

When is $c_0(\tau)$ complemented in tensor products of $\ell_p(I)$ and X ?

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ABSTRACT. Let X be a Banach space, I an infinite set, τ an infinite cardinal and $p \in [1, \infty)$. In contrast to a classical c_0 result due independently to Cembranos and Freniche, we prove that if the cofinality of τ is greater than the cardinality of I , then the injective tensor product $\ell_p(I) \widehat{\otimes}_\varepsilon X$ contains a complemented copy of $c_0(\tau)$ if and only if X does. This result is optimal for every regular cardinal τ .

On the other hand, we provide a generalization of a c_0 result of Oya by proving that if τ is an infinite cardinal, then the projective tensor product $\ell_p(I) \widehat{\otimes}_\pi X$ contains a complemented copy of $c_0(\tau)$ if and only if X does.

These results are obtained via useful descriptions of tensor products as convenient generalized sequence spaces.

1. Introduction

We use standard set-theoretical and Banach space theory terminology as may be found, e.g., in [14] and [15] respectively. We denote by B_X the closed unit ball of the Banach space X . If X and Y are Banach spaces, we denote by $\mathcal{L}(X, Y)$ the space of all bounded linear operators from X to Y and by $\mathcal{K}(X, Y)$ the subspace of all compact linear operators. We say that Y contains a copy (resp. a complemented copy) of X , and write $X \hookrightarrow Y$ (resp. $X \hookrightarrow_c Y$), if X is isomorphic to a subspace (resp. complemented subspace) of Y . If X and Y are isometrically isomorphic Banach spaces, we write $X \equiv Y$.

We shall denote the projective and injective tensor norms by $\|\cdot\|_\pi$ and $\|\cdot\|_\varepsilon$ respectively. The projective (resp. injective) tensor product of X and Y is the completion of $X \otimes Y$ with respect to $\|\cdot\|_\pi$ (resp. $\|\cdot\|_\varepsilon$) and will be denoted by $X \widehat{\otimes}_\pi Y$ (resp. $X \widehat{\otimes}_\varepsilon Y$).

Given X a Banach space, I an infinite set and $(x_i)_{i \in I} \in X^I$, the notation $x = \sum_i x_i$ means that, for every $\varepsilon > 0$, there exists a finite subset $\emptyset \neq F_0 \subset I$ such that, for every finite subset F of I with $F_0 \subset F$, we have $\|x - \sum_{i \in F} x_i\| < \varepsilon$.

For a non-empty set Γ , $c_0(\Gamma)$ denotes the Banach space of all real-valued maps f on Γ with the property that for each $\varepsilon > 0$, the set $\{\gamma \in \Gamma : |f(\gamma)| \geq \varepsilon\}$ is finite, equipped with the supremum norm. We will refer to $c_0(\Gamma)$ as $c_0(\tau)$ when the cardinality of Γ (denoted by $|\Gamma|$) is equal to τ . This space will be denoted by c_0 when $\tau = \aleph_0$.

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Given $p \in [1, \infty)$, we denote

$$\ell_p(I, X) = \left\{ (x_i)_{i \in I} \in X^I : \sum_{i \in I} \|x_i\|^p < \infty \right\},$$

equipped with the complete norm

$$\|\bar{x}\|_p = \left[\sum_{i \in I} \|x_i\|^p \right]^{\frac{1}{p}},$$

for each $\bar{x} = (x_i)_{i \in I} \in \ell_p(I, X)$. These spaces will be denoted by $\ell_p(I)$ when $X = \mathbb{R}$. If $I = \mathbb{N}$, we will write ℓ_p .

By $\ell_\infty(I)$ we will denote the Banach space of all bounded real-valued maps on I , with the supremum norm. This space will be denoted by ℓ_∞ when $I = \mathbb{N}$.

Recall that if τ is an infinite cardinal then the *cofinality* of τ , denoted by $\text{cf}(\tau)$, is the least cardinal α such that there exists a family of ordinals $\{\beta_j : j \in \alpha\}$ satisfying $\beta_j < \tau$ for all $j \in \alpha$, and $\sup\{\beta_j : j \in \alpha\} = \tau$. A cardinal τ is said to be *regular* when $\text{cf}(\tau) = \tau$; otherwise, it is said to be *singular*.

An immediate consequence of the classical Cembranos-Freniche Theorem [3, Main Theorem], [11, Corollary 2.5] is the following result.

THEOREM 1.1. *For each $p \in [1, \infty)$ we have $c_0 \xrightarrow{c} \ell_p \widehat{\otimes}_\varepsilon \ell_\infty$.*

However, it is well known that $c_0 \not\xrightarrow{c} \ell_\infty$ (see, e. g., [9, Corollary 11, p. 156]). On the other hand, Oya proved the following theorem.

THEOREM 1.2 ([18], Theorem 2). *If X is a Banach space and $p \in [1, \infty)$, then*

$$c_0 \xrightarrow{c} \ell_p \widehat{\otimes}_\pi X \iff c_0 \xrightarrow{c} X.$$

These facts motivate the following problems.

PROBLEM 1.3. *Let X be a Banach space, I be an infinite set, $p \in [1, \infty)$ and τ be an infinite cardinal. What assumptions on τ and I yield*

$$c_0(\tau) \xrightarrow{c} \ell_p(I) \widehat{\otimes}_\varepsilon X \iff c_0(\tau) \xrightarrow{c} X?$$

PROBLEM 1.4. *Given X a Banach space, I an infinite set, $p \in [1, \infty)$ and τ an infinite cardinal, is it true that*

$$c_0(\tau) \xrightarrow{c} \ell_p(I) \widehat{\otimes}_\pi X \iff c_0(\tau) \xrightarrow{c} X?$$

In this paper, we provide a partial solution to Problem 1.3 and a complete solution to Problem 1.4. More precisely, we will prove that for every Banach space X , infinite set I , $p \in [1, \infty)$ and infinite cardinal τ , one has

$$c_0(\tau) \xrightarrow{c} \ell_p(I) \widehat{\otimes}_\pi X \iff c_0(\tau) \xrightarrow{c} X.$$

Additionally, if the $\text{cf}(\tau) > |I|$, then one also has

$$(1.1) \quad c_0(\tau) \xrightarrow{c} \ell_p(I) \widehat{\otimes}_\varepsilon X \iff c_0(\tau) \xrightarrow{c} X.$$

REMARK 1.5. The equivalence (1.1) cannot be extended to the case $p = \infty$. Setting $I = \mathbb{N}$ and $\tau = \aleph_1$, by [13, Theorem 5.3] we know that

$$c_0(\aleph_1) \xrightarrow{c} \ell_\infty \widehat{\otimes}_\varepsilon \ell_\infty(\aleph_1),$$

although $c_0(\aleph_1) \not\xrightarrow{c} \ell_\infty(\aleph_1)$ by [9, Corollary 11, p. 156].

REMARK 1.6. (1.1) is optimal for every infinite regular cardinal κ . Indeed, setting $I = \kappa$, again by [13, Theorem 4.5] we have

$$c_0(\kappa) \xrightarrow{c} \ell_p(\kappa) \widehat{\otimes}_\varepsilon \ell_\infty(\kappa),$$

but $c_0(\kappa) \not\xrightarrow{c} \ell_\infty(\tau)$ once again by [9, Corollary 11, p. 156].

2. Preliminary results and notations

We will denote by $(e_j)_{j \in \tau}$ the unit-vector basis of $c_0(\tau)$ or $\ell_p(\tau)$ for $p \in [1, \infty)$, that is, $e_j(j) = 1$ and $e_j(k) = 0$ for each $j, k \in \tau$, $j \neq k$. If Γ is a subset of τ , we identify $c_0(\Gamma)$ with the closed subspace of $c_0(\tau)$ consisting of the maps g on τ such that $g(j) = 0$ for each $j \in \tau \setminus \Gamma$.

