



HAL
open science

CONTROLLABILITY OF PERIODIC BILINEAR QUANTUM SYSTEMS ON INFINITE GRAPHS

Kaïs Ammari, Alessandro Duca

► **To cite this version:**

Kaïs Ammari, Alessandro Duca. CONTROLLABILITY OF PERIODIC BILINEAR QUANTUM SYSTEMS ON INFINITE GRAPHS. 2019. hal-02169344v1

HAL Id: hal-02169344

<https://hal.science/hal-02169344v1>

Preprint submitted on 1 Jul 2019 (v1), last revised 7 Sep 2020 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

CONTROLLABILITY OF PERIODIC BILINEAR QUANTUM SYSTEMS ON INFINITE GRAPHS

KAÏS AMMARI AND ALESSANDRO DUCA

ABSTRACT. In this work, we study the controllability of the bilinear Schrödinger equation on infinite graphs for periodic quantum states. We consider the equation (BSE) $i\partial_t\psi = -\Delta\psi + u(t)B\psi$ in the Hilbert space L_p^2 composed by functions defined on an infinite graph \mathcal{G} verifying periodic boundary conditions on the infinite edges. The Laplacian $-\Delta$ is equipped with specific boundary conditions, B is a bounded symmetric operator and $u \in L^2((0, T), \mathbb{R})$ with $T > 0$. We present the well-posedness of the (BSE) in suitable subspaces of L_p^2 . In such spaces, we study the global exact controllability and we provide examples involving for instance tadpole graphs and star graphs with infinite spokes.

CONTENTS

1. Introduction and preliminaries	1
2. Infinite tadpole graph	3
3. Star graph with infinite spokes	7
4. Generic graphs	11
Appendix A. Global approximate controllability	14
References	15

1. INTRODUCTION AND PRELIMINARIES

Let \mathcal{G} be a generic graph composed by N finite edges $\{e_j\}_{1 \leq j \leq N}$ of lengths $\{L_j\}_{1 \leq j \leq N}$ and \tilde{N} half-lines $\{e_j\}_{N+1 \leq j \leq N+\tilde{N}}$. Each edge e_j with $1 \leq j \leq N$ is associated to a coordinate starting from 0 and going to L_j , while e_j with $N+1 \leq j \leq N+\tilde{N}$ is parametrized with a coordinate starting from 0 and going to $+\infty$. We consider \mathcal{G} as domain of functions

$$f := (f^1, \dots, f^{N+\tilde{N}}) : \mathcal{G} \rightarrow \mathbb{C}, \quad f^j : e_j \rightarrow \mathbb{C}, \quad 1 \leq j \leq N+\tilde{N}.$$

Let $\{L_j\}_{N+1 \leq j \leq N+\tilde{N}} \subset \mathbb{R}^+$. We consider the Hilbert space

$$(1) \quad L_p^2 = \left(\prod_{j=1}^N L^2(e_j, \mathbb{C}) \right) \times \left(\prod_{j=N+1}^{N+\tilde{N}} L_p^2(e_j, \mathbb{C}) \right), \quad \text{with}$$

$$L_p^2(e_j, \mathbb{C}) = \left\{ f \in L_{loc}^2(e_j, \mathbb{C}) : f(\cdot) = f(\cdot + 2\pi k L_j), \quad \forall k \in \mathbb{N}^* \right\}, \quad N+1 \leq j \leq N+\tilde{N}.$$

2010 *Mathematics Subject Classification.* 35Q40, 93B05, 93C05.

Key words and phrases. Bilinear control, infinite graph.

The Hilber spaces L_p^2 is equipped with the norm $\|\cdot\|_{L_p^2}$ induced by the scalar product

$$\langle \psi, \varphi \rangle_{L_p^2} = \sum_{j=1}^{N+\tilde{N}} \int_0^{L_j} \overline{\psi^j(x)} \varphi^j(x) dx, \quad \forall \psi, \varphi \in L_p^2.$$

We introduce the spaces $H_p^s := L_p^2 \cap \left(\left(\prod_{j=1}^N H^s(e_j, \mathbb{C}) \right) \times \left(\prod_{j=N+1}^{N+\tilde{N}} H_{loc}^s(e_j, \mathbb{C}) \right) \right)$ with $s > 0$ equipped with the norm $\|\cdot\|_{H_p^s}$ such that, for every $f = (f^1, \dots, f^{N+\tilde{N}}) \in H_p^s$,

$$(2) \quad \|f\|_{H_p^s}^2 = \sum_{l=1}^N \|f^l\|_{H^s(e_l, \mathbb{C})}^2 + \sum_{l=N+1}^{N+\tilde{N}} \|f^l\|_{H^s((0, L_l), \mathbb{C})}^2.$$

For $T > 0$, we consider the bilinear Schrödinger equation in L_p^2

$$(BSE) \quad \begin{cases} i\partial_t \psi(t) = -A\psi(t) + u(t)B\psi(t), & t \in (0, T), \\ \psi(0, x) = \psi_0(x). \end{cases}$$

The operator A is a Laplacian equipped with suitable boundary conditions such that $D(A)$ is contained in H_p^2 . The operator B is a bounded symmetric operator in L_p^2 and $u \in L^2((0, T), \mathbb{R})$ with $T > 0$. We respectively denote

$$\varphi := (\varphi_k)_{k \in \mathbb{N}^*}, \quad (\mu_k)_{k \in \mathbb{N}^*}$$

an orthonormal system of L_p^2 made by eigenfunctions of A and the corresponding eigenvalues. For $s > 0$, we define the spaces $\mathcal{H}(\varphi) := \overline{\text{span}\{\varphi_k\} | k \in \mathbb{N}^*}^{L_p^2}$ and

$$(3) \quad \begin{aligned} H_{\mathcal{G}}^s(\varphi) &:= \{\psi \in \mathcal{H}(\varphi) \mid \sum_{k \in \mathbb{N}^*} |k^s \langle \varphi_k, \psi \rangle_{L_p^2}|^2 < \infty\}, \\ h^s &:= \left\{ (a_k)_{k \in \mathbb{N}^*} \in \ell^2(\mathbb{C}) \mid \sum_{k \in \mathbb{N}^*} |k^s a_k|^2 < \infty \right\}. \end{aligned}$$

We respectively equip $H_{\mathcal{G}}^s(\varphi)$ and h^s with the norms $\|\cdot\|_{(s)} = \left(\sum_{k \in \mathbb{N}^*} |k^s \langle \varphi_k, \cdot \rangle_{L_p^2}|^2 \right)^{\frac{1}{2}}$ and

$$\|\mathbf{x}\|_{(s)} = \left(\sum_{k \in \mathbb{N}^*} |k^s x_k|^2 \right)^{\frac{1}{2}}, \quad \forall \mathbf{x} := (x_k)_{k \in \mathbb{N}^*} \in h^s.$$

Let Γ_T^u be the unitary propagator (when it is defined) corresponding to the dynamics of (BSE) in the time interval $[0, T]$.

Definition 1.1. The bilinear Schrödinger equation (BSE) is said to be globally exactly controllable in $H_{\mathcal{G}}^s(\varphi)$ for suitable φ and $s > 0$ when, for every $\psi_1, \psi_2 \in H_{\mathcal{G}}^s(\varphi)$ such that $\|\psi_1\|_{L_p^2} = \|\psi_2\|_{L_p^2}$, there exist $T > 0$ and $u \in L^2((0, T), \mathbb{R})$ such that

$$\Gamma_T^u \psi_1 = \psi_2.$$

The bilinear Schrödinger equation (BSE) is said to be energetically controllable with respect to some energetic levels $(\mu_k)_{k \in \mathbb{N}^*}$ when, for every $m, n \in \mathbb{N}^*$, there exist $T > 0$ and $u \in L^2((0, T), \mathbb{R})$ such that

$$\Gamma_T^u \varphi_m = \varphi_n.$$

The aim of the work is to study the global exact controllability of the (BSE) on infinite graphs for states in suitable spaces $H_{\mathcal{G}}^s(\varphi)$ with $s > 0$. From such result, we deduce the energetic controllability with respect to $(\mu_k)_{k \in \mathbb{N}^*}$.

Graph type models have been largely used to study complicated phenomena with simpler settings and we focus our attention on the metric graphs. When a metric graph is equipped with a self-adjoint operator as a Schrödinger Hamiltonian with specific boundary conditions on the vertices, it is called quantum graph (we refer the reader for non-linear phenomena to [1, 9, 10, 16, 17, 18, 19, 20, 21]).

The controllability of the bilinear Schrödinger equation on bounded intervals has been widely studied in the literature starting by the seminal work on bilinear systems of Ball, Mardsen and Slemrod [4]. For further results, we refer to [5, 6, 7, 8, 14, 15].

The paper is organized as follows. In Sections 2 and Section 3, we prove the global exact controllability for tadpole and star-shaped with infinite spokes graphs, respectively. In the last section, we generalize the previous results to some generic infinite graphs.

2. INFINITE TADPOLE GRAPH

Let \mathcal{T} be an *infinite tadpole graph* composed by two edges e_1 and e_2 . The self-closing edge e_1 , the “head”, is connected to e_2 in the vertex v and it is parametrized in the clockwise direction with a coordinate going from 0 to 1 (the length of e_1). The “tail” e_2 is an half-line equipped with a coordinate starting from 0 in v and going to $+\infty$. The tadpole graph presents a natural symmetry axis that we denote by r .

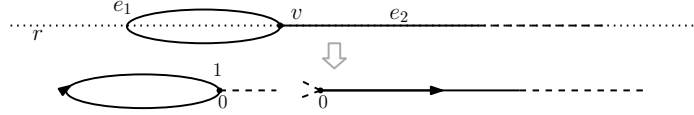


Figure 1. The parametrization of the infinite tadpole graph and its natural symmetric axis r .

