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When is $c_0(\tau)$ complemented in tensor products of $\ell_p(I)$ and X?

Vinícius Morelli Cortes, Elói Medina Galego, and Christian Samuel

ABSTRACT. Let X be a Banach space, I an infinite set, τ an infinite cardinal and $p \in [1,\infty)$. In contrast to a classical c_0 result due independently to Cembranos and Freniche, we prove that if the cofinality of τ is greater than the cardinality of I, then the injective tensor product $\ell_p(I)\widehat{\otimes}_{\varepsilon} X$ contains a complemented copy of $c_0(\tau)$ if and only if X does. This result is optimal for every regular cardinal τ .

On the other hand, we provide a generalization of a c_0 result of Oya by proving that if τ is an infinite cardinal, then the projective tensor product $\ell_p(I)\widehat{\otimes}_{\pi}X$ contains a complemented copy of $c_0(\tau)$ if and only if X does.

These results are obtained via useful descriptions of tensor products as convenient generalized sequence spaces.

1. Introduction

We use standard set-theoretical and Banach space theory terminology as may be found, e.g., in [14] and [15] respectively. We denote by B_X the closed unit ball of the Banach space X. If X and Y are Banach spaces, we denote by $\mathcal{L}(X, Y)$ the space of all bounded linear operators from X to Y and by $\mathcal{K}(X, Y)$ the subspace of all compact linear operators. We say that Y contains a copy (resp. a complemented copy) of X, and write $X \hookrightarrow Y$ (resp. $X \stackrel{c}{\hookrightarrow} Y$), if X is isomorphic to a subspace (resp. complemented subspace) of Y. If X and Y are isometrically isomorphic Banach spaces, we write $X \equiv Y$.

We shall denote the projective and injective tensor norms by $\|\cdot\|_{\pi}$ and $\|\cdot\|_{\varepsilon}$ respectively. The projective (resp. injective) tensor product of X and Y is the completion of $X \otimes Y$ with respect to $\|\cdot\|_{\pi}$ (resp. $\|\cdot\|_{\varepsilon}$) and will be denoted by $X \widehat{\otimes}_{\pi} Y$ (resp. $X \widehat{\otimes}_{\varepsilon} Y$).

Given X a Banach space, I an infinite set and $(x_i)_{i\in I} \in X^I$, the notation $x = \sum_i x_i$ means that, for every $\varepsilon > 0$, there exists a finite subset $\emptyset \neq F_0 \subset I$ such that, for every finite subset F of I with $F_0 \subset F$, we have $||x - \sum_{i\in F} x_i|| < \varepsilon$.

For a non-empty set Γ , $c_0(\Gamma)$ denotes the Banach space of all real-valued maps f on Γ with the property that for each $\varepsilon > 0$, the set $\{\gamma \in \Gamma : |f(\gamma)| \ge \varepsilon\}$ is finite, equipped with the supremum norm. We will refer to $c_0(\Gamma)$ as $c_0(\tau)$ when the cardinality of Γ (denoted by $|\Gamma|$) is equal to τ . This space will be denoted by c_0 when $\tau = \aleph_0$.

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Given $p \in [1, \infty)$, we denote

$$\ell_p(I, X) = \left\{ (x_i)_{i \in I} \in X^I : \sum_{i \in I} \|x_i\|^p < \infty \right\},\$$

equipped with the complete norm

$$\|\overline{x}\|_p = \left[\sum_{i \in I} \|x_i\|^p\right]^{\frac{1}{p}},$$

for each $\overline{x} = (x_i)_{i \in I} \in \ell_p(I, X)$. These spaces will be denoted by $\ell_p(I)$ when $X = \mathbb{R}$. If $I = \mathbb{N}$, we will write ℓ_p .

By $\ell_{\infty}(I)$ we will denote the Banach space of all bounded real-valued maps on I, with the supremum norm. This space will be denoted by ℓ_{∞} when $I = \mathbb{N}$.

Recall that if τ is an infinite cardinal then the *cofinality* of τ , denoted by $cf(\tau)$, is the least cardinal α such that there exists a family of ordinals $\{\beta_j : j \in \alpha\}$ satisfying $\beta_j < \tau$ for all $j \in \alpha$, and $\sup\{\beta_j : j \in \alpha\} = \tau$. A cardinal τ is said to be *regular* when $cf(\tau) = \tau$; otherwise, it is said to be *singular*.

An immediate consequence of the classical Cembranos-Freniche Theorem [3, Main Theorem], [11, Corollary 2.5] is the following result.

THEOREM 1.1. For each $p \in [1, \infty)$ we have $c_0 \stackrel{c}{\hookrightarrow} \ell_p \widehat{\otimes}_{\varepsilon} \ell_{\infty}$.

However, it is well known that $c_0 \not\hookrightarrow \ell_{\infty}$ (see, e. g., [9, Corollary 11, p. 156]). On the other hand, Oya proved the following theorem.

THEOREM 1.2 ([18], Theorem 2). If X is a Banach space and $p \in [1, \infty)$, then

$$c_0 \stackrel{c}{\hookrightarrow} \ell_p \widehat{\otimes}_\pi X \iff c_0 \stackrel{c}{\hookrightarrow} X.$$

These facts motivate the following problems.

PROBLEM 1.3. Let X be a Banach space, I be an infinite set, $p \in [1, \infty)$ and τ be an infinite cardinal. What assumptions on τ and I yield

$$c_0(\tau) \stackrel{c}{\hookrightarrow} \ell_p(I) \widehat{\otimes}_{\varepsilon} X \iff c_0(\tau) \stackrel{c}{\hookrightarrow} X?$$

PROBLEM 1.4. Given X a Banach space, I an infinite set, $p \in [1, \infty)$ and τ an infinite cardinal, is it true that

$$c_0(\tau) \stackrel{c}{\hookrightarrow} \ell_p(I) \widehat{\otimes}_{\pi} X \iff c_0(\tau) \stackrel{c}{\hookrightarrow} X?$$

In this paper, we provide a partial solution to Problem 1.3 and a complete solution to Problem 1.4. More precisely, we will prove that for every Banach space X, infinite set $I, p \in [1, \infty)$ and infinite cardinal τ , one has

$$c_0(\tau) \stackrel{c}{\hookrightarrow} \ell_p(I) \widehat{\otimes}_{\pi} X \iff c_0(\tau) \stackrel{c}{\hookrightarrow} X.$$

Additionally, if the $cf(\tau) > |I|$, then one also has

(1.1)
$$c_0(\tau) \stackrel{c}{\hookrightarrow} \ell_p(I) \widehat{\otimes}_{\varepsilon} X \iff c_0(\tau) \stackrel{c}{\hookrightarrow} X$$

REMARK 1.5. The equivalence (1.1) cannot be extended to the case $p = \infty$. Setting $I = \mathbb{N}$ and $\tau = \aleph_1$, by [13, Theorem 5.3] we know that

$$c_0(\aleph_1) \stackrel{c}{\hookrightarrow} \ell_\infty \widehat{\otimes}_{\varepsilon} \ell_\infty(\aleph_1),$$

although $c_0(\aleph_1) \not\hookrightarrow \ell_{\infty}(\aleph_1)$ by [9, Corollary 11, p. 156].

REMARK 1.6. (1.1) is optimal for every infinite regular cardinal κ . Indeed, setting $I = \kappa$, again by [13, Theorem 4.5] we have

$$c_0(\kappa) \stackrel{c}{\hookrightarrow} \ell_p(\kappa) \widehat{\otimes}_{\varepsilon} \ell_{\infty}(\kappa),$$

but $c_0(\kappa) \not\xrightarrow{c} \ell_{\infty}(\tau)$ once again by [9, Corollary 11, p. 156].