We begin by recalling the following classical result by H.P. Rosenthal.

THEOREM 2.1. ([19, Remark following Theorem 3.4]) *Let X be a Banach space and τ an infinite cardinal. Let $T : c_0(\tau) \rightarrow X$ be a bounded linear operator such that*

$$\inf\{\|T(e_j)\| : j \in \tau\} > 0.$$

Then there exists a subset $\Gamma \subset \tau$ such that $|\Gamma| = \tau$ and $T|_{c_0(\Gamma)}$ is an isomorphism onto its image.

We recall that a family $(x_j^*)_{j \in \tau}$ in the dual space X^* is said to be *weak*-null* if for each $x \in X$ we have

$$(x_j^*(x))_{j \in \tau} \in c_0(\tau).$$

Recall also that a family $(x_j)_{j \in \tau}$ in a Banach space X is said to be *equivalent* to the canonical basis of $c_0(\tau)$ if there exists $T : c_0(\tau) \rightarrow X$ an isomorphism onto its image satisfying $T(e_j) = x_j$, for each $j \in \tau$.

The main characterization of complemented copies of $c_0(\tau)$ we will use is the following result obtained in [7].

THEOREM 2.2. *Let X be a Banach space and τ be an infinite cardinal. The following are equivalent:*

- (1) *X contains a complemented copy of $c_0(\tau)$.*
- (2) *There exist a family $(x_j)_{j \in \tau}$ equivalent to the unit-vector basis of $c_0(\tau)$ in X and a weak*-null family $(x_j^*)_{j \in \tau}$ in X^* such that, for each $j, k \in \tau$,*

$$x_j^*(x_k) = \delta_{jk}.$$

- (3) *There exist a family $(x_j)_{j \in \tau}$ equivalent to the unit-vector basis of $c_0(\tau)$ in X and a weak*-null family $(x_j^*)_{j \in \tau}$ in X^* such that*

$$\inf_{j \in \tau} |x_j^*(x_j)| > 0.$$

We will also use the following two simple lemmas.

LEMMA 2.3. *Let I be an infinite set and J be a non-empty set. Let $\{I_j\}_{j \in J}$ be a family of subsets of I such that $\bigcup_{j \in J} I_j = I$. If $\text{cf}(|I|) > |J|$, then there exists $j_0 \in J$ such that $|I_{j_0}| = |I|$.*

PROOF. Suppose that the conclusion does not hold. Then we have $|I_j| < |I|$ for each $i \in J$ and thus, by the definition of cofinality,

$$\sup\{|I_j| : j \in J\} < |I|.$$

Since $|J| < \text{cf}(|I|) \leq |I|$, we obtain

$$|I| = \left| \bigcup_{j \in J} I_j \right| \leq \max(|J|, \sup\{|I_j| : j \in J\}) < |I|,$$

a contradiction that finishes the proof. \square

LEMMA 2.4. *Let I be an infinite set and J be a non-empty, finite set. For each $i \in I$, let $(a_{i,j})_{j \in J}$ be a family of positive real numbers. Suppose that there exists $\delta > 0$ satisfying, for all $i \in I$,*

$$\sum_{j \in J} a_{i,j} \geq \delta.$$

Then there exist $j_0 \in J$ and $I' \subset I$ such that $|I'| = |I|$ and

$$a_{i,j_0} \geq \frac{\delta}{|J|}, \quad \forall i \in I'.$$

PROOF. By hypothesis, for each $i \in I$ there exists $j(i) \in J$ such that

$$a_{i,j(i)} \geq \frac{\delta}{|J|}.$$

Let $J' = \{j(i) : i \in I\}$ and for each $j \in J'$, consider $I_j = \{i \in I : j(i) = j\}$. Since $I = \bigcup_{j \in J'} I_j$ is infinite and J' is finite, by Lemma 2.3 there exists $j_0 \in J'$ such that $|I_{j_0}| = |I|$. Setting $I' = I_{j_0}$, the proof is complete. \square

3. Spaces $s\ell_p(I, X)$, $F_p(I, X)$ and $\ell_p\langle I, X \rangle$

In this section we shall introduce some generalized sequence spaces which will be of interest.

Let X be a Banach space, I be an infinite set and $p \in [1, \infty)$. Following [21], we denote

$$s\ell_p(I, X) = \left\{ (x_i)_{i \in I} \in X^I : \sum_{i \in I} |x^*(x_i)|^p < \infty, \text{ for all } x^* \in X^* \right\}.$$

It is straightforward to check that the function

$$\|\bar{x}\|_{s\ell_p} = \sup \left\{ \left[\sum_{i \in I} |x^*(x_i)|^p \right]^{\frac{1}{p}} : x^* \in B_{X^*} \right\},$$

where $\bar{x} = (x_i)_{i \in I} \in s\ell_p(I, X)$, is a complete norm on $s\ell_p(I, X)$. For each $i \in I$, we denote by $R_i : X \rightarrow X^I$ the canonical inclusion defined by $R_i(x) = (\delta_{ik}x)_{k \in I}$, for each $x \in X$. Our interest lies in the closed subspace

$$F_p(I, X) = \left\{ \bar{x} = (x_i)_{i \in I} \in s\ell_p(I, X) : \bar{x} = \sum_{i \in I} R_i(x_i) \right\}$$

of $s\ell_p(I, X)$.

Given $p \in (1, \infty)$ and $q \in (1, \infty)$ the conjugate index of p , there exists a canonical isometric isomorphism between $s\ell_p(I, X)$ and $\mathcal{L}(\ell_q(I), X)$ which maps an

element $\bar{x} = (x_i)_{i \in I} \in s\ell_p(I, X)$ to the bounded linear operator $S \in \mathcal{L}(\ell_q(I), X)$ defined by

$$S(\bar{a}) = \sum_{i \in I} a_i x_i,$$

for all $\bar{a} = (a_i)_{i \in I} \in \ell_q(I)$. This isometry identifies $F_p(I, X)$ and the subspace $\mathcal{K}(\ell_q(I), X)$ of compact linear operators from $\ell_q(I)$ to X .

In a similar way, in the case $p = 1$ there exists an isometric isomorphism between $sl_1(I, X)$ and $\mathcal{L}(c_0(I), X)$ which identifies $F_1(I, X)$ and $\mathcal{K}(c_0(I), X)$.

Next, let $p \in (1, \infty)$ and $q \in (1, \infty)$ be the conjugate index of p . Following [1, 5], we denote

$$\ell_p\langle I, X \rangle = \left\{ \bar{x} = (x_i)_{i \in I} \in X^I : \sum_{i \in I} |x_i^*(x_i)| < \infty, \text{ for all } (x_i^*)_{i \in I} \in s\ell_q(I, X^*) \right\}.$$

It is straightforward to check that the function

$$\|\bar{x}\|_{\ell_p\langle I, X \rangle} = \sup \left\{ \sum_{i \in I} |x_i^*(x_i)| : \bar{x}^* = (x_i^*)_{i \in I} \in B_{s\ell_q(I, X^*)} \right\},$$

where $\bar{x} = (x_i)_{i \in I} \in \ell_p\langle I, X \rangle$, is a complete norm on $\ell_p\langle I, X \rangle$.

We will summarize the main properties of those spaces in our next results. Our statements and proofs are simple modifications of those found in [1, 2, 5].

PROPOSITION 3.1. *Let X be a Banach space, I be an infinite set and $p \in [1, \infty)$.*

- (1) $\ell_p(I, X) \subset F_p(I, X)$ and $\|\bar{x}\|_{s\ell_p} \leq \|\bar{x}\|_p$, for all $\bar{x} \in \ell_p(I, X)$;
- (2) For $p > 1$, $\ell_p\langle I, X \rangle \subset \ell_p(I, X)$ and $\|\bar{x}\|_p \leq \|\bar{x}\|_{\ell_p\langle I, X \rangle}$, for all $\bar{x} \in \ell_p\langle I, X \rangle$.

