square$

#### 4. UNIQUENESS OF SOBOLEV LIFTINGS

The role of this section is to provide tools for checking that analytical obstructions are indeed obstructions. Roughly speaking, the question we address here is the following. Assume that  $u : \mathcal{M} \rightarrow \mathcal{N}$  has *some* “bad” lifting  $\tilde{u}$ . How to make sure that *all other* possible liftings are also “bad”?

We present two types of results. The former ones (Proposition 4.1, Proposition 4.2, Corollary 4.3) are valid in particular in the case of the universal coverings of compact connected manifolds. The latter ones (Proposition 4.4, Corollary 4.5) are valid for more general coverings, but require more assumptions on the bad lifting. Although, strictly speaking, it is possible to prove Theorem 4 using only Corollary 4.5, we find instructive to provide two different proofs, relying on different topological assumptions and analytical arguments.

Throughout this section, we make the following assumptions.

- (4.1)  $\pi \in C^\infty(\widetilde{\mathcal{N}}, \mathcal{N})$  is a Riemannian covering,
- (4.2)  $\widetilde{\mathcal{N}}$  and  $\mathcal{N}$  are connected,
- (4.3)  $\mathcal{M}$  is a relatively compact connected open subset of some  $m$ -dimensional Riemannian manifold  $\mathcal{M}'$ .

This includes as special cases the interior of a smooth compact manifold and bounded open sets in  $\mathbb{R}^m$ . (However, if we restrict to open sets in  $\mathbb{R}^m$ , boundedness is not essential.) Our assumption on  $\mathcal{M}$  emphasizes the fact that the smoothness of the boundary of  $\mathcal{M}$  plays no role here.

A subset of  $\mathcal{M}$  is negligible if it is, near each point and in local coordinates, the image of a negligible set for the  $m$ -dimensional Lebesgue measure.

The uniqueness results are obtained under the assumption

$$(4.4) \quad sp \geq 1,$$

which is the relevant one for uniqueness [6]. In view of the applications we have in mind, we also assume that

$$(4.5) \quad 0 < s < 1,$$

but this latter assumption is not necessary for the validity of the results below.

Uniqueness being a local matter, we consider maps in  $W_{loc}^{s,p}(\mathcal{M})$ . By a standard argument, it then suffices to prove uniqueness for maps in  $W^{s,p}(B)$ , with  $B$  a ball in  $\mathbb{R}^m$ .

**Proposition 4.1.** *Assume (4.1)–(4.5) and, in addition  $\text{inj}(\mathcal{N}) > 0$ .*

*Let  $\tilde{u}, \tilde{v} \in W_{loc}^{s,p}(\mathcal{M}, \tilde{\mathcal{N}})$  be such that  $\pi \circ \tilde{u} = \pi \circ \tilde{v}$  on  $\mathcal{M}$ . Then either  $\tilde{u} = \tilde{v}$  a.e. on  $\mathcal{M}$  or  $\tilde{u} \neq \tilde{v}$  a.e. on  $\mathcal{M}$ .*

*Proof.* As explained above, we may assume that  $\mathcal{M}$  is a ball and  $\tilde{u}, \tilde{v} \in W^{s,p}(\mathcal{M}, \tilde{\mathcal{N}})$ .

Let us note that, if  $\varphi : [0, \infty) \rightarrow \mathbb{R}$  is an  $L$ -Lipschitz function, then

$$(4.6) \quad f : \mathcal{M} \rightarrow \mathbb{R}, \quad f(x) := \varphi(d_{\tilde{\mathcal{N}}}(\tilde{u}(x), \tilde{v}(x))), \quad \forall x \in \mathcal{M},$$

satisfies

$$|f(x) - f(y)| \leq L |d_{\tilde{\mathcal{N}}}(\tilde{u}(x), \tilde{v}(x)) - d_{\tilde{\mathcal{N}}}(\tilde{u}(y), \tilde{v}(y))| \leq L [d_{\tilde{\mathcal{N}}}(\tilde{u}(x), \tilde{u}(y)) + d_{\tilde{\mathcal{N}}}(\tilde{v}(x), \tilde{v}(y))],$$

and thus  $f \in W^{s,p}(\mathcal{M}, \mathbb{R})$ .

