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# When is $c_{0}(\tau)$ complemented in tensor products of $\ell_{p}(I)$ and $X$ ? 

Vinícius Morelli Cortes, Elói Medina Galego, and Christian Samuel


#### Abstract

Let $X$ be a Banach space, $I$ an infinite set, $\tau$ an infinite cardinal and $p \in[1, \infty)$. In contrast to a classical $c_{0}$ result due independently to Cembranos and Freniche, we prove that if the cofinality of $\tau$ is greater than the cardinality of $I$, then the injective tensor product $\ell_{p}(I) \widehat{\otimes}_{\varepsilon} X$ contains a complemented copy of $c_{0}(\tau)$ if and only if $X$ does. This result is optimal for every regular cardinal $\tau$.

On the other hand, we provide a generalization of a $c_{0}$ result of Oya by proving that if $\tau$ is an infinite cardinal, then the projective tensor product $\ell_{p}(I) \widehat{\otimes}_{\pi} X$ contains a complemented copy of $c_{0}(\tau)$ if and only if $X$ does.

These results are obtained via useful descriptions of tensor products as convenient generalized sequence spaces.


## 1. Introduction

We use standard set-theoretical and Banach space theory terminology as may be found, e.g., in $[\mathbf{1 4}]$ and $[\mathbf{1 5}]$ respectively. We denote by $B_{X}$ the closed unit ball of the Banach space X . If $X$ and $Y$ are Banach spaces, we denote by $\mathcal{L}(X, Y)$ the space of all bounded linear operators from $X$ to $Y$ and by $\mathcal{K}(X, Y)$ the subspace of all compact linear operators. We say that $Y$ contains a copy (resp. a complemented copy) of $X$, and write $X \hookrightarrow Y$ (resp. $X \stackrel{c}{\hookrightarrow} Y$ ), if $X$ is isomorphic to a subspace (resp. complemented subspace) of $Y$. If $X$ and $Y$ are isometrically isomorphic Banach spaces, we write $X \equiv Y$.

We shall denote the projective and injective tensor norms by $\|\cdot\|_{\pi}$ and $\|\cdot\|_{\varepsilon}$ respectively. The projective (resp. injective) tensor product of $X$ and $Y$ is the completion of $X \otimes Y$ with respect to $\|\cdot\|_{\pi}$ (resp. $\|\cdot\|_{\varepsilon}$ ) and will be denoted by $X \widehat{\otimes}_{\pi} Y$ (resp. $X \widehat{\otimes}_{\varepsilon} Y$ ).

Given $X$ a Banach space, $I$ an infinite set and $\left(x_{i}\right)_{i \in I} \in X^{I}$, the notation $x=\sum_{i} x_{i}$ means that, for every $\varepsilon>0$, there exists a finite subset $\varnothing \neq F_{0} \subset I$ such that, for every finite subset $F$ of $I$ with $F_{0} \subset F$, we have $\left\|x-\sum_{i \in F} x_{i}\right\|<\varepsilon$.

For a non-empty set $\Gamma, c_{0}(\Gamma)$ denotes the Banach space of all real-valued maps $f$ on $\Gamma$ with the property that for each $\varepsilon>0$, the set $\{\gamma \in \Gamma:|f(\gamma)| \geq \varepsilon\}$ is finite, equipped with the supremum norm. We will refer to $c_{0}(\Gamma)$ as $c_{0}(\tau)$ when the cardinality of $\Gamma$ (denoted by $|\Gamma|)$ is equal to $\tau$. This space will be denoted by $c_{0}$ when $\tau=\aleph_{0}$.

[^0]Given $p \in[1, \infty)$, we denote

$$
\ell_{p}(I, X)=\left\{\left(x_{i}\right)_{i \in I} \in X^{I}: \sum_{i \in I}\left\|x_{i}\right\|^{p}<\infty\right\}
$$

equipped with the complete norm

$$
\|\bar{x}\|_{p}=\left[\sum_{i \in I}\left\|x_{i}\right\|^{p}\right]^{\frac{1}{p}},
$$

for each $\bar{x}=\left(x_{i}\right)_{i \in I} \in \ell_{p}(I, X)$. These spaces will be denoted by $\ell_{p}(I)$ when $X=\mathbb{R}$. If $I=\mathbb{N}$, we will write $\ell_{p}$.

By $\ell_{\infty}(I)$ we will denote the Banach space of all bounded real-valued maps on $I$, with the supremum norm. This space will be denoted by $\ell_{\infty}$ when $I=\mathbb{N}$.

Recall that if $\tau$ is an infinite cardinal then the cofinality of $\tau$, denoted by $\operatorname{cf}(\tau)$, is the least cardinal $\alpha$ such that there exists a family of ordinals $\left\{\beta_{j}: j \in \alpha\right\}$ satisfying $\beta_{j}<\tau$ for all $j \in \alpha$, and $\sup \left\{\beta_{j}: j \in \alpha\right\}=\tau$. A cardinal $\tau$ is said to be regular when $\operatorname{cf}(\tau)=\tau$; otherwise, it is said to be singular.

An immediate consequence of the classical Cembranos-Freniche Theorem [3, Main Theorem], [11, Corollary 2.5] is the following result.

Theorem 1.1. For each $p \in[1, \infty)$ we have $c_{0} \stackrel{c}{\leftrightarrows} \ell_{p} \widehat{\otimes}_{\varepsilon} \ell_{\infty}$.
However, it is well known that $c_{0} \stackrel{\leftrightarrows}{\leftrightarrows} \ell_{\infty}$ (see, e. g., [9, Corollary 11, p. 156]). On the other hand, Oya proved the following theorem.

Theorem 1.2 ([18], Theorem 2). If $X$ is a Banach space and $p \in[1, \infty)$, then

$$
c_{0} \stackrel{c}{\hookrightarrow} \ell_{p} \widehat{\otimes}_{\pi} X \Longleftrightarrow c_{0} \stackrel{c}{\hookrightarrow} X .
$$

These facts motivate the following problems.
Problem 1.3. Let $X$ be a Banach space, $I$ be an infinite set, $p \in[1, \infty)$ and $\tau$ be an infinite cardinal. What assumptions on $\tau$ and $I$ yield

$$
c_{0}(\tau) \stackrel{c}{\hookrightarrow} \ell_{p}(I) \widehat{\otimes}_{\varepsilon} X \Longleftrightarrow c_{0}(\tau) \stackrel{c}{\hookrightarrow} X ?
$$

Problem 1.4. Given $X$ a Banach space, I an infinite set, $p \in[1, \infty)$ and $\tau$ an infinite cardinal, is it true that

$$
c_{0}(\tau) \stackrel{c}{\hookrightarrow} \ell_{p}(I) \widehat{\otimes}_{\pi} X \Longleftrightarrow c_{0}(\tau) \stackrel{c}{\hookrightarrow} X ?
$$

In this paper, we provide a partial solution to Problem 1.3 and a complete solution to Problem 1.4. More precisely, we will prove that for every Banach space $X$, infinite set $I, p \in[1, \infty)$ and infinite cardinal $\tau$, one has

$$
c_{0}(\tau) \stackrel{c}{\hookrightarrow} \ell_{p}(I) \widehat{\otimes}_{\pi} X \Longleftrightarrow c_{0}(\tau) \stackrel{c}{\hookrightarrow} X .
$$

Additionally, if the $\operatorname{cf}(\tau)>|I|$, then one also has

$$
\begin{equation*}
c_{0}(\tau) \stackrel{c}{\hookrightarrow} \ell_{p}(I) \widehat{\otimes}_{\varepsilon} X \Longleftrightarrow c_{0}(\tau) \stackrel{c}{\hookrightarrow} X . \tag{1.1}
\end{equation*}
$$

Remark 1.5. The equivalence (1.1) cannot be extended to the case $p=\infty$. Setting $I=\mathbb{N}$ and $\tau=\aleph_{1}$, by [13, Theorem 5.3] we know that

$$
c_{0}\left(\aleph_{1}\right) \stackrel{c}{\hookrightarrow} \ell_{\infty} \widehat{\otimes}_{\varepsilon} \ell_{\infty}\left(\aleph_{1}\right),
$$

although $c_{0}\left(\aleph_{1}\right) \stackrel{c}{\leftrightarrows} \ell_{\infty}\left(\aleph_{1}\right)$ by [9, Corollary 11, p. 156].

REMARK 1.6. (1.1) is optimal for every infinite regular cardinal $\kappa$. Indeed, setting $I=\kappa$, again by [13, Theorem 4.5] we have

$$
c_{0}(\kappa) \stackrel{c}{\hookrightarrow} \ell_{p}(\kappa) \widehat{\otimes}_{\varepsilon} \ell_{\infty}(\kappa),
$$

but $c_{0}(\kappa) \stackrel{c}{\leftrightarrows} \ell_{\infty}(\tau)$ once again by [9, Corollary 11, p. 156].

## 2. Preliminary results and notations

We will denote by $\left(e_{j}\right)_{j \in \tau}$ the unit-vector basis of $c_{0}(\tau)$ or $\ell_{p}(\tau)$ for $p \in[1, \infty)$, that is, $e_{j}(j)=1$ and $e_{j}(k)=0$ for each $j, k \in \tau, j \neq k$. If $\Gamma$ is a subset of $\tau$, we identify $c_{0}(\Gamma)$ with the closed subspace of $c_{0}(\tau)$ consisting of the maps $g$ on $\tau$ such that $g(j)=0$ for each $j \in \tau \backslash \Gamma$.