PROOF. Statement (1) follows immediately from the definitions of $\ell_p(I, X)$ and $s\ell_p(I, X)$. Let us show (2). Given $\bar{x} = (x_i)_{i \in I} \in \ell_p\langle I, X \rangle$, we fix, for every $i \in I$, $x_i^* \in B_{X^*}$ such that $\|x_i\| = x_i^*(x_i)$. For every $(\lambda_i)_{i \in I} \in B_{\ell_q(I)}$ we have $(\lambda_i x_i^*)_{i \in I} \in B_{s\ell_q(I, X^*)}$ and so

$$\sum_{i \in I} |\lambda_i x_i^*(x_i)| = \sum_{i \in I} |\lambda_i| \|x_i\| \leq \|\bar{x}\|_{\ell_p\langle I, X \rangle}.$$

Therefore

$$\left(\sum_{i \in I} \|x_i\|^p \right)^{\frac{1}{p}} = \|\bar{x}\|_p \leq \|\bar{x}\|_{\ell_p\langle I, X \rangle}$$

and the proof is complete. \square

We will denote the canonical inclusion of $x \in X$ in X^{**} by \hat{x} . The proof of the following proposition is straightforward.

PROPOSITION 3.2. *Let X be a Banach space, I be an infinite set, $p \in [1, \infty)$, $\bar{x} = (x_i)_{i \in I} \in X^I$ and $\bar{y} = (\hat{x}_i)_{i \in I} \in (X^{**})^I$. Then $\bar{x} \in s\ell_p(I, X)$ if, and only if, $\bar{y} \in s\ell_p(I, X^{**})$. Furthermore, $\|\bar{x}\|_{s\ell_p} = \|\bar{y}\|_{s\ell_p}$.*

4. Generalized sequence spaces and tensor products

The goal of this section is to establish a connection between the generalized sequence spaces defined in the previous section and the projective and injective tensor products $\ell_p(I) \widehat{\otimes}_\pi X$ and $\ell_p(I) \widehat{\otimes}_\varepsilon X$.

It is obvious that for every $\bar{a} = (a_i)_{i \in I} \in \mathbb{R}^I$ and $x \in X$ we have $(a_i x)_{i \in I} \in X^I$ and furthermore the operator $\ell_p(I) \times X \ni (\bar{a}, x) \mapsto (a_i x)_{i \in I} \in X^I$ is bilinear. This bilinear operator induces a linear operator $T : \ell_p(I) \otimes X \rightarrow X^I$ [20, Proposition 1.4]. It is also obvious that $\text{Im}(T) \subset \ell_p \langle I, X \rangle \cap s\ell_p(I, X)$. We summarize the main properties of this operator in the following proposition.

PROPOSITION 4.1. *Given $p \in [1, \infty)$, we have:*

- (1) $\text{Im}(T)$ contains all finitely non-zero families in $\ell_p(I, X)$;
- (2) $\|T(u)\|_{s\ell_p(X)} = \|u\|_\varepsilon$, for each $u \in \ell_p(I) \otimes X$;
- (3) If $p \in (1, \infty)$, then $\|T(u)\|_{\ell_p \langle X \rangle} \leq \|u\|_\pi$, for each $u \in \ell_p(I) \otimes X$.

PROOF. Given $x \in X$ and $i \in I$, notice that $T(e_i \otimes x) = (\delta_{ij} x)_{j \in I}$. This shows statement (1).

Next, let $u \in \ell_p(I) \otimes X$ be given and fix a representation $u = \sum_{n=1}^m a_n \otimes x_n$, where $a_n = (a_i^n)_{i \in I}$. Then we have

$$\begin{aligned} \|T(u)\|_{s\ell_p(X)} &= \sup \left\{ \left[\sum_{i \in I} \left| \sum_{n=1}^m a_i^n x^*(x_n) \right|^p \right]^{\frac{1}{p}} : x^* \in B_{X^*} \right\} \\ &= \sup \left\{ \left\| \sum_{n=1}^m x^*(x_n) a_n \right\|_p : x^* \in B_{X^*} \right\} = \|u\|_\varepsilon. \end{aligned}$$

This proves (2).

Finally, suppose that $p \in (1, \infty)$ and let us show (3). Let $q \in (1, \infty)$ be the conjugate index of p . Given $\bar{x}^* = (x_i^*)_{i \in I} \in B_{s\ell_q(I, X^*)}$, $a = (a_i)_{i \in I} \in \ell_p(I)$, and $x \in X$, by Hölder's Inequality we have

$$\sum_{i \in I} |a_i x_i^*(x)| \leq \left(\sum_{i \in I} |a_i|^p \right)^{\frac{1}{p}} \left(\sum_{i \in I} |x_i^*(x)|^q \right)^{\frac{1}{q}} \leq \|a\|_p \|x\|.$$

Hence, the bilinear operator $\ell_p(I) \times X \ni (a, x) \mapsto (a_i x)_{i \in I} \in \ell_p \langle I, X \rangle$ has norm at most 1. By [20, Theorem 2.9], we obtain

$$\|T(u)\|_{\ell_p \langle X \rangle} \leq \|u\|_\pi,$$

for every $u \in \ell_p(I) \otimes X$, as desired. \square

COROLLARY 4.2. *Let X be a Banach space, I be an infinite set and $p \in [1, \infty)$. Let $T : \ell_p(I) \otimes X \rightarrow X^I$ be the linear map defined on Proposition 4.1.*

- (1) T admits an unique linear extension T_ε to $\ell_p(I) \widehat{\otimes}_\varepsilon X$, which is an isometry onto $F_p(I, X)$;
- (2) If $p \in (1, \infty)$, then T admits an unique linear extension of same norm T_π from $\ell_p(I) \widehat{\otimes}_\pi X$ to $\ell_p \langle I, X \rangle$.

PROOF. Let us first show (1). By the density of $\ell_p(I) \otimes X$ in $\ell_p(I) \widehat{\otimes}_\varepsilon X$ and Proposition 4.1, T admits an unique linear extension $T_\varepsilon : \ell_p(I) \widehat{\otimes}_\varepsilon X \rightarrow F_p(I, X)$,

which is an isometry onto its image. Given $\bar{x} = (x_i)_{i \in I} \in F_p(I, X)$, by Proposition 4.1 we have

$$x = \sum_{i \in I} R_i(x_i) = \sum_{i \in I} T(e_i \otimes x_i) \in \overline{\text{Im}(T)} = \text{Im}(T_\varepsilon).$$

This proves that T_ε is onto $F_p(I, X)$.

Similarly, if $p \in (1, \infty)$, the density of $\ell_p(I) \otimes X$ in $\ell_p(I) \widehat{\otimes}_\pi X$ and Proposition 4.1 imply that T admits a unique linear extension $T_\pi : \ell_p(I) \widehat{\otimes}_\pi X \rightarrow \ell_p(I, X)$ satisfying $\|T_\pi\| = \|T\| \leq 1$. \square

As in the countable case, we will show that T_π is a linear isometry from $\ell_p(I) \widehat{\otimes}_\pi X$ onto $\ell_p(I, X)$. We will use the following key lemma.