2. Preliminary results and notations

We will denote by $(e_j)_{j\in\tau}$ the unit-vector basis of $c_0(\tau)$ or $\ell_p(\tau)$ for $p \in [1,\infty)$, that is, $e_j(j) = 1$ and $e_j(k) = 0$ for each $j, k \in \tau, j \neq k$. If Γ is a subset of τ , we identify $c_0(\Gamma)$ with the closed subspace of $c_0(\tau)$ consisting of the maps g on τ such that g(j) = 0 for each $j \in \tau \setminus \Gamma$.

We begin by recalling the following classical result by H.P. Rosenthal.

THEOREM 2.1. ([19, Remark following Theorem 3.4]) Let X be a Banach space and τ an infinite cardinal. Let $T : c_0(\tau) \to X$ be a bounded linear operator such that

$$\inf\{\|T(e_j)\| : j \in \tau\} > 0.$$

Then there exists a subset $\Gamma \subset \tau$ such that $|\Gamma| = \tau$ and $T_{|c_0(\Gamma)}$ is an isomorphism onto its image.

We recall that a family $(x_j^*)_{j \in \tau}$ in the dual space X^* is said to be *weak*-null* if for each $x \in X$ we have

$$(x_j^*(x))_{j\in\tau}\in c_0(\tau).$$

Recall also that a family $(x_j)_{j\in\tau}$ in a Banach space X is said to be *equivalent* to the canonical basis of $c_0(\tau)$ if there exists $T: c_0(\tau) \to X$ an isomorphism onto its image satisfying $T(e_j) = x_j$, for each $j \in \tau$.

The main characterization of complemented copies of $c_0(\tau)$ we will use is the following result obtained in [7].

THEOREM 2.2. Let X be a Banach space and τ be an infinite cardinal. The following are equivalent:

- (1) X contains a complemented copy of $c_0(\tau)$.
- (2) There exist a family $(x_j)_{j\in\tau}$ equivalent to the unit-vector basis of $c_0(\tau)$ in X and a weak^{*}-null family $(x_i^*)_{j\in\tau}$ in X^* such that, for each $j, k \in \tau$,

$$x_j^*(x_k) = \delta_{jk}.$$

(3) There exist a family $(x_j)_{j\in\tau}$ equivalent to the unit-vector basis of $c_0(\tau)$ in X and a weak^{*}-null family $(x_i^*)_{j\in\tau}$ in X^* such that

$$\inf_{j\in\tau}|x_j^*(x_j)|>0.$$

We will also use the following two simple lemmas.

LEMMA 2.3. Let I be an infinite set and J be a non-empty set. Let $\{I_j\}_{j\in J}$ be a family of subsets of I such that $\bigcup_{j\in J} I_j = I$. If cf(|I|) > |J|, then there exists $j_0 \in J$ such that $|I_{j_0}| = |I|$.

PROOF. Suppose that the conclusion does not hold. Then we have $|I_j| < |I|$ for each $i \in J$ and thus, by the definition of cofinality,

$$\sup\{|I_j|: j \in J\} < |I|.$$

Since $|J| < cf(|I|) \le |I|$, we obtain

$$|I| = \left| \bigcup_{j \in J} I_j \right| \le \max(|J|, \sup\{|I_j| : j \in J\}) < |I|,$$

a contradiction that finishes the proof.

LEMMA 2.4. Let I be an infinite set and J be a non-empty, finite set. For each $i \in I$, let $(a_{i,j})_{j \in J}$ be a family of positive real numbers. Suppose that there exists $\delta > 0$ satisfying, for all $i \in I$,

$$\sum_{j\in J} a_{i,j} \ge \delta.$$

Then there exist $j_0 \in J$ and $I' \subset I$ such that |I'| = |I| and

$$a_{i,j_0} \ge \frac{\delta}{|J|}, \ \forall i \in I'.$$

PROOF. By hypothesis, for each $i \in I$ there exists $j(i) \in J$ such that

$$a_{i,j(i)} \ge \frac{\delta}{|J|}.$$

Let $J' = \{j(i) : i \in I\}$ and for each $j \in J'$, consider $I_j = \{i \in I : j(i) = j\}$. Since $I = \bigcup_{j \in J'} I_j$ is infinite and J' is finite, by Lemma 2.3 there exists $j_0 \in J'$ such that $|I_{j_0}| = |I|$. Setting $I' = I_{j_0}$, the proof is complete.

3. Spaces $s\ell_p(I,X)$, $F_p(I,X)$ and $\ell_p\langle I,X\rangle$

In this section we shall introduce some generalized sequence spaces which will be of interest.

Let X be a Banach space, I be an infinite set and $p \in [1, \infty)$. Following [21], we denote

$$s\ell_p(I,X) = \left\{ (x_i)_{i \in I} \in X^I : \sum_{i \in I} |x^*(x_i)|^p < \infty, \text{ for all } x^* \in X^* \right\}.$$

It is straightforward to check that the function

$$\|\overline{x}\|_{s\ell_p} = \sup\left\{\left[\sum_{i\in I} |x^*(x_i)|^p\right]^{\frac{1}{p}} : x^* \in B_{X^*}\right\},\$$

where $\overline{x} = (x_i)_{i \in I} \in s\ell_p(I, X)$, is a complete norm on $s\ell_p(I, X)$. For each $i \in I$, we denote by $R_i : X \to X^I$ the canonical inclusion defined by $R_i(x) = (\delta_{ik}x)_{k \in I}$, for each $x \in X$. Our interest lies in the closed subspace

$$F_p(I,X) = \left\{ \overline{x} = (x_i)_{i \in I} \in s\ell_p(I,X) : \overline{x} = \sum_{i \in I} R_i(x_i) \right\}$$

of $s\ell_p(I, X)$.

Given $p \in (1,\infty)$ and $q \in (1,\infty)$ the conjugate index of p, there exists a canonical isometric isomorphism between $s\ell_p(I,X)$ and $\mathcal{L}(\ell_q(I),X)$ which maps an

element $\overline{x} = (x_i)_{i \in I} \in s\ell_p(I, X)$ to the bounded linear operator $S \in \mathcal{L}(\ell_q(I), X)$ defined by

$$S(\overline{a}) = \sum_{i \in I} a_i x_i,$$

for all $\overline{a} = (a_i)_{i \in I} \in \ell_q(I)$. This isometry identifies $F_p(I, X)$ and the subspace $\mathcal{K}(\ell_q(I), X)$ of compact linear operators from $\ell_q(I)$ to X.

In a similar way, in the case p = 1 there exists an isometric isomorphism between $sl_1(I, X)$ and $\mathcal{L}(c_0(I), X)$ which identifies $F_1(I, X)$ and $\mathcal{K}(c_0(I), X)$.

Next, let $p \in (1, \infty)$ and $q \in (1, \infty)$ be the conjugate index of p. Following [1, 5], we denote

$$\ell_p \langle I, X \rangle = \left\{ \overline{x} = (x_i)_{i \in I} \in X^I : \sum_{i \in I} |x_i^*(x_i)| < \infty, \text{ for all } (x_i^*)_{i \in I} \in s\ell_q(I, X^*) \right\}.$$

It is straightforward to check that the function

$$\|\overline{x}\|_{\ell_p\langle X\rangle} = \sup\left\{\sum_{i\in I} |x_i^*(x_i)| : \overline{x}^* = (x_i^*)_{i\in I} \in B_{s\ell_q(I,X^*)}\right\},\$$

where $\overline{x} = (x_i)_{i \in I} \in \ell_p \langle I, X \rangle$, is a complete norm on $\ell_p \langle I, X \rangle$.

We will summarize the main properties of those spaces in our next results. Our statements and proofs are simple modifications of those found in [1, 2, 5].

PROPOSITION 3.1. Let X be a Banach space, I be an infinite set and $p \in [1, \infty)$.