We begin by recalling the following classical result by H.P. Rosenthal.
Theorem 2.1. ([19, Remark following Theorem 3.4]) Let $X$ be a Banach space and $\tau$ an infinite cardinal. Let $T: c_{0}(\tau) \rightarrow X$ be a bounded linear operator such that

$$
\inf \left\{\left\|T\left(e_{j}\right)\right\|: j \in \tau\right\}>0
$$

Then there exists a subset $\Gamma \subset \tau$ such that $|\Gamma|=\tau$ and $T_{\mid c_{0}(\Gamma)}$ is an isomorphism onto its image.

We recall that a family $\left(x_{j}^{*}\right)_{j \in \tau}$ in the dual space $X^{*}$ is said to be weak ${ }^{*}$-null if for each $x \in X$ we have

$$
\left(x_{j}^{*}(x)\right)_{j \in \tau} \in c_{0}(\tau) .
$$

Recall also that a family $\left(x_{j}\right)_{j \in \tau}$ in a Banach space $X$ is said to be equivalent to the canonical basis of $c_{0}(\tau)$ if there exists $T: c_{0}(\tau) \rightarrow X$ an isomorphism onto its image satisfying $T\left(e_{j}\right)=x_{j}$, for each $j \in \tau$.

The main characterization of complemented copies of $c_{0}(\tau)$ we will use is the following result obtained in [7].

Theorem 2.2. Let $X$ be a Banach space and $\tau$ be an infinite cardinal. The following are equivalent:
(1) $X$ contains a complemented copy of $c_{0}(\tau)$.
(2) There exist a family $\left(x_{j}\right)_{j \in \tau}$ equivalent to the unit-vector basis of $c_{0}(\tau)$ in $X$ and a weak ${ }^{*}$-null family $\left(x_{j}^{*}\right)_{j \in \tau}$ in $X^{*}$ such that, for each $j, k \in \tau$,

$$
x_{j}^{*}\left(x_{k}\right)=\delta_{j k}
$$

(3) There exist a family $\left(x_{j}\right)_{j \in \tau}$ equivalent to the unit-vector basis of $c_{0}(\tau)$ in $X$ and a weak ${ }^{*}$-null family $\left(x_{j}^{*}\right)_{j \in \tau}$ in $X^{*}$ such that

$$
\inf _{j \in \tau}\left|x_{j}^{*}\left(x_{j}\right)\right|>0
$$

We will also use the following two simple lemmas.
Lemma 2.3. Let $I$ be an infinite set and $J$ be a non-empty set. Let $\left\{I_{j}\right\}_{j \in J}$ be a family of subsets of $I$ such that $\bigcup_{j \in J} I_{j}=I$. If $\operatorname{cf}(|I|)>|J|$, then there exists $j_{0} \in J$ such that $\left|I_{j_{0}}\right|=|I|$.

Proof. Suppose that the conclusion does not hold. Then we have $\left|I_{j}\right|<|I|$ for each $i \in J$ and thus, by the definition of cofinality,

$$
\sup \left\{\left|I_{j}\right|: j \in J\right\}<|I|
$$

Since $|J|<\operatorname{cf}(|I|) \leq|I|$, we obtain

$$
|I|=\left|\bigcup_{j \in J} I_{j}\right| \leq \max \left(|J|, \sup \left\{\left|I_{j}\right|: j \in J\right\}\right)<|I|
$$

a contradiction that finishes the proof.
Lemma 2.4. Let $I$ be an infinite set and $J$ be a non-empty, finite set. For each $i \in I$, let $\left(a_{i, j}\right)_{j \in J}$ be a family of positive real numbers. Suppose that there exists $\delta>0$ satisfying, for all $i \in I$,

$$
\sum_{j \in J} a_{i, j} \geq \delta
$$

Then there exist $j_{0} \in J$ and $I^{\prime} \subset I$ such that $\left|I^{\prime}\right|=|I|$ and

$$
a_{i, j_{0}} \geq \frac{\delta}{|J|}, \forall i \in I^{\prime}
$$

Proof. By hypothesis, for each $i \in I$ there exists $j(i) \in J$ such that

$$
a_{i, j(i)} \geq \frac{\delta}{|J|}
$$

Let $J^{\prime}=\{j(i): i \in I\}$ and for each $j \in J^{\prime}$, consider $I_{j}=\{i \in I: j(i)=j\}$. Since $I=\bigcup_{j \in J^{\prime}} I_{j}$ is infinite and $J^{\prime}$ is finite, by Lemma 2.3 there exists $j_{0} \in J^{\prime}$ such that $\left|I_{j_{0}}\right|=|I|$. Setting $I^{\prime}=I_{j_{0}}$, the proof is complete.
3. Spaces $s \ell_{p}(I, X), F_{p}(I, X)$ and $\ell_{p}\langle I, X\rangle$

In this section we shall introduce some generalized sequence spaces which will be of interest.

Let $X$ be a Banach space, $I$ be an infinite set and $p \in[1, \infty)$. Following [21], we denote

$$
s \ell_{p}(I, X)=\left\{\left(x_{i}\right)_{i \in I} \in X^{I}: \sum_{i \in I}\left|x^{*}\left(x_{i}\right)\right|^{p}<\infty, \text { for all } x^{*} \in X^{*}\right\}
$$

It is straightforward to check that the function

$$
\|\bar{x}\|_{s \ell_{p}}=\sup \left\{\left[\sum_{i \in I}\left|x^{*}\left(x_{i}\right)\right|^{p}\right]^{\frac{1}{p}}: x^{*} \in B_{X^{*}}\right\}
$$

where $\bar{x}=\left(x_{i}\right)_{i \in I} \in s \ell_{p}(I, X)$, is a complete norm on $s \ell_{p}(I, X)$. For each $i \in I$, we denote by $R_{i}: X \rightarrow X^{I}$ the canonical inclusion defined by $R_{i}(x)=\left(\delta_{i k} x\right)_{k \in I}$, for each $x \in X$. Our interest lies in the closed subspace

$$
F_{p}(I, X)=\left\{\bar{x}=\left(x_{i}\right)_{i \in I} \in s \ell_{p}(I, X): \bar{x}=\sum_{i \in I} R_{i}\left(x_{i}\right)\right\}
$$

of $s \ell_{p}(I, X)$.
Given $p \in(1, \infty)$ and $q \in(1, \infty)$ the conjugate index of $p$, there exists a canonical isometric isomorphism between $s \ell_{p}(I, X)$ and $\mathcal{L}\left(\ell_{q}(I), X\right)$ which maps an
element $\bar{x}=\left(x_{i}\right)_{i \in I} \in s \ell_{p}(I, X)$ to the bounded linear operator $S \in \mathcal{L}\left(\ell_{q}(I), X\right)$ defined by

$$
S(\bar{a})=\sum_{i \in I} a_{i} x_{i}
$$

for all $\bar{a}=\left(a_{i}\right)_{i \in I} \in \ell_{q}(I)$. This isometry identifies $F_{p}(I, X)$ and the subspace $\mathcal{K}\left(\ell_{q}(I), X\right)$ of compact linear operators from $\ell_{q}(I)$ to $X$.

In a similar way, in the case $p=1$ there exists an isometric isomorphism between $s l_{1}(I, X)$ and $\mathcal{L}\left(c_{0}(I), X\right)$ which identifies $F_{1}(I, X)$ and $\mathcal{K}\left(c_{0}(I), X\right)$.

Next, let $p \in(1, \infty)$ and $q \in(1, \infty)$ be the conjugate index of $p$. Following $[\mathbf{1}, \mathbf{5}]$, we denote

$$
\ell_{p}\langle I, X\rangle=\left\{\bar{x}=\left(x_{i}\right)_{i \in I} \in X^{I}: \sum_{i \in I}\left|x_{i}^{*}\left(x_{i}\right)\right|<\infty, \text { for all }\left(x_{i}^{*}\right)_{i \in I} \in s \ell_{q}\left(I, X^{*}\right)\right\}
$$

It is straightforward to check that the function

$$
\|\bar{x}\|_{\ell_{p}\langle X\rangle}=\sup \left\{\sum_{i \in I}\left|x_{i}^{*}\left(x_{i}\right)\right|: \bar{x}^{*}=\left(x_{i}^{*}\right)_{i \in I} \in B_{s \ell_{q}\left(I, X^{*}\right)}\right\}
$$

where $\bar{x}=\left(x_{i}\right)_{i \in I} \in \ell_{p}\langle I, X\rangle$, is a complete norm on $\ell_{p}\langle I, X\rangle$.

We will summarize the main properties of those spaces in our next results. Our statements and proofs are simple modifications of those found in $[\mathbf{1}, \mathbf{2}, \mathbf{5}]$.

Proposition 3.1. Let $X$ be a Banach space, $I$ be an infinite set and $p \in[1, \infty)$.
(1) $\ell_{p}(I, X) \subset F_{p}(I, X)$ and $\|\bar{x}\|_{s \ell_{p}} \leq\|\bar{x}\|_{p}$, for all $\bar{x} \in \ell_{p}(I, X)$;
(2) For $p>1, \ell_{p}\langle I, X\rangle \subset \ell_{p}(I, X)$ and $\|\bar{x}\|_{p} \leq\|\bar{x}\|_{\ell_{p}\langle X\rangle}$, for all $\bar{x} \in \ell_{p}\langle I, X\rangle$.

