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ON A PROOF OF THE TREE CONJECTURE FOR TRIANGLE TILING BILLIARDS.

OLGA PARIS-ROMASKEVICH

To Manya and Katya, to the moments we shared around mathematics on one winter day in Moscow.

ABSTRACT. Tiling billiards model a movement of light in heterogeneous medium consisting of homogeneous cells in which the coefficient of refraction between two cells is equal to -1 . The dynamics of such billiards depends strongly on the form of an underlying tiling. In this work we consider periodic tilings by triangles (and cyclic quadrilaterals), and define natural foliations associated to tiling billiards in these tilings. By studying these foliations we manage to prove the Tree Conjecture for triangle tiling billiards that was stated in the work by Baird-Smith, Davis, Fromm and Iyer, as well as its generalization that we call Density property.

1. INTRODUCTION

*To dwellers in a wood, almost every species of tree
has its voice as well as its feature.
Thomas Hardy, Under the Greenwood Tree*

A tiling billiard is a model of movement of light in a heterogeneous medium that is constructed as a union of homogeneous pieces [13], [11]. In this work, we only consider tiling billiards with refraction coefficient equal to -1 . The reader can fix the number -1 in their memory as a refraction ratio between any two neighbouring homogeneous chunks in the medium till the end of the article.

A mathematical formalization is the following. Take any polygonal tiling of a plane and define a billiard in it such that a point particle moves in a straight line till a moment when it reaches a border of a tile. Then it passes to a neighboring tile, and its direction follows Snell's law with a local refraction coefficient equal to -1 . One is interested in the dynamics of particles in such a class of dynamical systems, the so-called **tiling billiards** [11]. Of course, the dynamics of a tiling billiard depends very strongly on the tiling on which it is defined, see Figure 1 for examples.

Tiling billiards were first introduced by Mascarenhas and Fluegel [20] in the context of the physics of light. The mathematical study of these billiards was proposed in [11]. The study of tiling billiards is quite a new subject in mathematics, although it has already showed itself as very rich and interesting from the point of view of dynamics, see [11, 7, 9, 16]. The study of tiling billiards stays for now a highly unexplored area. The only non-trivial examples of tiling billiards for which the dynamics has been more or less understood are that of a tiling billiard on a trihexagonal tiling [9] and on a periodic triangle tiling [7, 16].

We would like to make a side note that the case of refraction coefficient equal to -1 doesn't (yet?) correspond to any physical reality. Although, this hypothesis is not as unnatural as one could think. Recent progress in meta-materials has showed [24, 25] the existence of (artificially constructed) metamaterials with negativ refraction index (around -0.6). Most of usual plastic or glass materials have coefficients of refraction bigger than 1.

This work considers two tilings (periodic triangle tiling and cyclic quadrilateral tiling) that have many common features. Each of these two periodic tilings consists of congruent triangles (or **cyclic**