Let L_p^2 be composed by functions which are periodic on the tail with period 1, *i.e.* $L_2 = 1$. We consider the bilinear Schrödinger equation (BSE) in L_p^2 with $A = -\Delta$ the Laplacian equipped with *Neumann-Kirchhoff* boundary conditions in the vertex v , *i.e.*

$$D(A) = \left\{ \psi = (\psi^1, \psi^2) \in H_p^2 : \psi \in C^0(\mathcal{T}, \mathbb{C}), \quad \frac{\partial \psi^1}{\partial x}(0) + \frac{\partial \psi^1}{\partial x}(1) + \frac{\partial \psi^2}{\partial x}(0) = 0 \right\}.$$

We assume the control field $B : \psi = (\psi^1, \psi^2) \mapsto (V^1 \psi^1, V^2 \psi^2)$ being such that

$$V^1(x) = x^2(x-1)^2, \quad V^2(x) = \sum_{n \in \mathbb{N}} (x-n)^2(x-n-1)^2 \chi_{[n, n+1]}(x).$$

In this framework, the (BSE) corresponds to the two following Cauchy systems respectively in $L^2(e_1, \mathbb{C})$ and $L_p^2(e_2, \mathbb{C})$

$$(BSEt) \quad \begin{cases} i\partial_t \psi^1 = -\Delta \psi^1 + u V^1 \psi^1, & \begin{cases} i\partial_t \psi^2 = -\Delta \psi^2 + u V^2 \psi^2, \\ \psi^2(0) = \psi_0^2. \end{cases} \\ \psi^1(0) = \psi_0^1, \end{cases}$$

Let $\varphi := (\varphi_k)_{k \in \mathbb{N}^*}$ be an orthonormal system of L_p^2 made by eigenfunctions of A and corresponding to the eigenvalues $(\mu_k)_{k \in \mathbb{N}^*}$ such that, for every $k \in \mathbb{N}^* \setminus \{1\}$,

$$\begin{cases} \varphi_k = (\cos(2(k-1)\pi x), \cos(2(k-1)\pi x)), & \mu_k = 4(k-1)^2 \pi^2, \\ \varphi_1 = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}), & \mu_1 = 0. \end{cases}$$

Remark 2.1. We notice that each $f = (f^1, f^2) \in L_p^2$ belongs to $\mathcal{H}(\varphi)$ if and only if:

- f^1 is symmetric with respect to the symmetry axis r of \mathcal{T} ;
- f^2 has period 2π and $f^2|_{[2n\pi, 2(n+1)\pi]} \equiv f^1$ for every $n \in \mathbb{N}$.

Proposition 2.2. *Let $\psi_0 \in H_{\mathcal{T}}^4(\varphi)$ and $u \in L^2((0, T), \mathbb{R})$. There exists a unique mild solution of the (BSEt) in $H_{\mathcal{T}}^4(\varphi)$, i.e. a function $\psi \in C_0([0, T], H_{\mathcal{T}}^4(\varphi))$ such that*

$$(4) \quad \psi(t) = e^{i\Delta t} \psi_0 - i \int_0^t e^{i\Delta(t-s)} u(s) B \psi(s) ds.$$

In conclusion, the flow of (BSEt) on $\mathcal{H}(\varphi)$ can be extended as a unitary flow Γ_T^u with respect to the L_p^2 -norm such that $\Gamma_T^u \psi_0 = \psi(t)$ for any solution ψ of (BSEt) with initial data $\psi_0 \in \mathcal{H}(\varphi)$.

Proof. 1) Unitary flow. We consider Remark 2.1. For every $f = (f^1, f^2) \in \mathcal{H}(\varphi)$, we notice that $(Bf)^1$ inherits from f^1 the property of being symmetric with respect to the symmetry axis r , while $(Bf)^2|_{[2n\pi, 2(n+1)\pi]} \equiv (Bf)^1$ for every $n \in \mathbb{N}$ as $f^2|_{[2n\pi, 2(n+1)\pi]} \equiv f^1$ for every $n \in \mathbb{N}$. Now, $(Bf)^2$ has period 2π and $(Bf)^2(x) = (Bf)(2(n+1)\pi - x)$ for every $n \in \mathbb{N}$ and $x \in [2n\pi, (2n+1)\pi]$. Thus, $Bf = (Vf^1, Vf^2) \in \mathcal{H}(\varphi)$ for every $f = (f^1, f^2) \in \mathcal{H}(\varphi)$ and the control field B preserves the space $\mathcal{H}(\varphi)$.

The space $\mathcal{H}(\varphi)$ is an Hilbert space where the operator A is self-adjoint and B is bounded symmetric. Thanks to [4, Theorem 2.5], the (BSEt) admits and unique solution $\psi \in C^0([0, T], \mathcal{H}(\varphi))$ for every $T > 0$ and $\psi_0 \in \mathcal{H}(\varphi)$.

The flow of (BSEt) is unitary in $\mathcal{H}(\varphi)$ thanks to the following arguments. If $u \in C^0((0, T), \mathbb{R})$, then $\psi \in C^1((0, T), \mathcal{H}(\varphi))$ and $\partial_t \|\psi(t)\|_{L_p^2}^2 = 0$ from (BSEt). Thus $\|\psi(t)\|_{L_p^2} = \|\psi_0\|_{L_p^2}$. The generalization for $u \in L^2((0, T), \mathbb{R})$ follows from a classical density argument, which ensures that the flow of the dynamics of the (BSEt) is unitary in $\mathcal{H}(\varphi)$.

2) Regularity of the integral term in the mild solution. The remaining part of the proof follows from the arguments leading to [8, Lemma 1; Proposition 2] (also adopted in the proof of [2, Proposition 2.1]). Let $\psi \in C^0([0, T], H_{\mathcal{T}}^4(\varphi))$ with $T > 0$. We notice $B\psi(s) \in H_p^4 \cap H_{\mathcal{T}}^2(\varphi)$ for almost every $s \in (0, t)$ and $t \in (0, T)$. Let $G(t) = \int_0^t e^{i\Delta(t-s)} u(s) B\psi(s, x) ds$ so that

$$\|G(t)\|_{(4)} = \left(\sum_{k \in \mathbb{N}^*} \left| k^4 \int_0^t e^{i\mu_k s} \langle \varphi_k, u(s) B\psi(s, \cdot) \rangle_{L_p^2} ds \right|^2 \right)^{\frac{1}{2}}.$$

For $f(s, \cdot) := u(s) B\psi(s, \cdot)$ such that $f = (f^1, f^2)$ and $k \in \mathbb{N}^* \setminus \{1\}$, we have

$$\begin{aligned} \langle \varphi_k, f(s) \rangle_{L_p^2} &= -\frac{1}{\mu_k} \left(\int_0^1 \varphi_k^1(y) \partial_x^2 f^1(s, y) dy + \int_0^1 \varphi_k^2(y) \partial_x^2 f^2(s, y) dy \right) \\ &= -\frac{2}{\mu_k} \int_0^1 \varphi_k^1(y) \partial_x^2 f^1(s, y) dy = \frac{1}{4(k-1)^3 \pi^3} \int_0^1 \sin(2(k-1)\pi x) \partial_x^3 f^1(s, y) dy \\ &= \frac{1}{8(k-1)^4 \pi^4} \left(\partial_x^3 f^1(s, 1) - \partial_x^3 f^1(s, 0) - \int_0^1 \cos(2(k-1)\pi x) \partial_x^4 f^1(s, y) dy \right). \end{aligned}$$

In the last relations, we considered that $\varphi_k^1(\cdot) \partial_x^2 f^1(s, \cdot)|_{[0,1]} = \varphi_k^2(\cdot) \partial_x^2 f^2(s, \cdot)|_{[0,1]}$ as $\partial_x^2 f^1(s, \cdot)|_{[0,1]} = \partial_x^2 f^2(s, \cdot)|_{[0,1]}$. We recall the norm of H_p^4 provided in (2). Equivalently to the first point of the proof of [2, Proposition 2.1], there exists $C_1 > 0$ such that

$$\|G(t)\|_{(4)} \leq C_1 \left(\left\| \int_0^t (\partial_x^3 f^1(s, 1) - \partial_x^3 f^1(s, 0)) e^{i\mu(\cdot)s} ds \right\|_{\ell^2} + \sqrt{t} \|f\|_{L^2((0,t), H_p^4)} \right).$$

Thanks [11, Proposition B.6], there exist $C_2(t) > 0$ uniformly bounded for t in bounded intervals such that $\|G(t)\|_{(4)} \leq C_2(t)\|f(\cdot, \cdot)\|_{L^2((0,t), H_p^4)}$. For every $t \in [0, T]$, the last inequality shows that $G(t) \in H_{\mathcal{T}}^4(\varphi)$ and the provided upper bound is uniform. The Dominated Convergence Theorem leads to $G \in C^0([0, T], H_{\mathcal{T}}^4(\varphi))$.