(1) $\ell_p(I,X) \subset F_p(I,X)$ and $\|\overline{x}\|_{s\ell_p} \leq \|\overline{x}\|_p$, for all $\overline{x} \in \ell_p(I,X)$;

2) For
$$p > 1$$
, $\ell_p \langle I, X \rangle \subset \ell_p (I, X)$ and $\|\overline{x}\|_p \leq \|\overline{x}\|_{\ell_p \langle X \rangle}$, for all $\overline{x} \in \ell_p \langle I, X \rangle$

PROOF. Statement (1) follows immediately from the definitions of $\ell_p(I, X)$ and $s\ell_p(I, X)$. Let us show (2). Given $\overline{x} = (x_i)_{i \in I} \in \ell_p\langle I, X \rangle$, we fix, for every $i \in I, x_i^* \in B_{X^*}$ such that $||x_i|| = x_i^*(x_i)$. For every $(\lambda_i)_{i \in I} \in B_{\ell_q(I)}$ we have $(\lambda_i x_i^*)_{i \in I} \in B_{sl_q(X^*)}$ and so

$$\sum_{i \in I} |\lambda_i x_i^*(x_i)| = \sum_{i \in I} |\lambda_i| \, \|x_i\| \le \|\overline{x}\|_{\ell_p \langle X \rangle}.$$

Therefore

$$\left(\sum_{i\in I} \|x_i\|^p\right)^{\frac{1}{p}} = \|\overline{x}\|_p \le \|\overline{x}\|_{\ell_p\langle X\rangle}$$

and the proof is complete.

We will denote the canonical inclusion of $x \in X$ in X^{**} by \hat{x} . The proof of the following proposition is straightforward.

PROPOSITION 3.2. Let X be a Banach space, I be an infinite set, $p \in [1, \infty)$, $\overline{x} = (x_i)_{i \in I} \in X^I$ and $\overline{y} = (\widehat{x_i})_{i \in I} \in (X^{**})^I$. Then $\overline{x} \in s\ell_p(I, X)$ if, and only if, $\overline{y} \in s\ell_p(I, X^{**})$. Furthermore, $\|\overline{x}\|_{s\ell_p} = \|\overline{y}\|_{s\ell_p}$.

4. Generalized sequence spaces and tensor products

The goal of this section is to establish a connection between the generalized sequence spaces defined in the previous section and the projective and injective tensor products $\ell_p(I)\widehat{\otimes}_{\pi}X$ and $\ell_p(I)\widehat{\otimes}_{\varepsilon}X$.

It is obvious that for every $\overline{a} = (a_i)_{i \in I} \in \mathbb{R}^I$ and $x \in X$ we have $(a_i x)_{i \in I} \in X^I$ and furthermore the operator $\ell_p(I) \times X \ni (\overline{a}, x) \mapsto (a_i x)_{i \in I} \in X^I$ is bilinear. This bilinear operator induces a linear operator $T : \ell_p(I) \otimes X \to X^I$ [20, Proposition 1.4]. It is also obvious that $\operatorname{Im}(T) \subset \ell_p\langle I, X \rangle \cap s\ell_p(I, X)$. We summarize the main properties of this operator in the following proposition.

PROPOSITION 4.1. Given $p \in [1, \infty)$, we have:

- (1) Im(T) contains all finitely non-zero families in $\ell_p(I, X)$;
- (2) $||T(u)||_{s\ell_p(X)} = ||u||_{\varepsilon}$, for each $u \in \ell_p(I) \otimes X$;
- (3) If $p \in (1,\infty)$, then $||T(u)||_{\ell_p\langle X\rangle} \leq ||u||_{\pi}$, for each $u \in \ell_p(I) \otimes X$.

PROOF. Given $x \in X$ and $i \in I$, notice that $T(e_i \otimes x) = (\delta_{ij}x)_{j \in I}$. This shows statement (1).

Next, let $u \in \ell_p(I) \otimes X$ be given and fix a representation $u = \sum_{n=1}^m a_n \otimes x_n$, where $a_n = (a_i^n)_{i \in I}$. Then we have

$$||T(u)||_{s\ell_p(X)} = \sup\left\{ \left[\sum_{i \in I} \left| \sum_{n=1}^m a_i^n x^*(x_n) \right|^p \right]^{\frac{1}{p}} : x^* \in B_{X^*} \right\}$$
$$= \sup\left\{ \left\| \sum_{n=1}^m x^*(x_n) a_n \right\|_p : x^* \in B_{X^*} \right\} = ||u||_{\varepsilon}.$$

This proves (2).

Finally, suppose that $p \in (1, \infty)$ and let us show (3). Let $q \in (1, \infty)$ be the conjugate index of p. Given $\overline{x}^* = (x_i^*)_{i \in I} \in B_{s\ell_q(I,X^*)}$, $a = (a_i)_{i \in I} \in \ell_p(I)$, and $x \in X$, by Hölder's Inequality we have

$$\sum_{i \in I} |a_i x_i^*(x)| \le \left(\sum_{i \in I} |a_i|^p\right)^{\frac{1}{p}} \left(\sum_{i \in I} |x_i^*(x)|^q\right)^{\frac{1}{q}} \le ||a||_p ||x||$$

Hence, the bilinear operator $\ell_p(I) \times X \ni (a, x) \mapsto (a_i x)_{i \in I} \in \ell_p \langle I, X \rangle$ has norm at most 1. By [20, Theorem 2.9], we obtain

$$||T(u)||_{\ell_p\langle X\rangle} \le ||u||_{\pi},$$

for every $u \in \ell_p(I) \otimes X$, as desired.

COROLLARY 4.2. Let X be a Banach space, I be an infinite set and $p \in [1, \infty)$. Let $T : \ell_p(I) \otimes X \to X^I$ be the linear map defined on Proposition 4.1.

(1) T admits an unique linear extension T_{ε} to $\ell_p(I) \widehat{\otimes}_{\varepsilon} X$, which is an isometry onto $F_p(I, X)$;

(2) If $p \in (1, \infty)$, then T admits an unique linear extension of same norm T_{π} from $\ell_p(I)\widehat{\otimes}_{\pi}X$ to $\ell_p\langle I, X \rangle$.

PROOF. Let us first show (1). By the density of $\ell_p(I) \otimes X$ in $\ell_p(I) \widehat{\otimes}_{\varepsilon} X$ and Proposition 4.1, T admits an unique linear extension $T_{\varepsilon} : \ell_p(I) \widehat{\otimes}_{\varepsilon} X \to F_p(I, X)$, which is an isometry onto its image. Given $\overline{x} = (x_i)_{i \in I} \in F_p(I, X)$, by Proposition 4.1 we have

$$x = \sum_{i \in I} R_i(x_i) = \sum_{i \in I} T(e_i \otimes x_i) \in \overline{\mathrm{Im}(T)} = \mathrm{Im}(T_{\varepsilon}) \; .$$

This proves that T_{ε} is onto $F_p(I, X)$.

Similarly, if $p \in (1, \infty)$, the density of $\ell_p(I) \otimes X$ in $\ell_p(I) \widehat{\otimes}_{\pi} X$ and Proposition 4.1 imply that T admits an unique linear extension $T_{\pi} : \ell_p(I) \widehat{\otimes}_{\pi} X \to \ell_p \langle I, X \rangle$ satisfying $||T_{\pi}|| = ||T|| \leq 1$.

As in the countable case, we will show that T_{π} is a linear isometry from $\ell_p(I)\widehat{\otimes}_{\pi}X$ onto $\ell_p\langle I, X \rangle$. We will use the following key lemma.