Proof. Statement (1) follows immediately from the definitions of $\ell_{p}(I, X)$ and $s \ell_{p}(I, X)$. Let us show (2). Given $\bar{x}=\left(x_{i}\right)_{i \in I} \in \ell_{p}\langle I, X\rangle$, we fix, for every $i \in I, x_{i}^{*} \in B_{X^{*}}$ such that $\left\|x_{i}\right\|=x_{i}^{*}\left(x_{i}\right)$. For every $\left(\lambda_{i}\right)_{i \in I} \in B_{\ell_{q}(I)}$ we have $\left(\lambda_{i} x_{i}^{*}\right)_{i \in I} \in B_{s l_{q}\left(X^{*}\right)}$ and so

$$
\sum_{i \in I}\left|\lambda_{i} x_{i}^{*}\left(x_{i}\right)\right|=\sum_{i \in I}\left|\lambda_{i}\right|\left\|x_{i}\right\| \leq\|\bar{x}\|_{\ell_{p}\langle X\rangle}
$$

Therefore

$$
\left(\sum_{i \in I}\left\|x_{i}\right\|^{p}\right)^{\frac{1}{p}}=\|\bar{x}\|_{p} \leq\|\bar{x}\|_{\ell_{p}\langle X\rangle}
$$

and the proof is complete.
We will denote the canonical inclusion of $x \in X$ in $X^{* *}$ by $\widehat{x}$. The proof of the following proposition is straightforward.

Proposition 3.2. Let $X$ be a Banach space, $I$ be an infinite set, $p \in[1, \infty)$, $\bar{x}=\left(x_{i}\right)_{i \in I} \in X^{I}$ and $\bar{y}=\left(\widehat{x_{i}}\right)_{i \in I} \in\left(X^{* *}\right)^{I}$. Then $\bar{x} \in s \ell_{p}(I, X)$ if, and only if, $\bar{y} \in \operatorname{s\ell _{p}}\left(I, X^{* *}\right)$. Furthermore, $\|\bar{x}\|_{s \ell_{p}}=\|\bar{y}\|_{s \ell_{p}}$.

## 4. Generalized sequence spaces and tensor products

The goal of this section is to establish a connection between the generalized sequence spaces defined in the previous section and the projective and injective tensor products $\ell_{p}(I) \widehat{\otimes}_{\pi} X$ and $\ell_{p}(I) \widehat{\otimes}_{\varepsilon} X$.

It is obvious that for every $\bar{a}=\left(a_{i}\right)_{i \in I} \in \mathbb{R}^{I}$ and $x \in X$ we have $\left(a_{i} x\right)_{i \in I} \in X^{I}$ and furthermore the operator $\ell_{p}(I) \times X \ni(\bar{a}, x) \mapsto\left(a_{i} x\right)_{i \in I} \in X^{I}$ is bilinear. This bilinear operator induces a linear operator $T: \ell_{p}(I) \otimes X \rightarrow X^{I}[\mathbf{2 0}$, Proposition 1.4]. It is also obvious that $\operatorname{Im}(T) \subset \ell_{p}\langle I, X\rangle \cap s \ell_{p}(I, X)$. We summarize the main properties of this operator in the following proposition.

Proposition 4.1. Given $p \in[1, \infty)$, we have:
(1) $\operatorname{Im}(T)$ contains all finitely non-zero families in $\ell_{p}(I, X)$;
(2) $\|T(u)\|_{s \ell_{p}(X)}=\|u\|_{\varepsilon}$, for each $u \in \ell_{p}(I) \otimes X$;
(3) If $p \in(1, \infty)$, then $\|T(u)\|_{\ell_{p}\langle X\rangle} \leq\|u\|_{\pi}$, for each $u \in \ell_{p}(I) \otimes X$.

Proof. Given $x \in X$ and $i \in I$, notice that $T\left(e_{i} \otimes x\right)=\left(\delta_{i j} x\right)_{j \in I}$. This shows statement (1).

Next, let $u \in \ell_{p}(I) \otimes X$ be given and fix a representation $u=\sum_{n=1}^{m} a_{n} \otimes x_{n}$, where $a_{n}=\left(a_{i}^{n}\right)_{i \in I}$. Then we have

$$
\begin{aligned}
\|T(u)\|_{s \ell_{p}(X)} & =\sup \left\{\left[\sum_{i \in I}\left|\sum_{n=1}^{m} a_{i}^{n} x^{*}\left(x_{n}\right)\right|^{p}\right]^{\frac{1}{p}}: x^{*} \in B_{X^{*}}\right\} \\
& =\sup \left\{\left\|\sum_{n=1}^{m} x^{*}\left(x_{n}\right) a_{n}\right\|_{p}: x^{*} \in B_{X^{*}}\right\}=\|u\|_{\varepsilon} .
\end{aligned}
$$

This proves (2).
Finally, suppose that $p \in(1, \infty)$ and let us show (3). Let $q \in(1, \infty)$ be the conjugate index of $p$. Given $\bar{x}^{*}=\left(x_{i}^{*}\right)_{i \in I} \in B_{s \ell_{q}\left(I, X^{*}\right)}, a=\left(a_{i}\right)_{i \in I} \in \ell_{p}(I)$, and $x \in X$, by Hölder's Inequality we have

$$
\sum_{i \in I}\left|a_{i} x_{i}^{*}(x)\right| \leq\left(\sum_{i \in I}\left|a_{i}\right|^{p}\right)^{\frac{1}{p}}\left(\sum_{i \in I}\left|x_{i}^{*}(x)\right|^{q}\right)^{\frac{1}{q}} \leq\|a\|_{p}\|x\|
$$

Hence, the bilinear operator $\ell_{p}(I) \times X \ni(a, x) \mapsto\left(a_{i} x\right)_{i \in I} \in \ell_{p}\langle I, X\rangle$ has norm at most 1. By [20, Theorem 2.9], we obtain

$$
\|T(u)\|_{\ell_{p}\langle X\rangle} \leq\|u\|_{\pi}
$$

for every $u \in \ell_{p}(I) \otimes X$, as desired.
Corollary 4.2. Let $X$ be a Banach space, $I$ be an infinite set and $p \in[1, \infty)$. Let $T: \ell_{p}(I) \otimes X \rightarrow X^{I}$ be the linear map defined on Proposition 4.1.
(1) $T$ admits an unique linear extension $T_{\varepsilon}$ to $\ell_{p}(I) \widehat{\otimes}_{\varepsilon} X$, which is an isometry onto $F_{p}(I, X)$;
(2) If $p \in(1, \infty)$, then $T$ admits an unique linear extension of same norm $T_{\pi}$ from $\ell_{p}(I) \widehat{\otimes}_{\pi} X$ to $\ell_{p}\langle I, X\rangle$.
Proof. Let us first show (1). By the density of $\ell_{p}(I) \otimes X$ in $\ell_{p}(I) \widehat{\otimes}_{\varepsilon} X$ and Proposition 4.1, $T$ admits an unique linear extension $T_{\varepsilon}: \ell_{p}(I) \widehat{\otimes}_{\varepsilon} X \rightarrow F_{p}(I, X)$,

WHEN IS $c_{0}(\tau)$ COMPLEMENTED IN TENSOR PRODUCTS OF $\ell_{p}(I)$ AND $X$ ?
which is an isometry onto its image. Given $\bar{x}=\left(x_{i}\right)_{i \in I} \in F_{p}(I, X)$, by Proposition 4.1 we have

$$
x=\sum_{i \in I} R_{i}\left(x_{i}\right)=\sum_{i \in I} T\left(e_{i} \otimes x_{i}\right) \in \overline{\operatorname{Im}(T)}=\operatorname{Im}\left(T_{\varepsilon}\right) .
$$

This proves that $T_{\varepsilon}$ is onto $F_{p}(I, X)$.
Similarly, if $p \in(1, \infty)$, the density of $\ell_{p}(I) \otimes X$ in $\ell_{p}(I) \widehat{\otimes}_{\pi} X$ and Proposition 4.1 imply that $T$ admits an unique linear extension $T_{\pi}: \ell_{p}(I) \widehat{\otimes}_{\pi} X \rightarrow \ell_{p}\langle I, X\rangle$ satisfying $\left\|T_{\pi}\right\|=\|T\| \leq 1$.

As in the countable case, we will show that $T_{\pi}$ is a linear isometry from $\ell_{p}(I) \widehat{\otimes}_{\pi} X$ onto $\ell_{p}\langle I, X\rangle$. We will use the following key lemma.