3) Conclusion. As $\text{Ran}(B|_{H_{\mathcal{T}}^4(\varphi)}) \subseteq H_p^4 \cap H_{\mathcal{T}}^2(\varphi) \subseteq H_p^4$, we have $B \in \mathcal{L}(H_{\mathcal{T}}^4(\varphi), H_p^4)$ thanks to the arguments of [12, Remark 2.1]. Let $\psi_0 \in H_{\mathcal{T}}^4(\varphi)$. We consider the map

$$F : \psi \in C^0([0, T], H_{\mathcal{T}}^4(\varphi)) \mapsto \phi \in C^0([0, T], H_{\mathcal{T}}^4(\varphi)),$$

$$\phi(t) = F(\psi)(t) = e^{i\Delta t}\psi_0 - \int_0^t e^{i\Delta(t-s)}u(s)B\psi(s)ds, \quad \forall t \in [0, T].$$

For every $\psi_1, \psi_2 \in C^0([0, T], H_{\mathcal{T}}^4(\varphi))$, from the first point of the proof, there exists $C(t) > 0$ uniformly bounded for t lying on bounded intervals such that

$$\begin{aligned} \|F(\psi_1) - F(\psi_2)\|_{L^\infty((0,T), H_{\mathcal{T}}^4(\varphi))} &\leq \left\| \int_0^{\cdot} e^{i\Delta(\cdot-s)}u(s)B(\psi_1(s) - \psi_2(s))ds \right\|_{L^\infty((0,T), H_{\mathcal{T}}^4(\varphi))} \\ &\leq C(T)\|u\|_{L^2((0,T), \mathbb{R})} \|B\|_{\mathcal{L}(H_{\mathcal{T}}^4(\varphi), H_p^4)} \|\psi_1 - \psi_2\|_{L^\infty((0,T), H_{\mathcal{T}}^4(\varphi))}. \end{aligned}$$

If $\|u\|_{L^2((0,T), \mathbb{R})}$ is small enough, then F is a contraction and Banach Fixed Point Theorem yields the existence of $\psi \in C^0([0, T], H_{\mathcal{T}}^4(\varphi))$ such that $F(\psi) = \psi$. When $\|u\|_{L^2((0,t), \mathbb{R})}$ is not sufficiently small, we decompose $(0, T)$ with a sufficiently thin partition $\{t_j\}_{0 \leq j \leq n}$ with $n \in \mathbb{N}^*$ such that each $\|u\|_{L^2([t_{j-1}, t_j], \mathbb{R})}$ is so small such that F , defined on the interval $[t_{j-1}, t_j]$, is a contraction. \square

By recalling the definitions of global exact controllability and energetic controllability provided in Definition 1.1, we present the following result.

Theorem 2.3. *The (BSEt) is globally exactly controllable in $H_{\mathcal{T}}^4(\varphi)$ and energetically controllable in $(\mu_k)_{k \in \mathbb{N}^*}$.*

Proof. The statement is proved by using the arguments leading to [2, Theorem 2.2].

1) Local exact controllability. We notice that $\varphi_1(T) = e^{-i\mu_1 T}\varphi_1 = \varphi_1$ with $T > 0$ as the first eigenvalue μ_1 is equal to 0. For $\epsilon, T > 0$, we define

$$O_{\epsilon, T}^4 := \{\psi \in H_{\mathcal{T}}^4(\varphi) \mid \|\psi\|_{L_p^2} = 1, \|\psi - \varphi_1\|_{(4)} < \epsilon\}.$$

We ensure there exist $T, \epsilon > 0$ so that, for every $\psi \in O_{\epsilon, T}^4$, there exists $u \in L^2((0, T), \mathbb{R})$ such that $\psi = \Gamma_T^u \varphi_1$. The result can be proved by showing the surjectivity of the map

$$\Gamma_T^{(\cdot)} \varphi_1 : u \in L^2((0, T), \mathbb{R}) \mapsto \psi \in O_{\epsilon, T}^4, \quad \Gamma_t^{(\cdot)} \varphi_1 = \sum_{k \in \mathbb{N}^*} \varphi_k(t) \langle \varphi_k(t), \Gamma_t^{(\cdot)} \varphi_1 \rangle_{L_p^2},$$

for $T > 0$ large enough. We recall the definition of h^4 provided in (3). Let α be the map defined as the sequence with elements $\alpha_k(u) = \langle \varphi_k(T), \Gamma_T^u \varphi_1 \rangle_{L_p^2}$ for $k \in \mathbb{N}^*$ such that

$$\alpha : L^2((0, T), \mathbb{R}) \longrightarrow Q := \{\mathbf{x} := (x_k)_{k \in \mathbb{N}^*} \in h^4(\mathbb{C}) \mid \|\mathbf{x}\|_{\ell^2} = 1\}.$$

The local exact controllability follows from the local surjectivity of α in a neighborhood of $\alpha(0) = \delta = (\delta_{k,1})_{k \in \mathbb{N}^*}$ with respect to the h^4 -norm. To this end, we consider the Generalized Inverse Function Theorem ([13, Theorem 1; p. 240]) and we study the surjectivity of $\gamma(v) := (d_u \alpha(0)) \cdot v$ the Fréchet derivative of α . Let $B_{k,1} := \langle \varphi_k, B \varphi_1 \rangle_{L_p^2}$ with

$k \in \mathbb{N}^*$. The map γ is the sequence of elements $\gamma_k(v) := -i \int_0^T v(\tau) e^{i(\mu_k - \mu_1)\tau} d\tau B_{k,1}$ with $k \in \mathbb{N}^*$ so that

$$\gamma : L^2((0, T), \mathbb{R}) \longrightarrow T_\delta Q = \{\mathbf{x} := (x_k)_{k \in \mathbb{N}^*} \in h^4(\mathbb{C}) \mid ix_1 \in \mathbb{R}\}.$$

The surjectivity of γ corresponds to the solvability of the moments problem

$$(5) \quad x_k B_{k,1}^{-1} = -i \int_0^T u(\tau) e^{i(\mu_k - \mu_1)\tau} d\tau, \quad \forall \{x_k\}_{k \in \mathbb{N}^*} \in T_\delta Q \subset h^4.$$

By direct computation, there exists $C > 0$ such that $|\langle \varphi_1, B\varphi_1 \rangle_{L_p^2}| \geq \frac{C}{k^4}$ for every $k \in \mathbb{N}^*$ and

$$(x_k B_{k,1}^{-1})_{k \in \mathbb{N}^*} \in \ell^2, \quad ix_1 B_{k,1}^{-1} \in \mathbb{R}.$$

In conclusion, the solvability of (5) is guaranteed by [11, Proposition B.5] since

$$(ix_k B_{k,1}^{-1})_{k \in \mathbb{N}^*} \in \{(c_k)_{k \in \mathbb{N}^*} \in \ell^2 \mid c_1 \in \mathbb{R}\}, \quad \inf_{k \in \mathbb{N}^*} |\mu_{k+1} - \mu_k| = 4\pi^2.$$

2) Global exact controllability. Let $T, \epsilon > 0$ be so that **1)** is valid. Thanks to Remark A.3, for any $\psi_1, \psi_2 \in H_\mathcal{T}^4(\varphi)$ such that $\|\psi_1\|_{L_p^2} = \|\psi_2\|_{L_p^2} = p$, there exist $T_1, T_2 > 0$, $u_1 \in L^2((0, T_1), \mathbb{R})$ and $u_2 \in L^2((0, T_2), \mathbb{R})$ such that

$$\|\Gamma_{T_1}^{u_1} p^{-1} \psi_1 - \varphi_1\|_{(4)} < \epsilon, \quad \|\Gamma_{T_2}^{u_2} p^{-1} \psi_2 - \varphi_1\|_{(4)} < \epsilon$$

and $p^{-1} \Gamma_{T_1}^{u_1} \psi_1, p^{-1} \Gamma_{T_2}^{u_2} \psi_2 \in O_{\epsilon, T}^4$. From **1)**, there exist $u_3, u_4 \in L^2((0, T), \mathbb{R})$ such that

$$\Gamma_T^{u_3} \Gamma_{T_1}^{u_1} \psi_1 = \Gamma_T^{u_4} \Gamma_{T_2}^{u_2} \psi_2 = p\varphi_1 \implies \exists T > 0, \tilde{u} \in L^2((0, \tilde{T}), \mathbb{R}) : \Gamma_{\tilde{T}}^{\tilde{u}} \psi_1 = \psi_2.$$

3) Energetic controllability. The energetic controllability follows as $\varphi_k \in H_\mathcal{T}^4(\varphi)$ for every $k \in \mathbb{N}^*$. \square

Let $\Phi := (\phi_k)_{k \in \mathbb{N}^*}$ be an orthonormal system of L_p^2 made by eigenfunctions of A and corresponding to the eigenvalues $(\lambda_k)_{k \in \mathbb{N}^*}$ such that

$$\phi_k = (\sqrt{2} \sin(2k\pi x), 0), \quad \lambda_k = 4k^2\pi^2, \quad \forall k \in \mathbb{N}^*.$$

We notice that the results [2, Theorem 2.1; Theorem 2.2] are still valid in the framework of the section and they lead to the following proposition.

Proposition 2.4. *Let the (BSEt) be considered with B being a bounded symmetric operator in L_p^2 such that $B : \psi \mapsto (V\psi^1, 0)$ with $V(x) = x(1-x)$. The (BSEt) is globally exactly controllable in $H_\mathcal{T}^3(\Phi)$ and energetically controllable in $(\lambda_k)_{k \in \mathbb{N}^*}$.*

The techniques leading to Proposition 2.2, Theorem 2.3 and Proposition 2.4 also imply the following corollary.

Corollary 2.5. *Let the (BSEt) be considered with B be a bounded symmetric operator in L_p^2 such that $B : \psi = (\psi^1, \psi^2) \mapsto (V^1\psi^1, V^2\psi^2)$ with*

$$V^1(x) = x(1-x) + x^2(x-1)^2, \quad V^2(x) = \sum_{n \in \mathbb{N}} (x-n)^2(x-n-1)^2 \chi_{[n, n+1]}(x).$$

The (BSEt) is globally exactly controllable in $H_\mathcal{T}^4(\varphi)$ and $H_\mathcal{T}^3(\Phi)$.