Set  $\ell := \min\{1, \text{inj}(\mathcal{N})\}$  and  $\varphi : [0, \infty) \rightarrow \mathbb{R}$ ,  $\varphi(t) := \min\{t/\ell, 1\}$ ,  $\forall t \geq 0$ . The assumption  $\pi \circ \tilde{u} = \pi \circ \tilde{v}$  implies, via Lemma 2.1, that the corresponding function  $f$  in (4.6) satisfies  $f(\mathcal{M}) \subseteq \{0, 1\}$ . Under the assumptions  $sp \geq 1$  and  $\mathcal{M}$  connected, the space  $W^{s,p}(\mathcal{M}, \{0, 1\})$  contains only constant a.e. functions [6, theorem B.1] (see also [5, lemma A.1; 7; 9; 13, lemma 1.1]). Thus either  $f = 0$  a.e. on  $\mathcal{M}$ , or  $f = 1$  a.e. on  $\mathcal{M}$ , whence the conclusion.  $\square$

**Proposition 4.2.** *Assume (4.1)–(4.5) and, in addition, that  $\text{inj}(\mathcal{N}) > 0$  and that  $\pi$  is a normal covering.*

*If  $\tilde{u}, \tilde{v} \in W_{loc}^{s,p}(\mathcal{M}, \tilde{\mathcal{N}})$  and if  $\pi \circ \tilde{u} = \pi \circ \tilde{v}$  on  $\mathcal{M}$ , then there exists  $\tau \in \text{Aut}(\pi)$  such that  $\tilde{v} = \tau \circ \tilde{u}$  a.e. on  $\mathcal{M}$ .*

In the case where  $\pi$  is the universal covering of a compact connected Riemannian manifold, Proposition 4.2 is due to Bethuel and Chiron [4, Lemma A.4].

*Proof of Proposition 4.2.* For each deck transformation  $\tau \in \text{Aut}(\pi)$ , we define the measurable set

$$A_\tau := \{x \in \mathcal{M}; \tilde{v}(x) = \tau \circ \tilde{u}(x)\}.$$

Since the covering  $\pi$  is normal, we have

$$\mathcal{M} = \bigcup_{\tau \in \text{Aut}(\pi)} A_\tau.$$

Due to the at most countability of  $\text{Aut}(\pi)$ , there exists  $\tau \in \text{Aut}(\pi)$  such that  $A_\tau$  is non-negligible. For this  $\tau$ , combining the equality  $\pi \circ (\tau \circ \tilde{u}) = \pi \circ \tilde{u} = \pi \circ \tilde{v}$  on  $\mathcal{M}$  with the fact that  $\tau \circ \tilde{u} \in W^{s,p}(\mathcal{M}, \tilde{\mathcal{N}})$  and with the previous proposition, we obtain  $\tilde{v} = \tau \circ \tilde{u}$  a.e. in  $\mathcal{M}$ .  $\square$

**Corollary 4.3.** *Assume (4.1)–(4.5) and, in addition, that  $\text{inj}(\mathcal{N}) > 0$  and that  $\pi$  is a normal covering.*

*Let  $\tilde{u} \in W_{loc}^{s,p}(\mathcal{M}, \tilde{\mathcal{N}}) \setminus W^{s,p}(\mathcal{M}, \tilde{\mathcal{N}})$  and set  $u := \pi \circ \tilde{u}$ . Then  $u$  has no lifting  $\tilde{v} \in W^{s,p}(\mathcal{M}, \tilde{\mathcal{N}})$ .*

*Proof.* Argue by contradiction. By Proposition 4.2, there exists some  $\tau \in \text{Aut}(\pi)$  such that  $\tilde{u} = \tau^{-1} \circ \tilde{v}$  a.e. on  $\mathcal{M}$ . This leads to the contradiction  $\tilde{u} \in W^{s,p}(\mathcal{M}, \tilde{\mathcal{N}})$ .  $\square$

We now turn to uniqueness results involving solely the assumptions (4.1)–(4.5).

**Proposition 4.4.** *Assume (4.1)–(4.5).*

*Let  $\tilde{u}, \tilde{v} \in W_{loc}^{s,p}(\mathcal{M}, \tilde{\mathcal{N}})$  be such that  $\pi \circ \tilde{u} = \pi \circ \tilde{v}$  on  $\mathcal{M}$ . Assume, moreover, that  $\tilde{u}$  is continuous. Then either  $\tilde{u} = \tilde{v}$  a.e. on  $\mathcal{M}$  or  $\tilde{u} \neq \tilde{v}$  a.e. on  $\mathcal{M}$ .*

*Proof.* Assume that the set  $C := \{y \in \mathcal{M}; \tilde{u}(y) = \tilde{v}(y)\}$  is non-negligible. By continuity of  $\tilde{u}$ , for each  $x \in \mathcal{M}$ , there exist  $\varepsilon = \varepsilon(x) > 0$  and  $r = r(x) > 0$  such that  $(\pi \circ \tilde{u})(B_\varepsilon(x))$  is contained in an evenly covered geodesic ball  $U = U(x)$  of radius  $r$ . We consider the set

$$D := \{x \in \mathcal{M}; C \cap B_\varepsilon(x) \text{ is non-negligible}\}.$$

By the assumption on  $C$ , the set  $D$  is non-empty. We claim that

$$x \in D \implies [\text{the set } B_\varepsilon(x) \setminus C \text{ is negligible}].$$

This claim clearly implies that the set  $D$  is both open and closed, and thus, by connectedness, that  $D = \mathcal{M}$ , whence (via the claim) the conclusion of the proposition. It therefore remains to establish the claim.