LEMMA 4.3. Let X be a Banach space, I be an infinite set and $p \in (1, \infty)$. Given $u \in \ell_p(I) \widehat{\otimes}_{\pi} X$, there exists a family $(x_i)_{i \in I}$ in X such that

$$u = \sum_{i \in I} e_i \otimes x_i.$$

PROOF. Let $q \in (1, \infty)$ be the conjugate index of p and $(e_i^*)_{i \in I}$ be the unit basis of $\ell_q(I)$. For each $i \in I$ consider the bounded linear operator $\pi_i : \ell_p(I) \to \ell_p(I)$ defined by

$$\pi_i(a) = e_i^*(a)e_i,$$

for each $a = (a_j)_{j \in I} \in \ell_p(I)$, and for every finite subset $\emptyset \neq F \subset I$ write

$$P_F = \sum_{j \in F} \pi_j$$

Let $u \in \ell_p(I) \widehat{\otimes}_{\pi} X$ be given and let us prove that

(4.1)
$$u = \sum_{i \in I} (\pi_i \otimes I)(u)$$

By [20, Proposition 2.8] there exist bounded sequences $(a_n)_{n\geq 1}$ in $\ell_p(I)$ and $(y_n)_{n\geq 1}$ in X satisfying

$$u = \sum_{n=1}^{\infty} a_n \otimes y_n$$
 and $\sum_{n=1}^{\infty} ||a_n||_p ||y_n|| < \infty.$

We may suppose that $||y_n|| = 1$ for every $n \ge 1$. Given $\varepsilon > 0$, take $N \ge 1$ such that

$$\sum_{n=N+1}^{\infty} \|a_n\|_p \le \frac{\varepsilon}{3}$$

It is clear that

$$\left\|u-\sum_{n=1}^N a_n\otimes y_n\right\|_{\pi}\leq \frac{\varepsilon}{3}.$$

There exists a finite subset $\emptyset \neq F_0 \subset I$ such that for every finite subset $F \subset I$ satisfying $F_0 \subset F$ and every $1 \leq n \leq N$ we have

$$\|P_F(a_n) - a_n\|_p \le \frac{\varepsilon}{3N}.$$

Thus,

$$\begin{aligned} \|u - (P_F \otimes I)(u)\| &\leq \left\|u - \sum_{n=1}^N a_n \otimes y_n\right\|_{\pi} \\ &+ \left\|\sum_{n=1}^N a_n \otimes y_n - \sum_{n=1}^N P_F(a_n) \otimes y_n\right\|_{\pi} \\ &+ \left\|\sum_{n=1}^N P_F(a_n) \otimes y_n - (P_F \otimes I)(u)\right\|_{\pi} \leq \varepsilon. \end{aligned}$$

This shows that (4.1) holds.

Finally, since

$$(\pi_i \otimes I)(u) = \sum_{n=1}^{\infty} \pi_i(a_n) \otimes y_n = \sum_{n=1}^{\infty} e_i^*(a_n) e_i \otimes y_n = e_i \otimes \left(\sum_{n=1}^{\infty} e_i^*(a_n) y_n\right),$$

for each $i \in I$, it follows that

$$x_i = \sum_{n=1}^{\infty} e_i^*(a_n) y_n$$

satisfies the desired properties.

We are now in a position to show that T_{π} is an isometry onto $\ell_p(I, X)$.

THEOREM 4.4. Let X be a Banach space, I be an infinite set and $p \in (1, \infty)$. The operator T_{π} is an isometric isomorphism from $\ell_p(I)\widehat{\otimes}_{\pi}X$ onto $\ell_p\langle I, X \rangle$.

PROOF. We have shown that

$$||T_{\pi}(u)||_{\ell_p\langle X\rangle} \le ||u||_{\pi},$$

for every $u \in \ell_p(I)\widehat{\otimes}_{\pi} X$. In order to prove that T_{π} is isometric, it is sufficient to prove the reverse inequality on span { $e_i \otimes x : i \in I, x \in X$ }. Given $v \in \text{span} \{ e_i \otimes x : i \in I, x \in X \}$, there exist a finite subset $\emptyset \neq I_0 \subset I$ and a finite family $(y_i)_{i \in I_0}$ of elements of X such that

$$v = \sum_{i \in I_0} e_i \otimes y_i.$$

By [20, p. 24] and the Hahn-Banach Theorem, there exists a bounded linear operator $b \in \mathcal{L}(\ell_p(I), X^*) \equiv [\ell_p(I)\widehat{\otimes}_{\pi}X]^*$ satisfying

$$||b|| = 1$$
 and $||v||_{\pi} = \sum_{i \in I_0} b(e_i)(y_i).$

Setting $y_i = 0$ for each $i \in I \setminus I_0$, it is clear that $T_{\pi}(v) = (y_i)_{i \in I}$ and

$$||T_{\pi}(v)||_{\ell_p\langle X\rangle} \ge ||v||_{\pi},$$

as desired.

Let us show next that T_{π} is onto. Given $\overline{x} = (x_i)_{i \in I} \in \ell_p \langle I, X \rangle$, we will show that the family $(e_i \otimes x_i)_{i \in I}$ is summable in $\ell_p(I) \widehat{\otimes}_{\pi} X$ and

(4.2)
$$\overline{x} = T_{\pi} \left(\sum_{i \in I} e_i \otimes x_i \right).$$

For every $\overline{x}^* = (x_i^*)_{i \in I} \in s\ell_q(I, X^*)$ we have

$$\left|\sum_{i\in I} x_i^*(x_i)\right| \le \|\overline{x}\|_{\ell_p\langle X\rangle} \|\overline{x}^*\|_{sl_q(X^*)}.$$

Hence, the linear functional θ defined on $sl_q(I, X^*) \equiv \mathcal{L}(\ell_p(I), X^*)$ by

$$\theta(\overline{x}^*) = \sum_{i \in I} x_i^*(x_i),$$

for each $\overline{x}^* = (x_i^*)_{i \in I} \in s\ell_q(I, X^*)$, is bounded. Let us denote by θ_r its restriction to the subspace $\mathcal{K}(\ell_p(I), X^*)$.

Let $q \in (1, \infty)$ be the conjugate index of p. Recall that the space $\ell_q(I)$ has the approximation property [20, p. 73], and therefore

(4.3)
$$\mathcal{K}(\ell_p(I), X^*) \equiv \ell_q(I)\widehat{\otimes}_{\varepsilon} X^*,$$

by [20, Corollary 4.13]. Moreover, since $\ell_p(I)$ also has the Radon-Nikodým property [20, Corollary 5.45], by (4.3) and [20, Theorem 5.33] we know that

$$\ell_p(I)\widehat{\otimes}_{\pi}X^{**} \equiv [\ell_q(I)\widehat{\otimes}_{\varepsilon}X^*]^* \equiv [\mathcal{K}(\ell_p(I),X^*)]^*.$$

It follows from Lemma 4.3 that there exists a family $(x_i^{**})_{i \in I}$ in X^{**} such that

$$\theta_r = \sum_{i \in I} e_i \otimes x_i^{**}$$

We observe that for every $i \in I$ and for every $x^* \in X^*$ we have

$$x_i^{**}(x^*) = \theta_r(e_i^* \otimes x^*) = x^*(x_i),$$

and so $x_i^{**} = \hat{x}_i$. Thus, the family $(e_i \otimes \hat{x}_i)_{i \in I}$ is summable in $\ell_p(I) \widehat{\otimes}_{\pi} X^{**}$ and hence, the family $(e_i \otimes x_i)_{i \in I}$ in $\ell_p(I) \widehat{\otimes}_{\pi} X$, by [8, Corollary 14, p. 238]. Finally, it is clear that (4.2) holds.

REMARK 4.5. We observe that $T_{\pi}(e_i \otimes x_i) = R_i(x_i)$, for each $i \in I$. Therefore, for each $\overline{x} = (x_i)_{i \in I} \in \ell_p \langle I, X \rangle$, the family $(R_i(x_i))_{i \in I}$ is summable in $\ell_p \langle I, X \rangle$ and

$$\overline{x} = \sum_{i \in I} R_i(x_i)$$

REMARK 4.6. By Proposition 3.1, every family $\overline{x} = (x_i)_{i \in I} \in \ell_p \langle I, X \rangle$ has only countably many non-zero coordinates. Hence, by the previous remark, the subspace

$$L = \{ \overline{y} = (y_i)_{i \in I} \in \ell_p \langle I, X \rangle : \{ i \in I : y_i = 0 \} \text{ is finite} \}$$

is dense in $\ell_p \langle I, X \rangle$.