3. STAR GRAPH WITH INFINITE SPOKES

Let \mathcal{S} be a *star graph* graph composed by N finite edges $\{e_j\}_{1 \leq j \leq N}$ of lengths $\{L_j\}_{1 \leq j \leq N}$ and \tilde{N} half-lines $\{e_j\}_{N+1 \leq j \leq N+\tilde{N}}$. The edges are connected in the internal vertex v , while the $\{v_j\}_{1 \leq j \leq N}$ are the external vertices of \mathcal{S} (those vertices of \mathcal{S} connected with only one edge). Each e_j with $1 \leq j \leq N$ is associated to a coordinate starting from 0 in v_j and going to L_j , while e_j with $N+1 \leq j \leq N+\tilde{N}$ is parametrized with a coordinate starting from 0 in v and going to infinite.

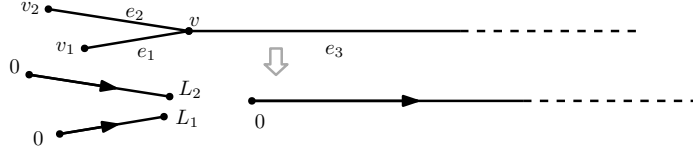


Figure 2. The parametrization of a star graph with $\tilde{N} = 1$ infinite spoke and $N = 2$ finite spokes.

Let L_p^2 be defined in (1). This space is composed by functions which are periodic on the infinite edges with periods $\{L_j\}_{N+1 \leq j \leq N+\tilde{N}}$. We consider the bilinear Schrödinger equation (BSE) in L_p^2 and the Laplacian $A = -\Delta$ being equipped with *Neumann-Kirchhoff* boundary conditions in v and *Neumann* boundary conditions in $\{v_j\}_{1 \leq j \leq N}$, i.e.

$$D(A) = \left\{ \psi = (\psi^1, \dots, \psi^{N+\tilde{N}}) \in H_p^2 : \sum_{j=1}^N \frac{\partial \psi^j}{\partial x}(L_j) = \sum_{j=N+1}^{N+\tilde{N}} \frac{\partial \psi^j}{\partial x}(0), \right. \\ \left. \psi \in C^0(\mathcal{S}, \mathbb{C}), \quad \frac{\partial \psi^j}{\partial x}(v_j) = 0 \quad \forall 1 \leq j \leq N \right\}.$$

Let $B : \psi \in L_p^2 \mapsto B\psi = ((B\psi)^1, \dots, (B\psi)^{N+\tilde{N}})$ be a bounded symmetric operator. The (BSE) corresponds to the following Cauchy systems in $L^2(e_j, \mathbb{C})$ when $1 \leq j \leq N$ and in $L_p^2(e_j, \mathbb{C})$ when $N+1 \leq j \leq N+\tilde{N}$ (defined in (1))

$$(BSEs) \quad \begin{cases} i\partial_t \psi^j(t) = -\Delta \psi^j(t) + u(t)(B\psi)^j(t), & t \in (0, T), \\ \psi^j(0) = \psi_0^j. \end{cases}$$

Let $L_{N+1}/L_j \in \mathbb{Q}$ for every $N+2 \leq j \leq N+\tilde{N}$. We denote by $l_j \in \mathbb{N}^*$ the smallest natural number such that

$$(6) \quad l_j \frac{L_{N+1}}{L_j} \in \mathbb{N}^*, \quad \text{with } 1 \leq j \leq N+\tilde{N}.$$

Let $n_k := (k-1) \prod_{j=N+1}^{N+\tilde{N}} l_j \frac{L_{N+1}}{L_j} \in \mathbb{N}$ for every $k \in \mathbb{N}^*$. We notice

$$\bigcap_{j=N+1}^{N+\tilde{N}} \left\{ \frac{2m\pi}{L_j} \right\}_{m \in \mathbb{N}} = \left\{ \frac{2n_k\pi}{L_{N+1}} \right\}_{k \in \mathbb{N}^*}.$$

Assumptions A. The numbers $\{L_j\}_{1 \leq j \leq N+\tilde{N}}$ are such that every ratios $\frac{L_{N+1}}{L_j} \in \mathbb{Q}$ for any $N+2 \leq j \leq N+\tilde{N}$. In addition, there exist $J \subseteq \mathbb{N}^*$ with $|J| = +\infty$ and

$\{c_j\}_{N+1 \leq j \leq N+\tilde{N}}$ with $c_j \in [0, L_j]$ for any $N+1 \leq j \leq N+\tilde{N}$ such that

$$\sum_{j=1}^N \tan\left(\frac{2n_k\pi}{L_{N+1}}L_j\right) = \sum_{j=N+1}^{N+\tilde{N}} \tan\left(\frac{2n_k\pi}{L_{N+1}}c_j\right), \quad \forall k \in J.$$

In conclusion, for $(\mu_k)_{k \in \mathbb{N}^*}$ the sequence obtained by reordering $\left\{\frac{4n_k^2\pi^2}{L_{N+1}^2}\right\}_{k \in J}$, there exist $C > 0$ such that $\mu_k \leq Ck^2$ for every $k \in \mathbb{N}^*$.

Remark. Let $\mu_k \sim k^2$ and $c \in \mathbb{R}^+$ be such that $0 \notin \sigma(A+c, \mathcal{H}(\varphi))$ (the spectrum of $A+c$ in the Hilbert space $\mathcal{H}(\varphi)$). For every $s > 0$, there exists $C_1, C_2 > 0$ such that

$$C_1 \|\psi\|_{(s)} \leq \| |A+c|^{s/2} \psi \|_{L_p^2} \leq C_2 \|\psi\|_{(s)}, \quad \forall \psi \in H_{\mathcal{S}}^s(\varphi).$$

When Assumptions A are satisfied, we define $(\varphi_k)_{k \in \mathbb{N}^*}$ an orthonormal system of L_p^2 made by eigenfunctions of $(\mu_k)_{k \in \mathbb{N}^*}$ such that

$$(7) \quad \varphi_1^j = \frac{1}{\sqrt{(N+\tilde{N})L_j}}, \quad \forall 1 \leq j \leq N+\tilde{N},$$

when $\mu_1 = 0$, while for every $k \in \mathbb{N}^*$ such that $\mu_k \neq 0$ (if $\mu_k = 0$, then $k = 1$),

$$(8) \quad \begin{cases} \varphi_k^1 = \alpha_k \cos(\sqrt{\mu_k}x), \\ \varphi_k^j = \alpha_k \frac{\cos(\sqrt{\mu_k}L_1)}{\cos(\sqrt{\mu_k}L_j)} \cos(\sqrt{\mu_k}x), & \forall 2 \leq j \leq N, \\ \varphi_k^j = \alpha_k \frac{\cos(\sqrt{\mu_k}L_1)}{\cos(\sqrt{\mu_k}c_j)} \cos(\sqrt{\mu_k}(x+c_j)), & \forall N+1 \leq j \leq N+\tilde{N} \end{cases}$$

with $\alpha_k \in \mathbb{C}$ such that $\|\varphi_k\|_{L_p^2} = 1$ and for every $k \in \mathbb{N}^*$.

Lemma 3.1. Let \mathcal{S} be a star graph satisfying Assumptions A. The sequence $(\varphi_k)_{k \in \mathbb{N}^*}$ is an orthonormal system of L_p^2 made by eigenfunctions of the Laplacian A corresponding to the eigenvalues $(\mu_k)_{k \in \mathbb{N}^*}$.

Proof. Any eigenfunction $f = (f^1, \dots, f^{N+\tilde{N}})$ of A corresponding to an eigenvalue $\mu \neq 0$ has to be such that f^j has period $\frac{2\pi}{L_j}$ for every $N+1 \leq j \leq N+\tilde{N}$. Thus,

$$\sqrt{\mu} \in \bigcap_{j=N+1}^{N+\tilde{N}} \left\{ \frac{2m\pi}{L_j} \right\}_{m \in \mathbb{N}} \supseteq \left\{ \frac{2n_k\pi}{L_{N+1}} \right\}_{k \in J}.$$

Thanks to the Neumann boundary conditions in $\{v_j\}_{j \leq N}$ and to the periodicity conditions in $\{e_j\}_{N+1 \leq j \leq N+\tilde{N}}$, there exists $c_j \in [0, L_j]$ for any $N+1 \leq j \leq N+\tilde{N}$ such that

$$\begin{cases} f^j = \alpha_j \cos(\sqrt{\mu}x), & 1 \leq j \leq N, \\ f^j = \alpha_j \cos(\sqrt{\mu}(x+c_j)) + \beta_j \sin(\sqrt{\mu}(x+c_j)), & N+1 \leq j \leq N+\tilde{N}, \end{cases}$$

with suitable $\{\alpha_j\}_{j \neq N+\tilde{N}}, \{\beta_j\}_{N+1 \leq j \neq N+\tilde{N}} \subset \mathbb{C}$. The Neumann-Kirchhoff boundary condition in v yields that $f \in C^0(\mathcal{S}, \mathbb{C})$ and then

$$\begin{cases} \alpha_1 \cos(\sqrt{\mu}L_1) = \alpha_j \cos(\sqrt{\mu}L_j), & \forall 2 \leq j \leq N, \\ \alpha_1 \cos(\sqrt{\mu}L_1) = \alpha_j \cos(c_j) + \beta_j \sin(c_j), & \forall N+1 \leq j \leq N+\tilde{N}. \end{cases}$$

When $\beta_j = 0$ for every $N+1 \leq j \leq N+\tilde{N}$, the last identities implies the validity of (8). The second condition characterizing the Neumann-Kirchhoff boundary conditions is verified when $\beta_j = 0$ thanks to the definition of the numbers $c_j \in [0, L_j]$ for every

$N + 1 \leq j \leq N + \tilde{N}$. As a consequence, $(\varphi_k)_{k \in \mathbb{N}^*}$ is an orthonormal sequence of L_p^2 made by eigenfunctions of A . \square

Equivalently to Proposition 2.2, we have the following well-posedness result.