Lemma 4.3. Let $X$ be a Banach space, $I$ be an infinite set and $p \in(1, \infty)$. Given $u \in \ell_{p}(I) \widehat{\otimes}_{\pi} X$, there exists a family $\left(x_{i}\right)_{i \in I}$ in $X$ such that

$$
u=\sum_{i \in I} e_{i} \otimes x_{i}
$$

Proof. Let $q \in(1, \infty)$ be the conjugate index of $p$ and $\left(e_{i}^{*}\right)_{i \in I}$ be the unit basis of $\ell_{q}(I)$. For each $i \in I$ consider the bounded linear operator $\pi_{i}: \ell_{p}(I) \rightarrow \ell_{p}(I)$ defined by

$$
\pi_{i}(a)=e_{i}^{*}(a) e_{i}
$$

for each $a=\left(a_{j}\right)_{j \in I} \in \ell_{p}(I)$, and for every finite subset $\varnothing \neq F \subset I$ write

$$
P_{F}=\sum_{j \in F} \pi_{j}
$$

Let $u \in \ell_{p}(I) \widehat{\otimes}_{\pi} X$ be given and let us prove that

$$
\begin{equation*}
u=\sum_{i \in I}\left(\pi_{i} \otimes I\right)(u) \tag{4.1}
\end{equation*}
$$

By [20, Proposition 2.8] there exist bounded sequences $\left(a_{n}\right)_{n \geq 1}$ in $\ell_{p}(I)$ and $\left(y_{n}\right)_{n \geq 1}$ in $X$ satisfying

$$
u=\sum_{n=1}^{\infty} a_{n} \otimes y_{n} \text { and } \sum_{n=1}^{\infty}\left\|a_{n}\right\|_{p}\left\|y_{n}\right\|<\infty
$$

We may suppose that $\left\|y_{n}\right\|=1$ for every $n \geq 1$. Given $\varepsilon>0$, take $N \geq 1$ such that

$$
\sum_{n=N+1}^{\infty}\left\|a_{n}\right\|_{p} \leq \frac{\varepsilon}{3}
$$

It is clear that

$$
\left\|u-\sum_{n=1}^{N} a_{n} \otimes y_{n}\right\|_{\pi} \leq \frac{\varepsilon}{3}
$$

There exists a finite subset $\varnothing \neq F_{0} \subset I$ such that for every finite subset $F \subset I$ satisfying $F_{0} \subset F$ and every $1 \leq n \leq N$ we have

$$
\left\|P_{F}\left(a_{n}\right)-a_{n}\right\|_{p} \leq \frac{\varepsilon}{3 N}
$$

Thus,

$$
\begin{aligned}
\left\|u-\left(P_{F} \otimes I\right)(u)\right\| \leq & \left\|u-\sum_{n=1}^{N} a_{n} \otimes y_{n}\right\|_{\pi} \\
& +\left\|\sum_{n=1}^{N} a_{n} \otimes y_{n}-\sum_{n=1}^{N} P_{F}\left(a_{n}\right) \otimes y_{n}\right\|_{\pi} \\
& +\left\|\sum_{n=1}^{N} P_{F}\left(a_{n}\right) \otimes y_{n}-\left(P_{F} \otimes I\right)(u)\right\|_{\pi} \leq \varepsilon .
\end{aligned}
$$

This shows that (4.1) holds.
Finally, since

$$
\left(\pi_{i} \otimes I\right)(u)=\sum_{n=1}^{\infty} \pi_{i}\left(a_{n}\right) \otimes y_{n}=\sum_{n=1}^{\infty} e_{i}^{*}\left(a_{n}\right) e_{i} \otimes y_{n}=e_{i} \otimes\left(\sum_{n=1}^{\infty} e_{i}^{*}\left(a_{n}\right) y_{n}\right),
$$

for each $i \in I$, it follows that

$$
x_{i}=\sum_{n=1}^{\infty} e_{i}^{*}\left(a_{n}\right) y_{n}
$$

satisfies the desired properties.
We are now in a position to show that $T_{\pi}$ is an isometry onto $\ell_{p}\langle I, X\rangle$.
Theorem 4.4. Let $X$ be a Banach space, I be an infinite set and $p \in(1, \infty)$. The operator $T_{\pi}$ is an isometric isomorphism from $\ell_{p}(I) \widehat{\otimes}_{\pi} X$ onto $\ell_{p}\langle I, X\rangle$.

Proof. We have shown that

$$
\left\|T_{\pi}(u)\right\|_{\ell_{p}\langle X\rangle} \leq\|u\|_{\pi},
$$

for every $u \in \ell_{p}(I) \widehat{\otimes}_{\pi} X$. In order to prove that $T_{\pi}$ is isometric, it is sufficient to prove the reverse inequality on $\operatorname{span}\left\{e_{i} \otimes x: i \in I, x \in X\right\}$. Given $v \in \operatorname{span}\left\{e_{i} \otimes x\right.$ : $i \in I, x \in X\}$, there exist a finite subset $\varnothing \neq I_{0} \subset I$ and a finite family $\left(y_{i}\right)_{i \in I_{0}}$ of elements of $X$ such that

$$
v=\sum_{i \in I_{0}} e_{i} \otimes y_{i} .
$$

By [20, p. 24] and the Hahn-Banach Theorem, there exists a bounded linear operator $b \in \mathcal{L}\left(\ell_{p}(I), X^{*}\right) \equiv\left[\ell_{p}(I) \widehat{\otimes}_{\pi} X\right]^{*}$ satisfying

$$
\|b\|=1 \text { and }\|v\|_{\pi}=\sum_{i \in I_{0}} b\left(e_{i}\right)\left(y_{i}\right) .
$$

Setting $y_{i}=0$ for each $i \in I \backslash I_{0}$, it is clear that $T_{\pi}(v)=\left(y_{i}\right)_{i \in I}$ and

$$
\left\|T_{\pi}(v)\right\|_{\ell_{p}\langle X\rangle} \geq\|v\|_{\pi},
$$

as desired.
Let us show next that $T_{\pi}$ is onto. Given $\bar{x}=\left(x_{i}\right)_{i \in I} \in \ell_{p}\langle I, X\rangle$, we will show that the family $\left(e_{i} \otimes x_{i}\right)_{i \in I}$ is summable in $\ell_{p}(I) \widehat{\otimes}_{\pi} X$ and

$$
\begin{equation*}
\bar{x}=T_{\pi}\left(\sum_{i \in I} e_{i} \otimes x_{i}\right) . \tag{4.2}
\end{equation*}
$$

For every $\bar{x}^{*}=\left(x_{i}^{*}\right)_{i \in I} \in s \ell_{q}\left(I, X^{*}\right)$ we have

$$
\left|\sum_{i \in I} x_{i}^{*}\left(x_{i}\right)\right| \leq\|\bar{x}\|_{\ell_{p}\langle X\rangle}\left\|\bar{x}^{*}\right\|_{s l_{q}\left(X^{*}\right)}
$$

Hence, the linear functional $\theta$ defined on $s l_{q}\left(I, X^{*}\right) \equiv \mathcal{L}\left(\ell_{p}(I), X^{*}\right)$ by

$$
\theta\left(\bar{x}^{*}\right)=\sum_{i \in I} x_{i}^{*}\left(x_{i}\right)
$$

for each $\bar{x}^{*}=\left(x_{i}^{*}\right)_{i \in I} \in s \ell_{q}\left(I, X^{*}\right)$, is bounded. Let us denote by $\theta_{r}$ its restriction to the subspace $\mathcal{K}\left(\ell_{p}(I), X^{*}\right)$.

Let $q \in(1, \infty)$ be the conjugate index of $p$. Recall that the space $\ell_{q}(I)$ has the approximation property $[\mathbf{2 0}$, p. 73] , and therefore

$$
\begin{equation*}
\mathcal{K}\left(\ell_{p}(I), X^{*}\right) \equiv \ell_{q}(I) \widehat{\otimes}_{\varepsilon} X^{*} \tag{4.3}
\end{equation*}
$$

by [20, Corollary 4.13]. Moreover, since $\ell_{p}(I)$ also has the Radon-Nikodým property [20, Corollary 5.45], by (4.3) and [20, Theorem 5.33] we know that

$$
\ell_{p}(I) \widehat{\otimes}_{\pi} X^{* *} \equiv\left[\ell_{q}(I) \widehat{\otimes}_{\varepsilon} X^{*}\right]^{*} \equiv\left[\mathcal{K}\left(\ell_{p}(I), X^{*}\right)\right]^{*}
$$

It follows from Lemma 4.3 that there exists a family $\left(x_{i}^{* *}\right)_{i \in I}$ in $X^{* *}$ such that

$$
\theta_{r}=\sum_{i \in I} e_{i} \otimes x_{i}^{* *}
$$

We observe that for every $i \in I$ and for every $x^{*} \in X^{*}$ we have

$$
x_{i}^{* *}\left(x^{*}\right)=\theta_{r}\left(e_{i}^{*} \otimes x^{*}\right)=x^{*}\left(x_{i}\right)
$$

and so $x_{i}^{* *}=\widehat{x_{i}}$. Thus, the family $\left(e_{i} \otimes \widehat{x_{i}}\right)_{i \in I}$ is summable in $\ell_{p}(I) \widehat{\otimes}_{\pi} X^{* *}$ and hence, the family $\left(e_{i} \otimes x_{i}\right)_{i \in I}$ in $\ell_{p}(I) \widehat{\otimes}_{\pi} X$, by [8, Corollary 14, p. 238]. Finally, it is clear that (4.2) holds.

REmARK 4.5. We observe that $T_{\pi}\left(e_{i} \otimes x_{i}\right)=R_{i}\left(x_{i}\right)$, for each $i \in I$. Therefore, for each $\bar{x}=\left(x_{i}\right)_{i \in I} \in \ell_{p}\langle I, X\rangle$, the family $\left(R_{i}\left(x_{i}\right)\right)_{i \in I}$ is summable in $\ell_{p}\langle I, X\rangle$ and

$$
\bar{x}=\sum_{i \in I} R_{i}\left(x_{i}\right)
$$

Remark 4.6. By Proposition 3.1, every family $\bar{x}=\left(x_{i}\right)_{i \in I} \in \ell_{p}\langle I, X\rangle$ has only countably many non-zero coordinates. Hence, by the previous remark, the subspace

$$
L=\left\{\bar{y}=\left(y_{i}\right)_{i \in I} \in \ell_{p}\langle I, X\rangle:\left\{i \in I: y_{i}=0\right\} \text { is finite }\right\}
$$

is dense in $\ell_{p}\langle I, X\rangle$.

## 5. Dual spaces of $\ell_{p}\langle I, X\rangle$ and $F_{p}(I, X)$

We now turn our attention to convenient identifications of the spaces $\ell_{p}\langle I, X\rangle$ and $F_{p}(I, X)$ and of their duals.

