A proof of Sophie Germain primes conjecture

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Abstract In this paper a proof of the existence of an infinite number of Sophie Germain primes is going to be presented. In order to do that, the basic formula for prime numbers was analyzed with the intention of finding out when this formula would produce a Sophie Germain prime and when not. It will be shown that the number of Sophie Germain primes is infinite. Originally very difficult problem (in observational space) has been transformed into a sampler one (in generative space) that can be solved. It will be also shown that Sophie Germain primes could be obtained through two stage recursive process, and that will be used to obtain a reasonable estimation of the number of Sophie Germain primes.

1 Introduction

A prime p is a Sophie Germain prime if $2p + 1$ is a prime too [1]. In that case the prime number $2p + 1$ is called safe prime. These special primes have applications in public key cryptography, pseudorandom number generation, and primality testing; see, for example [2, 4, 6]. Originally, they have been used in the investigation of cases of Fermat's last theorem [3]. It has been conjectured that infinitely many Sophie Germain primes exist, but this was unproven (see for instance, [5]).

In this paper it is going to be proved that an infinite number of Sophie Germain primes exists . The problem is addressed in generative space, which means that prime numbers are not going to be analyzed directly, but rather their representatives, that can be used to produce them. In the second part of the paper it will be shown that Sohie Germain primes could be generated by two stage recursive process (sieve), and that will be used to generate formula for the number of Sophie Germain primes.

Remark 1: *Prime numbers 2 and 3 are in a sense special primes, since they do not share some of the common features of all other prime numbers. For instance, every prime number, apart from 2*

and 3, can be expressed in the form 6l + 1 or 6l - 1, where l ϵ N. So, in this paper, most of the time, prime numbers bigger than 3 are analyzed. It has to be said that both 2 and 3 are Sophie \Germain primes, but it has no impact on the conclusion of the paper.

Remark 2: *In this paper any infinite series in the form $c_1*l \pm c_2$ is going to be called a thread defined by number c_1 (in literature these forms are known as linear factors – however, it seems that the term thread is probably better choice in this context). Here c_1 and c_2 are numbers that belong to the set of natural numbers (c_2 can also be zero and usually is smaller than c_1) and l represents an infinite series of consecutive natural numbers in the form (1, 2, 3, ...).*

2 Proof of the conjecture

It is easy to check that any prime number (apart from 2 and 3) can be expressed in the form $6l + 1$ or $6s – 1$ ($l, s \in N$). Having that in mind, it is easy to conclude that numbers in the form $6l + 1$ could never be Sophie Germain primes since their safe primes are in the form

$$2(6l + 1) + 1 = 12l + 3 = 3(4l + 1),$$

and that is a composite number divisible by 3. Hence, the prime number that can potentially be a Sophie Germain prime must be in the form $6s – 1$. The safe prime will then be in the form $6(2s) – 1$. We denote any composite number (number that is represented as a product of prime numbers bigger than 3) with *CPN5*. A number in the form $6l + 1$ is denoted with *mpl*, while a number in the form $6s - 1$ is denoted with *mps* ($l, s \in N$). That means that any composite number *CPN5* can be expressed in the form *mpl x mpl*, *mps x mps* or *mpl x mps*.

If we represent all composite numbers in *mps* form with $6k – 1$ ($k \in N$) the following must hold

$$k = \frac{CPN5 + 1}{6}. \tag{1}$$

Since *CPN5* should be in the *mps* form, *CPN5* can be generally expressed as a product *mpl* x *mps.* Let *mpl* and *mps* be defined as

$$mpl = 6x + 1 \text{ and } mps = 6y – 1 \ (x, y \in N).$$

That leads to

$$CPN5 = mpl\ x\ mps = 6(6xy - x + y)\ -1, \tag{2}$$

or due to symmetry

$$mpl = 6y + 1 \text{ and } mps = 6x - 1\ (x, y \in N),$$

which leads to

$$CPN5 = mpl\ x\ mps = 6(6xy + x - y)\ -1. \tag{3}$$

If (2) or (3) is replaced in (1), forms of k that cannot produce a Sophie Germain prime number will be obtained. Those forms are expressed by the following equations

$$k = (6x - 1)y + x \tag{4a}$$

$$k = (6x + 1)\ y - x, \tag{4b}$$

where $x, y \in N$. These equations are equivalent (they will produce the same numbers) and can be used interchangeably. What should be noticed is the following: a thread defined by (4a) is spread over all threads defined by (4b), and vice versa.