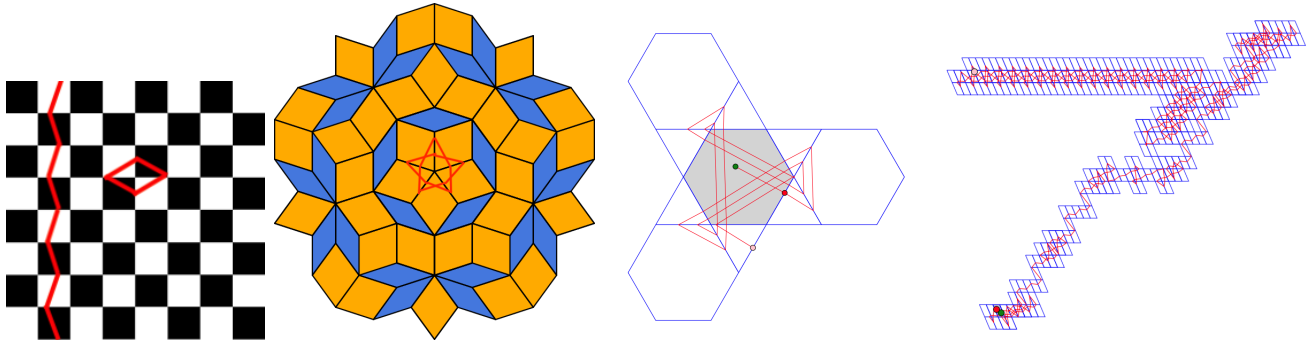


FIGURE 1. Several examples of tiling billiards with different trajectory behavior and several examples of questions one could ask. From left to right: 1. A **square tiling** with trivial behavior of trajectories: all trajectories are either of period 4 or escape to infinity along verticals or horizontals in a periodic way. This billiard has a property that all of its bounded trajectories are closed. Does this property persist for a larger class of tilings? 2. **Penrose tiling** and a (very simple) periodic trajectory in it. How often one finds periodic trajectories in Penrose tilings? Penrose tiling billiards have not been studied yet. 3. **Trihexagonal tiling billiard** was studied in [9], and its trajectories exhibit ergodic properties. 4. A very simple **parallelogram tiling** with seemingly not as simple tiling billiard has not yet been studied. Do (almost all?) its trajectories escape linearly to infinity?

quadrilaterals¹) and has a property that each two neighbouring tiles are centrally symmetric to each other with respect to the middle of their common side, see Figure 2. Moreover, in these two tilings one can cover the tiles into two colors in such a way that neighbouring tiles have different colors and that tiles with the same color can be identified by a translation. Of course, a periodic tiling by quadrilaterals is always possible, whatever the form of a quadrilateral. Although we are mostly interested in the special case of cyclic quadrilaterals, and for a reason that will be clear right away. The cyclic tiles are the only ones in this class that admit the *folding construction* that we discuss in 3.

2. MOTIVATION: SYMBOLIC DYNAMICS OF TRIANGLE TILING BILLIARDS.

Consider a **periodic triangle tiling** obtained by cutting a plane by three families of equidistant parallel lines. One normalizes the tiles in order that the circumcircle of a tile has its radius equal to 1. This is possible since the dynamics of a tiling billiard is equivariant under homothety.

This tiling was introduced in [11] and studied in more detail in [7] and subsequently in [16] by the author in collaboration with Pascal Hubert. Our work gives a description of the qualitative behavior of generic trajectories as well as sheds some light on non-generic behaviors. Although a more precise understanding of symbolic dynamics of the trajectories (both generic and exceptional) is far from being complete. One of the goals of this work was to advance the understanding of such symbolic behavior.

In this Section we list some previously discovered results on triangle tiling billiards, and state the Tree conjecture on the symbolic dynamics of closed trajectories that was the main motivation for the present work.

¹A cyclic quadrilateral is a quadrilateral inscribed into a circle.

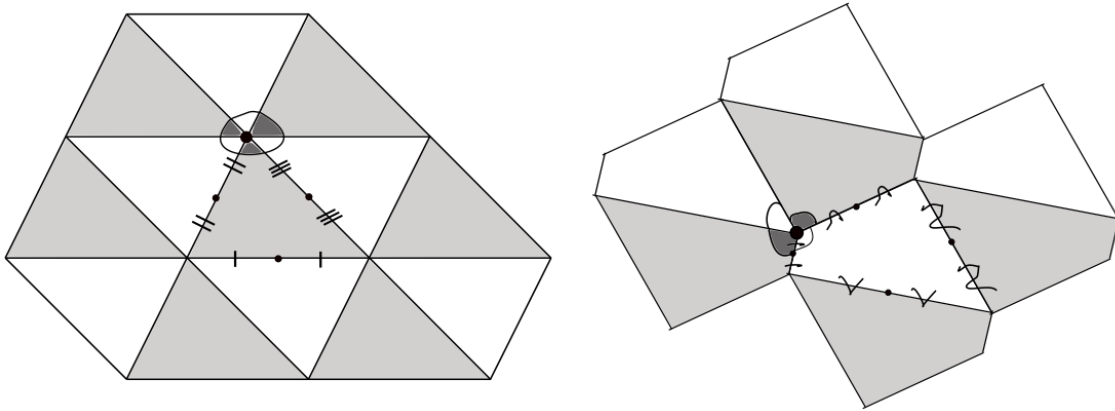


FIGURE 2. A triangle tiling and cyclic quadrilateral tiling share two important properties. First, they are colorable in two alternating colors. Second, the equilibrium of the angles is preserved. Indeed, in each vertex of a tiling a sum of angles of each of the colors is equal to π . For the periodic triangle tiling the relation of the angles is trivial, $\alpha + \beta + \gamma = \alpha + \beta + \gamma = \pi$ and for the quadrilateral tiling the relation on the equilibrium of the angles is $\alpha + \gamma = \beta + \delta = \pi$, and is equivalent to the fact that the quadrilateral is cyclic. Moreover, each two neighbouring tiles are centrally symmetric with respect to the middle of the common edge for both of the tilings.