Theorem 5.1. Let $X$ be a Banach space, $I$ be an infinite set, $p \in(1, \infty)$ and $q \in(1, \infty)$ be the conjugate index of $p$. Then there exists an isometric isomorphism from $\ell_{p}\langle I, X\rangle^{*}$ onto $s \ell_{q}\left(I, X^{*}\right)$ which maps $\varphi \in \ell_{p}\langle I, X\rangle^{*}$ to $\left(x_{i}^{*}\right)_{i \in I} \in s \ell_{q}\left(I, X^{*}\right)$, where

$$
\varphi(\bar{x})=\sum_{i \in I} x_{i}^{*}\left(x_{i}\right)
$$

for every $\bar{x}=\left(x_{i}\right)_{i \in I} \in \ell_{p}\langle I, X\rangle$.

Proof. Let $T_{\pi}$ be the canonical isometric isomorphism from $\ell_{p}(I) \widehat{\otimes}_{\pi} X$ onto $\ell_{p}\langle I, X\rangle$ considered in Corollary 4.2. Then $T_{\pi}^{*}$ is also an isometric isomorphism from $\ell_{p}\langle I, X\rangle^{*}$ onto $\left[\ell_{p}(I) \widehat{\otimes}_{\pi} X\right]^{*}$. By [20, p. 24], we have

$$
\left[\ell_{p}(I) \widehat{\otimes}_{\pi} X\right]^{*} \equiv \mathcal{L}\left(\ell_{p}(I), X^{*}\right) \equiv s \ell_{q}\left(I, X^{*}\right)
$$

Given $\varphi \in \ell_{p}\langle I, X\rangle^{*}$, let $\left(x_{i}^{*}\right)_{i \in I} \in s \ell_{q}\left(I, X^{*}\right)$ be the unique element corresponding to $T_{\pi}^{*}(\varphi) \in\left[\ell_{p}(I) \widehat{\otimes}_{\pi} X\right]^{*}$ via the above identifications. By Remark 4.5, for each

$$
\bar{x}=\left(x_{i}\right)_{i \in I}=\sum_{i \in I} R_{i}\left(x_{i}\right)=\sum_{i \in I} T_{\pi}\left(e_{i} \otimes x_{i}\right) \in \ell_{p}\langle I, X\rangle,
$$

we have

$$
\varphi(\bar{x})=\sum_{i \in I}\left(\varphi \circ T_{\pi}\right)\left(e_{i} \otimes x_{i}\right)=\sum_{i \in I} T_{\pi}^{*}(\varphi)\left(e_{i} \otimes x_{i}\right)=\sum_{j \in I} x_{i}^{*}\left(x_{i}\right),
$$

and the proof is complete.
Remark 5.2. For every $\varphi=\left(x_{i}^{*}\right)_{i \in I} \in \ell_{p}\langle I, X\rangle^{*}$, we have

$$
\|\varphi\|=\sup \left\{\sum_{i \in I}\left|x_{i}^{*}\left(x_{i}\right)\right|:\left(x_{i}\right)_{i \in I} \in B_{\ell_{p}\langle X\rangle}\right\} .
$$

ThEOREM 5.3. Let $X$ be a Banach space, $I$ be an infinite set, $p \in(1, \infty)$ and $q \in(1, \infty)$ be the conjugate index of $p$. Then there exists an isometric isomorphism from $F_{p}(I, X)^{*}$ onto $\ell_{q}\left\langle I, X^{*}\right\rangle$ which maps $\psi \in F_{p}(I, X)^{*}$ to $\left(x_{i}^{*}\right)_{i \in I} \in \ell_{q}\left\langle I, X^{*}\right\rangle$, where

$$
\psi(\bar{x})=\sum_{i \in I} x_{i}^{*}\left(x_{i}\right)
$$

for every $\bar{x}=\left(x_{i}\right)_{i \in I} \in F_{p}(I, X)$.
Proof. Let $T_{\varepsilon}$ be the canonical isometric isomorphism from $\ell_{p}(I) \widehat{\otimes}_{\varepsilon} X$ onto $F_{p}(I, X)$ considered in Corollary 4.2. Since the space $\ell_{q}(I)$ has both the approximation property and the Radon-Nikodým property, by [20, Theorem 5.33] we know that

$$
\left[\ell_{p}(I) \widehat{\otimes}_{\varepsilon} X\right]^{*} \equiv \ell_{q}(I) \widehat{\otimes}_{\pi} X^{*}
$$

By Theorem 4.4 and Remark 4.5, the operator

$$
\ell_{q}\left\langle I, X^{*}\right\rangle \ni\left(x_{i}^{*}\right)_{i \in I} \mapsto \sum_{i \in I} e_{i} \otimes x_{i}^{*} \in \ell_{q}(I) \widehat{\otimes}_{\pi} X^{*}
$$

is also an isometric isomorphism.
Given $\psi \in F_{p}(I, X)^{*}$, there exists an unique $\left(x_{i}^{*}\right)_{i \in I} \in \ell_{q}\left\langle I, X^{*}\right\rangle$ corresponding to $T_{\varepsilon}^{*}(\varphi) \in\left[\ell_{p}(I) \widehat{\otimes}_{\varepsilon} X\right]^{*}$ via the above identifications. For each

$$
\bar{x}=\left(x_{i}\right)_{i \in I}=\sum_{i \in I} R_{i}\left(x_{i}\right)=\sum_{i \in I} T_{\varepsilon}\left(e_{i} \otimes x_{i}\right) \in F_{p}(I, X),
$$

we have

$$
\psi(\bar{x})=\sum_{i \in I}\left(\psi \circ T_{\varepsilon}\right)\left(e_{i} \otimes x_{i}\right)=\sum_{i \in I} T_{\varepsilon}^{*}(\psi)\left(e_{i} \otimes x_{i}\right)=\sum_{j \in I} x_{i}^{*}\left(x_{i}\right),
$$

and the proof is complete.
6. Complemented copies of $c_{0}(\tau)$ in $\ell_{p}(I) \widehat{\otimes}_{\pi} X$ and $\ell_{p}(I) \widehat{\otimes}_{\varepsilon} X$

This section is devoted to the study of the complemented copies of $c_{0}(\tau)$ in the projective and injective tensor products $\ell_{p}(I) \widehat{\otimes}_{\pi} X$ and $\ell_{p}(I) \widehat{\otimes}_{\varepsilon} X$. We begin with the former space.

Lemma 6.1. Let $X$ be a Banach space, $I$ be an infinite set, $p \in(1, \infty)$ and $\tau$ be an infinite cardinal. Let $\left(\overline{x_{j}}\right)_{j \in \tau}=\left(\left(x_{i, j}\right)_{i \in I}\right)_{j \in \tau}$ be a family of $\ell_{p}\langle I, X\rangle$ equivalent to the unit basis of $c_{0}(\tau)$ and let $\left(\varphi_{j}\right)_{j \in \tau}=\left(\left(\varphi_{i, j}\right)_{i \in I}\right)_{j \in \tau}$ be a bounded family of $\ell_{p}\langle I, X\rangle^{*}$. Then, for each $\varepsilon>0$, there exists a finite subset $F_{\varepsilon} \subset I$ satisfying, for every $j \in \tau$,

$$
\left|\sum_{i \in I \backslash F_{\varepsilon}} \varphi_{i, j}\left(x_{i, j}\right)\right|<\varepsilon
$$

Proof. Let $M=\sup _{j \in \tau}\left\|\varphi_{j}\right\|$. We may assume $\left\|\overline{x_{j}}\right\|_{\ell_{p}\langle X\rangle}=1$ for every $j \in \tau$. Suppose that the conclusion does not hold; then there exists $\varepsilon>0$ such that, for every finite subset $F \subset I$, there exists $j \in \tau$ with

$$
\left|\sum_{i \notin F} \varphi_{i, j}\left(x_{i, j}\right)\right| \geq \varepsilon
$$

We will show that this leads to a contradiction.
We construct by induction a sequence $\left(j_{k}\right)_{k \geq 1}$ in $\tau$ and a sequence of pairwise disjoint finite subsets $\left(F_{k}\right)_{k \geq 1}$ of $I$ such that

$$
\begin{equation*}
\left|\sum_{i \in F_{k}} \varphi_{i, j_{k}}\left(x_{i, j_{k}}\right)\right| \geq \frac{\varepsilon}{2}, \forall k \geq 1 \tag{6.1}
\end{equation*}
$$

Put $F_{0}=\varnothing$. By hypothesis, there exists $j_{1} \in \tau$ such that

$$
\left|\sum_{i \in I} \varphi_{i, j_{1}}\left(x_{i, j_{1}}\right)\right| \geq \varepsilon
$$

Since

$$
\sum_{i \in I}\left|\varphi_{i, j_{1}}\left(x_{i, j_{1}}\right)\right| \leq M
$$

there exists a finite subset $\varnothing \neq F_{1} \subset I$ such that

$$
\sum_{i \in I \backslash F_{1}}\left|\varphi_{i, j_{1}}\left(x_{i, j_{1}}\right)\right| \leq \frac{\varepsilon}{2}
$$

and hence

$$
\sum_{i \in F_{1}}\left|\varphi_{i, j_{1}}\left(x_{i, j_{1}}\right)\right| \geq \frac{\varepsilon}{2}
$$

Next, let $k \geq 1$ be given and suppose we have constructed $j_{1}, \ldots, j_{k}$ and $F_{1}, \ldots, F_{k}$ as desired. Put $A=\cup_{l=1}^{k} F_{l}$. By hypothesis, there exists $j_{k+1} \in \tau$ such that

$$
\left|\sum_{i \in I \backslash A} \varphi_{i, j_{k+1}}\left(x_{i, j_{k+1}}\right)\right| \geq \varepsilon
$$