Let  $x \in D$ . Write  $\pi^{-1}(U(x))$  as a disjoint union,  $\pi^{-1}(U(x)) = \bigcup_{i \in I} V_i$ , with  $\pi : V_i \rightarrow U(x)$  a diffeomorphism. Since  $\tilde{u}$  is continuous, there exists some  $j \in I$  such that  $\tilde{u}(B_\varepsilon(x)) \subset V_j$ . Let  $\varphi(t) := \min\{t/r, 1\}$ ,  $\forall t \geq 0$ , and set  $f(y) := \varphi(d_{\tilde{\mathcal{N}}}(\tilde{u}(y), \tilde{v}(y)))$ ,  $\forall y \in B_\varepsilon(x)$ . As in the proof of Proposition 4.1, we have  $f \in W^{s,p}(B_\varepsilon(x), \{0, 1\})$ , and thus  $f$  is constant. Since the set  $f^{-1}(\{0\})$  is non-negligible (by definition of the set  $D$ ), we find that  $f = 0$  a.e. on  $B_\varepsilon(x)$ , and thus  $\tilde{u} = \tilde{v}$  a.e. in  $B_\varepsilon(x)$ , as claimed.  $\square$

In the spirit of Corollary 4.3, we have the following consequence of Proposition 4.4.

**Corollary 4.5.** *Assume (4.1)–(4.5).*

*Let  $\tilde{u} \in W_{loc}^{s,p}(\mathcal{M}, \tilde{\mathcal{N}}) \setminus W^{s,p}(\mathcal{M}, \tilde{\mathcal{N}})$  be a continuous map and set  $u := \pi \circ \tilde{u}$ .*

*If  $u$  has a lifting  $\tilde{v} \in W^{s,p}(\mathcal{M}, \tilde{\mathcal{N}})$ , then  $\tilde{u} \neq \tilde{v}$  a.e.*

## 5. ANALYTICAL SINGULARITY

In this section, we prove Theorem 4. In what follows, we assume that

$$(5.1) \quad 0 < s < 1, \quad p = 1/s,$$

$$(5.2) \quad m \geq 2.$$

**5.1. The basic ingredient.** We start by proving the existence of smooth maps  $\tilde{u} : \mathbb{R}^m \rightarrow \tilde{\mathcal{N}}$  such that  $|\tilde{u}|_{W^{s,p}(B_1)}$  is arbitrarily large, while  $|\pi \circ \tilde{u}|_{W^{s,p}(\mathbb{R}^m)}$  is arbitrarily small.

**Lemma 5.1.** *Assume (5.1)–(5.2). Let  $r > 0$  and  $x_0 \in \mathbb{R}^m$ .*

*Let  $\pi \in C^\infty(\tilde{\mathcal{N}}, \mathcal{N})$  be a Riemannian covering, with  $\tilde{\mathcal{N}}$  connected.*

*Given  $\tilde{b}, \tilde{b}' \in \tilde{\mathcal{N}}$  such that  $\tilde{b} \neq \tilde{b}'$  but  $\pi(\tilde{b}) = \pi(\tilde{b}')$ , and given  $\varepsilon, M > 0$ , there exists some  $\tilde{u} \in C^\infty(\mathbb{R}^m, \tilde{\mathcal{N}})$  such that*

- (i)  $\tilde{u}(x) = \tilde{b}$  when  $|x - x_0| \geq r$ ,
- (ii)  $\tilde{u}(x) = \tilde{b}'$  near  $x_0$ ,
- (iii)  $|\tilde{u}|_{W^{s,p}(B_r(x_0))} > M$ ,
- (iv)  $|\pi \circ \tilde{u}|_{W^{s,p}(\mathbb{R}^m)} < \varepsilon$ .

*Proof.* With no loss of generality, we let  $x_0 = 0$  and  $r = 1$ .