Also, in order to have Sophie Germain pair (which consists of Sophie Germain prime p and safe prime $2p + 1$), a safe prime which is generated by $2k$ cannot be composite. If the safe number is composite the following equation holds

$$k = \frac{CPN5 + 1}{2 \cdot 6}, \tag{5}$$

where *CPN5* is a composite number in the *mps* form. Using the same analysis as in the previous case and replacing for instance (2) in (5), additional cases in which k cannot be used to produce Sophie Germain prime are obtained. They are defined by the following equation

$$k = \begin{matrix} (6x+1)y - \frac{x}{2}, & x\ is\ even \\ (6x+1)y - 3x - \frac{x+1}{2}, & x\ is\ odd \end{matrix}, \tag{6a}$$

where $x, y \in N$. Alternatively, it is possible to use the equation (3) and replace it in (5) and then the following equation holds

$$k = \begin{matrix} (6x-1)y + \frac{x}{2}, & x \text{ is even} \\ (6x-1)y - 3x + \frac{x+1}{2}, & x \text{ is odd} \end{matrix} . \quad (6b)$$

A different equation that produces the same numbers as the equation (6a) is obtained. In the text that follows we denote with (6b') the equation (6b) that contains only the threads defined for $x = 1$ and $x = 2$.

Equations (4a) and (6a) (and other alternatives like (4b) and (6b)) give a sufficient and necessary condition for k, so that it cannot be used for the generation of prime pairs in the form $(p, 2p + 1)$. In order to prove that there are infinitely many prime pairs in the form $(p, 2p + 1)$ it is necessary to prove that infinitely many natural numbers cannot be expressed neither in form (4a) nor in form (6a). Here, we will consider equations (4a, 6a and 6b'). The reason for inclusion of (6b') will become clear later in the text.

First, the forms of (4a, 6a, and 6b') for some values of x will be checked. Equations (4a, 6a, 6b') create sufficient (not necessary) condition that k defined by them cannot generate the Sophie Germain primes.

Case x = 1: $k = 7y - 4$, $k = 5y + 1$, $k = 5y - 2$,

Case x = 2: $k = 13y - 1$, $k = 11y + 2$, $k = 11y + 1$;

Case x = 3: $k = 19y - 11$, $k = 17y + 3$,

Case x = 4: $k = 5(5y) - 2$, $k = 23y + 4$,

Case x = 5: $k = 31y - 18$, $k = 29y + 5$,

Case x = 6: $k = 37y - 3$, $k = 35y + 6 = 5(7y + 1) + 1$,

Case x = 7: $k = 43y - 25$, $k = 41y + 7$,

Case x = 8: $k = 7(7y) - 4$, $k = 47y + 8$.

It can be seen that k is represented by the threads that are defined by prime numbers bigger than 3. From examples (cases $x = 4$ and $x = 8$), it can be seen that if $(6x - 1)$ or $(6x + 1)$ represent a

composite number, k that is represented by thread defined by that number also has a representation by the thread defined by one of the prime factors of that composite number. That can be proved easily in the general case, by direct calculation, using representations similar to (2) and (3). Here, only one case is going to be analyzed. All other cases can be analyzed analogously. In this case, assume that

$$(6x - 1) = (6l + 1)(6s - 1) \tag{7}$$

where (l, $s \in N$). Equation (7) leads to

$$x = 6ls - l + s. \tag{8}$$

Considering (8) and using the following representation of k that includes a thread defined by ($6x$-1)

$$k = (6x - 1)y + x,$$

the simple calculation leads to

$$k = (6l + 1)(6s - 1)y + 6ls - l + s = (6l + 1)(6s-1)y + s(6l + 1) - l$$

or

$$k = (6l+1)((6s-1)y + s) - l$$

which means

$$k = (6l + 1)f - l,$$

and that represents the already exiting form of the representation of k for the factor ($6l$ + 1), where

$$f = (6s - 1)y + s.$$

Here the equivalence of equations (4a) and (4b) is used.