LEMMA 4.3. *Let X be a Banach space, I be an infinite set and $p \in (1, \infty)$. Given $u \in \ell_p(I) \widehat{\otimes}_\pi X$, there exists a family $(x_i)_{i \in I}$ in X such that*

$$u = \sum_{i \in I} e_i \otimes x_i.$$

PROOF. Let $q \in (1, \infty)$ be the conjugate index of p and $(e_i^*)_{i \in I}$ be the unit basis of $\ell_q(I)$. For each $i \in I$ consider the bounded linear operator $\pi_i : \ell_p(I) \rightarrow \ell_p(I)$ defined by

$$\pi_i(a) = e_i^*(a)e_i,$$

for each $a = (a_j)_{j \in I} \in \ell_p(I)$, and for every finite subset $\emptyset \neq F \subset I$ write

$$P_F = \sum_{j \in F} \pi_j.$$

Let $u \in \ell_p(I) \widehat{\otimes}_\pi X$ be given and let us prove that

$$(4.1) \quad u = \sum_{i \in I} (\pi_i \otimes I)(u).$$

By [20, Proposition 2.8] there exist bounded sequences $(a_n)_{n \geq 1}$ in $\ell_p(I)$ and $(y_n)_{n \geq 1}$ in X satisfying

$$u = \sum_{n=1}^{\infty} a_n \otimes y_n \quad \text{and} \quad \sum_{n=1}^{\infty} \|a_n\|_p \|y_n\| < \infty.$$

We may suppose that $\|y_n\| = 1$ for every $n \geq 1$. Given $\varepsilon > 0$, take $N \geq 1$ such that

$$\sum_{n=N+1}^{\infty} \|a_n\|_p \leq \frac{\varepsilon}{3}.$$

It is clear that

$$\left\| u - \sum_{n=1}^N a_n \otimes y_n \right\|_\pi \leq \frac{\varepsilon}{3}.$$

There exists a finite subset $\emptyset \neq F_0 \subset I$ such that for every finite subset $F \subset I$ satisfying $F_0 \subset F$ and every $1 \leq n \leq N$ we have

$$\|P_F(a_n) - a_n\|_p \leq \frac{\varepsilon}{3N}.$$

Thus,

$$\begin{aligned} \|u - (P_F \otimes I)(u)\| &\leq \left\| u - \sum_{n=1}^N a_n \otimes y_n \right\|_{\pi} \\ &\quad + \left\| \sum_{n=1}^N a_n \otimes y_n - \sum_{n=1}^N P_F(a_n) \otimes y_n \right\|_{\pi} \\ &\quad + \left\| \sum_{n=1}^N P_F(a_n) \otimes y_n - (P_F \otimes I)(u) \right\|_{\pi} \leq \varepsilon. \end{aligned}$$

This shows that (4.1) holds.

Finally, since

$$(\pi_i \otimes I)(u) = \sum_{n=1}^{\infty} \pi_i(a_n) \otimes y_n = \sum_{n=1}^{\infty} e_i^*(a_n) e_i \otimes y_n = e_i \otimes \left(\sum_{n=1}^{\infty} e_i^*(a_n) y_n \right),$$

for each $i \in I$, it follows that

$$x_i = \sum_{n=1}^{\infty} e_i^*(a_n) y_n$$

satisfies the desired properties. \square

We are now in a position to show that T_{π} is an isometry onto $\ell_p\langle I, X \rangle$.

THEOREM 4.4. *Let X be a Banach space, I be an infinite set and $p \in (1, \infty)$. The operator T_{π} is an isometric isomorphism from $\ell_p(I) \widehat{\otimes}_{\pi} X$ onto $\ell_p\langle I, X \rangle$.*

PROOF. We have shown that

$$\|T_{\pi}(u)\|_{\ell_p\langle X \rangle} \leq \|u\|_{\pi},$$

for every $u \in \ell_p(I) \widehat{\otimes}_{\pi} X$. In order to prove that T_{π} is isometric, it is sufficient to prove the reverse inequality on $\text{span}\{e_i \otimes x : i \in I, x \in X\}$. Given $v \in \text{span}\{e_i \otimes x : i \in I, x \in X\}$, there exist a finite subset $\emptyset \neq I_0 \subset I$ and a finite family $(y_i)_{i \in I_0}$ of elements of X such that

$$v = \sum_{i \in I_0} e_i \otimes y_i.$$

By [20, p. 24] and the Hahn-Banach Theorem, there exists a bounded linear operator $b \in \mathcal{L}(\ell_p(I), X^*) \equiv [\ell_p(I) \widehat{\otimes}_{\pi} X]^*$ satisfying

$$\|b\| = 1 \text{ and } \|v\|_{\pi} = \sum_{i \in I_0} b(e_i)(y_i).$$

Setting $y_i = 0$ for each $i \in I \setminus I_0$, it is clear that $T_{\pi}(v) = (y_i)_{i \in I}$ and

$$\|T_{\pi}(v)\|_{\ell_p\langle X \rangle} \geq \|v\|_{\pi},$$

as desired.

Let us show next that T_{π} is onto. Given $\bar{x} = (x_i)_{i \in I} \in \ell_p\langle I, X \rangle$, we will show that the family $(e_i \otimes x_i)_{i \in I}$ is summable in $\ell_p(I) \widehat{\otimes}_{\pi} X$ and

$$(4.2) \quad \bar{x} = T_{\pi} \left(\sum_{i \in I} e_i \otimes x_i \right).$$

For every $\bar{x}^* = (x_i^*)_{i \in I} \in sl_q(I, X^*)$ we have

$$\left| \sum_{i \in I} x_i^*(x_i) \right| \leq \|\bar{x}\|_{\ell_p\langle X \rangle} \|\bar{x}^*\|_{sl_q(X^*)}.$$

Hence, the linear functional θ defined on $sl_q(I, X^*) \equiv \mathcal{L}(\ell_p(I), X^*)$ by

$$\theta(\bar{x}^*) = \sum_{i \in I} x_i^*(x_i),$$

for each $\bar{x}^* = (x_i^*)_{i \in I} \in sl_q(I, X^*)$, is bounded. Let us denote by θ_r its restriction to the subspace $\mathcal{K}(\ell_p(I), X^*)$.

Let $q \in (1, \infty)$ be the conjugate index of p . Recall that the space $\ell_q(I)$ has the approximation property [20, p. 73], and therefore

$$(4.3) \quad \mathcal{K}(\ell_p(I), X^*) \equiv \ell_q(I) \widehat{\otimes}_\varepsilon X^*,$$

by [20, Corollary 4.13]. Moreover, since $\ell_p(I)$ also has the Radon-Nikodým property [20, Corollary 5.45], by (4.3) and [20, Theorem 5.33] we know that

$$\ell_p(I) \widehat{\otimes}_\pi X^{**} \equiv [\ell_q(I) \widehat{\otimes}_\varepsilon X^*]^* \equiv [\mathcal{K}(\ell_p(I), X^*)]^*.$$

It follows from Lemma 4.3 that there exists a family $(x_i^{**})_{i \in I}$ in X^{**} such that

$$\theta_r = \sum_{i \in I} e_i \otimes x_i^{**}.$$

We observe that for every $i \in I$ and for every $x^* \in X^*$ we have

$$x_i^{**}(x^*) = \theta_r(e_i^* \otimes x^*) = x^*(x_i),$$

and so $x_i^{**} = \widehat{x}_i$. Thus, the family $(e_i \otimes \widehat{x}_i)_{i \in I}$ is summable in $\ell_p(I) \widehat{\otimes}_\pi X^{**}$ and hence, the family $(e_i \otimes x_i)_{i \in I}$ in $\ell_p(I) \widehat{\otimes}_\pi X$, by [8, Corollary 14, p. 238]. Finally, it is clear that (4.2) holds. \square

REMARK 4.5. We observe that $T_\pi(e_i \otimes x_i) = R_i(x_i)$, for each $i \in I$. Therefore, for each $\bar{x} = (x_i)_{i \in I} \in \ell_p\langle I, X \rangle$, the family $(R_i(x_i))_{i \in I}$ is summable in $\ell_p\langle I, X \rangle$ and

$$\bar{x} = \sum_{i \in I} R_i(x_i).$$

REMARK 4.6. By Proposition 3.1, every family $\bar{x} = (x_i)_{i \in I} \in \ell_p\langle I, X \rangle$ has only countably many non-zero coordinates. Hence, by the previous remark, the subspace

$$L = \{\bar{y} = (y_i)_{i \in I} \in \ell_p\langle I, X \rangle : \{i \in I : y_i = 0\} \text{ is finite}\}$$

is dense in $\ell_p\langle I, X \rangle$.