Proposition 3.2. *Let the star graph \mathcal{S} satisfy Assumptions A and B be bounded symmetric operator in L_p^2 such that*

$$B : \mathcal{H}(\varphi) \longrightarrow \mathcal{H}(\varphi), \quad B : H_{\mathcal{S}}^2(\varphi) \longrightarrow H_{\mathcal{S}}^2(\varphi), \quad B : H_{\mathcal{S}}^3(\varphi) \longrightarrow H_p^3 \cap H_{\mathcal{S}}^2(\varphi).$$

Let $\psi_0 \in H_{\mathcal{S}}^3(\varphi)$ and $u \in L^2((0, T), \mathbb{R})$. There exists a unique mild solution $\psi \in C^0([0, T], H_{\mathcal{S}}^3)$ of the (BSEs) (defined in (4)). The flow of (BSEs) on $\mathcal{H}(\varphi)$ can be extended as a unitary flow Γ_T^u with respect to the L_p^2 -norm such that $\Gamma_T^u \psi_0 = \psi(t)$ for any solution ψ of (BSEs) with initial data $\psi_0 \in \mathcal{H}(\varphi)$.

Proof. The proof follows from the same arguments adopted in Proposition 2.2. First, we notice that A is self-adjoint in $\mathcal{H}(\varphi)$ and B is bounded symmetric since $B : \mathcal{H}(\varphi) \rightarrow \mathcal{H}(\varphi)$. Second, we can define an unitary flow for the dynamics of the equation in $\mathcal{H}(\varphi)$ as in the proof of the mentioned proposition.

1) Regularity of the integral term in the mild solution. Let $\psi \in C^0([0, T], H_{\mathcal{S}}^3)$ with $T > 0$. We notice $B\psi(s) \in H_p^3 \cap H_{\mathcal{S}}^2$ for almost every $s \in (0, t)$ and $t \in (0, T)$. We assume that $\mu_1 \neq 0$, but the proof is equivalent in the generic case. Let $G(t) = \int_0^t e^{i\Delta(t-s)} u(s) B\psi(s, x) ds$ be so that

$$\|G(t)\|_{(3)} = \left(\sum_{k \in \mathbb{N}^*} \left| k^3 \int_0^t e^{i\mu_k s} \langle \varphi_k, u(s) B\psi(s, \cdot) \rangle_{L_p^2} ds \right|^2 \right)^{\frac{1}{2}}.$$

Let $f(s, \cdot) := u(s) B\psi(s, \cdot)$. We define $\partial_x f(s) = (\partial_x f^1(s), \dots, \partial_x f^N(s))$ the derivative of $f(s)$ and $P(\varphi_k) = (P(\varphi_k^1), \dots, P(\varphi_k^N))$ the primitive of φ_k so that $P(\varphi_k) = -\frac{1}{\mu_k} \partial_x \varphi_k$ for every $k \in \mathbb{N}^*$. Thanks to the validity of Assumptions A, we have $\mu_k \sim k^2$. As in the first point of the proof of Proposition 2.2, for every $k \in \mathbb{N}^*$,

$$\begin{aligned} \left| k^3 \int_0^t e^{i\mu_k s} \langle \varphi_k, f(s) \rangle_{L_p^2} ds \right| &\leq \frac{C_1}{k} \sum_{j=1}^{N+\tilde{N}} \left(\left| \partial_x \varphi_k^j(L_j) \int_0^t e^{i\mu_k s} \partial_x^2 f^j(s, L_j) ds \right| \right. \\ &\left. + \left| \partial_x \varphi_k^j(0) \int_0^t e^{i\mu_k s} \partial_x^2 f^j(s, 0) ds \right| + \left| \int_0^t e^{i\mu_k s} \int_0^{L_j} \partial_x \varphi_k^j(y) \partial_x^3 f^j(s, y) dy ds \right| \right). \end{aligned}$$

The argument of [2, Remark 3.4] yields that $\overline{\partial_x^3 f(s, \cdot) \in \text{span}\{\mu_k^{-1/2} \partial_x \varphi_k : k \in \mathbb{N}^*\}}^{L^2}$ for almost every $s \in (0, t)$ and $t \in (0, T)$ and there exists $C_2 > 0$ such that

$$\begin{aligned} \|G(t)\|_{(3)} &\leq C_2 \sum_{j=1}^{N+\tilde{N}} \left(\left\| \int_0^t \partial_x^2 f^j(s, 0) e^{i\mu(\cdot)s} ds \right\|_{\ell^2} + \left\| \int_0^t \partial_x^2 f^j(s, L_j) e^{i\mu(\cdot)s} ds \right\|_{\ell^2} \right) \\ &\quad + C_3 \left\| \int_0^t \langle \mu(\cdot)^{-1/2} \partial_x \varphi(\cdot)(s), \partial_x^3 f(s) \rangle_{L_p^2} e^{i\mu(\cdot)s} ds \right\|_{\ell^2}. \end{aligned}$$

We recall the norm of H_p^3 provided in (2). From [11, Proposition B.6], there exist $C_3(t) > 0$ uniformly bounded for t in bounded intervals such that $\|G\|_{(3)} \leq C_5(t) \|f(\cdot, \cdot)\|_{L^2((0, t), H_p^3)}$. The provided upper bounds are uniform and the Dominated Convergence Theorem leads to $G \in C^0([0, T], H_{\mathcal{S}}^3(\varphi))$.

2) Conclusion. We proceed as in the second point of the proof of Proposition 2.2. Let $\psi_0 \in H^3_{\mathcal{F}}(\varphi)$. We consider the map $F : \psi \in C^0([0, T], H^3_{\mathcal{F}}(\varphi)) \mapsto \phi \in C^0([0, T], H^3_{\mathcal{F}}(\varphi))$ with

$$\phi(t) = F(\psi)(t) = e^{i\Delta t}\psi_0 - \int_0^t e^{i\Delta(t-s)}u(s)B\psi(s)ds, \quad \forall t \in [0, T].$$

Let $L^\infty(H^3_{\mathcal{F}}(\varphi)) := L^\infty((0, T), H^3_{\mathcal{F}}(\varphi))$. For every $\psi_1, \psi_2 \in C^0([0, T], H^3_{\mathcal{F}}(\varphi))$, thanks to **1)**, there exists $C(t) > 0$ uniformly bounded for t lying on bounded intervals such that

$$\|F(\psi_1) - F(\psi_2)\|_{L^\infty(H^3_{\mathcal{F}}(\varphi))} \leq C(T)\|u\|_{L^2((0, T), \mathbb{R})} \|B\|_{\mathcal{L}(H^3_{\mathcal{F}}(\varphi), H^3_{\mathcal{F}}(\varphi))} \|\psi_1 - \psi_2\|_{L^\infty(H^3_{\mathcal{F}}(\varphi))}.$$

The Banach Fixed Point Theorem leads to the claim as in the mentioned proof. \square

By recalling the definitions of global exact controllability and energetic controllability provided in Definition 1.1, we present the following result.

Theorem 3.3. *Let the hypotheses of Proposition 3.2 be satisfied. We also assume that*

- (1) *there exists $C > 0$ such that $|\langle \varphi_k, B\varphi_1 \rangle_{L^2_p} \geq \frac{C}{k^3}$ for every $k \in \mathbb{N}^*$;*
- (2) *for every $(j, k), (l, m) \in \mathbb{N}^2$ such that $(j, k) \neq (l, m)$, $j \neq l$ and $k \neq m$ satisfying $\mu_j - \mu_k = \mu_l - \mu_m$, it holds*

$$\langle \varphi_j, B\varphi_j \rangle_{L^2_p} - \langle \varphi_k, B\varphi_k \rangle_{L^2_p} - \langle \varphi_l, B\varphi_l \rangle_{L^2_p} + \langle \varphi_m, B\varphi_m \rangle_{L^2_p} \neq 0.$$

The (BSEs) is globally exactly controllable in $H^3_{\mathcal{F}}(\varphi)$ and energetically controllable in $(\mu_k)_{k \in \mathbb{N}^}$.*

Proof. 1) Local exact controllability. The statement follows as Theorem 2.3. First, for $\epsilon, T > 0$, the local exact controllability in $O^3_{\epsilon, T} := \{\psi \in H^3_{\mathcal{F}} \mid \|\psi\|_{L^2_p} = 1, \|\psi - \varphi_1(T)\|_{(3)} < \epsilon\}$ with $\varphi_1(T) = e^{-i\mu_1 T}\varphi_1$ is ensured by proving the surjectivity of the map

$$\gamma : L^2((0, T), \mathbb{R}) \longrightarrow T_\delta Q = \{\mathbf{x} := (x_k)_{k \in \mathbb{N}^*} \in h^3(\mathbb{C}) \mid ix_1 \in \mathbb{R}\}$$

the sequence of elements $\gamma_k(v) := -i \int_0^T v(\tau) e^{i(\mu_k - \mu_1)\tau} d\tau B_{k,1}$ with $B_{k,1} := \langle \varphi_k, B\varphi_1 \rangle_{L^2_p}$ $k \in \mathbb{N}^*$. The surjectivity of γ corresponds to the solvability of the moments problem

$$(9) \quad x_k B_{k,1}^{-1} = -i \int_0^T u(\tau) e^{i(\mu_k - \mu_1)\tau} d\tau, \quad \forall (x_k)_{k \in \mathbb{N}^*} \in T_\delta Q \subset h^3.$$

As there exists $C > 0$ such that $|\langle \varphi_k, B\varphi_1 \rangle_{L^2_p}| \geq \frac{C}{k^3}$ for every $k \in \mathbb{N}^*$, we have $(x_k B_{k,1}^{-1})_{k \in \mathbb{N}^*} \in \ell^2$ and $ix_1 B_{k,1}^{-1} \in \mathbb{R}$. The solvability of (9) is guaranteed by [11, Proposition B.5] since $\inf_{k \in \mathbb{N}^*} |\mu_{k+1} - \mu_k| \geq \pi^2(\max\{L_j^2 : N+1 \leq j \leq N+\tilde{N}\})^{-1} > 0$.