There exists a finite subset $\varnothing \neq F_{k+1} \subset(I \backslash A)$ such that

$$
\sum_{i \in I \backslash\left(A \cup F_{k+1}\right)}\left|\varphi_{i, j_{k+1}}\left(x_{i, j_{k+1}}\right)\right| \leq \frac{\varepsilon}{2}
$$

and so

$$
\left|\sum_{i \in F_{k+1}} \varphi_{i, j_{k+1}}\left(x_{i, j_{k+1}}\right)\right| \geq \frac{\varepsilon}{2}
$$

It follows from the construction that $j_{l} \neq j_{k}$ for $l \neq k$.
For each $k \geq 1$ and $i \in I$ we denote

$$
\psi_{i, k}= \begin{cases}\varphi_{i, j_{k}} & \text { if } i \in F_{k} \\ 0 & \text { if } i \in I \backslash F_{k}\end{cases}
$$

and $\psi_{k}=\left(\psi_{i, k}\right)_{i \in I}$. It is obvious that $\left(\psi_{k}\right)_{k \geq 1}$ is a bounded sequence in $F_{q}\left(I, X^{*}\right) \equiv$ $\ell_{p}\left\langle I, X^{* *}\right\rangle$, where $q \in(1, \infty)$ is the conjugate index of $p$. We claim that $\left(\psi_{k}\right)_{k \geq 1}$ is a weakly-null sequence. Indeed, by Remark 4.6, the subspace

$$
L=\left\{\theta^{* *}=\left(\theta_{i}^{* *}\right)_{i \in I} \in \ell_{p}\left\langle I, X^{* *}\right\rangle:\left\{i \in I: \theta_{i}^{* *}=0\right\} \text { is finite }\right\}
$$

is dense in $\ell_{p}\left\langle I, X^{* *}\right\rangle$. Given $\theta^{* *}=\left(\theta_{i}^{* *}\right)_{i \in I} \in L$, there exists $k_{0} \geq 1$ such that $\theta_{i}^{* *}=0$, for all $i \in F_{k}, k \geq k_{0}$, and hence,

$$
\lim _{k \rightarrow \infty} \theta^{* *}\left(\psi_{k}\right)=\lim _{k \rightarrow \infty} \sum_{i \in F_{k}} \theta_{i}^{* *}\left(\psi_{i, k}\right)=0
$$

The density of $L$ then establishes our claim.
The sequence $\left(\overline{x_{j_{k}}}\right)_{k \geq 1}$ is equivalent to the unit basis of $c_{0}$ and furthermore

$$
\left|\psi_{k}\left(\overline{x_{j_{k}}}\right)\right| \geq \frac{\varepsilon}{2}
$$

for each $k \geq 1$, in contradiction with the Dunford-Pettis Property of $c_{0}[\mathbf{1 2}$, p. 596].

We are now in a position to prove the first main result of this section.
Theorem 6.2. Given $X$ a Banach space, $I$ an infinite set, $p \in(1, \infty)$ and $\tau$ an infinite cardinal, we have

$$
c_{0}(\tau) \stackrel{c}{\hookrightarrow} \ell_{p}(I) \widehat{\otimes}_{\pi} X \Longleftrightarrow c_{0}(\tau) \stackrel{c}{\hookrightarrow} X .
$$

Proof. Suppose first that $c_{0}(\tau) \stackrel{c}{\hookrightarrow} \ell_{p}(I) \widehat{\otimes}_{\pi} X \equiv \ell_{p}\langle I, X\rangle$. By Theorem 2.2, there exist $\left(\overline{x_{j}}\right)_{j \in \tau}=\left(\left(x_{i, j}\right)_{i \in I}\right)_{j \in \tau}$ equivalent to the canonical basis of $c_{0}(\tau)$ in $\ell_{p}\langle I, X\rangle$, and $\left(\varphi_{j}\right)_{j \in \tau}$ a weak ${ }^{*}$-null family in $\ell_{p}\langle I, X\rangle^{*}$ satisfying $\varphi_{j}\left(x_{j}\right)=1$ for all $j \in \tau$. By Theorem 5.1, for each $j \in \tau$, there exists a family $\left(\varphi_{i, j}\right)_{i \in I}$ in $X^{*}$ such that

$$
1=\varphi_{j}\left(\overline{x_{j}}\right)=\sum_{i \in I} \varphi_{i, j}\left(x_{i, j}\right)
$$

By the Uniform Boundedness Principle, $\left(\varphi_{j}\right)_{j \in \tau}$ is bounded, and so there exists $M>0$ such that $\left\|\varphi_{j}\right\| \leq M$ for each $j \in \tau$.

An appeal to Lemma 6.1 yields a finite subset $F \subset I$ satisfying

$$
\left|\sum_{i \in I \backslash F} \varphi_{i, j}\left(x_{i, j}\right)\right|<\frac{1}{2}, \forall j \in \tau
$$

Therefore, $F \neq \varnothing$ and

$$
\sum_{i \in F}\left|\varphi_{i, j}\left(x_{i, j}\right)\right|>\frac{1}{2}, \forall j \in \tau
$$

Next, Lemma 2.4 implies that there exist $i_{0} \in F$ and $\tau_{1} \subset \tau$ satisfying $\left|\tau_{1}\right|=\tau$ and

$$
\begin{equation*}
\left|\varphi_{i_{0}, j}\left(x_{i_{0}, j}\right)\right|>\frac{1}{2|F|}, \forall j \in \tau_{1} \tag{6.2}
\end{equation*}
$$

We have

$$
\frac{1}{2|F|}<\left|\varphi_{i_{0}, j}\left(x_{i_{0}, j}\right)\right| \leq M\left\|x_{i_{0}, j}\right\|, \forall j \in \tau_{1}
$$

and thus

$$
\begin{equation*}
\left\|x_{i_{0}, j}\right\|>\frac{1}{2 M|F|}, \forall j \in \tau_{1} \tag{6.3}
\end{equation*}
$$

Consider the bounded linear operator $\pi_{i_{0}}: \ell_{p}\langle I, X\rangle \rightarrow X$ defined by $\pi_{i_{0}}(\bar{x})=$ $x_{i_{0}}$ for each $\bar{x}=\left(x_{i}\right)_{i \in I} \in \ell_{p}\langle I, X\rangle$. By hypothesis, there exists $T: c_{0}(\tau) \rightarrow \ell_{p}\langle I, X\rangle$ an isomorphism onto its image such that $T\left(e_{j}\right)=\overline{x_{j}}$, for all $j \in \tau$. By (6.3), we have

$$
\left\|\left(\pi_{i_{0}} \circ T\right)\left(e_{j}\right)\right\|>\frac{1}{2 M|F|}, \forall j \in \tau_{1}
$$

and therefore, by Theorem 2.1, there exists $\tau_{2} \subset \tau_{1}$ such that $\left|\tau_{2}\right|=\left|\tau_{1}\right|=\tau$ and $\pi_{i_{0}} \circ T_{\mid c_{0}\left(\tau_{2}\right)}$ is an isomorphism onto its image. Thus, $\left(x_{i_{0}, j}\right)_{j \in \tau_{2}}$ is equivalent to the unit basis of $c_{0}\left(\tau_{2}\right)$ in $X$. The family $\left(\varphi_{j}\right)_{j \in \tau}$ is weak*-null and hence, $\left(\varphi_{i_{0}, j}\right)_{j \in \tau_{2}}$ is a weak ${ }^{*}$-null family in $X^{*}$. Theorem 2.2 and (6.2) then imply that $c_{0}(\tau) \stackrel{c}{\hookrightarrow} X$.

For the converse, it is sufficient to notice that $\ell_{p}\langle I, X\rangle$ contains a complemented copy of $X$.

We now turn our attention to complemented copies of $c_{0}(\tau)$ in the injective tensor product $\ell_{p}(I) \widehat{\otimes}_{\varepsilon} X$.

Theorem 6.3. Let $X$ be a Banach space, $I$ be an infinite set, $p \in[1, \infty)$ and $\tau$ be an infinite cardinal. If $\operatorname{cf}(\tau)>|I|$, then

$$
c_{0}(\tau) \stackrel{c}{\hookrightarrow} \ell_{p}(I) \widehat{\otimes}_{\varepsilon} X \Longleftrightarrow c_{0}(\tau) \stackrel{c}{\hookrightarrow} X .
$$

Proof. Suppose first that $c_{0}(\tau) \stackrel{c}{\hookrightarrow} \ell_{p}(I) \widehat{\otimes}_{\pi} X \equiv F_{p}(I, X)$. By Theorem 2.2, there exists $\left(\overline{x_{j}}\right)_{j \in \tau}=\left(\left(x_{i, j}\right)_{i \in I}\right)_{j \in \tau}$ equivalent to the unit basis of $c_{0}(\tau)$ in $F_{p}(I, X)$, and a weak*-null family $\left(\varphi_{j}\right)_{j \in \tau}=\left(\left(\varphi_{i, j}\right)_{i \in I}\right)_{j \in \tau}$ of $F_{p}(I, X)^{*}$ satisfying

$$
1=\varphi_{j}\left(\overline{x_{j}}\right)=\sum_{i \in I} \varphi_{i, j}\left(x_{i, j}\right), \forall j \in \tau
$$