Assume that we are able to prove the lemma for *some fixed*  $\varepsilon_0$  and *every*  $M > 0$ . Let  $0 < \varepsilon < \varepsilon_0$ . Let  $\tilde{u}$  as above, corresponding to  $\varepsilon_0$  and to  $M' := (\varepsilon M)/\varepsilon_0$ . We define  $\lambda > 1$  by the equation  $\lambda^{m-1} = \varepsilon_0/\varepsilon$ , and we set  $\tilde{v}(x) := \tilde{u}(\lambda x)$ ,  $\forall x \in \mathbb{R}^m$ . By scaling,  $\tilde{v}$  satisfies items (i)–(iv) (for  $\varepsilon$  and  $M$ ). It therefore suffices to establish the existence of  $\tilde{u}$  satisfying (i)–(iv) for *some*  $\varepsilon_0 > 0$  and *arbitrary*  $M > 0$ .

Since the manifold  $\widetilde{\mathcal{N}}$  is connected, there exists a map  $\gamma \in C^\infty(\mathbb{R}, \widetilde{\mathcal{N}})$  such that  $\gamma(t) = \widetilde{b}$  if  $t \leq 0$  and  $\gamma(t) = \widetilde{b}'$  if  $t \geq 1$ . We define, for every  $\delta \in (0, 1)$ , the map  $\tilde{u}_\delta \in C^\infty(\mathbb{R}^m, \widetilde{\mathcal{N}})$  through the formula

$$\tilde{u}_\delta(x) = \gamma\left(\frac{1-2|x|}{\delta}\right), \quad \forall x \in \mathbb{R}^m.$$

Clearly,  $\tilde{u}_\delta$  satisfies (i) and (ii). In view of the above discussion, in order to complete the proof of the lemma it suffices to prove that

$$(5.3) \quad \lim_{\delta \rightarrow 0} |\tilde{u}_\delta|_{W^{s,p}(B_1)} = \infty,$$

$$(5.4) \quad \limsup_{\delta \rightarrow 0} |\pi \circ \tilde{u}_\delta|_{W^{s,p}(\mathbb{R}^m)} < \infty.$$

We note that

$$\lim_{\delta \rightarrow 0} \tilde{u}_\delta = \tilde{u} \text{ a.e. in } \mathbb{R}^m,$$

where

$$\tilde{u}(x) := \begin{cases} \tilde{b}', & \text{if } x \in B_{1/2} \\ \tilde{b}, & \text{if } x \in \mathbb{R}^m \setminus B_{1/2} \end{cases},$$

and that  $\tilde{u} \notin W^{s,p}(B_1, \widetilde{\mathcal{N}})$  (see the proof of Proposition 4.1). This implies (5.3).

In order to prove (5.4), we set  $u_\delta := \pi \circ \tilde{u}_\delta$  and we note the following:

$$(5.5) \quad u_\delta \equiv \pi(\tilde{b}) \text{ in } \mathbb{R}^m \setminus U_\delta, \text{ where } U_\delta := \{x \in \mathbb{R}^m; (1-\delta)/2 < |x| < 1/2\},$$

$$(5.6) \quad u_\delta \text{ is } \frac{C}{\delta} \text{ -- Lipschitz, with } C \text{ independent of } \delta,$$

$$(5.7) \quad d_{\mathcal{N}}(u_\delta(x), u_\delta(y)) \leq C, \text{ with } C \text{ independent of } \delta.$$

Combining (5.5)–(5.7), we find (using the assumption  $sp = 1$ ) that<sup>6</sup>

$$\begin{aligned} |u_\delta|_{W^{s,p}(\mathbb{R}^m)}^p &\lesssim \iint_{U_\delta \times \mathbb{R}^m} \frac{d_{\mathcal{N}}(u_\delta(x), u_\delta(y))^p}{|x-y|^{m+1}} dx dy \\ &\lesssim \iint_{x \in U_\delta, |x-y| \leq \delta} \frac{|x-y|^p / \delta^p}{|x-y|^{m+1}} dx dy + \iint_{x \in U_\delta, |x-y| > \delta} \frac{1}{|x-y|^{m+1}} dx dy \\ &\lesssim \frac{1}{\delta} \int_{U_\delta} dx = \frac{1}{\delta} |U_\delta| \lesssim 1, \end{aligned}$$

whence (5.4).

The proof of Lemma 5.1 is complete. □

**5.2. The analytic obstruction.** Using Lemma 5.1, we construct an analytic singularity adapted to the case of the universal covering.