Now, it is going to be proved that the number of natural numbers that cannot be represented by the models (4a, 6a, 6b') is infinite.

From (4a, 6a, 6b') it can be seen that all numbers that can be represented in the form

$$5y - 2,\ 5y + 1,$$

cannot be used for generation of Sophie Germain primes. In other words, a ratio r_1 = 2/5 of all natural numbers cannot be used for generation of Sophie Germain primes. The ratio c_1 = 1 - 2/5 = 3/5 of all natural numbers cannot be represented by those two patterns and they still contain some numbers that can be used for representation of Sophie Germain primes.

What does this analysis actually tell us?

1. We know that all natural numbers can be represented by five threads defined by number 5:

   $5y$, $5y - 1$, $5y - 2$, $5y - 3$ and $5y - 4$ ($y \in N$). By doing that we can simply disentangle the numbers (actually the threads that "contain" numbers) that cannot be used for generation of Sophie Germain primes from the numbers that potentially can be used for generation of Sophie Germain primes. So, threads $5y - 2$ and $5y - 4$ almost exclusively contain numbers that cannot be used for generation of Sophie Germain primes (the only exception is number 1 that belongs to thread $5y - 4$ and can generate a pair of Sophie Germain primes). The other three threads $5y$, $5y - 1$ and $5y$ - 3, potentially contain numbers that can be used for generation of Sophie Germain primes. The only reason why some of the numbers that belong to those threads would actually not be available for generation of Sophie Germain primes, is that those threads are entangled with some threads in (4a, 6a, 6b') that are defined by prime numbers bigger than 5.

2. Can we use the fact that we have 3/5 of numbers that potentially can be used for the generation of Sophie Germain primes? Fortunately the answer is YES. Let us denote with $\pi TP(n)$ the number of Sophie Germain primes smaller than some number n, $n \in N$. Now, for $n \leq 31$, we can say the following (we should be aware that n belongs to the observational space)

$$\pi TP(n) > (c_1 * n/6). \quad (9)$$

   Why is it so? The reason is following: the numbers in generative space that are smaller or

equal than number 5 that cannot be used for generation of Sophie Germain primes can belong only to those threads in (4a, 6a, 6b') that are defined by prime numbers not bigger than 5. In current situation these are only threads that are defined by number 5. Number 5 in generative space defines numbers 29 and 31 in observational space, so it is safe to say that equation (9) is correct. So, why we have inequality sign instead of equality sign? The reason is simple – one of the threads defined by (4a, 6a, 6b') is in the form $5y + 1$ and generates numbers that are bigger than 5. That form will produce the same numbers as the form $5y$-4 (y ϵ N, and $y > 1$). So, starting from number 2, out of every 5 consecutive numbers at least 2 numbers cannot be used for the generation of Sophie Germain primes. We can see that within first five numbers there is an exception - four numbers that can be used for generation of Sophie Germain primes exists, while estimation $c_1 * n/6$ will suggest that exist three numbers, or

$$\pi TP(31) = 4 > (c_1 * 31/6) = 3.1.$$

3. Here we are going to explain the most sensitive point in the previous part: why is it true that numbers that cannot be used for generation of the Sophie Germain primes and that are not bigger than 5, can belong only to those threads in (4a, 6a, 6b') that are defined by the number 5?

   From (4a) we can see clear answer for the following groups of threads:

$$(6x - 1)y + x,$$

since these threads generate numbers that are bigger than prime number that defines thread. However, numbers generated by threads defined by (6a)

$$k = \begin{matrix} (6x+1)\,y - \dfrac{x}{2}, & x \text{ is even} \\ (6x+1)\,y - 3x - \dfrac{x+1}{2}, & x \text{ is odd} \end{matrix}, \quad ,$$

can generate numbers smaller than the prime number that defines thread. Here, we will show

that it happens only for $y = 1$ and that will actually not affect the number of the numbers that can be used for the generation of Sophie Germain primes, that are smaller than 5.