For each $j \in \tau$, there exists a finite subset $\varnothing \neq F_{j} \subset I$ such that

$$
\begin{equation*}
\sum_{i \in F_{j}}\left|\varphi_{i, j}\left(x_{i, j}\right)\right|>\frac{1}{2} \tag{6.4}
\end{equation*}
$$

Let $\mathcal{F}=\left\{F_{j}: j \in \tau\right\}$, and for each $F \in \mathcal{F}$, consider $\tau(F)=\left\{j \in \tau: F_{j}=F\right\}$. Since $\tau=\bigcup_{F \in \mathcal{F}} \tau_{F}$ and $\operatorname{cf}(\tau)>|I| \geq|\mathcal{F}|$, by Lemma 2.3 there exists $G_{0} \in \mathcal{F}$ such that $\left|\tau\left(G_{0}\right)\right|=\tau$. Setting $\tau_{1}=\tau\left(G_{0}\right)$, by (6.4) we obtain

$$
\sum_{i \in G_{0}}\left|\varphi_{i, j}\left(x_{i, j}\right)\right|>\frac{1}{2}, \forall j \in \tau_{1}
$$

Next, an appeal to Lemma 2.4 yields $i_{0} \in G_{0}$ and $\tau_{2} \subset \tau_{1}$ satisfying $\left|\tau_{2}\right|=\left|\tau_{1}\right|=\tau$ and

$$
\begin{equation*}
\left|\varphi_{i_{0}, j}\left(x_{i_{0}, j}\right)\right|>\frac{1}{2\left|G_{0}\right|}, \forall j \in \tau_{2} \tag{6.5}
\end{equation*}
$$

By the Uniform Boundedness Principle, $\left(\varphi_{j}\right)_{j \in \tau}$ is bounded, and so there exists $M>0$ such that $\left\|\varphi_{j}\right\| \leq M$, for each $j \in \tau$. Therefore we have

$$
\frac{1}{2\left|G_{0}\right|}<\left|\varphi_{i_{0}, j}\left(x_{i_{0}, j}\right)\right| \leq M\left\|x_{i_{0}, j}\right\|, \forall j \in \tau_{2}
$$

and thus

$$
\begin{equation*}
\left\|x_{i_{0}, j}\right\|>\frac{1}{2 M\left|G_{0}\right|}, \forall j \in \tau_{2} \tag{6.6}
\end{equation*}
$$

Consider the bounded linear operator $\pi_{i_{0}}: s \ell_{p}(I, X) \rightarrow X$ defined by $\pi_{i_{0}}(\bar{x})=$ $x_{i_{0}}$ for each $\bar{x}=\left(x_{i}\right)_{i \in I} \in s \ell_{p}(I, X)$. By hypothesis, there exists $T: c_{0}(\tau) \rightarrow$ $F_{p}(I, X)$ an isomorphism onto its image such that $T\left(e_{j}\right)=\overline{x_{j}}$, for all $i \in \tau$. By (6.6), we have

$$
\left\|\left(\pi_{i_{0}} \circ T\right)\left(e_{j}\right)\right\|>\frac{1}{2 M\left|G_{0}\right|}, \forall j \in \tau_{2}
$$

and therefore, by Theorem 2.1, there exists $\tau_{3} \subset \tau_{2}$ such that $\left|\tau_{3}\right|=\left|\tau_{2}\right|=\tau$ and $\pi_{i_{0}} \circ T_{\mid c_{0}\left(\tau_{3}\right)}$ is an isomorphism onto its image. Thus, $\left(x_{i_{0}, j}\right)_{j \in \tau_{3}}$ is equivalent to the unit basis of $c_{0}\left(\tau_{3}\right)$ in $X$. The family $\left(\varphi_{j}\right)_{j \in \tau}$ is weak*-null and hence, $\left(\varphi_{i_{0}, j}\right)_{j \in \tau_{3}}$ is weak*-null in $X^{*}$. Theorem 2.2 and (6.5) then imply $c_{0}(\tau) \stackrel{c}{\hookrightarrow} X$.

For the converse, it suffices to notice that $F_{p}(I, X)$ contains a complemented copy of $X$.

Our next step is to obtain a result analogue to Theorem 6.3 without assumptions on the cofinality of $\tau$. In this direction, we begin with the following lemma.

Lemma 6.4. Let $X$ be a Banach space, $I$ be an infinite set, $p \in(1, \infty)$ and $\tau$ be an infinite cardinal. Let $\left(\overline{x_{j}}\right)_{j \in \tau}=\left(\left(x_{i, j}\right)_{i \in I}\right)_{j \in \tau}$ be a family of $F_{p}(I, X)$ equivalent to the unit basis of $c_{0}(\tau)$ and let $\left(\varphi_{j}\right)_{j \in \tau}=\left(\left(\varphi_{i, j}\right)_{i \in I}\right)_{j \in \tau}$ be a bounded family of $F_{p}(I, X)^{*} \equiv \ell_{q}\left\langle I, X^{*}\right\rangle$, where $q \in(1, \infty)$ is the conjugate index of $p$. Suppose that

$$
\mathcal{K}\left(\ell_{q}(I), X^{* *}\right)=\mathcal{L}\left(\ell_{q}(I), X^{* *}\right)
$$

Then, for each $\varepsilon>0$, there exists a finite subset $F_{\varepsilon} \subset I$ satisfying

$$
\left|\sum_{i \in I \backslash F_{\varepsilon}} \varphi_{i, j}\left(x_{i, j}\right)\right|<\varepsilon, \forall j \in \tau
$$

Proof. By hypothesis and Theorem 5.3, there exists $M>0$ such that

$$
\begin{equation*}
M \geq\left\|\varphi_{j}\right\|=\left\|\left(\varphi_{i, j}\right)_{i \in I}\right\|_{\ell_{q}\left\langle X^{*}\right\rangle}, \forall j \in \tau \tag{6.7}
\end{equation*}
$$

Suppose by contradiction that the result does not hold. As in the proof of Lemma 6.1, there exist $\varepsilon>0$, a sequence of distinct indexes $\left(j_{k}\right)_{k \geq 1}$ of $\tau$, and a sequence $\left(F_{k}\right)_{k \geq 1}$ of finite, non-empty and pairwise disjoint subsets of $I$ satisfying

$$
\begin{equation*}
\left|\sum_{i \in F_{k}} \varphi_{i, j_{k}}\left(x_{i, j_{k}}\right)\right|>\varepsilon, \forall k \geq 1 \tag{6.8}
\end{equation*}
$$

Consider $\psi_{k}=\left(\psi_{i, k}\right)_{i \in I} \in \ell_{q}\left\langle I, X^{*}\right\rangle$, where

$$
\psi_{i, k}=\left\{\begin{array}{cl}
\varphi_{i, j_{k}} & \text { if } i \in F_{k} \\
0 & \text { if } i \in I \backslash F_{k} .
\end{array}\right.
$$

By (6.7) we have

$$
\begin{equation*}
\left\|\psi_{k}\right\|_{\ell_{q}\left\langle X^{*}\right\rangle} \leq\left\|\left(\varphi_{i, j_{k}}\right)_{i \in I}\right\|_{\ell_{q}\left\langle X^{*}\right\rangle}=\left\|\varphi_{j_{k}}\right\| \leq M, \forall k \geq 1 \tag{6.9}
\end{equation*}
$$

We claim that $\left(\psi_{k}\right)_{k \geq 1}$ is a weakly-null sequence in $\ell_{q}\left\langle I, X^{*}\right\rangle$. Recall that

$$
\ell_{q}\left\langle I, X^{*}\right\rangle^{*} \equiv s \ell_{p}\left(I, X^{* *}\right) \equiv \mathcal{L}\left(\ell_{q}(I), X^{* *}\right),
$$

and

$$
F_{p}\left(I, X^{* *}\right) \equiv \mathcal{K}\left(\ell_{q}(I), X^{* *}\right)
$$

Thus, by hypothesis, we have

$$
\ell_{q}\left\langle I, X^{*}\right\rangle^{*} \equiv F_{p}\left(I, X^{* *}\right)
$$

The subspace

$$
L=\left\{\theta^{* *}=\left(\theta_{i}^{* *}\right)_{i \in I} \in F_{p}\left(I, X^{* *}\right):\left\{i \in I: \theta_{i}^{* *}=0\right\} \text { is finite }\right\}
$$

is dense in $F_{p}\left(I, X^{* *}\right)$. Observe that if $\theta^{* *}=\left(\theta_{i}^{* *}\right)_{i \in I} \in L$, there exists $k_{0} \geq 1$ such that $\theta_{i}^{* *}=0$, for all $i \in F_{k}, k \geq k_{0}$, and hence,

$$
\lim _{k \rightarrow \infty} \theta^{* *}\left(\psi_{k}\right)=\lim _{k \rightarrow \infty} \sum_{i \in F_{k}} \theta_{i}^{* *}\left(\psi_{i, k}\right)=0
$$

The density of $L$ then establishes our claim.
Now, notice that

$$
\left|\psi_{k}\left(\overline{x_{j_{k}}}\right)\right|=\left|\sum_{i \in I} \psi_{i, k}\left(x_{i, j_{k}}\right)\right|=\left|\sum_{i \in F_{k}} \varphi_{i, j_{k}}\left(x_{i, j_{k}}\right)\right|>\frac{\varepsilon}{2}, \quad \forall k \geq 1,
$$

by construction. On the other hand, since the sequence $\left(\psi_{k}\right)_{k \geq 1}$ is weakly-null in $F_{p}(I, X)^{*}$ and $\left(\overline{x_{j_{k}}}\right)_{k \geq 1}$ is, by hypothesis, equivalent to the unit basis of $c_{0}$, by the Dunford-Pettis Property of $c_{0}[\mathbf{1 2}, \mathrm{p} .596]$ we know that $\psi_{k}\left(x_{j_{k}}\right) \longrightarrow 0$, a contradiction that finishes the proof.