**Lemma 5.2.** *Assume (5.1)–(5.2).*

*Let  $\pi \in C^\infty(\widetilde{\mathcal{N}}, \mathcal{N})$  be a non-trivial Riemannian covering, with  $\widetilde{\mathcal{N}}$  connected.*

*Let  $\mathcal{M} \subset \mathbb{R}^m$  be a connected open set and let  $a \in \overline{\mathcal{M}}$ .*

*Then there exists a map  $\tilde{u} : \mathbb{R}^m \rightarrow \widetilde{\mathcal{N}}$  such that*

- (i)  $\tilde{u} \in C^\infty(\mathbb{R}^m \setminus \{a\}, \widetilde{\mathcal{N}})$ ,
- (ii)  $\tilde{u} \notin W^{s,p}(\mathcal{M}, \widetilde{\mathcal{N}})$ ,
- (iii)  $\pi \circ \tilde{u} \in W^{s,p}(\mathbb{R}^m, \mathcal{N})$ ,

---

<sup>6</sup>Here and in the sequel,  $|U|$  denotes the Lebesgue measure of the set  $U \subset \mathbb{R}^m$ .

(iv)  $\pi \circ \tilde{u}$  is a strong limit in  $W^{s,p}(\mathbb{R}^m, \mathcal{N})$  of maps in  $C^\infty(\mathbb{R}^m, \mathcal{N})$ .

Before proceeding to the proof of the lemma, we explain the meaning of items (ii) and (iv). In (ii), the  $W^{s,p}$  semi-norm involves the Euclidean distance in  $\mathbb{R}^m$ , not the geodesic distance on  $\mathcal{M}$ . The meaning of item (iv) is the following. We embed  $\mathcal{N}$  into some  $\mathbb{R}^\nu$ . Then there exist a sequence  $(u^j) \subset C^\infty(\mathbb{R}^m, \mathcal{N})$  and some  $b \in \mathcal{N}$  such that  $u^j - b, \pi \circ \tilde{u} - b \in W^{s,p}(\mathcal{M}, \mathbb{R}^\nu)$  and  $u^j - u \rightarrow 0$  in  $W^{s,p}(\mathbb{R}^m, \mathbb{R}^\nu)$  as  $j \rightarrow \infty$ .

*Proof of Lemma 5.2.* Since  $m \geq 2$  and  $a \in \overline{\mathcal{M}}$ , there exists a sequence of closed balls  $(\overline{B}_{\rho_k}(a_k))_{k \geq 0}$  such that:

- (a)  $\overline{B}_{\rho_k}(a_k) \subset \mathcal{M} \setminus \{a\}, \forall k$ ,
- (b) the balls are mutually disjoint,
- (c)  $a_k \rightarrow a$  (and thus  $\rho_k \rightarrow 0$ ) as  $k \rightarrow \infty$ ,
- (d) there exists a sequence  $r_j \searrow 0$  such that  $\{x \in \mathbb{R}^m; |x - a| = r_j\} \cap \overline{B}_{\rho_k}(a_k) = \emptyset, \forall j, \forall k$ .

Since, by assumption, the cover  $\pi$  is non-trivial, there exist  $\tilde{b}$  and  $\tilde{b}'$  as in Lemma 5.1. Let  $(\varepsilon_k)_{k \geq 0}$  be a sequence of positive numbers to be defined later. Let, for every  $k \geq 0$ ,  $\tilde{u}_k$  be the map corresponding, as in Lemma 5.1, to  $B_{\rho_k}(a_k)$ ,  $\varepsilon_k$  and  $M := k + 1$ . We set, for each  $x \in \mathbb{R}^m$ ,

$$\tilde{u}(x) := \begin{cases} \tilde{u}_k(x), & \text{if } x \in B_{\rho_k}(a_k) \text{ for some } k \geq 0 \\ \tilde{b}, & \text{otherwise} \end{cases}.$$

Clearly, (i) holds. Also clearly,

$$|\tilde{u}|_{W^{s,p}(\mathcal{M})}^p \geq |\tilde{u}|_{W^{s,p}(B_{\rho_k}(a_k))}^p \geq k + 1, \quad \forall k \geq 0,$$

and thus assertion (ii) holds. By the countable patching property of Sobolev maps [22, Lemma 2.3], we have (using the assumption  $0 < s < 1$ ),

$$(5.8) \quad |\pi \circ \tilde{u}|_{W^{s,p}(\mathbb{R}^m)}^p \leq 2^p \sum_{k \geq 0} \rho_k^{m-1} |\pi \circ \tilde{u}_k|_{W^{s,p}(\mathbb{R}^m)}^p < 2^p \sum_{k \geq 0} \rho_k^{m-1} \varepsilon_k.$$

We now choose  $\varepsilon_k$  such that  $\sum_{k \geq 0} \rho_k^{m-1} \varepsilon_k < \infty$  and obtain (iii).