5. Dual spaces of $\ell_p \langle I, X \rangle$ and $F_p(I, X)$

We now turn our attention to convenient identifications of the spaces $\ell_p \langle I, X \rangle$ and $F_p(I, X)$ and of their duals.

THEOREM 5.1. Let X be a Banach space, I be an infinite set, $p \in (1, \infty)$ and $q \in (1, \infty)$ be the conjugate index of p. Then there exists an isometric isomorphism from $\ell_p \langle I, X \rangle^*$ onto $s\ell_q(I, X^*)$ which maps $\varphi \in \ell_p \langle I, X \rangle^*$ to $(x_i^*)_{i \in I} \in s\ell_q(I, X^*)$, where

$$\varphi(\overline{x}) = \sum_{i \in I} x_i^*(x_i),$$

for every $\overline{x} = (x_i)_{i \in I} \in \ell_p \langle I, X \rangle$.

PROOF. Let T_{π} be the canonical isometric isomorphism from $\ell_p(I)\widehat{\otimes}_{\pi}X$ onto $\ell_p\langle I, X\rangle$ considered in Corollary 4.2. Then T_{π}^* is also an isometric isomorphism from $\ell_p\langle I, X\rangle^*$ onto $[\ell_p(I)\widehat{\otimes}_{\pi}X]^*$. By [**20**, p. 24], we have

$$[\ell_p(I)\widehat{\otimes}_{\pi}X]^* \equiv \mathcal{L}(\ell_p(I), X^*) \equiv s\ell_q(I, X^*).$$

Given $\varphi \in \ell_p \langle I, X \rangle^*$, let $(x_i^*)_{i \in I} \in s \ell_q(I, X^*)$ be the unique element corresponding to $T^*_{\pi}(\varphi) \in [\ell_p(I)\widehat{\otimes}_{\pi}X]^*$ via the above identifications. By Remark 4.5, for each

$$\overline{x} = (x_i)_{i \in I} = \sum_{i \in I} R_i(x_i) = \sum_{i \in I} T_{\pi}(e_i \otimes x_i) \in \ell_p \langle I, X \rangle,$$

we have

$$\varphi(\overline{x}) = \sum_{i \in I} (\varphi \circ T_{\pi})(e_i \otimes x_i) = \sum_{i \in I} T_{\pi}^*(\varphi)(e_i \otimes x_i) = \sum_{j \in I} x_i^*(x_i),$$

and the proof is complete.

REMARK 5.2. For every $\varphi = (x_i^*)_{i \in I} \in \ell_p \langle I, X \rangle^*$, we have

$$\|\varphi\| = \sup\left\{\sum_{i\in I} |x_i^*(x_i)| : (x_i)_{i\in I} \in B_{\ell_p\langle X\rangle}\right\}.$$

THEOREM 5.3. Let X be a Banach space, I be an infinite set, $p \in (1, \infty)$ and $q \in (1, \infty)$ be the conjugate index of p. Then there exists an isometric isomorphism from $F_p(I, X)^*$ onto $\ell_q\langle I, X^* \rangle$ which maps $\psi \in F_p(I, X)^*$ to $(x_i^*)_{i \in I} \in \ell_q \langle I, X^* \rangle$, where

$$\psi(\overline{x}) = \sum_{i \in I} x_i^*(x_i),$$

for every $\overline{x} = (x_i)_{i \in I} \in F_p(I, X)$.

PROOF. Let T_{ε} be the canonical isometric isomorphism from $\ell_p(I)\widehat{\otimes}_{\varepsilon}X$ onto $F_p(I, X)$ considered in Corollary 4.2. Since the space $\ell_q(I)$ has both the approximation property and the Radon-Nikodým property, by [20, Theorem 5.33] we know that

$$[\ell_p(I)\widehat{\otimes}_{\varepsilon}X]^* \equiv \ell_q(I)\widehat{\otimes}_{\pi}X^*.$$

By Theorem 4.4 and Remark 4.5, the operator

$$\ell_q \langle I, X^* \rangle \ni (x_i^*)_{i \in I} \mapsto \sum_{i \in I} e_i \otimes x_i^* \in \ell_q(I) \widehat{\otimes}_{\pi} X^*$$

is also an isometric isomorphism.

Given $\psi \in F_p(I, X)^*$, there exists an unique $(x_i^*)_{i \in I} \in \ell_q \langle I, X^* \rangle$ corresponding to $T^*_{\varepsilon}(\varphi) \in [\ell_p(I) \widehat{\otimes}_{\varepsilon} X]^*$ via the above identifications. For each

$$\overline{x} = (x_i)_{i \in I} = \sum_{i \in I} R_i(x_i) = \sum_{i \in I} T_{\varepsilon}(e_i \otimes x_i) \in F_p(I, X),$$

we have

$$\psi(\overline{x}) = \sum_{i \in I} (\psi \circ T_{\varepsilon})(e_i \otimes x_i) = \sum_{i \in I} T_{\varepsilon}^*(\psi)(e_i \otimes x_i) = \sum_{j \in I} x_i^*(x_i),$$

and the proof is complete.

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6. Complemented copies of $c_0(\tau)$ in $\ell_p(I)\widehat{\otimes}_{\pi}X$ and $\ell_p(I)\widehat{\otimes}_{\varepsilon}X$

This section is devoted to the study of the complemented copies of $c_0(\tau)$ in the projective and injective tensor products $\ell_p(I)\widehat{\otimes}_{\pi}X$ and $\ell_p(I)\widehat{\otimes}_{\varepsilon}X$. We begin with the former space.

LEMMA 6.1. Let X be a Banach space, I be an infinite set, $p \in (1, \infty)$ and τ be an infinite cardinal. Let $(\overline{x_j})_{j\in\tau} = ((x_{i,j})_{i\in I})_{j\in\tau}$ be a family of $\ell_p\langle I, X \rangle$ equivalent to the unit basis of $c_0(\tau)$ and let $(\varphi_j)_{j\in\tau} = ((\varphi_{i,j})_{i\in I})_{j\in\tau}$ be a bounded family of $\ell_p\langle I, X \rangle^*$. Then, for each $\varepsilon > 0$, there exists a finite subset $F_{\varepsilon} \subset I$ satisfying, for every $j \in \tau$,

$$\left|\sum_{i\in I\setminus F_{\varepsilon}}\varphi_{i,j}(x_{i,j})\right|<\varepsilon.$$

PROOF. Let $M = \sup_{j \in \tau} \|\varphi_j\|$. We may assume $\|\overline{x_j}\|_{\ell_p\langle X \rangle} = 1$ for every $j \in \tau$. Suppose that the conclusion does not hold; then there exists $\varepsilon > 0$ such that, for every finite subset $F \subset I$, there exists $j \in \tau$ with

$$\left|\sum_{i \notin F} \varphi_{i,j}(x_{i,j})\right| \ge \varepsilon$$

We will show that this leads to a contradiction.

We construct by induction a sequence $(j_k)_{k\geq 1}$ in τ and a sequence of pairwise disjoint finite subsets $(F_k)_{k\geq 1}$ of I such that

(6.1)
$$\left|\sum_{i\in F_k}\varphi_{i,j_k}(x_{i,j_k})\right| \ge \frac{\varepsilon}{2}, \ \forall k\ge 1.$$

L.