2.1. Triangle tiling billiards: known results. The results of this paragraph come entirely from either [7] or [16].²

Denote the angles of a tile in a periodic triangle tiling by α, β, γ and its corresponding sides by a, b, c . We suppose that any triangle in the tiling is oriented in such a way that a counterclockwise tour of its boundary reads abc . Such a tiling can be colored in two alternating colors (grey and white). We call the grey tiles positively oriented, and the white tiles negatively oriented.

A **symbolic code** of a curve on the plane with respect to a periodic triangle tiling is defined as a word in the alphabet $\mathcal{A}_\Delta = \{ab, ba, bc, cb, ca, ac\}$ which corresponds to the sequence of connected sides. For example, a code of a curve making a tour of a vertex in a tiling in a clock-wise manner is $ab\ bc\ ca\ ab\ bc\ ca$. Our reader is probably asking herself: why we wouldn't just write $abcabc$? Actually, this new, one could think, redundant notation for a symbolic code of a periodic trajectory, happens to be much more efficient in order to understand symbolic dynamics, as the reader will see in the following. We thank Pierre Dehornoy for this representation.

The state of art on the behavior of triangle tiling billiards can be summarized in a following

Theorem 1. [7, 16] *Consider a triangle tiling billiard. Then the following holds.*

1. *Every trajectory passes by each tile at most once. Additionally, the oriented distance between a segment of a trajectory in some tile and its circumcenter is preserved along a trajectory.*
2. *All bounded trajectories are simple closed curves.*
3. *All bounded trajectories are stable under small perturbation (form of a tile, initial condition), i.e. they deform to bounded trajectories with the same symbolic code.*
4. *The period of any closed trajectory belongs to $\{4n + 2, n \in \mathbf{N}^*\}$.*
5. *As a stronger statement, the symbolic code $w \in \mathcal{A}_\Delta^{\mathbf{N}}$ has its smallest period $s \in \mathcal{A}_\Delta$ of odd length, $w = \overline{s^2}$.*

²Even though the article [16] has been published (and even, appeared online) before [7], we have studied in detail an early draft by Baird-Smith, Davis, Fromm and Iyer, and our work is based on the ideas and results from their work.

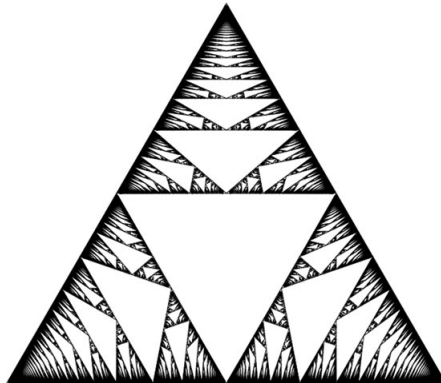


FIGURE 3. Rauzy gasket $\mathcal{R} \subset \Delta$ is a fractal set homeomorphic to Sierpinski carpet.

The statements 1.–3. have been proven and 4. has been conjectured in [7], and 4.–5. have been proven in [16]. The first three statements are simple consequences of an important folding idea, see Section 3. The proof of the two last statements uses in a crucial way the relation of triangle tiling billiards with interval exchange transformations with flips that was discovered in [7].

A trajectory is **escaping to infinity** (or simply **escaping**) if it is not closed. This definition makes sense since from the point 1. of Theorem 1, any trajectory which is not closed, is not "spiraling" in and out in a bounded domain of a plane but genuinely escapes to infinity. A trajectory is **linearly escaping** if it escapes to infinity and stays in a bounded distance from some line on the plane. As proven in [16], most of the trajectories of triangle tiling billiards are either closed (open property) or linearly escaping (a property which is dense in open sets). In order to precise this statement, we need one more definition.