Finally, it remains to prove item (iv). For *scalar* functions, this follows from (iii), but some care is needed for manifold-valued maps. With  $r_j$  as in (d), set  $u := \pi \circ \tilde{u} : \mathbb{R}^m \rightarrow \mathcal{N}$ ,  $b := \pi(\tilde{b})$  and define

$$u^j(x) := \begin{cases} u(x), & \text{if } |x - a| \geq r_j \\ b, & \text{if } |x - a| < r_j \end{cases}.$$

Clearly,  $u^j \in C^\infty(\mathbb{R}^m, \mathcal{N})$ ,  $u^j - b, u - b \in W^{s,p}(\mathbb{R}^m, \mathbb{R}^\nu)$  and  $u^j - u \rightarrow 0$  a.e. and in  $L^p(\mathbb{R}^m)$  as  $j \rightarrow \infty$ . It thus suffices to prove that  $|u - u^j|_{W^{s,p}(\mathbb{R}^m)} \rightarrow 0$  as  $j \rightarrow \infty$ . For this purpose, we note that  $|u - u^j|_{W^{s,p}(\mathbb{R}^m)} = |v^j|_{W^{s,p}(\mathbb{R}^m)}$ , where

$$(5.9) \quad v^j := u - u^j + b = \begin{cases} \pi \circ \tilde{u}_k, & \text{in } B_{\rho_k}(a_k), \text{ if } B_{\rho_k}(a_k) \subset B_{r_j}(a) \\ b, & \text{elsewhere} \end{cases}.$$

By (5.8) and the choice of  $\varepsilon_k$ , we have  $|v^j|_{W^{s,p}(\mathbb{R}^m)} \rightarrow 0$  as  $j \rightarrow \infty$ .

The proof of Lemma 5.2 is complete.  $\square$

*Proof of Theorem 4 for the universal covering of connected, non-simply-connected, compact Riemannian manifolds  $\mathcal{N}$ .* When  $\mathcal{M}$  is a smooth bounded open set in  $\mathbb{R}^m$ , we first note that, on  $\mathcal{M} \times \mathcal{M}$ , the geodesic distance  $d_{\mathcal{M}}$  is equivalent to the Euclidean distance in  $\mathbb{R}^m$ . It then suffices to combine Lemma 5.2 with Corollary 4.3 (applied in the connected set  $\mathcal{M} \setminus \{a\}$ ). The case of a

manifold reduces to this special case, since the analytical singularity constructed in Lemma 5.2 is constant outside an arbitrarily small neighborhood of  $a$ .  $\square$

**5.3. A variant of the analytic obstruction.** In the general case, Theorem 4 can be obtained via a suitable variant of Lemma 5.2.

**Lemma 5.3.** *Assume (5.1)–(5.2).*

*Let  $\pi \in C^\infty(\widetilde{\mathcal{N}}, \mathcal{N})$  be a non-trivial Riemannian covering, with  $\widetilde{\mathcal{N}}$  connected.*

*Let  $b \in \mathcal{N}$  and write  $\pi^{-1}(\{b\}) = \{\tilde{b}_i; i \in I\}$ .*

*Let  $\mathcal{M} \subset \mathbb{R}^m$  be a connected open set and let  $a \in \overline{\mathcal{M}}$ .*

*Then there exist a family  $(U_i)_{i \in I}$  of open sets and a family  $(\tilde{u}_i)_{i \in I}$  of maps such that*

- (i)  $U_i \subsetneq \mathcal{M} \setminus \{a\}$ ,  $\forall i \in I$ ,
- (ii)  $U_i \cap \overline{\bigcup_{j \neq i} U_j} = \emptyset$ ,  $\forall i \in I$ ,
- (iii)  $\mathcal{M} \setminus \overline{\bigcup_{j \neq i} U_j}$  is connected,  $\forall i \in I$ ,
- (iv)  $\tilde{u}_i \in C^\infty(\mathbb{R}^m \setminus \{a\}, \widetilde{\mathcal{N}})$ ,  $\forall i \in I$ ,
- (v)  $\tilde{u}_i \equiv \tilde{b}_i$  in  $\mathbb{R}^m \setminus (U_i \cup \{a\})$ ,  $\forall i \in I$ ,
- (vi)  $\tilde{u}_i \notin W^{s,p}(U_i, \widetilde{\mathcal{N}})$ ,  $\forall i \in I$ ,
- (vii) if we set

$$u := \begin{cases} \pi \circ \tilde{u}_i, & \text{in } U_i \\ b, & \text{in } \mathbb{R}^m \setminus \bigcup_{i \in I} U_i \end{cases},$$