Put $F_0 = \emptyset$. By hypothesis, there exists $j_1 \in \tau$ such that

$$\left|\sum_{i\in I}\varphi_{i,j_1}(x_{i,j_1})\right|\geq\varepsilon.$$

Since

$$\sum_{\in I} |\varphi_{i,j_1}(x_{i,j_1})| \le M,$$

there exists a finite subset $\emptyset \neq F_1 \subset I$ such that

1

$$\sum_{i\in I\setminus F_1} |\varphi_{i,j_1}(x_{i,j_1})| \le \frac{\varepsilon}{2},$$

and hence

$$\sum_{i \in F_1} |\varphi_{i,j_1}(x_{i,j_1})| \ge \frac{\varepsilon}{2}$$

Next, let $k \ge 1$ be given and suppose we have constructed j_1, \ldots, j_k and F_1, \ldots, F_k as desired. Put $A = \bigcup_{l=1}^k F_l$. By hypothesis, there exists $j_{k+1} \in \tau$ such that

$$\left|\sum_{i\in I\setminus A}\varphi_{i,j_{k+1}}(x_{i,j_{k+1}})\right|\geq \varepsilon.$$

There exists a finite subset $\emptyset \neq F_{k+1} \subset (I \setminus A)$ such that

$$\sum_{i \in I \setminus (A \cup F_{k+1})} |\varphi_{i,j_{k+1}}(x_{i,j_{k+1}})| \le \frac{\varepsilon}{2}$$

and so

$$\left|\sum_{i\in F_{k+1}}\varphi_{i,j_{k+1}}(x_{i,j_{k+1}})\right|\geq \frac{\varepsilon}{2}.$$

It follows from the construction that $j_l \neq j_k$ for $l \neq k$.

For each $k \ge 1$ and $i \in I$ we denote

$$\psi_{i,k} = \begin{cases} \varphi_{i,j_k} & \text{if } i \in F_k \\ 0 & \text{if } i \in I \setminus F_k, \end{cases}$$

and $\psi_k = (\psi_{i,k})_{i \in I}$. It is obvious that $(\psi_k)_{k \ge 1}$ is a bounded sequence in $F_q(I, X^*) \equiv \ell_p \langle I, X^{**} \rangle$, where $q \in (1, \infty)$ is the conjugate index of p. We claim that $(\psi_k)_{k \ge 1}$ is a weakly-null sequence. Indeed, by Remark 4.6, the subspace

$$L = \{\theta^{**} = (\theta_i^{**})_{i \in I} \in \ell_p \langle I, X^{**} \rangle : \{i \in I : \theta_i^{**} = 0\} \text{ is finite} \}$$

is dense in $\ell_p(I, X^{**})$. Given $\theta^{**} = (\theta_i^{**})_{i \in I} \in L$, there exists $k_0 \ge 1$ such that $\theta_i^{**} = 0$, for all $i \in F_k$, $k \ge k_0$, and hence,

$$\lim_{k \to \infty} \theta^{**}(\psi_k) = \lim_{k \to \infty} \sum_{i \in F_k} \theta^{**}_i(\psi_{i,k}) = 0$$

The density of L then establishes our claim.

The sequence $(\overline{x_{j_k}})_{k\geq 1}$ is equivalent to the unit basis of c_0 and furthermore

$$|\psi_k(\overline{x_{j_k}})| \ge \frac{\varepsilon}{2},$$

for each $k \ge 1$, in contradiction with the Dunford-Pettis Property of c_0 [12, p. 596].

We are now in a position to prove the first main result of this section.

THEOREM 6.2. Given X a Banach space, I an infinite set, $p \in (1, \infty)$ and τ an infinite cardinal, we have

$$c_0(\tau) \stackrel{c}{\hookrightarrow} \ell_p(I) \widehat{\otimes}_{\pi} X \iff c_0(\tau) \stackrel{c}{\hookrightarrow} X.$$

PROOF. Suppose first that $c_0(\tau) \stackrel{c}{\hookrightarrow} \ell_p(I) \widehat{\otimes}_{\pi} X \equiv \ell_p \langle I, X \rangle$. By Theorem 2.2, there exist $(\overline{x_j})_{j \in \tau} = ((x_{i,j})_{i \in I})_{j \in \tau}$ equivalent to the canonical basis of $c_0(\tau)$ in $\ell_p \langle I, X \rangle$, and $(\varphi_j)_{j \in \tau}$ a weak*-null family in $\ell_p \langle I, X \rangle^*$ satisfying $\varphi_j(x_j) = 1$ for all $j \in \tau$. By Theorem 5.1, for each $j \in \tau$, there exists a family $(\varphi_{i,j})_{i \in I}$ in X^* such that

$$1 = \varphi_j(\overline{x_j}) = \sum_{i \in I} \varphi_{i,j}(x_{i,j}).$$

By the Uniform Boundedness Principle, $(\varphi_j)_{j \in \tau}$ is bounded, and so there exists M > 0 such that $\|\varphi_j\| \leq M$ for each $j \in \tau$.

An appeal to Lemma 6.1 yields a finite subset $F \subset I$ satisfying

1

$$\left|\sum_{i\in I\setminus F}\varphi_{i,j}(x_{i,j})\right| < \frac{1}{2}, \ \forall j\in\tau.$$

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Therefore, $F \neq \emptyset$ and

$$\sum_{i \in F} |\varphi_{i,j}(x_{i,j})| > \frac{1}{2}, \ \forall j \in \tau.$$

Next, Lemma 2.4 implies that there exist $i_0 \in F$ and $\tau_1 \subset \tau$ satisfying $|\tau_1| = \tau$ and

(6.2)
$$|\varphi_{i_0,j}(x_{i_0,j})| > \frac{1}{2|F|}, \forall j \in \tau_1.$$

We have

$$\frac{1}{2|F|} < |\varphi_{i_0,j}(x_{i_0,j})| \le M ||x_{i_0,j}||, \ \forall j \in \tau_1,$$

and thus

(6.3)
$$||x_{i_0,j}|| > \frac{1}{2M|F|}, \ \forall j \in \tau_1.$$

Consider the bounded linear operator $\pi_{i_0} : \ell_p \langle I, X \rangle \to X$ defined by $\pi_{i_0}(\overline{x}) = x_{i_0}$ for each $\overline{x} = (x_i)_{i \in I} \in \ell_p \langle I, X \rangle$. By hypothesis, there exists $T : c_0(\tau) \to \ell_p \langle I, X \rangle$ an isomorphism onto its image such that $T(e_j) = \overline{x_j}$, for all $j \in \tau$. By (6.3), we have

$$\|(\pi_{i_0} \circ T)(e_j)\| > \frac{1}{2M|F|}, \ \forall j \in \tau_1,$$

and therefore, by Theorem 2.1, there exists $\tau_2 \subset \tau_1$ such that $|\tau_2| = |\tau_1| = \tau$ and $\pi_{i_0} \circ T_{|c_0(\tau_2)}$ is an isomorphism onto its image. Thus, $(x_{i_0,j})_{j\in\tau_2}$ is equivalent to the unit basis of $c_0(\tau_2)$ in X. The family $(\varphi_j)_{j\in\tau}$ is weak*-null and hence, $(\varphi_{i_0,j})_{j\in\tau_2}$ is a weak*-null family in X*. Theorem 2.2 and (6.2) then imply that $c_0(\tau) \stackrel{c}{\to} X$.

For the converse, it is sufficient to notice that $\ell_p \langle I, X \rangle$ contains a complemented copy of X.

We now turn our attention to complemented copies of $c_0(\tau)$ in the injective tensor product $\ell_p(I)\widehat{\otimes}_{\varepsilon} X$.