Let $\Delta = \{(x_1, x_2, x_3) | x_i \geq 0, \sum_i x_i = 1\} \subset \mathbf{R}^3$. If $x_j > \frac{1}{2}$ for some j , one maps a triple (x_1, x_2, x_3) to a new one where $x'_j := 2x_j - 1$ (and the other two coordinates stay unchanged) and normalizes to get back to Δ_2 . Equivalently, one subtracts the sum of two smaller coordinates from the biggest one. A subset $\mathcal{R} \subset \Delta$ of triples on which such operation can be applied infinitely, is called the **Rauzy gasket** and was defined and studied in [3, 4]. See Figure 3.

This set \mathcal{R} , seemingly unnatural if introduced as above, appears to be a set of parameters for the set of *interesting* maps in different (or maybe not as different?...nobody knows!) dynamical contexts, see for example the work of Avila-Hubert-Skripchenko on systems of isometries [5], Dynnikov-DeLeo [12] on sections of 3-periodic surfaces as well as that of Arnoux-Rauzy [3] on 6–interval exchange transformations on the circle. This set \mathcal{R} is still not completely understood, for example no one knows its exact Hausdorff dimension. Now we provide its relation to the triangle tiling billiards which is precisely stated and proven in [16].

Let $\bar{\mathcal{R}}$ be the set of triangles (forms of tiles in a periodic triangle tiling) with angles α, β, γ such that the point $p := (1 - \frac{2}{\pi}\alpha, 1 - \frac{2}{\pi}\beta, 1 - \frac{2}{\pi}\gamma) \in \mathbf{R}_+^3$ belongs to the Rauzy gasket \mathcal{R} . A trajectory of a triangle tiling billiard is called **exceptional** if first, the tiles of the corresponding tiling belong to the set $\bar{\mathcal{R}}$ and, second, this trajectory passes through the circumcenter of some tile (and hence, by point 1. of Theorem 1, of *any* tile).

Theorem 2. [16] *If a trajectory is not closed and not exceptional than it is linearly escaping. Moreover, the set of exceptional trajectories has zero measure and almost all³ of them escape non-linearly.*

³This *almost all* is not taken with respect to the Lebesgue measure (since $\text{Leb}(\bar{\mathcal{R}}) = \text{Leb}(\mathcal{R}) = 0$ by [4]) but with respect to the Avila-Hubert-Skripchenko measure [5, 6] on the Rauzy gasket. We think that the word *almost* can be taken away, i.e. we think that all of the exceptional trajectories are non-linearly escaping.

Exceptional trajectories of triangle tiling billiards are of great interest because of their relationship (see [7]) with arithmetic orbits of a famous Arnoux-Rauzy family of interval exchange transformations defined in [3]. These orbits and similar algebraic objects were studied, among others, in [15, 21, 19]. The relationship between triangle tiling billiards and the Arnoux-Rauzy family comes from the [Proposition 15, [16]] which states that the tiling billiards give a family of natural square roots of the Arnoux-Rauzy family which are fully flipped interval exchange transformations on 3 intervals on the circle.

We hope to develop the connection we discovered between exceptional tiling billiard trajectories and arithmetic orbits of Arnoux-Rauzy minimal maps in the future work. We think that the understanding of the behavior of exceptional trajectories (and their density properties) can come from approaching these trajectories by bigger and bigger periodic trajectories. Which leads us to the next Section, where we present our main result on the symbolic behavior of periodic trajectories.

2.2. Tree conjecture: formulation and motivation. The Tree conjecture concerns the symbolic behavior of *any* periodic trajectory of a triangle tiling billiard, see Figure 4.

Denote by $\Lambda = (V, E)$ an abstract graph such that the set V consists of the points in the plane which are vertices of tiles, two vertices in V are connected by an edge in E if they are connected in the tiling. The abstract graph Λ comes with its embedding in the plane. In other words, Λ is a graph we see when we look at the tiling. Then the Tree conjecture can be formulated as follows.

Conjecture 1 (Tree conjecture for triangle tilings). *Take any periodic closed trajectory δ of a triangle tiling billiard. It incloses some bounded domain $U \subset \mathbf{R}^2$ in the plane, $\partial U = \delta$. Then the graph $G := U \cap \Lambda$ (as a subgraph of Λ) is a tree. In other words, a trajectory δ passes by all the tiles that intersect U .*

This conjecture was first formulated in [7] and proven there for the case of *obtuse* periodic triangle tilings (a graph G in question is in this case a chain).