*then  $u \in C^\infty(\mathbb{R}^m \setminus \{a\}, \mathcal{N})$  and  $u \in W^{s,p}(\mathbb{R}^m, \mathcal{N})$ ,*

*(viii)  $u$  is the strong limit in  $W^{s,p}(\mathbb{R}^m, \mathcal{N})$  of maps in  $C^\infty(\mathbb{R}^m, \mathcal{N})$ .*

*Proof.* Our construction is again based on a family of balls, but this time indexed over  $k \geq 0$  and  $i \in I$  (we recall that the set  $I$  is at most countable). The requirements on the closed balls  $(\overline{B}_{\rho_{k,i}}(a_{k,i}))_{k \geq 0, i \in I}$  are the following:

- (a)  $\overline{B}_{\rho_{k,i}}(a_{k,i}) \subset \mathcal{M} \setminus \{a\}$ ,  $\forall k, \forall i$ ,
- (b) the balls are mutually disjoint,
- (c)  $a_{k,i} \rightarrow a$  (and thus  $\rho_{k,i} \rightarrow 0$ ) as  $k + i \rightarrow \infty$ ,
- (d) there exists a sequence  $r_j \searrow 0$  such that  $\{x \in \mathbb{R}^m; |x - a| = r_j\} \cap \overline{B}_{\rho_{k,i}}(a_{k,i}) = \emptyset$ ,  $\forall j, \forall k$ ,  $\forall i$ .

Set  $U_i := \bigcup_{k \geq 0} B_{\rho_{k,i}}(a_{k,i})$ . Clearly,  $\overline{U_i} = \bigcup_{k \geq 0} \overline{B}_{\rho_{k,i}}(a_{k,i}) \cup \{a\}$ , and (i) and (ii) hold. By a straightforward argument, assumptions (b) and (c), combined with the fact that  $\mathcal{M}$  is connected and  $m \geq 2$ , imply (iii). (Actually, we have the more general property that  $\mathcal{M} \setminus \overline{\bigcup_{j \in J} U_j}$  is connected,  $\forall J \subseteq I$ .)

We next define  $\tilde{u}_i$ ,  $i \in I$ . Since the covering  $\pi$  is non-trivial, we can consider, for each  $i$ , some  $j = j(i) \in I \setminus \{i\}$ . Let, for every  $k$ ,  $\tilde{u}_{k,i}$  correspond, as in Lemma 5.1, to  $\tilde{b} := \tilde{b}_i$ ,  $\tilde{b}' := \tilde{b}_j$ , to the ball  $B_{\rho_{k,i}}(a_{k,i})$ , and to the numbers  $\varepsilon_{k,i}$  and  $M := k + 1$ . By analogy with the proof of Lemma 5.2, we require that  $\sum_{k \geq 0, i \in I} \rho_{k,i}^{m-1} \varepsilon_{k,i} < \infty$ . We set

$$\tilde{u}_i(x) := \begin{cases} \tilde{u}_{k,i}(x), & \text{if } x \in B_{\rho_{k,i}}(a_{k,i}) \text{ for some } k \geq 0 \\ \tilde{b}_i, & \text{otherwise} \end{cases}.$$

Following the proof of Lemma 5.2, we find that (iv) through (viii) hold.

The proof of Lemma 5.3 is complete.  $\square$

*Proof of Theorem 4 in the general case.* Again, we may assume that  $\mathcal{M}$  is an open set in  $\mathbb{R}^m$ . Let  $u$  be as in Lemma 5.3. Argue by contradiction and assume that  $u = \pi \circ \tilde{u}$  for some

$\tilde{u} \in W^{s,p}(\mathcal{M}, \widetilde{\mathcal{N}})$ . Let  $i \in I$ . By Corollary 4.5 applied to  $u$  in the connected open set  $\mathcal{M} \setminus \overline{\bigcup_{j \neq i} U_j}$ , for the smooth lifting  $\tilde{u}_i$ , we have  $\tilde{u} \neq \tilde{u}_i$  a.e. in the set  $V := \mathcal{M} \setminus \overline{\bigcup_{j \in I} U_j}$ . Thus, a.e. in  $V$ , we have  $\tilde{u}(x) \notin \{\tilde{b}_i; i \in I\}$ . This contradicts the facts that  $V$  has positive measure and  $\pi \circ \tilde{u}(x) = b$ ,  $\forall x \in V$ .  $\square$