5. Dual spaces of $\ell_p\langle I, X \rangle$ and $F_p(I, X)$

We now turn our attention to convenient identifications of the spaces $\ell_p\langle I, X \rangle$ and $F_p(I, X)$ and of their duals.

THEOREM 5.1. *Let X be a Banach space, I be an infinite set, $p \in (1, \infty)$ and $q \in (1, \infty)$ be the conjugate index of p . Then there exists an isometric isomorphism from $\ell_p\langle I, X \rangle^*$ onto $sl_q(I, X^*)$ which maps $\varphi \in \ell_p\langle I, X \rangle^*$ to $(x_i^*)_{i \in I} \in sl_q(I, X^*)$, where*

$$\varphi(\bar{x}) = \sum_{i \in I} x_i^*(x_i),$$

for every $\bar{x} = (x_i)_{i \in I} \in \ell_p\langle I, X \rangle$.

PROOF. Let T_π be the canonical isometric isomorphism from $\ell_p(I)\widehat{\otimes}_\pi X$ onto $\ell_p\langle I, X \rangle$ considered in Corollary 4.2. Then T_π^* is also an isometric isomorphism from $\ell_p\langle I, X \rangle^*$ onto $[\ell_p(I)\widehat{\otimes}_\pi X]^*$. By [20, p. 24], we have

$$[\ell_p(I)\widehat{\otimes}_\pi X]^* \equiv \mathcal{L}(\ell_p(I), X^*) \equiv s\ell_q(I, X^*).$$

Given $\varphi \in \ell_p\langle I, X \rangle^*$, let $(x_i^*)_{i \in I} \in s\ell_q(I, X^*)$ be the unique element corresponding to $T_\pi^*(\varphi) \in [\ell_p(I)\widehat{\otimes}_\pi X]^*$ via the above identifications. By Remark 4.5, for each

$$\bar{x} = (x_i)_{i \in I} = \sum_{i \in I} R_i(x_i) = \sum_{i \in I} T_\pi(e_i \otimes x_i) \in \ell_p\langle I, X \rangle,$$

we have

$$\varphi(\bar{x}) = \sum_{i \in I} (\varphi \circ T_\pi)(e_i \otimes x_i) = \sum_{i \in I} T_\pi^*(\varphi)(e_i \otimes x_i) = \sum_{j \in I} x_j^*(x_i),$$

and the proof is complete. \square

REMARK 5.2. For every $\varphi = (x_i^*)_{i \in I} \in \ell_p\langle I, X \rangle^*$, we have

$$\|\varphi\| = \sup \left\{ \sum_{i \in I} |x_i^*(x_i)| : (x_i)_{i \in I} \in B_{\ell_p\langle I, X \rangle} \right\}.$$

THEOREM 5.3. *Let X be a Banach space, I be an infinite set, $p \in (1, \infty)$ and $q \in (1, \infty)$ be the conjugate index of p . Then there exists an isometric isomorphism from $F_p(I, X)^*$ onto $\ell_q\langle I, X^* \rangle$ which maps $\psi \in F_p(I, X)^*$ to $(x_i^*)_{i \in I} \in \ell_q\langle I, X^* \rangle$, where*

$$\psi(\bar{x}) = \sum_{i \in I} x_i^*(x_i),$$

for every $\bar{x} = (x_i)_{i \in I} \in F_p(I, X)$.

PROOF. Let T_ε be the canonical isometric isomorphism from $\ell_p(I)\widehat{\otimes}_\varepsilon X$ onto $F_p(I, X)$ considered in Corollary 4.2. Since the space $\ell_q(I)$ has both the approximation property and the Radon-Nikodým property, by [20, Theorem 5.33] we know that

$$[\ell_p(I)\widehat{\otimes}_\varepsilon X]^* \equiv \ell_q(I)\widehat{\otimes}_\pi X^*.$$

By Theorem 4.4 and Remark 4.5, the operator

$$\ell_q\langle I, X^* \rangle \ni (x_i^*)_{i \in I} \mapsto \sum_{i \in I} e_i \otimes x_i^* \in \ell_q(I)\widehat{\otimes}_\pi X^*$$

is also an isometric isomorphism.

Given $\psi \in F_p(I, X)^*$, there exists a unique $(x_i^*)_{i \in I} \in \ell_q\langle I, X^* \rangle$ corresponding to $T_\varepsilon^*(\psi) \in [\ell_p(I)\widehat{\otimes}_\varepsilon X]^*$ via the above identifications. For each

$$\bar{x} = (x_i)_{i \in I} = \sum_{i \in I} R_i(x_i) = \sum_{i \in I} T_\varepsilon(e_i \otimes x_i) \in F_p(I, X),$$

we have

$$\psi(\bar{x}) = \sum_{i \in I} (\psi \circ T_\varepsilon)(e_i \otimes x_i) = \sum_{i \in I} T_\varepsilon^*(\psi)(e_i \otimes x_i) = \sum_{j \in I} x_j^*(x_i),$$

and the proof is complete. \square

6. Complemented copies of $c_0(\tau)$ in $\ell_p(I) \widehat{\otimes}_\pi X$ and $\ell_p(I) \widehat{\otimes}_\varepsilon X$

This section is devoted to the study of the complemented copies of $c_0(\tau)$ in the projective and injective tensor products $\ell_p(I) \widehat{\otimes}_\pi X$ and $\ell_p(I) \widehat{\otimes}_\varepsilon X$. We begin with the former space.

LEMMA 6.1. *Let X be a Banach space, I be an infinite set, $p \in (1, \infty)$ and τ be an infinite cardinal. Let $(\overline{x_j})_{j \in \tau} = ((x_{i,j})_{i \in I})_{j \in \tau}$ be a family of $\ell_p(I, X)$ equivalent to the unit basis of $c_0(\tau)$ and let $(\varphi_j)_{j \in \tau} = ((\varphi_{i,j})_{i \in I})_{j \in \tau}$ be a bounded family of $\ell_p(I, X)^*$. Then, for each $\varepsilon > 0$, there exists a finite subset $F_\varepsilon \subset I$ satisfying, for every $j \in \tau$,*

$$\left| \sum_{i \in I \setminus F_\varepsilon} \varphi_{i,j}(x_{i,j}) \right| < \varepsilon.$$

PROOF. Let $M = \sup_{j \in \tau} \|\varphi_j\|$. We may assume $\|\overline{x_j}\|_{\ell_p(I, X)} = 1$ for every $j \in \tau$. Suppose that the conclusion does not hold; then there exists $\varepsilon > 0$ such that, for every finite subset $F \subset I$, there exists $j \in \tau$ with

$$\left| \sum_{i \notin F} \varphi_{i,j}(x_{i,j}) \right| \geq \varepsilon.$$

We will show that this leads to a contradiction.