2) Global exact controllability and energetic controllability. The global exact controllability in $H^3_{\mathcal{F}}(\varphi)$ is ensured as in the second point of the proof of Theorem 2.3 by considering Remark A.4 instead of Remark A.3. In conclusion, the energetic controllability follows as $\varphi_k \in H^3_{\mathcal{F}}(\varphi)$ for every $k \in \mathbb{N}^*$. \square

Remark. *Let $\{L_j\}_{1 \leq j \leq N+\tilde{N}}$ be such that every ratios $\frac{L_{N+1}}{L_j} \in \mathbb{Q}$ for any $1 \leq j \leq N+\tilde{N}$. We notice that Assumptions A are satisfied with $c_j = 0$ for every $N+1 \leq j \leq N+\tilde{N}$. Indeed, for $\tilde{n}_k := (k-1) \prod_{j=1}^{N+\tilde{N}} l_j \frac{L_{N+1}}{L_j}$ with $k \in \mathbb{N}^*$ and l_j from (6), we have*

$$\left\{ \frac{4\tilde{n}_k^2 \pi^2}{L_{N+1}^2} \right\}_{k \in \mathbb{N}^*}$$

is a sequence of eigenvalues of A that we denote by $(\mu_k)_{k \in \mathbb{N}^*}$. The corresponding eigenfunctions $(\varphi_k)_{k \in \mathbb{N}^*}$ are defined in (7) and (8). In this framework,

$$\mu_k \sim k^2, \quad \tan(\mu_k L_j) = 0 \quad \forall k \in \mathbb{N}^*.$$

The validity of Assumptions A is ensured with $c_j = 0$ for every $N + 1 \leq j \leq N + \tilde{N}$. We consider the operator $\tilde{B} : \psi \mapsto (V^1 \psi^1, \dots, V^{N+\tilde{N}} \psi^{N+\tilde{N}})$ being such that

$$\begin{cases} V^j(x) = x^2(x - L_j)^2, & \forall 1 \leq j \leq N, \\ V^j(x) = \sum_{n \in \mathbb{N}} (x - nL_j)^2 (x - (n-1)L_j)^2 \chi_{[nL_j, (n+1)L_j]}(x), & \forall N + 1 \leq j \leq N + \tilde{N}. \end{cases}$$

If we assume B being such that $B\psi = \sum_{j=1}^{+\infty} \langle \varphi_j, \tilde{B}\psi \rangle_{L_2} \varphi_j$, then the global exact controllability is guaranteed in $H^3_{\mathcal{F}}(\varphi)$, which leads to the energetic controllability in $(\mu_k)_{k \in \mathbb{N}^*}$ of the bilinear Schrödinger equation (BSEs). The result follows from Theorem 3.3.

4. GENERIC GRAPHS

In this section, we study the controllability of the (BSE) for a generic graph \mathcal{G} made by N finite edges $\{e_j\}_{1 \leq j \leq N}$ of lengths $\{L_j\}_{1 \leq j \leq N}$, \tilde{N} half-lines $\{e_j\}_{N+1 \leq j \leq N+\tilde{N}}$ and M vertices $\{v_j\}_{1 \leq j \leq M}$. We call V_e and V_i the external and the internal vertices of \mathcal{G} , i.e.

$$V_e := \{v \in \{v_j\}_{1 \leq j \leq M} \mid \exists! e \in \{e_j\}_{1 \leq j \leq N} : v \in e\}, \quad V_i := \{v_j\}_{1 \leq j \leq M} \setminus V_e.$$

For every v vertex of \mathcal{G} , we denote $N(v) := \{l \in \{1, \dots, N\} \mid v \in e_l\}$. We consider the bilinear Schrödinger equation (BSE) in L^2_p for a generic graph \mathcal{G} with $\{L_j\}_{N+1 \leq j \leq N+\tilde{N}}$. Let $A = -\Delta$ being equipped with *Neumann-Kirchhoff* boundary conditions in every internal vertex $v \in V_i$, i.e. every function $f \in D(A)$ is continuous in v and

$$\sum_{e \ni v} \frac{\partial f}{\partial x_e}(v) = 0.$$

The derivatives are assumed to be taken in the directions away from the vertex (outgoing directions). In addition, the external vertices V_e are equipped with *Dirichlet* or *Neumann* type boundary conditions. We respectively call (\mathcal{NK}) , (\mathcal{D}) and (\mathcal{N}) the *Neumann-Kirchhoff*, *Dirichlet* and *Neumann* boundary conditions characterizing $D(A)$.

We say that \mathcal{G} is equipped with one of the previous boundaries in a vertex v , when each $f \in D(A)$ satisfies it in v . We say that \mathcal{G} is equipped with (\mathcal{D}) (or (\mathcal{N})) when, for every $f \in D(A)$, the function f satisfies (\mathcal{D}) (or (\mathcal{N})) in every $v \in V_e$ and verifies (\mathcal{NK}) in every $v \in V_i$. In addition, the graph \mathcal{G} is equipped with (\mathcal{DN}) when, for every $f \in D(A)$ and $v \in V_e$, the function f satisfies (\mathcal{D}) (or (\mathcal{N})) in v and f verifies (\mathcal{NK}) in every $v \in V_i$.

Let $\varphi := (\varphi_k)_{k \in \mathbb{N}^*}$ be an orthonormal system of L^2_p made by eigenfunctions of A and let $(\mu_k)_{k \in \mathbb{N}^*}$ be the corresponding eigenvalues. Let $[r]$ be the entire part of $r \in \mathbb{R}$. We define $\mathcal{G}(\varphi) = \bigcup_{k \in \mathbb{N}^*} \text{supp}(\varphi_k)$ and, for $s > 0$,

$$\begin{aligned} H^s_{\mathcal{NK}}(\varphi) &:= \left\{ \psi \in \mathcal{H}(\varphi) \cap H^s_p \mid \partial_x^{2n_2} \psi \text{ continuous in } v, \sum_{e \in N(v)} \partial_{x_e}^{2n_1+1} \psi(v) = 0, \right. \\ &\quad \left. \forall n_1, n_2 \in \mathbb{N}^* \cup \{0\}, n_1 < [(s+1)/2], n_2 < [s/2], \forall v \in V_i \right\}. \end{aligned}$$

Let $V_e(\varphi)$ and $V_i(\varphi)$ respectively be the external and internal vertices of $\mathcal{G}(\varphi)$.

Remark 4.1. We notice the following facts.

- $\mathcal{G}(\varphi)$ is a sub-graph of \mathcal{G} that can be either infinite, or finite according to the choice of the orthonormal family φ .

- The functions belonging to $\mathcal{H}(\varphi)$, $H_{\mathcal{G}}^s(\varphi)$ and $H_{\mathcal{N}\mathcal{K}}^s(\varphi)$ can be considered as functions with domain $\mathcal{G}(\varphi)$.
- $\mathcal{G}(\varphi)$ shares some external and internal vertices with \mathcal{G} , while he has new external vertices, which are $V_e(\varphi) \setminus V_e$.
- Each $\varphi_k|_{\mathcal{G}(\varphi)}$ is an eigenfunction of a Laplacian \tilde{A} defined on $\mathcal{H}(\varphi)$ as follows. The domain $D(\tilde{A})$ is composed by those functions in H_p^2 with support in $\mathcal{G}(\varphi)$ such that they satisfy Dirichlet boundary conditions in the vertices $V_e(\varphi) \setminus V_e$, while they verify the same boundary conditions defining $D(A)$ in the vertices $V_i(\varphi)$ and in $V_e(\varphi) \cap V_e$.

From now on, when we claim that the vertices of $\mathcal{G}(\varphi)$ are equipped with any type of boundary conditions, this is done in the meaning of Remark 4.1. Let $\eta > 0$, $a \geq 0$ and

$$I := \{(j, k) \in (\mathbb{N}^*)^2 : j \neq k\}.$$

Assumptions I (φ, η). Let B be a bounded and symmetric operator in L_p^2 .

- (1) There exists $C > 0$ such that $|\langle \varphi_k, B\varphi_1 \rangle_{L_p^2}| \geq \frac{C}{k^{2+\eta}}$ for every $k \in \mathbb{N}^*$.
- (2) For every $(j, k), (l, m) \in I$ such that $(j, k) \neq (l, m)$ and $\mu_j - \mu_k = \mu_l - \mu_m$, it holds $\langle \varphi_j, B\varphi_j \rangle_{L_p^2} - \langle \varphi_k, B\varphi_k \rangle_{L_p^2} - \langle \varphi_l, B\varphi_l \rangle_{L_p^2} + \langle \varphi_m, B\varphi_m \rangle_{L_p^2} \neq 0$.