Our interest in the Tree conjecture comes from its relationship to the density properties of other interesting and already studied objects, putting tiling billiards in a larger perspective. Indeed, the *Tree conjecture* is a first step in our approach of density properties of the fractal curves mentioned above, especially of the arithmetic orbits of Arnoux-Yoccoz map (and other minimal maps in the Arnoux-Rauzy family). In future work, we hope to adapt the methods used in the proof of the Tree Conjecture united with some additional arguments, in order to prove the existence of exceptional trajectories in triangle tiling billiards passing by all triangles of the tiling. The Tree Conjecture is interesting in itself since it gives a partial description of the symbolic dynamics of tiling billiards.

The main result of this work is

Theorem 3. *The Tree Conjecture for triangle tilings is true.*

We suspect the analogue of the Tree Conjecture to hold for cyclic quadrilateral tilings as well but we haven't manage to prove it yet, see the discussion in paragraph 6.1. The Tree Conjecture has a stronger form that we call Density property (even if obviously the trajectories are in no way dense on the plane), see paragraph 6.2.

The idea of the proof of the *Tree Conjecture* on both of the tilings is as follows. In order to study the symbolic behavior of one trajectory, it is helpful to study an entire *foliation* of trajectories that comes with it. We construct these very special foliations and then study them in detail. Thanks to this study, the *Tree conjecture* (which deals with *global* behavior of trajectories) is reduced to the *Flower conjecture* which deals with the *local* behavior of separatrices in associated foliations.

The methods we propose here can be generalized for a much bigger class of tilings but we have chosen deliberately to consider the simplest cases of periodic (triangle and cyclic quadrilateral)