We construct by induction a sequence $(j_k)_{k \geq 1}$ in τ and a sequence of pairwise disjoint finite subsets $(F_k)_{k \geq 1}$ of I such that

$$(6.1) \quad \left| \sum_{i \in F_k} \varphi_{i,j_k}(x_{i,j_k}) \right| \geq \frac{\varepsilon}{2}, \quad \forall k \geq 1.$$

Put $F_0 = \emptyset$. By hypothesis, there exists $j_1 \in \tau$ such that

$$\left| \sum_{i \in I} \varphi_{i,j_1}(x_{i,j_1}) \right| \geq \varepsilon.$$

Since

$$\sum_{i \in I} |\varphi_{i,j_1}(x_{i,j_1})| \leq M,$$

there exists a finite subset $\emptyset \neq F_1 \subset I$ such that

$$\sum_{i \in I \setminus F_1} |\varphi_{i,j_1}(x_{i,j_1})| \leq \frac{\varepsilon}{2},$$

and hence

$$\sum_{i \in F_1} |\varphi_{i,j_1}(x_{i,j_1})| \geq \frac{\varepsilon}{2}.$$

Next, let $k \geq 1$ be given and suppose we have constructed j_1, \dots, j_k and F_1, \dots, F_k as desired. Put $A = \cup_{l=1}^k F_l$. By hypothesis, there exists $j_{k+1} \in \tau$ such that

$$\left| \sum_{i \in I \setminus A} \varphi_{i,j_{k+1}}(x_{i,j_{k+1}}) \right| \geq \varepsilon.$$

There exists a finite subset $\emptyset \neq F_{k+1} \subset (I \setminus A)$ such that

$$\sum_{i \in I \setminus (A \cup F_{k+1})} |\varphi_{i, j_{k+1}}(x_{i, j_{k+1}})| \leq \frac{\varepsilon}{2},$$

and so

$$\left| \sum_{i \in F_{k+1}} \varphi_{i, j_{k+1}}(x_{i, j_{k+1}}) \right| \geq \frac{\varepsilon}{2}.$$

It follows from the construction that $j_l \neq j_k$ for $l \neq k$.

For each $k \geq 1$ and $i \in I$ we denote

$$\psi_{i, k} = \begin{cases} \varphi_{i, j_k} & \text{if } i \in F_k \\ 0 & \text{if } i \in I \setminus F_k, \end{cases}$$

and $\psi_k = (\psi_{i, k})_{i \in I}$. It is obvious that $(\psi_k)_{k \geq 1}$ is a bounded sequence in $F_q(I, X^*) \equiv \ell_p(I, X^{**})$, where $q \in (1, \infty)$ is the conjugate index of p . We claim that $(\psi_k)_{k \geq 1}$ is a weakly-null sequence. Indeed, by Remark 4.6, the subspace

$$L = \{\theta^{**} = (\theta_i^{**})_{i \in I} \in \ell_p(I, X^{**}) : \{i \in I : \theta_i^{**} = 0\} \text{ is finite}\}$$

is dense in $\ell_p(I, X^{**})$. Given $\theta^{**} = (\theta_i^{**})_{i \in I} \in L$, there exists $k_0 \geq 1$ such that $\theta_i^{**} = 0$, for all $i \in F_k$, $k \geq k_0$, and hence,

$$\lim_{k \rightarrow \infty} \theta^{**}(\psi_k) = \lim_{k \rightarrow \infty} \sum_{i \in F_k} \theta_i^{**}(\psi_{i, k}) = 0.$$

The density of L then establishes our claim.

The sequence $(\overline{x_{j_k}})_{k \geq 1}$ is equivalent to the unit basis of c_0 and furthermore

$$|\psi_k(\overline{x_{j_k}})| \geq \frac{\varepsilon}{2},$$

for each $k \geq 1$, in contradiction with the Dunford-Pettis Property of c_0 [12, p. 596]. \square

We are now in a position to prove the first main result of this section.

THEOREM 6.2. *Given X a Banach space, I an infinite set, $p \in (1, \infty)$ and τ an infinite cardinal, we have*

$$c_0(\tau) \xrightarrow{c} \ell_p(I) \widehat{\otimes}_\pi X \iff c_0(\tau) \xrightarrow{c} X.$$

PROOF. Suppose first that $c_0(\tau) \xrightarrow{c} \ell_p(I) \widehat{\otimes}_\pi X \equiv \ell_p(I, X)$. By Theorem 2.2, there exist $(\overline{x_j})_{j \in \tau} = ((x_{i, j})_{i \in I})_{j \in \tau}$ equivalent to the canonical basis of $c_0(\tau)$ in $\ell_p(I, X)$, and $(\varphi_j)_{j \in \tau}$ a weak*-null family in $\ell_p(I, X)^*$ satisfying $\varphi_j(x_j) = 1$ for all $j \in \tau$. By Theorem 5.1, for each $j \in \tau$, there exists a family $(\varphi_{i, j})_{i \in I}$ in X^* such that

$$1 = \varphi_j(\overline{x_j}) = \sum_{i \in I} \varphi_{i, j}(x_{i, j}).$$

By the Uniform Boundedness Principle, $(\varphi_j)_{j \in \tau}$ is bounded, and so there exists $M > 0$ such that $\|\varphi_j\| \leq M$ for each $j \in \tau$.

An appeal to Lemma 6.1 yields a finite subset $F \subset I$ satisfying

$$\left| \sum_{i \in I \setminus F} \varphi_{i, j}(x_{i, j}) \right| < \frac{1}{2}, \quad \forall j \in \tau.$$

Therefore, $F \neq \emptyset$ and

$$\sum_{i \in F} |\varphi_{i,j}(x_{i,j})| > \frac{1}{2}, \quad \forall j \in \tau.$$

Next, Lemma 2.4 implies that there exist $i_0 \in F$ and $\tau_1 \subset \tau$ satisfying $|\tau_1| = \tau$ and

$$(6.2) \quad |\varphi_{i_0,j}(x_{i_0,j})| > \frac{1}{2|F|}, \quad \forall j \in \tau_1.$$

We have

$$\frac{1}{2|F|} < |\varphi_{i_0,j}(x_{i_0,j})| \leq M \|x_{i_0,j}\|, \quad \forall j \in \tau_1,$$

and thus

$$(6.3) \quad \|x_{i_0,j}\| > \frac{1}{2M|F|}, \quad \forall j \in \tau_1.$$

Consider the bounded linear operator $\pi_{i_0} : \ell_p\langle I, X \rangle \rightarrow X$ defined by $\pi_{i_0}(\bar{x}) = x_{i_0}$ for each $\bar{x} = (x_i)_{i \in I} \in \ell_p\langle I, X \rangle$. By hypothesis, there exists $T : c_0(\tau) \rightarrow \ell_p\langle I, X \rangle$ an isomorphism onto its image such that $T(e_j) = \bar{x}_j$, for all $j \in \tau$. By (6.3), we have

$$\|(\pi_{i_0} \circ T)(e_j)\| > \frac{1}{2M|F|}, \quad \forall j \in \tau_1,$$

and therefore, by Theorem 2.1, there exists $\tau_2 \subset \tau_1$ such that $|\tau_2| = |\tau_1| = \tau$ and $\pi_{i_0} \circ T|_{c_0(\tau_2)}$ is an isomorphism onto its image. Thus, $(x_{i_0,j})_{j \in \tau_2}$ is equivalent to the unit basis of $c_0(\tau_2)$ in X . The family $(\varphi_j)_{j \in \tau}$ is weak*-null and hence, $(\varphi_{i_0,j})_{j \in \tau_2}$ is a weak*-null family in X^* . Theorem 2.2 and (6.2) then imply that $c_0(\tau) \overset{c}{\hookrightarrow} X$.

For the converse, it is sufficient to notice that $\ell_p\langle I, X \rangle$ contains a complemented copy of X . \square

We now turn our attention to complemented copies of $c_0(\tau)$ in the injective tensor product $\ell_p(I) \widehat{\otimes}_\varepsilon X$.