Assumptions II (φ, η, a). We have $B : \mathcal{H}(\varphi) \rightarrow \mathcal{H}(\varphi)$ and $\text{Ran}(B|_{H_{\mathcal{G}}^2(\varphi)}) \subseteq H_{\mathcal{G}}^2(\varphi)$. In addition, one of the following points is satisfied.

- (1) When $\mathcal{G}(\varphi)$ is equipped with $(\mathcal{D}/\mathcal{N})$ and $a + \eta \in (0, 3/2)$, there exists $d \in [\max\{a + \eta, 1\}, 3/2)$ such that $\text{Ran}(B|_{H_{\mathcal{G}}^{2+d}(\varphi)}) \subseteq H_p^{2+d} \cap H_{\mathcal{G}}^2(\varphi)$.
- (2) When $\mathcal{G}(\varphi)$ is equipped with (\mathcal{N}) and $a + \eta \in (0, 7/2)$, there exist $d \in [\max\{a + \eta, 2\}, 7/2)$ and $d_1 \in (d, 7/2)$ such that $\text{Ran}(B|_{H_{\mathcal{N}\mathcal{K}}^{d_1}(\varphi)}) \subseteq H_{\mathcal{N}\mathcal{K}}^{d_1}(\varphi)$ and

$$\text{Ran}(B|_{H_{\mathcal{G}}^{2+d}(\varphi)}) \subseteq H_p^{2+d} \cap H_{\mathcal{N}\mathcal{K}}^{1+d}(\varphi) \cap H_{\mathcal{G}}^2(\varphi).$$

- (3) When \mathcal{G} is equipped with (\mathcal{D}) and $a + \eta \in (0, 5/2)$, there exists $d \in [\max\{a + \eta, 1\}, 5/2)$ such that $\text{Ran}(B|_{H_{\mathcal{G}}^{2+d}(\varphi)}) \subseteq H_p^{2+d} \cap H_{\mathcal{N}\mathcal{K}}^{1+d}(\varphi) \cap H_{\mathcal{G}}^2(\varphi)$. If $a + \eta \geq 2$, then there exists $d_1 \in (d, 5/2)$ such that $\text{Ran}(B|_{H_p^{d_1} \cap \mathcal{H}(\varphi)}) \subseteq H_p^{d_1} \cap \mathcal{H}(\varphi)$.

From now on, we omit the terms φ , η and a from the notations of Assumptions I and Assumptions II when their are not relevant.

We present *interpolation properties* for the spaces $H_{\mathcal{G}}^s(\varphi)$ with $s > 0$.

Proposition 4.2. Let $\varphi := (\varphi_k)_{k \in \mathbb{N}^*}$ be an orthonormal system of L_p^2 made by eigenfunctions of A .

1) If the graph $\mathcal{G}(\varphi)$ is equipped with $(\mathcal{D}/\mathcal{N})$, then

$$H_{\mathcal{G}}^{s_1+s_2}(\varphi) = H_{\mathcal{G}}^{s_1}(\varphi) \cap H_p^{s_1+s_2} \quad \text{for } s_1 \in \mathbb{N}, s_2 \in [0, 1/2).$$

2) If the graph $\mathcal{G}(\varphi)$ is equipped with (\mathcal{N}) , then

$$H_{\mathcal{G}}^{s_1+s_2}(\varphi) = H_{\mathcal{G}}^{s_1}(\varphi) \cap H_{\mathcal{N}\mathcal{K}}^{s_1+s_2}(\varphi) \quad \text{for } s_1 \in 2\mathbb{N}, s_2 \in [0, 3/2).$$

3) If the graph $\mathcal{G}(\varphi)$ is equipped with (\mathcal{D}) , then

$$H_{\mathcal{G}}^{s_1+s_2+1}(\varphi) = H_{\mathcal{G}}^{s_1+1}(\varphi) \cap H_{\mathcal{N}\mathcal{K}}^{s_1+s_2+1}(\varphi) \quad \text{for } s_1 \in 2\mathbb{N}, s_2 \in [0, 3/2).$$

Proof. Let us denote by $\{e_j\}_{j \leq N_1}$ the fine edges composing $\mathcal{G}(\varphi)$, while $\{e_j\}_{N_1+1 \leq j \leq N_1+\tilde{N}_1}$ the remaining ones. Let $L < \min\{L_k : k \in \{N_1+1, \dots, N_1+\tilde{N}_1\}\}$. We define $\tilde{\mathcal{G}}(\varphi)$ from $\mathcal{G}(\varphi)$ as follows. For every $N_1+1 \leq j \leq N_1+\tilde{N}_1$, we cut the edge e_j at distance L from the internal vertices of $\mathcal{G}(\varphi)$ where e_j is connected. We consider in $L^2(\tilde{\mathcal{G}}(\varphi), \mathbb{C})$ a self-adjoint Laplacian A_1 being defined as follows. Every internal vertex of $\tilde{\mathcal{G}}(\varphi)$ is equipped with Neumann-Kirchhoff boundary conditions. Every external vertex of $\tilde{\mathcal{G}}(\varphi)$ belonging to $V_e(\varphi)$, we consider the same boundary conditions defined for $\mathcal{G}(\varphi)$, while every other external vertex is equipped with (D) .



Figure 3. The figure represent the definition of $\tilde{\mathcal{G}}(\varphi)$ (on the right) for a graph $\mathcal{G}(\varphi)$ (on the left) with $N_1 = 11$ finite edges and $\tilde{N}_1 = 2$ infinite edges. We also underline the boundary conditions characterizing A_1 .

We denote by $H_{\tilde{\mathcal{G}}(\varphi)}^s := D(|A_1|^{\frac{s}{2}})$ for every $s > 0$. Afterwards, for every edge e_j with $N_1+1 \leq j \leq N_1+\tilde{N}_1$, we define a ring \tilde{e}_j having length L_j . We consider on $L^2(\tilde{e}_j, \mathbb{C})$ a self-adjoint Laplacian A_j with domain $D(A_j) = H^2(\tilde{e}_j, \mathbb{C})$ and we denote by $H_{\tilde{e}_j}^s := D(|A_j|^{\frac{s}{2}})$ for every $s > 0$. On $L^2([0, L], \mathbb{C})$, we consider a Dirichlet Laplacian A^D and Neumann Laplacian A^N , while we call, for every $s > 0$,

$$H_{e_j^D}^s := D(|A^D|^{\frac{s}{2}}), \quad H_{e_j^N}^s := D(|A^N|^{\frac{s}{2}}), \quad \forall N_1+1 \leq j \leq N_1+\tilde{N}_1.$$

For every $\psi = (\psi^1, \dots, \psi^{N_1+\tilde{N}_1}) \in H_{\tilde{\mathcal{G}}(\varphi)}^{s_1+s_2}(\varphi)$ with $s_1 \in \mathbb{N}$ and $s_2 \in [0, 1/2)$, there exist $\psi_1 = (\psi_1^1, \dots, \psi_1^{N_1+\tilde{N}_1}) \in H_{\tilde{\mathcal{G}}(\varphi)}^{s_1+s_2}$, $f^j \in H_{\tilde{e}_j}^{s_1+s_2}$ for every $N_1+1 \leq j \leq N_1+\tilde{N}_1$, $g^j \in H_{e_j^D}^{s_1+s_2}$ for every $N_1+1 \leq j \leq N_1+\tilde{N}_1$ and $h^j \in H_{e_j^N}^{s_1+s_2}$ for every $N_1+1 \leq j \leq N_1+\tilde{N}_1$ such that, for every $j \leq N_1$ and $N_1+1 \leq l \leq N_1+\tilde{N}_1$,

$$\begin{cases} \psi^j \equiv \psi_1^j, \\ \psi^l(x) = \psi_1^l(x) + f^l(x), & \forall x \in [0, L_l], \\ \psi^l(x) = f^l(x - [\frac{x}{L_l}]) + g^l(x - [\frac{x}{L_l}]) + h^l(x - [\frac{x}{L_l}]), & \forall x \in [L_l, +\infty). \end{cases}$$

Thanks to the last decomposition, the space $H_{\tilde{\mathcal{G}}(\varphi)}^{s_1+s_2}(\varphi)$ is equivalent to a subspace of

$$H_{\tilde{\mathcal{G}}(\varphi)}^{s_1+s_2} \times \prod_{j=N_1+1}^{N_1+\tilde{N}_1} (H_{\tilde{e}_j}^{s_1+s_2} \times H_{e_j^D}^{s_1+s_2} \times H_{e_j^N}^{s_1+s_2}).$$

Thanks to the first point of [11, Proposition 3.2], we have

$$\begin{cases} H_{\tilde{\mathcal{G}}(\varphi)}^{s_1+s_2} = H_{\tilde{\mathcal{G}}(\varphi)}^{s_1} \cap H^{s_1+s_2}(\tilde{\mathcal{G}}(\varphi), \mathbb{C}), \\ H_{\tilde{e}_j}^{s_1+s_2} = H_{\tilde{e}_j}^{s_1} \cap H^{s_1+s_2}((0, L_j), \mathbb{C}), & \forall N_1+1 \leq j \leq N_1+\tilde{N}_1, \\ H_{e_j^D}^{s_1+s_2} = H_{e_j^D}^{s_1} \cap H^{s_1+s_2}((0, L_j), \mathbb{C}), & \forall N_1+1 \leq j \leq N_1+\tilde{N}_1, \\ H_{e_j^N}^{s_1+s_2} = H_{e_j^N}^{s_1} \cap H^{s_1+s_2}((0, L_j), \mathbb{C}), & \forall N_1+1 \leq j \leq N_1+\tilde{N}_1. \end{cases}$$

The last relation implies that, for every $\psi \in H_{\mathcal{G}}^{s_1+s_2}(\varphi)$ with $s_1 \in \mathbb{N}$ and $s_2 \in [0, 1/2)$, there holds $\psi \in H_{\mathcal{G}}^{s_1}(\varphi) \cap H_p^{s_1+s_2}$ achieving the proof of the first point of the proposition, while the vice versa follows from the same techniques. In conclusion, the remaining statements equivalently follow from the second and third points of [11, Proposition 3.2]. \square

In the following theorem, we collect the well-posedness and the controllability result for the bilinear Schrödinger equation in this generic framework. The well-posedness is proved exactly as [2, Theorem 3.3] by using Proposition 4.2 instead of [2, Proposition 3.2]. The controllability result follows from the same arguments of [2, Theorem 3.6] by considering Proposition A.2 instead of [2, Theorem B.2].