So, if $y = 1$ and $x = 2x_1$ is even, thread produce numbers in the form

$$(6 \cdot 2x_1 + 1) - x_1 = 11x_1 + 1,$$

and that is actually thread defined by number 11 (this is thread generated by (6b') for $x = 2$). So, in this case new threads will not affect the number of Sophie Germain primes smaller than any previous prime number – in this case number 5. So, it does not affect threads generated by 5 and 7, and for all bigger primes it is already calculated since it is defined with prime 11. If $y > 1$, then the number $(6 \cdot 2x_1 + 1) \cdot y - x_1$ is bigger than prime number $(6 \cdot 2x_1 + 1)$ and by this also bigger than any previous prime number, which is in this case 5.

If $y = 1$ and $x = 2x_1 - 1$ is odd, thread produces numbers in the form

$$(6 \cdot (2x_1 - 1) + 1) - 3(2x_1 - 1) - x_1 = 5x_1 - 2,$$

and that is actually thread defined earlier by number 5 (thread generated by (6b') in the case $x = 1$). So, in this case new threads will not affect the number of Sophie Germain primes smaller than any previous prime number – in this case number 5. If $y > 1$, then the number $(6 \cdot (2x_1 - 1) + 1) \cdot y - 3(2x_1 - 1) - x_1$ is bigger than prime number $(6 \cdot (2x_1 - 1) + 1)$ and by this also bigger than any previous prime number, which is in this case 5.

Now, it is clear why (6b') is included in the analysis.

4. Now, we will consider one additional problem. In this moment we have solved the problem of estimation of Sophie Germain primes for the natural numbers smaller than 32. However, we do not know if (9) holds for numbers between 32 and 41 (which is the number in observational space that is connected to the next prime number in generative space). In this case problem is simple and can be solved easily. However, what will happen in the general

case when we do not know the value of the gap to the next prime number and we know that it can be large for large numbers? The solution comes from the fact that we know from [7] that although gaps between consecutive primes can be large, they cannot be arbitrarily large. From [7] we know that following equation holds starting from **some big enough number** (probably always – author is not aware of the literature that can support this statement)

$$p_{n+1} < 2 \cdot p_n. \tag{10}$$

Having that in mind, we can modify (9) so that it is valid in more general case (currently, case of interest is $n < 42$)

$$\pi\mathrm{TP}(n) > (c_1 \cdot n/12). \tag{11}$$

Since the number of the Sophie Germain primes can only be increased between ($6 \cdot p_n + 1$ and $6 \cdot p_{n+1}$) we know that (11) holds for all $n < 42$ since it will give estimation for n, that is actually valid for the value $n/2$.

Analysis performed here is valid in general case – it does not depend on how many threads defined by (4a, 6a, 6b') is analyzed. The only differences are:

- the interval on which approximation holds depends on the last prime number that is analyzed (if the last prime number that is analyzed is p_n, then the interval is $[p_n, 2 \cdot p_n]$ – in practice interval of interest is $[p_n, p_{n+1}]$). In that case for $n \in [p_n, p_{n+1}]$ the lower bound for $\pi\mathrm{TP}(n)$ is given by (c_n marks the ratio of threads that are still available for generation of twin primes after we consider all threads defined by prime bigger that 3 and smaller or equal to p_n)

$$\pi\mathrm{TP}(n) > (c_n \cdot n/12). \tag{11g}$$

- number of numbers that cannot be used for the generation of Sophie Germain primes and

that are smaller than the last prime that is analyzed, depends only on the threads defined by primes that are smaller or equal to that last prime that is analyzed.

Now, we denote prime numbers bigger than 3 as $p5$, where $p5(1) = 5$, $p5(2) = 7$ and so on. Also, we denote $p5 - 1$ (if $p5$ is a prime that defines one thread in (4a, 6a)) and $p5 - 2$ (if $p5$ is a prime defined also by (6b')) with $p5r$.