THEOREM 6.3. *Let X be a Banach space, I be an infinite set, $p \in [1, \infty)$ and τ be an infinite cardinal. If $\text{cf}(\tau) > |I|$, then*

$$c_0(\tau) \overset{c}{\hookrightarrow} \ell_p(I) \widehat{\otimes}_\varepsilon X \iff c_0(\tau) \overset{c}{\hookrightarrow} X.$$

PROOF. Suppose first that $c_0(\tau) \overset{c}{\hookrightarrow} \ell_p(I) \widehat{\otimes}_\pi X \equiv F_p(I, X)$. By Theorem 2.2, there exists $(\bar{x}_j)_{j \in \tau} = ((x_{i,j})_{i \in I})_{j \in \tau}$ equivalent to the unit basis of $c_0(\tau)$ in $F_p(I, X)$, and a weak*-null family $(\varphi_j)_{j \in \tau} = ((\varphi_{i,j})_{i \in I})_{j \in \tau}$ of $F_p(I, X)^*$ satisfying

$$1 = \varphi_j(\bar{x}_j) = \sum_{i \in I} \varphi_{i,j}(x_{i,j}), \quad \forall j \in \tau.$$

For each $j \in \tau$, there exists a finite subset $\emptyset \neq F_j \subset I$ such that

$$(6.4) \quad \sum_{i \in F_j} |\varphi_{i,j}(x_{i,j})| > \frac{1}{2}.$$

Let $\mathcal{F} = \{F_j : j \in \tau\}$, and for each $F \in \mathcal{F}$, consider $\tau(F) = \{j \in \tau : F_j = F\}$. Since $\tau = \bigcup_{F \in \mathcal{F}} \tau(F)$ and $\text{cf}(\tau) > |I| \geq |F|$, by Lemma 2.3 there exists $G_0 \in \mathcal{F}$ such that $|\tau(G_0)| = \tau$. Setting $\tau_1 = \tau(G_0)$, by (6.4) we obtain

$$\sum_{i \in G_0} |\varphi_{i,j}(x_{i,j})| > \frac{1}{2}, \quad \forall j \in \tau_1.$$

Next, an appeal to Lemma 2.4 yields $i_0 \in G_0$ and $\tau_2 \subset \tau_1$ satisfying $|\tau_2| = |\tau_1| = \tau$ and

$$(6.5) \quad |\varphi_{i_0,j}(x_{i_0,j})| > \frac{1}{2|G_0|}, \forall j \in \tau_2.$$

By the Uniform Boundedness Principle, $(\varphi_j)_{j \in \tau}$ is bounded, and so there exists $M > 0$ such that $\|\varphi_j\| \leq M$, for each $j \in \tau$. Therefore we have

$$\frac{1}{2|G_0|} < |\varphi_{i_0,j}(x_{i_0,j})| \leq M\|x_{i_0,j}\|, \forall j \in \tau_2,$$

and thus

$$(6.6) \quad \|x_{i_0,j}\| > \frac{1}{2M|G_0|}, \forall j \in \tau_2.$$

Consider the bounded linear operator $\pi_{i_0} : s\ell_p(I, X) \rightarrow X$ defined by $\pi_{i_0}(\bar{x}) = x_{i_0}$ for each $\bar{x} = (x_i)_{i \in I} \in s\ell_p(I, X)$. By hypothesis, there exists $T : c_0(\tau) \rightarrow F_p(I, X)$ an isomorphism onto its image such that $T(e_j) = \bar{x}_j$, for all $i \in \tau$. By (6.6), we have

$$\|(\pi_{i_0} \circ T)(e_j)\| > \frac{1}{2M|G_0|}, \forall j \in \tau_2,$$

and therefore, by Theorem 2.1, there exists $\tau_3 \subset \tau_2$ such that $|\tau_3| = |\tau_2| = \tau$ and $\pi_{i_0} \circ T|_{c_0(\tau_3)}$ is an isomorphism onto its image. Thus, $(x_{i_0,j})_{j \in \tau_3}$ is equivalent to the unit basis of $c_0(\tau_3)$ in X . The family $(\varphi_j)_{j \in \tau}$ is weak*-null and hence, $(\varphi_{i_0,j})_{j \in \tau_3}$ is weak*-null in X^* . Theorem 2.2 and (6.5) then imply $c_0(\tau) \xrightarrow{c} X$.

For the converse, it suffices to notice that $F_p(I, X)$ contains a complemented copy of X . \square

Our next step is to obtain a result analogue to Theorem 6.3 without assumptions on the cofinality of τ . In this direction, we begin with the following lemma.

LEMMA 6.4. *Let X be a Banach space, I be an infinite set, $p \in (1, \infty)$ and τ be an infinite cardinal. Let $(\bar{x}_j)_{j \in \tau} = ((x_{i,j})_{i \in I})_{j \in \tau}$ be a family of $F_p(I, X)$ equivalent to the unit basis of $c_0(\tau)$ and let $(\varphi_j)_{j \in \tau} = ((\varphi_{i,j})_{i \in I})_{j \in \tau}$ be a bounded family of $F_p(I, X)^* \equiv \ell_q(I, X^*)$, where $q \in (1, \infty)$ is the conjugate index of p . Suppose that*

$$\mathcal{K}(\ell_q(I), X^{**}) = \mathcal{L}(\ell_q(I), X^{**}).$$

Then, for each $\varepsilon > 0$, there exists a finite subset $F_\varepsilon \subset I$ satisfying

$$\left| \sum_{i \in I \setminus F_\varepsilon} \varphi_{i,j}(x_{i,j}) \right| < \varepsilon, \forall j \in \tau.$$

PROOF. By hypothesis and Theorem 5.3, there exists $M > 0$ such that

$$(6.7) \quad M \geq \|\varphi_j\| = \|(\varphi_{i,j})_{i \in I}\|_{\ell_q(X^*)}, \forall j \in \tau.$$

Suppose by contradiction that the result does not hold. As in the proof of Lemma 6.1, there exist $\varepsilon > 0$, a sequence of distinct indexes $(j_k)_{k \geq 1}$ of τ , and a sequence $(F_k)_{k \geq 1}$ of finite, non-empty and pairwise disjoint subsets of I satisfying

$$(6.8) \quad \left| \sum_{i \in F_k} \varphi_{i,j_k}(x_{i,j_k}) \right| > \varepsilon, \forall k \geq 1.$$

Consider $\psi_k = (\psi_{i,k})_{i \in I} \in \ell_q\langle I, X^* \rangle$, where

$$\psi_{i,k} = \begin{cases} \varphi_{i,j_k} & \text{if } i \in F_k \\ 0 & \text{if } i \in I \setminus F_k. \end{cases}$$

By (6.7) we have

$$(6.9) \quad \|\psi_k\|_{\ell_q\langle X^* \rangle} \leq \|(\varphi_{i,j_k})_{i \in I}\|_{\ell_q\langle X^* \rangle} = \|\varphi_{j_k}\| \leq M, \quad \forall k \geq 1.$$

We claim that $(\psi_k)_{k \geq 1}$ is a weakly-null sequence in $\ell_q\langle I, X^* \rangle$. Recall that

$$\ell_q\langle I, X^* \rangle^* \equiv s\ell_p(I, X^{**}) \equiv \mathcal{L}(\ell_q(I), X^{**}),$$

and

$$F_p(I, X^{**}) \equiv \mathcal{K}(\ell_q(I), X^{**}).$$

Thus, by hypothesis, we have

$$\ell_q\langle I, X^* \rangle^* \equiv F_p(I, X^{**}).$$

The subspace

$$L = \{\theta^{**} = (\theta_i^{**})_{i \in I} \in F_p(I, X^{**}) : \{i \in I : \theta_i^{**} = 0\} \text{ is finite}\}$$

is dense in $F_p(I, X^{**})$. Observe that if $\theta^{**} = (\theta_i^{**})_{i \in I} \in L$, there exists $k_0 \geq 1$ such that $\theta_i^{**} = 0$, for all $i \in F_k$, $k \geq k_0$, and hence,

$$\lim_{k \rightarrow \infty} \theta^{**}(\psi_k) = \lim_{k \rightarrow \infty} \sum_{i \in F_k} \theta_i^{**}(\psi_{i,k}) = 0.$$

The density of L then establishes our claim.