By recalling the definitions of global exact controllability and energetic controllability provided in Definition 1.1, we present the following result.

Theorem 4.3. 1) *Let the couple (A, B) satisfy Assumptions II $(\varphi, \eta, \tilde{d})$ with $\eta > 0$ and $\tilde{d} \geq 0$. Let d be introduced in Assumptions II and $\mu_k \sim k^2$. For every $\psi_0 \in H_{\mathcal{G}}^{2+d}(\varphi)$ and $u \in L^2((0, T), \mathbb{R})$ with $T > 0$. There exists a unique mild solution $\psi \in C_0([0, T], H_{\mathcal{G}}^s(\varphi))$ of the (BSE) (defined in (4)). In addition, the flow of (BSE) on $\mathcal{H}(\varphi)$ can be extended as a unitary flow Γ_T^u with respect to the L_p^2 -norm such that $\Gamma_T^u \psi_0 = \psi(t)$ for any solution ψ of (BSE) with initial data $\psi_0 \in \mathcal{H}(\varphi)$.*

2) *Let for every $\epsilon > 0$ exist $C > 0$ and $\tilde{d} \geq 0$ such that*

$$|\mu_{k+1} - \mu_k| \geq Ck^{-\tilde{d}}, \quad \forall k \in \mathbb{N}^*.$$

If (A, B) satisfies Assumptions I (φ, η) and Assumptions II $(\varphi, \eta, \tilde{d})$ for $\eta > 0$, then the (BSE) is globally exactly controllable in $H_{\mathcal{G}}^s(\varphi)$ for $s = 2 + d$ with d from Assumptions II and energetically controllable in $(\mu_k)_{k \in \mathbb{N}^}$.*

Acknowledgments. The second author has been financially supported by the ISDEEC project by ANR-16-CE40-0013.

APPENDIX A. GLOBAL APPROXIMATE CONTROLLABILITY

Definition A.1. The (BSE) is said to be globally approximately controllable in $H_{\mathcal{G}}^s(\varphi)$ with $s > 0$ if, for every $N \in \mathbb{N}^*$, $\psi_1, \dots, \psi_N \in H_{\mathcal{G}}^s(\varphi)$, $\widehat{\Gamma} \in U(\mathcal{H}(\varphi))$ such that $\widehat{\Gamma}\psi_1, \dots, \widehat{\Gamma}\psi_N \in H_{\mathcal{G}}^s(\varphi)$ and $\epsilon > 0$, then there exist $T > 0$ and $u \in L^2((0, T), \mathbb{R})$ such that

$$\|\widehat{\Gamma}\psi_k - \Gamma_T^u \psi_k\|_{(s)} < \epsilon, \quad \forall \leq k \leq N.$$

Proposition A.2. *Let the hypotheses of Theorem 4.3 be satisfied. The (BSE) is globally approximately controllable in $H_{\mathcal{G}}^s(\varphi)$ for $s = 2 + d$ with d from Assumptions II.*

Proof. The proof is the same of [2, Theorem B.2]. \square

Remark A.3. *As Proposition A.2, the (BSE) is globally approximately controllable in $H_{\mathcal{G}}^4(\varphi)$ when the hypotheses of Theorem 2.3 are verified. Indeed, for every $(j, k), (l, m) \in I := \{(j, k) \in (\mathbb{N}^*)^2 : j \neq k\}$ so that $(j, k) \neq (l, m)$ and such that $\mu_j - \mu_k - \mu_l + \mu_m = \pi^2(j^2 - k^2 - l^2 + m^2) = 0$, there exists $C > 0$ such that*

$$\langle \varphi_j, B\varphi_j \rangle_{L_p^2} - \langle \varphi_k, B\varphi_k \rangle_{L_p^2} - \langle \varphi_l, B\varphi_l \rangle_{L_p^2} + \langle \varphi_m, B\varphi_m \rangle_{L_p^2} = C(j^{-4} - k^{-4} - l^{-4} + m^{-4}) \neq 0.$$

In conclusion, $(A + u_0 B, B)$ admits a non-degenerate chain of connectedness. The arguments leading to Proposition A.2 ensure to the claim.

Remark A.4. *Equivalently to Remark A.3, the (BSEs) is globally approximately controllable in $H^3_{\mathcal{F}}(\varphi)$ when the hypotheses of Theorem 3.3 are verified. Indeed, for every $(j, k), (l, m) \in I$ so that $(j, k) \neq (l, m)$ and such that $\mu_j - \mu_k - \mu_l + \mu_m = 0$, we have*

$$\langle \varphi_j, B\varphi_j \rangle_{L^2_p} - \langle \varphi_k, B\varphi_k \rangle_{L^2_p} - \langle \varphi_l, B\varphi_l \rangle_{L^2_p} + \langle \varphi_m, B\varphi_m \rangle_{L^2_p} \neq 0.$$

REFERENCES

- [1] R. Adami, E. Serra and P. Tilli, Negative Energy Ground States for the L^2 -Critical NLSE on Metric Graphs, *Communications in Mathematical Physics*, **1** (2017), 352–387.
- [2] K. Ammari and A. Duca. Controllability of localized quantum states on infinite graphs through bilinear control fields. *submitted: <https://arxiv.org/abs/1811.04273>*, 2018.
- [3] S. Avdonin, G. Leugering, and V. Mikheylov. On an inverse problem for tree-like networks of elastic strings. *ZAMM, Z. Angew. Math. Mech.*, 90(2):136–150, 2010.
- [4] J. M. Ball, J. E. Marsden, and M. Slemrod. Controllability for distributed bilinear systems. *SIAM J. Control Optim.*, 20(4):575–597, 1982.
- [5] J. M. Ball and M. Slemrod, Feedback stabilization of distributed semilinear control systems, *Appl. Math. Optim.*, **5** (1979), 169–179.
- [6] J. M. Ball, On the asymptotic behaviour of generalized processes, with applications to nonlinear evolution equations, *J. Differential Equations.*, **27** (1978), 224–265.
- [7] K. Beauchard. Local controllability of a 1-D Schrödinger equation. *J. Math. Pures Appl. (9)*, 84(7):851–956, 2005.
- [8] K. Beauchard and C. Laurent. Local controllability of 1D linear and nonlinear Schrödinger equations with bilinear control. *J. Math. Pures Appl. (9)*, 94(5):520–554, 2010.
- [9] L. Berrahmoune, Stabilization and decay estimate for distributed bilinear systems, *Systems and Control Letters.*, **36** (1999), 167–171.
- [10] H. Bounit and H. Hammouri, Feedback stabilization for a class of distributed semilinear control systems, *Nonlinear Anal.*, **37** (1999), 953–969.
- [11] A. Duca. Bilinear quantum systems on compact graphs: well-posedness and global exact controllability. *submitted: <https://hal.archives-ouvertes.fr/hal-01830297>*, 2018.
- [12] A. Duca. Simultaneous global exact controllability in projection of infinite 1D bilinear Schrödinger equations. *submitted: <https://arxiv.org/abs/1703.00966>*, 2018.
- [13] D. G. Luenberger. *Optimization by vector space methods*. John Wiley & Sons, Inc., New York-London-Sydney, 1969.
- [14] M. Morancey. Simultaneous local exact controllability of 1D bilinear Schrödinger equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 31(3):501–529, 2014.
- [15] M. Morancey and V. Nersisyan. Simultaneous global exact controllability of an arbitrary number of 1D bilinear Schrödinger equations. *J. Math. Pures Appl. (9)*, 103(1):228–254, 2015.
- [16] M. Ouzahra, Strong stabilization with decay estimate of semilinear systems, *Systems and Control Letters*, **57** (2008), 813–815.
- [17] M. Ouzahra, Exponential stabilization of distributed semilinear systems by optimal control, *Journal of Mathematical Analysis and Applications.*, **380** (2011), 117–123.
- [18] M. Ouzahra, A. Tsouli and A. Boutoulout, Stabilisation and polynomial decay estimate for distributed semilinear systems, *International journal of Control*, **85** (2012), 451–456.
- [19] E. Serra and L. Tentarelli, Bound states of the NLS equation on metric graphs with localized nonlinearities, *J. Differential Equations*, **260** (2016), 5627–5644.
- [20] E. Serra and L. Tentarelli, On the lack of bound states for certain NLS equations on metric graphs, *Nonlinear Anal.*, **145** (2016), 68–82.
- [21] L. Tentarelli, NLS ground states on metric graphs with localized nonlinearities, *J. Math. Anal. Appl.*, **433** (2016), 291–304.

UR ANALYSIS AND CONTROL OF PDES, UR 13ES64, DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCES OF MONASTIR, UNIVERSITY OF MONASTIR, TUNISIA

Email address: `kais.ammari@fsm.rnu.tn`

INSTITUT FOURIER, UNIVERSITÉ GRENOBLE ALPES, 100 RUE DES MATHÉMATIQUES, 38610 GIÈRES, FRANCE

Email address: `alessandro.duca@unito.it`