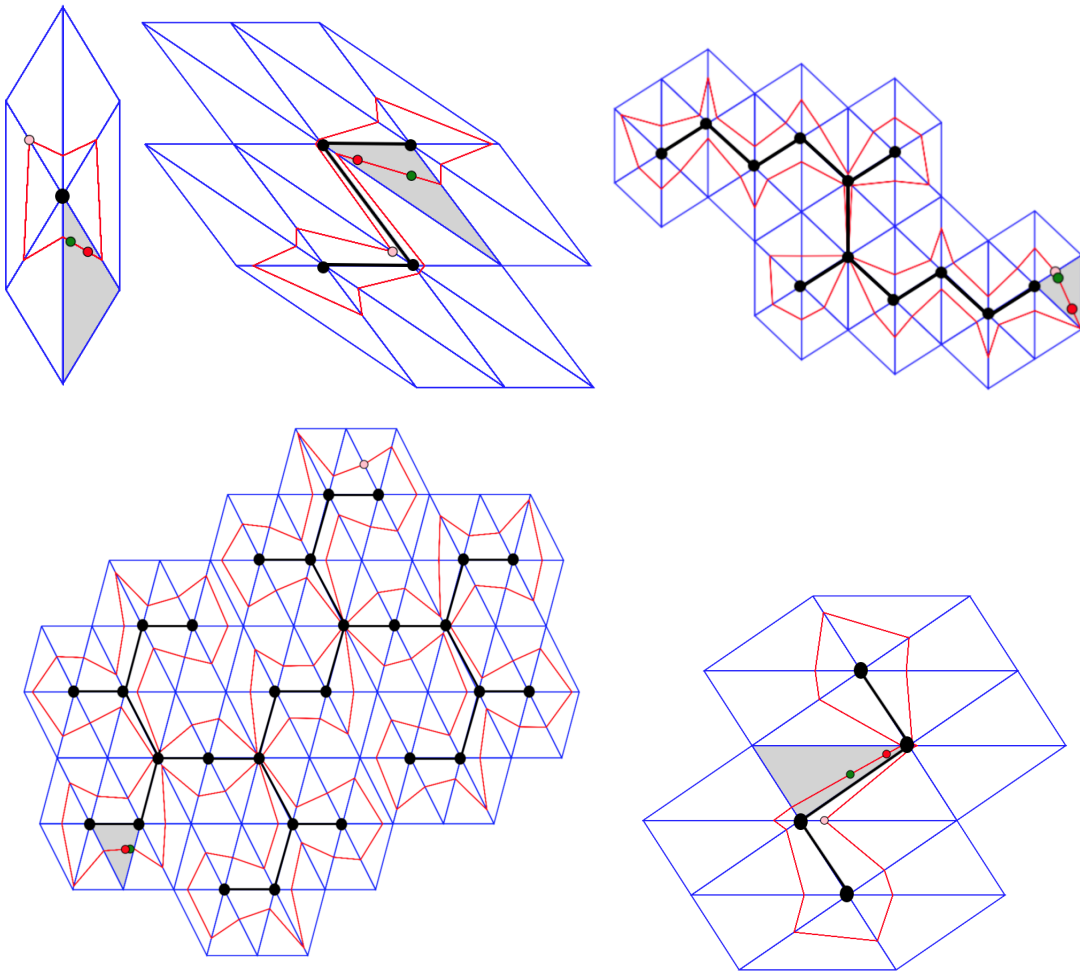


FIGURE 4. Several examples of triangle tiling trajectories and the corresponding trees. The simplest trajectory is a six-periodic trajectory and the corresponding graph is a simplest tree with one vertex. For the obtuse triangle tilings, the corresponding trees are always paths by a theorem in [7]. We then provide three more examples of trees for acute triangles: one sees that the forms of the trees can be quiet different. Pictures based on the program by Patrick Hooper and Alexander St Laurent, [14].

tilings in order to show the heart of the ideas. We postpone a detailed study of a more general setting to the work to come.

2.3. Plan of the paper. In the following, when said tiling billiard, we suppose that this is a tiling billiard in a periodic triangle or periodic cyclic quadrilateral tiling. We call these two tilings simply **triangle** and **quadrilateral tilings**.

In Section 3 we remind the folding construction for the triangle and quadrilateral tilings and apply it in order to explain basic properties of tiling billiards in these two tilings: all bounded trajectories are closed and stable under perturbation. In Section 4, we define tiling billiard foliations (paragraph 4.2), as well as singular trajectories of tiling billiards (paragraph 4.1). By using these foliations, we classify possible local behaviors of singular trajectories in triangle tiling billiards (paragraph 4.3). In Section 5 we finally prove the Tree Conjecture. We first reduce it to a so-called Flower conjecture that we introduce (paragraph 5.1). Second, we give a list of possible obstructions to the Flower Conjecture (paragraph 5.2). Third, we introduce the tools that help

us to deal with these cases (paragraph 5.3). Finally, we prove the Flower Conjecture, and hence the Tree Conjecture, by excluding the remaining cases (paragraph 5.4). In Section 6 we give possible generalizations and directions for future research. Namely, we state a more general form of the Tree Conjecture (Density property) and prove it for triangle tiling billiard trajectories which are not necessarily periodic (paragraph 6.2). Also we state two conjectures that can be seen as generalizations of Conjecture 1. One of them concerns the Tree Conjecture for cyclic quadrilateral tilings (paragraph 6.1) and another concerns the symbolic dynamics of fully flipped interval exchange transformations (paragraph 6.3).

3. FOLDING IN TRIANGLE AND CYCLIC QUADRILATERAL TILING BILLIARDS.

Tiling billiards on triangle and cyclic quadrilateral tilings have rigidity properties that can seem surprising at first sight. They are based on the folding construction (coming from [7]) which has a central place in this work. In this Section we present their construction, for triangle and cyclic quadrilateral tilings.

The material of this paragraph is essentially already contained in [7], although we present here our approach which is slightly more general. Baird-Smith-Davis-Fromm-Iyer use in an explicit way the structure of the periodic triangle tiling. Our proof implies that the properties discovered in [7] hold for a much larger class of tilings, see Remark 1. This change in perspective was very important for us since it has directly brought tiling billiard foliations that are the key tool of this work.