Now, notice that

$$|\psi_k(\overline{x_{j_k}})| = \left| \sum_{i \in I} \psi_{i,k}(x_{i,j_k}) \right| = \left| \sum_{i \in F_k} \varphi_{i,j_k}(x_{i,j_k}) \right| > \frac{\varepsilon}{2}, \quad \forall k \geq 1,$$

by construction. On the other hand, since the sequence $(\psi_k)_{k \geq 1}$ is weakly-null in $F_p(I, X)^*$ and $(\overline{x_{j_k}})_{k \geq 1}$ is, by hypothesis, equivalent to the unit basis of c_0 , by the Dunford-Pettis Property of c_0 [12, p. 596] we know that $\psi_k(x_{j_k}) \rightarrow 0$, a contradiction that finishes the proof. \square

THEOREM 6.5. *Let X be a Banach space, I be an infinite set, $p \in (1, \infty)$ and τ be an infinite cardinal. Suppose that*

$$\mathcal{K}(\ell_q(I), X^{**}) = \mathcal{L}(\ell_q(I), X^{**}),$$

where $q \in (1, \infty)$ is the conjugate index of p . Then

$$c_0(\tau) \xrightarrow{c} \ell_p(I) \widehat{\otimes}_\varepsilon X \iff c_0(\tau) \xrightarrow{c} X.$$

PROOF. Suppose first that $c_0(\tau) \xrightarrow{c} \ell_p(I) \widehat{\otimes}_\varepsilon X \equiv F_p(I, X)$. By Theorem 2.2, there exist $(\overline{x_j})_{j \in \tau} = ((x_{i,j})_{i \in I})_{j \in \tau}$ equivalent to the unit basis of $c_0(\tau)$ in $F_p(I, X)$, and a weak*-null family $(\varphi_j)_{j \in \tau} = ((\varphi_{i,j})_{i \in I})_{j \in \tau}$ of $F_p(I, X)^*$ satisfying

$$1 = \varphi_j(\overline{x_j}) = \sum_{i \in I} \varphi_{i,j}(x_{i,j}), \quad \forall j \in \tau.$$

By the Uniform Boundedness Principle, $(\varphi_j)_{j \in \tau}$ is bounded, and so there exists $M > 0$ such that

$$(6.10) \quad \|\varphi_j\| \leq M, \quad \forall j \in \tau.$$

An appeal to Lemma 6.4 yields a finite subset $F \subset I$ satisfying

$$\left| \sum_{i \in I \setminus F} \varphi_{i,j}(x_{i,j}) \right| < \frac{1}{2}, \quad \forall j \in \tau.$$

Therefore, $F \neq \emptyset$ and

$$\sum_{i \in F} |\varphi_{i,j}(x_{i,j})| \geq \left| \sum_{i \in F} \varphi_{i,j}(x_{i,j}) \right| \geq \left| \sum_{i \in I} \varphi_{i,j}(x_{i,j}) \right| - \left| \sum_{i \in I \setminus F} \varphi_{i,j}(x_{i,j}) \right| > \frac{1}{2}, \quad \forall j \in \tau.$$

Next, Lemma 2.4 implies that there exist $i_0 \in F$ and $\tau_1 \subset \tau$ satisfying $|\tau_1| = \tau$ and

$$(6.11) \quad |\varphi_{i_0,j}(x_{i_0,j})| > \frac{1}{2|F|}, \quad \forall j \in \tau_1.$$

By (6.10), we have

$$\frac{1}{2|F|} < |\varphi_{i_0,j}(x_{i_0,j})| \leq \|\varphi_{i_0,j}\| \|x_{i_0,j}\| \leq \|\varphi_j\| \|x_{i_0,j}\| \leq M \|x_{i_0,j}\|, \quad \forall j \in \tau_1,$$

and thus

$$(6.12) \quad \|x_{i_0,j}\| > \frac{1}{2M|F|}, \quad \forall j \in \tau_1.$$

Consider the bounded linear operator $\pi_{i_0} : F_p(I, X) \rightarrow X$ defined by $\pi_{i_0}(\bar{x}) = x_{i_0}$ for each $\bar{x} = (x_i)_{i \in I} \in F_p(I, X)$. By hypothesis, there exists $T : c_0(\tau) \rightarrow F_p(I, X)$ an isomorphism onto its image such that $T(e_j) = \bar{x}_j$, for all $j \in \tau$. By (6.12), we have

$$\|(\pi_{i_0} \circ T)(e_j)\| > \frac{1}{2M|F|}, \quad \forall j \in \tau_1,$$

and therefore, by Theorem 2.1, there exists $\tau_2 \subset \tau_1$ such that $|\tau_2| = |\tau_1| = \tau$ and $\pi_{i_0} \circ T|_{c_0(\tau_2)}$ is an isomorphism onto its image. Thus, $(x_{i_0,j})_{j \in \tau_2}$ is equivalent to the unit basis of $c_0(\tau_2)$ in X .

Next, given $x \in X$, we have

$$(\varphi_{i_0,j}(x))_{j \in \tau_2} \in c_0(\tau_2),$$

since, by hypothesis, $(\varphi_j)_{j \in \tau}$ is weak*-null. This proves that $(\varphi_{i_0,j})_{j \in \tau_2}$ is weak*-null in X^* . Theorem 2.2 and (6.11) then imply that $c_0(\tau) \xrightarrow{c} X$.

For the converse, it is sufficient to notice that $F_p(I, X)$ contains a complemented copy of X . \square

7. Final remarks and open problems

By adapting the proofs of Lemma 6.1 and Theorem 6.2, one can prove the following.

THEOREM 7.1. *Given X a Banach space, I an infinite set, $p \in [1, \infty)$ and τ an infinite cardinal, we have*

$$c_0(\tau) \xrightarrow{c} \ell_p(I, X) \iff c_0(\tau) \xrightarrow{c} X.$$

The above theorem extends to the uncountable case a c_0 result by Bombal [4, Theorem 4.3.1]. Moreover, since $\ell_1(I) \widehat{\otimes}_\pi X$ is isometrically isomorphic to $\ell_1(I, X)$ [20, p. 19], the next corollary follows immediately from Theorems 6.2 and 7.1.

COROLLARY 7.2. *Given X a Banach space, I an infinite set, $p \in [1, \infty)$ and τ an infinite cardinal, we have*

$$c_0(\tau) \xrightarrow{c} \ell_p(I) \widehat{\otimes}_\pi X \iff c_0(\tau) \xrightarrow{c} \ell_p(I, X) \iff c_0(\tau) \xrightarrow{c} X.$$

However, observe that the space $\ell_p(I, X)$ cannot be exchanged by the usual Banach space of Lebesgue integrable functions $L_p([0, 1], X)$ in the statement of Theorem 7.1, even in the case $\tau = \aleph_0$. Indeed, recall the following result due to Emmanuelle [10, Main Theorem].

THEOREM 7.3. *Let X be a Banach space and $p \in [1, \infty)$. Then*

$$c_0 \hookrightarrow X \implies c_0 \xrightarrow{c} L_p([0, 1], X).$$

Thus, we have

$$c_0 \xrightarrow{c} L_p([0, 1], \ell_\infty) \text{ but } c_0 \not\xrightarrow{c} \ell_\infty.$$

These facts arise naturally the following question.

PROBLEM 7.4. *Let X be a Banach space, $p \in [1, \infty)$ and τ be an infinite cardinal. What assumptions on τ yield*

$$c_0(\tau) \xrightarrow{c} \ell_p([0, 1], X) \iff c_0(\tau) \xrightarrow{c} X?$$

We do not know if Theorem 6.2 can be extended to the case $p = \infty$. That is:

PROBLEM 7.5. *Let X be a Banach space and τ be an infinite cardinal. Is it true that*

$$c_0(\tau) \xrightarrow{c} \ell_\infty \widehat{\otimes}_\pi X \implies c_0(\tau) \xrightarrow{c} X?$$

Finally, we also do not know if Theorem 6.3 is optimal for any cardinal τ . In particular, the following problem is unsolved.

PROBLEM 7.6. *Let X be a Banach space and $p \in [1, \infty)$. Does it follow that*

$$c_0(\aleph_\omega) \xrightarrow{c} \ell_p \widehat{\otimes}_\varepsilon X \implies c_0(\aleph_\omega) \xrightarrow{c} X?$$

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