## REFERENCES

- [1] R. A. Adams, *Sobolev spaces*, Pure and Applied Mathematics, vol. 65, Academic Press, New York-London, 1975.
- [2] J. M. Ball and A. Zărnescu, *Orientability and energy minimization in liquid crystal models*, Arch. Ration. Mech. Anal. **202** (2011), no. 2, 493–535, doi:10.1007/s00205-011-0421-3.
- [3] F. Bethuel, *A new obstruction to the extension problem for Sobolev maps between manifolds*, J. Fixed Point Theory Appl. **15** (2014), no. 1, 155–183, doi:10.1007/s11784-014-0185-0.
- [4] F. Bethuel and D. Chiron, *Some questions related to the lifting problem in Sobolev spaces*, Perspectives in nonlinear partial differential equations, Contemp. Math., vol. 446, Amer. Math. Soc., Providence, R.I., 2007, pp. 125–152.
- [5] F. Bethuel and F. Demengel, *Extensions for Sobolev mappings between manifolds*, Calc. Var. Partial Differential Equations **3** (1995), no. 4, 475–491, doi:10.1007/BF01187897.
- [6] J. Bourgain, H. Brezis, and P. Mironescu, *Lifting in Sobolev spaces*, J. Anal. Math. **80** (2000), 37–86.
- [7] ———, *Another look at Sobolev spaces*, Optimal control and partial differential equations, IOS, Amsterdam, 2001, pp. 439–455.
- [8] P. Bousquet, A. C. Ponce, and J. Van Schaftingen, *Strong approximation of fractional Sobolev maps*, J. Fixed Point Theory Appl. **15** (2014), no. 1, 133–153.
- [9] H. Brezis, *How to recognize constant functions. A connection with Sobolev spaces*, Uspekhi Mat. Nauk **57** (2002), no. 4(346), 59–74; English transl., Russian Math. Surveys **57** (2002), no. 4, 693–708.
- [10] H. Brezis and P. Mironescu, *Gagliardo–Nirenberg, composition and products in fractional Sobolev spaces*, J. Evol. Equ. **1** (2001), no. 4, 387–404.
- [11] ———, *Density in  $W^{s,p}(\Omega; N)$* , J. Funct. Anal. **269** (2015), no. 7, 2045–2109.
- [12] ———, *Sobolev maps with values into the circle*. In preparation.
- [13] R. Hardt, D. Kinderlehrer, and F. H. Lin, *The variety of configurations of static liquid crystals*, Variational methods (Paris, 1988), Progr. Nonlinear Differential Equations Appl., vol. 4, Birkhäuser, Boston, Mass., 1990, pp. 115–131.
- [14] A. Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002.
- [15] J. M. Lee, *Riemannian manifolds: An introduction to curvature*, Graduate Texts in Mathematics, vol. 176, Springer, New York, 1997.
- [16] ———, *Introduction to topological manifolds*, 2nd ed., Graduate Texts in Mathematics, vol. 202, Springer, New York, 2011.
- [17] P. Mironescu, *Lifting default for  $\mathbb{S}^1$ -valued maps*, C. R. Math. Acad. Sci. Paris **346** (2008), no. 19–20, 1039–1044, doi:10.1016/j.crma.2008.08.001.
- [18] ———, *Lifting of  $\mathbb{S}^1$ -valued maps in sums of Sobolev spaces*, available at <https://hal.archives-ouvertes.fr/hal-00747663>.
- [19] ———,  *$\mathbb{S}^1$ -valued Sobolev mappings*, Sovrem. Mat. Fundam. Napravl. **35** (2010), 86–100, doi:10.1007/s10958-010-0090-z (Russian); English transl., J. Math. Sci. (N.Y.) **170** (2010), no. 3, 340–355.
- [20] ———, *The role of the Hardy type inequalities in the theory of function spaces*, Rev. Roumaine Math. Pures Appl. **63** (2018), no. 4, 447–525.
- [21] P. Mironescu and J. Van Schaftingen, *The problem of extension of traces for Sobolev mappings into a manifold*. in preparation.
- [22] A. Monteil and J. Van Schaftingen, *Uniform boundedness principles for Sobolev maps into manifolds*, Ann. Inst. H. Poincaré Anal. Non Linéaire **36** (2019), no. 2, 417–449, doi:10.1016/j.anihpc.2018.06.002.
- [23] D. Mucci, *Strong density results in trace spaces of maps between manifolds*, Manuscripta Math. **128** (2009), no. 4, 421–441.
- [24] ———, *Sobolev maps into the projective line with bounded total variation*, Confluentes Math. **2** (2010), no. 2, 181–216, doi:10.1142/S179374421000017X.