Lemma 1 ([7]). *Consider a periodic triangle (cyclic quadrilateral) tiling, and some tile τ_0 in it. Then there exists a unique map of the plane $\mathcal{F} : \mathbf{R}^2 \rightarrow \mathcal{F}(\mathbf{R}^2) \subset \mathbf{R}^2$ such that for any tile τ the map $\mathcal{F}|_\tau$ is an isometry and $\mathcal{F}|_{\tau_0} = \text{id}$; and for any two tiles τ and τ' sharing an edge e their images have a common edge $\mathcal{F}(e)$ and are symmetric one to each other with respect to a symmetry along a line perpendicular to $\mathcal{F}(e)$. Two different folding maps (depending on τ_0) differ by a global isometry of \mathbf{R}^2 . Moreover, $\mathcal{F}(V) \subset \mathcal{C}$ for some circle \mathcal{C} . The set $\mathcal{F}^{-1}(\mathcal{C})$ is the union of all the circumcircles of all tiles in the tiling.*

Proof. Let τ be some tile in a tiling. We construct its image under \mathcal{F} in a following way. Take a **sequence** of tiles $\tau_0 = \tau_0, \tau_1, \dots, \tau_n = \tau$ **connecting** τ_0 to τ : the tiles τ_k and τ_{k+1} share an edge. In order to define $\mathcal{F}(\tau)$, we first fold by a global isometry the union $\tau_1 \cup \dots \cup \tau_n$ on τ_0 . This defines $\mathcal{F}(\tau_1)$. Then, we fold $\tau_k \cup \dots \cup \tau_n$ on τ_{k-1} for $k = 2, \dots, n$. At the end of the process, one defines a folded image of a tile τ . One easily sees that in restriction to τ such a process is an isometry (as a composition of isometries).

It is left to prove that such a definition doesn't depend on the sequence $\{\tau_k\}$ connecting τ_0 to τ . It is equivalent to proving that the image of τ_0 under the folding process we defined on a **loop** ($\tau_0 = \tau_N$) is equal to τ_0 . First, consider a shortest loop making a tour of some vertex v of τ_0 . When one folds one polygon on another in a tour around a vertex, the difference between grey and white angles in the vertex defines the displacement of the initial tile τ_0 with respect to its initial position. Since this difference is zero in both triangle and cyclic quadrilateral tilings (equilibrium of the angles), a tile comes back to its place. By breaking any loop into a sum of loops around vertices of the tiling, one finishes the proof. Clearly, two folding maps differ by an isometry.

Moreover, all of the vertices map to the same circle \mathcal{C} (a circumcircle of τ_0). It is true for two tiles sharing an edge, and hence for all tiles, see Figure 5. \square

We call the map \mathcal{F} a **folding map**, or simply, a **folding**. We call the image of the plane by a folding map an **origami**, denoted by $\mathcal{O} := \mathcal{F}(R^2)$.

Remark 1. The two tilings we study in this work belong to a much bigger class of tilings (for which the folding map can be defined) that is called *locally foldable tilings*. This class has been

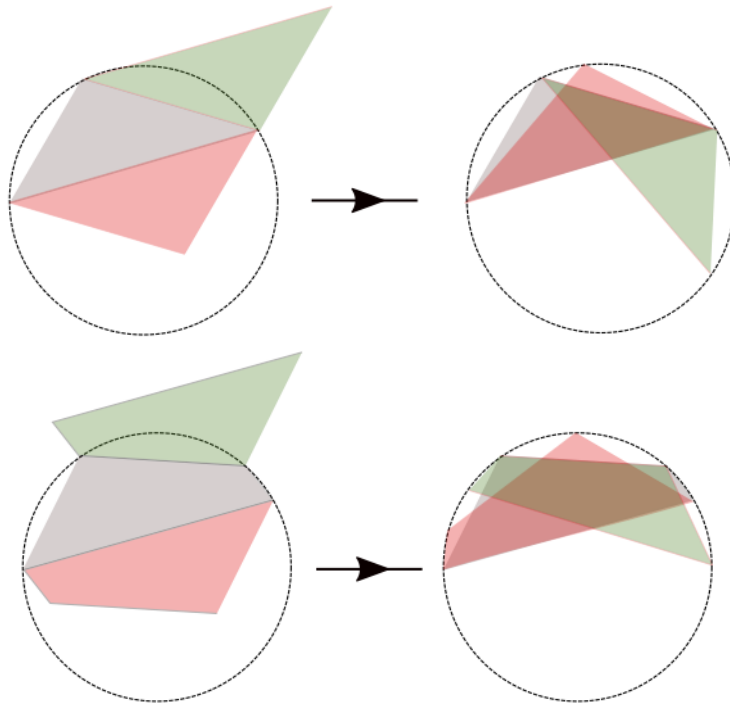


FIGURE 5. Folding on a circle for a patch of a triangle tiling (above) and a patch of a cyclic quadrilateral tiling (below). A tile τ_0 maps to itself under a folding, and the other tiles fold into a circumcircle of a grey tile. All the circumcircles of the tiles map to one circle \mathcal{C} under the folding. Indeed, one uses here the equality of angles based on the same arc of the circle.