THEOREM 6.3. Let X be a Banach space, I be an infinite set, $p \in [1, \infty)$ and τ be an infinite cardinal. If $cf(\tau) > |I|$, then

$$c_0(\tau) \stackrel{c}{\hookrightarrow} \ell_p(I) \widehat{\otimes}_{\varepsilon} X \iff c_0(\tau) \stackrel{c}{\hookrightarrow} X.$$

PROOF. Suppose first that $c_0(\tau) \stackrel{c}{\hookrightarrow} \ell_p(I) \widehat{\otimes}_{\pi} X \equiv F_p(I, X)$. By Theorem 2.2, there exists $(\overline{x_j})_{j \in \tau} = ((x_{i,j})_{i \in I})_{j \in \tau}$ equivalent to the unit basis of $c_0(\tau)$ in $F_p(I, X)$, and a weak*-null family $(\varphi_j)_{j \in \tau} = ((\varphi_{i,j})_{i \in I})_{j \in \tau}$ of $F_p(I, X)^*$ satisfying

$$1 = \varphi_j(\overline{x_j}) = \sum_{i \in I} \varphi_{i,j}(x_{i,j}), \ \forall j \in \tau.$$

For each $j \in \tau$, there exists a finite subset $\emptyset \neq F_j \subset I$ such that

(6.4)
$$\sum_{i \in F_j} |\varphi_{i,j}(x_{i,j})| > \frac{1}{2}$$

Let $\mathcal{F} = \{F_j : j \in \tau\}$, and for each $F \in \mathcal{F}$, consider $\tau(F) = \{j \in \tau : F_j = F\}$. Since $\tau = \bigcup_{F \in \mathcal{F}} \tau_F$ and $cf(\tau) > |I| \ge |\mathcal{F}|$, by Lemma 2.3 there exists $G_0 \in \mathcal{F}$ such that $|\tau(G_0)| = \tau$. Setting $\tau_1 = \tau(G_0)$, by (6.4) we obtain

$$\sum_{i \in G_0} |\varphi_{i,j}(x_{i,j})| > \frac{1}{2}, \ \forall j \in \tau_1.$$

Next, an appeal to Lemma 2.4 yields $i_0 \in G_0$ and $\tau_2 \subset \tau_1$ satisfying $|\tau_2| = |\tau_1| = \tau$ and

(6.5)
$$|\varphi_{i_0,j}(x_{i_0,j})| > \frac{1}{2|G_0|}, \forall j \in \tau_2$$

By the Uniform Boundedness Principle, $(\varphi_j)_{j \in \tau}$ is bounded, and so there exists M > 0 such that $\|\varphi_j\| \leq M$, for each $j \in \tau$. Therefore we have

$$\frac{1}{2|G_0|} < |\varphi_{i_0,j}(x_{i_0,j})| \le M \|x_{i_0,j}\|, \ \forall j \in \tau_2,$$

and thus

(6.6)
$$||x_{i_0,j}|| > \frac{1}{2M|G_0|}, \ \forall j \in \tau_2.$$

Consider the bounded linear operator $\pi_{i_0} : s\ell_p(I, X) \to X$ defined by $\pi_{i_0}(\overline{x}) = x_{i_0}$ for each $\overline{x} = (x_i)_{i \in I} \in s\ell_p(I, X)$. By hypothesis, there exists $T : c_0(\tau) \to F_p(I, X)$ an isomorphism onto its image such that $T(e_j) = \overline{x_j}$, for all $i \in \tau$. By (6.6), we have

$$\|(\pi_{i_0} \circ T)(e_j)\| > \frac{1}{2M|G_0|}, \ \forall j \in \tau_2,$$

and therefore, by Theorem 2.1, there exists $\tau_3 \subset \tau_2$ such that $|\tau_3| = |\tau_2| = \tau$ and $\pi_{i_0} \circ T_{|c_0(\tau_3)}$ is an isomorphism onto its image. Thus, $(x_{i_0,j})_{j \in \tau_3}$ is equivalent to the unit basis of $c_0(\tau_3)$ in X. The family $(\varphi_j)_{j \in \tau}$ is weak*-null and hence, $(\varphi_{i_0,j})_{j \in \tau_3}$ is weak*-null in X*. Theorem 2.2 and (6.5) then imply $c_0(\tau) \stackrel{c}{\hookrightarrow} X$.

For the converse, it suffices to notice that $F_p(I, X)$ contains a complemented copy of X.

Our next step is to obtain a result analogue to Theorem 6.3 without assumptions on the cofinality of τ . In this direction, we begin with the following lemma.

LEMMA 6.4. Let X be a Banach space, I be an infinite set, $p \in (1, \infty)$ and τ be an infinite cardinal. Let $(\overline{x_j})_{j\in\tau} = ((x_{i,j})_{i\in I})_{j\in\tau}$ be a family of $F_p(I, X)$ equivalent to the unit basis of $c_0(\tau)$ and let $(\varphi_j)_{j\in\tau} = ((\varphi_{i,j})_{i\in I})_{j\in\tau}$ be a bounded family of $F_p(I, X)^* \equiv \ell_q \langle I, X^* \rangle$, where $q \in (1, \infty)$ is the conjugate index of p. Suppose that

$$\mathcal{K}(\ell_q(I), X^{**}) = \mathcal{L}(\ell_q(I), X^{**})$$

Then, for each $\varepsilon > 0$, there exists a finite subset $F_{\varepsilon} \subset I$ satisfying

$$\left|\sum_{i\in I\setminus F_{\varepsilon}}\varphi_{i,j}(x_{i,j})\right|<\varepsilon, \ \forall j\in\tau.$$

PROOF. By hypothesis and Theorem 5.3, there exists M > 0 such that

(6.7)
$$M \ge \|\varphi_j\| = \|(\varphi_{i,j})_{i \in I}\|_{\ell_q \langle X^* \rangle}, \ \forall j \in \tau.$$

Suppose by contradiction that the result does not hold. As in the proof of Lemma 6.1, there exist $\varepsilon > 0$, a sequence of distinct indexes $(j_k)_{k\geq 1}$ of τ , and a sequence $(F_k)_{k\geq 1}$ of finite, non-empty and pairwise disjoint subsets of I satisfying

(6.8)
$$\left|\sum_{i\in F_k}\varphi_{i,j_k}(x_{i,j_k})\right| > \varepsilon, \ \forall k \ge 1.$$

Consider $\psi_k = (\psi_{i,k})_{i \in I} \in \ell_q \langle I, X^* \rangle$, where

$$\psi_{i,k} = \begin{cases} \varphi_{i,j_k} & \text{ if } i \in F_k \\ 0 & \text{ if } i \in I \setminus F_k. \end{cases}$$

By (6.7) we have

(6.9)
$$\|\psi_k\|_{\ell_q\langle X^*\rangle} \le \|(\varphi_{i,j_k})_{i\in I}\|_{\ell_q\langle X^*\rangle} = \|\varphi_{j_k}\| \le M, \ \forall k \ge 1.$$

We claim that $(\psi_k)_{k\geq 1}$ is a weakly-null sequence in $\ell_q(I, X^*)$. Recall that

$$\ell_q \langle I, X^* \rangle^* \equiv s \ell_p(I, X^{**}) \equiv \mathcal{L}(\ell_q(I), X^{**})$$

and

$$F_p(I, X^{**}) \equiv \mathcal{K}(\ell_q(I), X^{**}).$$

Thus, by hypothesis, we have

$$\ell_q \langle I, X^* \rangle^* \equiv F_p(I, X^{**}).$$

The subspace

$$L = \{\theta^{**} = (\theta_i^{**})_{i \in I} \in F_p(I, X^{**}) : \{i \in I : \theta_i^{**} = 0\} \text{ is finite} \}$$

is dense in $F_p(I, X^{**})$. Observe that if $\theta^{**} = (\theta_i^{**})_{i \in I} \in L$, there exists $k_0 \ge 1$ such that $\theta_i^{**} = 0$, for all $i \in F_k$, $k \ge k_0$, and hence,

$$\lim_{k \to \infty} \theta^{**}(\psi_k) = \lim_{k \to \infty} \sum_{i \in F_k} \theta_i^{**}(\psi_{i,k}) = 0.$$

The density of L then establishes our claim.

Now, notice that

$$|\psi_k(\overline{x_{j_k}})| = \left|\sum_{i \in I} \psi_{i,k}(x_{i,j_k})\right| = \left|\sum_{i \in F_k} \varphi_{i,j_k}(x_{i,j_k})\right| > \frac{\varepsilon}{2}, \ \forall k \ge 1,$$

by construction. On the other hand, since the sequence $(\psi_k)_{k\geq 1}$ is weakly-null in $F_p(I, X)^*$ and $(\overline{x_{j_k}})_{k\geq 1}$ is, by hypothesis, equivalent to the unit basis of c_0 , by the Dunford-Pettis Property of c_0 [12, p. 596] we know that $\psi_k(x_{j_k}) \longrightarrow 0$, a contradiction that finishes the proof.