Theorem 6.5. Let $X$ be a Banach space, $I$ be an infinite set, $p \in(1, \infty)$ and $\tau$ be an infinite cardinal. Suppose that

$$
\mathcal{K}\left(\ell_{q}(I), X^{* *}\right)=\mathcal{L}\left(\ell_{q}(I), X^{* *}\right)
$$

where $q \in(1, \infty)$ is the conjugate index of $p$. Then

$$
c_{0}(\tau) \stackrel{c}{\hookrightarrow} \ell_{p}(I) \widehat{\otimes}_{\varepsilon} X \Longleftrightarrow c_{0}(\tau) \stackrel{c}{\hookrightarrow} X .
$$

Proof. Suppose first that $c_{0}(\tau) \stackrel{c}{\hookrightarrow} \ell_{p}(I) \widehat{\otimes}_{\varepsilon} X \equiv F_{p}(I, X)$. By Theorem 2.2, there exist $\left(\overline{x_{j}}\right)_{j \in \tau}=\left(\left(x_{i, j}\right)_{i \in I}\right)_{j \in \tau}$ equivalent to the unit basis of $c_{0}(\tau)$ in $F_{p}(I, X)$, and a weak*-null family $\left(\varphi_{j}\right)_{j \in \tau}=\left(\left(\varphi_{i, j}\right)_{i \in I}\right)_{j \in \tau}$ of $F_{p}(I, X)^{*}$ satisfying

$$
1=\varphi_{j}\left(\overline{x_{j}}\right)=\sum_{i \in I} \varphi_{i, j}\left(x_{i, j}\right), \forall j \in \tau
$$

By the Uniform Boundedness Principle, $\left(\varphi_{j}\right)_{j \in \tau}$ is bounded, and so there exists $M>0$ such that

$$
\begin{equation*}
\left\|\varphi_{j}\right\| \leq M, \forall j \in \tau \tag{6.10}
\end{equation*}
$$

An appeal to Lemma 6.4 yields a finite subset $F \subset I$ satisfying

$$
\left|\sum_{i \in I \backslash F} \varphi_{i, j}\left(x_{i, j}\right)\right|<\frac{1}{2}, \forall j \in \tau
$$

Therefore, $F \neq \varnothing$ and

$$
\sum_{i \in F}\left|\varphi_{i, j}\left(x_{i, j}\right)\right| \geq\left|\sum_{i \in F} \varphi_{i, j}\left(x_{i, j}\right)\right| \geq\left|\sum_{i \in I} \varphi_{i, j}\left(x_{i, j}\right)\right|-\left|\sum_{i \in I \backslash F} \varphi_{i, j}\left(x_{i, j}\right)\right|>\frac{1}{2}, \forall j \in \tau
$$

Next, Lemma 2.4 implies that there exist $i_{0} \in F$ and $\tau_{1} \subset \tau$ satisfying $\left|\tau_{1}\right|=\tau$ and

$$
\begin{equation*}
\left|\varphi_{i_{0}, j}\left(x_{i_{0}, j}\right)\right|>\frac{1}{2|F|}, \forall j \in \tau_{1} \tag{6.11}
\end{equation*}
$$

By (6.10), we have

$$
\frac{1}{2|F|}<\left|\varphi_{i_{0}, j}\left(x_{i_{0}, j}\right)\right| \leq\left\|\varphi_{i_{0}, j}\right\|\left\|x_{i_{0}, j}\right\| \leq\left\|\varphi_{j}\right\|\left\|x_{i_{0}, j}\right\| \leq M\left\|x_{i_{0}, j}\right\|, \forall j \in \tau_{1}
$$

and thus

$$
\begin{equation*}
\left\|x_{i_{0}, j}\right\|>\frac{1}{2 M|F|}, \forall j \in \tau_{1} \tag{6.12}
\end{equation*}
$$

Consider the bounded linear operator $\pi_{i_{0}}: F_{p}(I, X) \rightarrow X$ defined by $\pi_{i_{0}}(\bar{x})=$ $x_{i_{0}}$ for each $\bar{x}=\left(x_{i}\right)_{i \in I} \in F_{p}(I, X)$. By hypothesis, there exists $T: c_{0}(\tau) \rightarrow$ $F_{p}(I, X)$ an isomorphism onto its image such that $T\left(e_{j}\right)=\overline{x_{j}}$, for all $j \in \tau$. By (6.12), we have

$$
\left\|\left(\pi_{i_{0}} \circ T\right)\left(e_{j}\right)\right\|>\frac{1}{2 M|F|}, \forall j \in \tau_{1}
$$

and therefore, by Theorem 2.1, there exists $\tau_{2} \subset \tau_{1}$ such that $\left|\tau_{2}\right|=\left|\tau_{1}\right|=\tau$ and $\pi_{i_{0}} \circ T_{\mid c_{0}\left(\tau_{2}\right)}$ is an isomorphism onto its image. Thus, $\left(x_{i_{0}, j}\right)_{j \in \tau_{2}}$ is equivalent to the unit basis of $c_{0}\left(\tau_{2}\right)$ in $X$.

Next, given $x \in X$, we have

$$
\left(\varphi_{i_{0}, j}(x)\right)_{j \in \tau_{2}} \in c_{0}\left(\tau_{2}\right)
$$

since, by hypothesis, $\left(\varphi_{j}\right)_{j \in \tau}$ is weak*-null. This proves that $\left(\varphi_{i_{0}, j}\right)_{j \in \tau_{2}}$ is weak*null in $X^{*}$. Theorem 2.2 and (6.11) then imply that $c_{0}(\tau) \stackrel{c}{\hookrightarrow} X$.

For the converse, it is sufficient to notice that $F_{p}(I, X)$ contains a complemented copy of $X$.

## 7. Final remarks and open problems

By adapting the proofs of Lemma 6.1 and Theorem 6.2, one can prove the following.

Theorem 7.1. Given $X$ a Banach space, $I$ an infinite set, $p \in[1, \infty)$ and $\tau$ an infinite cardinal, we have

$$
c_{0}(\tau) \stackrel{c}{\hookrightarrow} \ell_{p}(I, X) \Longleftrightarrow c_{0}(\tau) \stackrel{c}{\hookrightarrow} X .
$$

The above theorem extends to the uncountable case a $c_{0}$ result by Bombal [4, Theorem 4.3.1]. Moreover, since $\ell_{1}(I) \widehat{\otimes}_{\pi} X$ is isometrically isomorphic to $\ell_{1}(I, X)$ [20, p. 19], the next corollary follows immediately from Theorems 6.2 and 7.1.

Corollary 7.2. Given $X$ a Banach space, $I$ an infinite set, $p \in[1, \infty)$ and $\tau$ an infinite cardinal, we have

$$
c_{0}(\tau) \stackrel{c}{\hookrightarrow} \ell_{p}(I) \widehat{\otimes}_{\pi} X \Longleftrightarrow c_{0}(\tau) \stackrel{c}{\hookrightarrow} \ell_{p}(I, X) \Longleftrightarrow c_{0}(\tau) \stackrel{c}{\hookrightarrow} X .
$$

However, observe that the space $\ell_{p}(I, X)$ cannot be exchanged by the usual Banach space of Lebesque integrable functions $L_{p}([0,1], X)$ in the statement of Theorem 7.1, even in the case $\tau=\aleph_{0}$. Indeed, recall the following result due to Emmanuelle [10, Main Theorem].

Theorem 7.3. Let $X$ be a Banach space and $p \in[1, \infty)$. Then

$$
c_{0} \hookrightarrow X \Longrightarrow c_{0} \stackrel{c}{\hookrightarrow} L_{p}([0,1], X) .
$$

Thus, we have

$$
c_{0} \stackrel{c}{\hookrightarrow} L_{p}\left([0,1], \ell_{\infty}\right) \text { but } c_{0} \stackrel{c}{\hookrightarrow} \ell_{\infty} .
$$

These facts arise naturally the following question.
Problem 7.4. Let $X$ be a Banach space, $p \in[1, \infty)$ and $\tau$ be an infinite cardinal. What assumptions on $\tau$ yield

$$
c_{0}(\tau) \stackrel{c}{\hookrightarrow} \ell_{p}([0,1], X) \Longleftrightarrow c_{0}(\tau) \stackrel{c}{\hookrightarrow} X ?
$$

We do not know if Theorem 6.2 can be extended to the case $p=\infty$. That is:
Problem 7.5. Let $X$ be a Banach space and $\tau$ be an infinite cardinal. Is it true that

$$
c_{0}(\tau) \stackrel{c}{\hookrightarrow} \ell_{\infty} \widehat{\otimes}_{\pi} X \Longrightarrow c_{0}(\tau) \stackrel{c}{\hookrightarrow} X ?
$$

Finally, we also do not know if Theorem 6.3 is optimal for any cardinal $\tau$. In particular, the following problem is unsolved.

Problem 7.6. Let $X$ be a Banach space and $p \in[1, \infty)$. Does it follow that

$$
c_{0}\left(\aleph_{\omega}\right) \stackrel{c}{\hookrightarrow} \ell_{p} \widehat{\otimes}_{\varepsilon} X \Longrightarrow c_{0}\left(\aleph_{\omega}\right) \stackrel{c}{\hookrightarrow} X ?
$$

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