After step k (analysis of the threads in (4a, 6a, 6b') defined by first k primes bigger than 3) the ratio r_k of all numbers that cannot be used for generation of Sophie Germain primes and ratio c_k of all numbers still potentially available for the generation of Sophie Germain primes, are obtained. In the step $k + 1$ we will have (using some basic probabilistic rules)

$$r_{k+1}=r_k+\frac{p5(k+1)-\ p5r(k+1)}{p5(k+1)}-\frac{p5(k+1)-\ p5r(k+1)}{p5(k+1)}r_k.$$

After a few elementary calculations, the following equation is obtained

$$r_{k+1}=r_k+\frac{p5(k+1)-\ p5r(k+1)}{p5(k+1)}-(1-\ r_k).$$

Now, the following equation holds

$$c_{k+1}=1-\ r_{k+1}=1-\ r_k-\frac{p5(k+1)-\ p5r(k+1)}{p5(k+1)}-(1-\ r_k)=c_k-\frac{p5(k+1)-\ p5r(k+1)}{p5(k+1)}c_k.$$

or

$$c_{k+1}=\frac{p5r(k+1)}{p5(k+1)}c_k. \tag{12}$$

Equation (12) can also be written in the following form

$$c_{k+1}=\frac{\prod_{j=1}^{k+1} p5r(j)}{\prod_{j=1}^{k+1} p5(j)}. \tag{13}$$

If we now denote with $\alpha(j)$ value 1, if $p5(j)$ generates one thread and 2 if $p5(j)$ generates 2 threads

equation (13) can be written as

$$c_{k+1}=\prod_{j=1}^{k+1}\left(1-\frac{\alpha(j)}{p5(j)}\right). \tag{14}$$

Since α(*j*) can only take values 1 or 2 it is easy to conclude that following holds

$$c_{k+1}>\prod_{3<p5\leq p5(k+1)}\left(1-\frac{2}{p5}\right). \tag{15}$$

If denote by *p* any prime number, (15) can be written as

$$c_{k+1}>3\cdot\prod_{2<p\leq p5(k+1)}\left(1-\frac{2}{p}\right). \tag{16}$$

From [8, p. 68, eq. (2.32)] we know that following equation holds ($n1 \in N,\ n1 > 2$)

$$\prod_{2<p\leq n1}\left(1-\frac{2}{p}\right)=\frac{tc(2)}{(\ln(n1))^2}+O\left(e^{-2\sqrt{(\ln(n1))}}\right). \tag{17}$$

Having in mind (17), from (16) we can see that following equation holds (after slight change of notation; and having $n1 \in [p5(k+1), p5(k+2)]$)

$$c(n1)>\prod_{2<p\leq n1}\left(1-\frac{2}{p}\right)+2\cdot\prod_{2<p\leq n1}\left(1-\frac{2}{p}\right)>\frac{tc(2)}{(\ln(n1))^2}. \tag{18}$$

Now, having in mind (11g), we can write

$$\pi TP(n)>c(n1)\frac{n}{12}. \tag{19}$$

Since in our case we can write $n1 = n/6$ (from [7] we know that (19) holds for *n* big enough, if not always, and from the text before (17) – it must be $n \geq 18$) and knowing from [8] that tc(2) ≈ 0.83 > 0.8, from (19) we can write the following

$$\pi TP(n)>\frac{0.8}{\left(\ln\left(\frac{n}{6}\right)\right)^2}\frac{n}{12}=\frac{1}{15}\frac{n}{\left(\ln\left(\frac{n}{6}\right)\right)^2}. \tag{20}$$

(*Note:* The $n1 = n/6$ is not a natural number in general case. Trivial solution would be to rounded it

to nearest integer that is not bigger than $n/6$. Also, it is possible to show that (18) holds for real numbers, too, but it is beyond to scope of this paper.)

Since it is trivial to prove that the following equation holds

$$\lim_{n \to \infty} \frac{n}{\left(\ln\left(\frac{n}{6}\right)\right)^2} = \infty \quad ,$$

we can safely conclude that the number of Sophie Germain primes is infinite. That concludes the proof.