THEOREM 6.5. Let X be a Banach space, I be an infinite set, $p \in (1, \infty)$ and τ be an infinite cardinal. Suppose that

$$\mathcal{K}(\ell_q(I), X^{**}) = \mathcal{L}(\ell_q(I), X^{**}),$$

where $q \in (1, \infty)$ is the conjugate index of p. Then

$$c_0(\tau) \stackrel{c}{\hookrightarrow} \ell_p(I) \widehat{\otimes}_{\varepsilon} X \iff c_0(\tau) \stackrel{c}{\hookrightarrow} X.$$

PROOF. Suppose first that $c_0(\tau) \stackrel{c}{\hookrightarrow} \ell_p(I) \widehat{\otimes}_{\varepsilon} X \equiv F_p(I, X)$. By Theorem 2.2, there exist $(\overline{x_j})_{j \in \tau} = ((x_{i,j})_{i \in I})_{j \in \tau}$ equivalent to the unit basis of $c_0(\tau)$ in $F_p(I, X)$, and a weak*-null family $(\varphi_j)_{j \in \tau} = ((\varphi_{i,j})_{i \in I})_{j \in \tau}$ of $F_p(I, X)^*$ satisfying

$$1 = \varphi_j(\overline{x_j}) = \sum_{i \in I} \varphi_{i,j}(x_{i,j}), \ \forall j \in \tau.$$

By the Uniform Boundedness Principle, $(\varphi_j)_{j\in\tau}$ is bounded, and so there exists M>0 such that

(6.10)
$$\|\varphi_j\| \le M, \forall j \in \tau.$$

An appeal to Lemma 6.4 yields a finite subset $F \subset I$ satisfying

$$\left|\sum_{i\in I\setminus F}\varphi_{i,j}(x_{i,j})\right| < \frac{1}{2}, \ \forall j\in\tau.$$

Therefore, $F \neq \emptyset$ and

$$\sum_{i \in F} |\varphi_{i,j}(x_{i,j})| \ge \left| \sum_{i \in F} \varphi_{i,j}(x_{i,j}) \right| \ge \left| \sum_{i \in I} \varphi_{i,j}(x_{i,j}) \right| - \left| \sum_{i \in I \setminus F} \varphi_{i,j}(x_{i,j}) \right| > \frac{1}{2}, \forall j \in \tau.$$

Next, Lemma 2.4 implies that there exist $i_0 \in F$ and $\tau_1 \subset \tau$ satisfying $|\tau_1| = \tau$ and

(6.11)
$$|\varphi_{i_0,j}(x_{i_0,j})| > \frac{1}{2|F|}, \ \forall j \in \tau_1$$

By (6.10), we have

$$\frac{1}{2|F|} < |\varphi_{i_0,j}(x_{i_0,j})| \le \|\varphi_{i_0,j}\| \|x_{i_0,j}\| \le \|\varphi_j\| \|x_{i_0,j}\| \le M \|x_{i_0,j}\|, \ \forall j \in \tau_1,$$

(6.12)
$$||x_{i_0,j}|| > \frac{1}{2M|F|}, \forall j \in \tau_1.$$

Consider the bounded linear operator $\pi_{i_0}: F_p(I, X) \to X$ defined by $\pi_{i_0}(\overline{x}) = x_{i_0}$ for each $\overline{x} = (x_i)_{i \in I} \in F_p(I, X)$. By hypothesis, there exists $T: c_0(\tau) \to F_p(I, X)$ an isomorphism onto its image such that $T(e_j) = \overline{x_j}$, for all $j \in \tau$. By (6.12), we have

$$\|(\pi_{i_0} \circ T)(e_j)\| > \frac{1}{2M|F|}, \ \forall j \in \tau_1,$$

and therefore, by Theorem 2.1, there exists $\tau_2 \subset \tau_1$ such that $|\tau_2| = |\tau_1| = \tau$ and $\pi_{i_0} \circ T_{|c_0(\tau_2)}$ is an isomorphism onto its image. Thus, $(x_{i_0,j})_{j \in \tau_2}$ is equivalent to the unit basis of $c_0(\tau_2)$ in X.

Next, given $x \in X$, we have

$$(\varphi_{i_0,j}(x))_{j\in\tau_2}\in c_0(\tau_2),$$

since, by hypothesis, $(\varphi_j)_{j\in\tau}$ is weak*-null. This proves that $(\varphi_{i_0,j})_{j\in\tau_2}$ is weak*-null in X^* . Theorem 2.2 and (6.11) then imply that $c_0(\tau) \stackrel{c}{\hookrightarrow} X$.

For the converse, it is sufficient to notice that $F_p(I, X)$ contains a complemented copy of X.

7. Final remarks and open problems

By adapting the proofs of Lemma 6.1 and Theorem 6.2, one can prove the following.

THEOREM 7.1. Given X a Banach space, I an infinite set, $p \in [1,\infty)$ and τ an infinite cardinal, we have

$$c_0(\tau) \stackrel{c}{\hookrightarrow} \ell_p(I, X) \iff c_0(\tau) \stackrel{c}{\hookrightarrow} X.$$

The above theorem extends to the uncountable case a c_0 result by Bombal [4, Theorem 4.3.1]. Moreover, since $\ell_1(I) \widehat{\otimes}_{\pi} X$ is isometrically isomorphic to $\ell_1(I, X)$ [20, p. 19], the next corollary follows immediately from Theorems 6.2 and 7.1.

COROLLARY 7.2. Given X a Banach space, I an infinite set, $p \in [1, \infty)$ and τ an infinite cardinal, we have

$$c_0(\tau) \stackrel{c}{\hookrightarrow} \ell_p(I) \widehat{\otimes}_{\pi} X \iff c_0(\tau) \stackrel{c}{\hookrightarrow} \ell_p(I, X) \iff c_0(\tau) \stackrel{c}{\hookrightarrow} X.$$

However, observe that the space $\ell_p(I, X)$ cannot be exchanged by the usual Banach space of Lebesque integrable functions $L_p([0, 1], X)$ in the statement of Theorem 7.1, even in the case $\tau = \aleph_0$. Indeed, recall the following result due to Emmanuelle [10, Main Theorem].

THEOREM 7.3. Let X be a Banach space and $p \in [1, \infty)$. Then

$$c_0 \hookrightarrow X \implies c_0 \stackrel{c}{\hookrightarrow} L_p([0,1],X).$$

Thus, we have

$$c_0 \stackrel{c}{\hookrightarrow} L_p([0,1],\ell_\infty) \text{ but } c_0 \not\hookrightarrow \ell_\infty.$$

These facts arise naturally the following question.

PROBLEM 7.4. Let X be a Banach space, $p \in [1, \infty)$ and τ be an infinite cardinal. What assumptions on τ yield

$$c_0(\tau) \stackrel{c}{\hookrightarrow} \ell_p([0,1],X) \iff c_0(\tau) \stackrel{c}{\hookrightarrow} X?$$

We do not know if Theorem 6.2 can be extended to the case $p = \infty$. That is:

PROBLEM 7.5. Let X be a Banach space and τ be an infinite cardinal. Is it true that

$$c_0(\tau) \stackrel{c}{\hookrightarrow} \ell_{\infty} \widehat{\otimes}_{\pi} X \implies c_0(\tau) \stackrel{c}{\hookrightarrow} X?$$

Finally, we also do not know if Theorem 6.3 is optimal for any cardinal τ . In particular, the following problem is unsolved.

PROBLEM 7.6. Let X be a Banach space and $p \in [1, \infty)$. Does it follow that

$$c_0(\aleph_\omega) \stackrel{c}{\hookrightarrow} \ell_p \widehat{\otimes}_{\varepsilon} X \implies c_0(\aleph_\omega) \stackrel{c}{\hookrightarrow} X?$$

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