3. Estimation of the number of Sophie Germain primes

In this chapter we are going to analyze an alternative way of generating Sophie Germain primes. This will help us to give a reasonable estimation of the number of Sophie Germain primes that is smaller than some natural number n.

Prime numbers can be obtained in the following way:

First, we remove all even numbers (except 2) from the set of natural numbers. Then, it is necessary to remove the composite odd numbers from the rest of the numbers. In order to do that, the formula for the composite odd numbers is going to be analyzed. It is well known that odd numbers bigger than 1, here denoted by a, can be represented by the following formula

$$a = 2n + 1,$$

where $n \in N$. It is not difficult to prove that all composite odd numbers a_c can be represented by the following formula

$$a_c = 2(2ij + i + j) + 1 = 2((2j + 1)i + j) + 1. \tag{21}$$

where $i, j \in N$. It is simple to conclude that all composite numbers could be represented by product $(i + 1)(j + 1)$, where $i, j \in N$. If it is checked how that formula looks like for the odd numbers, after

simple calculation, equation (21) is obtained. This calculation is presented here. The form $2m + 1$, $m \in N$ will represent odd numbers that are composite. Then the following equation holds

$$2m+1=(i_1+1)(j_1+1) \quad ,$$

where $i_1, j_1 \in N$. Now, it is easy to see that the following equation holds

$$m=\frac{i_1 j_1+i_1+j_1}{2}.$$

In order to have $m \in N$, it is easy to check that i_1 and j_1 have to be in the forms

$$i_1 = 2i \text{ and } j_1 = 2j,$$

where $i, j \in N$. From that, it follows that m must be in the form

$$m = 2ij + i + j.$$

When all numbers represented by m are removed from the set of odd natural numbers bigger than 1, only the numbers that represent odd prime numbers are going to stay. In other words, only odd numbers that cannot be represented by (21) will stay. This process is equivalent to the sieve of Sundaram [9].

Let us denote the numbers used for the generation of odd prime numbers with *m2* (here we ignore number 2). Those are the numbers that are left after the implementation of Sundaram sieve. The number of those numbers that are smaller than some natural number n, is equivalent to the number of prime numbers smaller than n. If we denote with $\pi(n)$ number of primes smaller than n, the following equation holds

$$\pi(n) \approx \frac{n}{\ln\left(\frac{n}{2}\right) - \frac{1}{3}}.$$

Now, the following question can be asked:

What will happen when Sindaram's sieve is implemented once more, now on numbers m2?

The answer is that in the second step, the only numbers that are not going to be removed are prime numbers that are used for generation of some *mps* prime numbers. All of these numbers are in the *mps* form and actually represent Sophie Germain primes (it has been already explained that prime numbers in *mpl* form produce composite odd numbers divisible by 3, when formula $2 \cdot mpl + 1$ is applied on them). Since the same method is applied in the second step, intuitively can be concluded that the numbers left after the second Sundaram sieve, should be comparable to the following number *msg*

$$msg = \frac{\pi(n)}{\ln\left(\frac{\pi(n)}{2}\right) - \frac{1}{3}} \quad .$$

Of course, the result cannot be correct and it requires some compensation terms since the second Sundaram sieve is applied on an incomplete set, that is depleted by previously implemented Sundaram sieve.

Let us mark the number of Sophie Germain primes smaller than some natural number n with $\pi SGP(n)$. Good approximation of the $\pi SGP(n)$ is given by the following equation

$$\pi SGP(n) \approx (1 + 0.0129 \log(n)) \frac{n}{\left(\ln\left(\frac{n}{2}\right) - \frac{1}{3}\right)\left(\ln\left(\frac{n}{2}\right) - \ln\left(\ln\left(\frac{n}{2} - \frac{1}{3}\right)\right) + \frac{1}{3}\right)}.$$

This equation gives good approximation of the number of Sophie Germain primes smaller than natural number n, at least for the values of $n \leq 10^{14}$ (estimation for values $n = 10^3$ and $n = 10^4$ is actually correct).

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