known for centuries by origamists, and it also appears as one of the important tools in the discrete complex analysis for the dimer model. The arguments of Lemma 1 are not new, and are implicitly used for example in [17, 18, 8]. In this paper we concentrate ourselves on triangle and quadrilateral tilings. We hope that the general theory of tiling billiards in locally foldable tilings will emerge in the future.

Once the folding is defined, the statements 1.-3. of Theorem 1 follow directly from it, as was shown in [7]. We remind the proof of the first three points of Theorem 1 in the proof of a completely analogous theorem for cyclic quadrilaterals that follows in the next paragraph.

3.1. Basic orbit properties in triangle and cyclic quadrilateral tilings. In this paragraph we prove the results that were proven in [7] for triangle tiling billiards, for cyclic quadrilateral tilings. This result was announced without an explicit proof in [16].

The proof comes from completely copying the arguments in [7]. We repeat their argument here for completeness. The main idea is that a tiling billiard trajectory fold into the subsets of a *straight chord* in the circumcircle \mathcal{C} . In some sense the folding gives a first integral of a tiling billiard in triangle and cyclic quadrilateral tiling. In the context of Hamiltonian dynamics integrability means existence of (enough) additional integrals of motion, in other words the existence of the laws of preservation of energy. For tiling billiards we consider the direction of a trajectory (after folding, i.e. in "good" coordinates) is a first integral of the system. This folding reduces hence the dimension of the phase space.

Theorem 4. *The first three statements of Theorem 1 hold word by word for cyclic quadrilateral tilings.*

Proof. For any trajectory δ of a tiling billiard in a cyclic quadrilateral tiling, its image under the folding map is a segment $\mathcal{F}(\delta) \subset \mathcal{O}$ of a line $l \subset \mathbf{R}^2$, $\mathcal{F}(\delta) \subset l$. Indeed, for any trajectory that crosses an edge e , by folding one tile containing e onto its neighbour, the two pieces of the trajectory fold inside one segment.

We conclude that for any tile τ the segment $\mathcal{F}(\delta)$ crosses $\mathcal{F}(\tau)$ at most once (or never). Obviously, any bounded trajectory at some moment comes back to the same quadrilateral, hence it has no choice as to repeat itself and close up. We use here in a crucial way that the folding map \mathcal{F} is well-defined.

A bounded trajectory is a *simple closed curve*. Indeed, a trajectory can't intersect itself in a transverse way in some tile (and a non-transverse way is periodicity...) since in each tile τ it is contained in a fixed segment $\mathcal{F}^{-1}(\mathcal{F}(\tau) \cap l)$.

Moreover, each periodic trajectory is stable under a small enough perturbation since a perturbed trajectory crosses the same sides (the sides with the same names), i.e. its symbolic code is not changed. Hence it has to close up, since it comes back to the *same* quadrilateral and, by continuity, the same position. \square

4. TILING BILLIARD FOLIATIONS.

As we have seen in Section 3, one can define a folding map for triangle and quadrilateral tilings. Moreover, any tiling billiard trajectory folds under this map into a segment of a line in the origami.

Now take an origami \mathcal{O} in a disk bounded by \mathcal{C} , slice up a disk in a union of non-intersecting segments (by either a family of parallel chords, or a family of chords emanating from one point on the boundary of \mathcal{C}), and pull this slicing back to the tiled plane by \mathcal{F}^{-1} .

This process defines two (parallel and radial) families of *orientable foliations* on the plane with a tiling that are compatible with a folding. The leaves of these foliations are tiling billiard trajectories, and one can include a trajectory in two naturally defined foliations, and understand its behavior as a behavior of a trajectory in a family, and not to look at it as some lonely object.

4.1. What happens when a trajectory hits a corner of a tile. In a classic setting of a billiard in a bounded domain with piecewise smooth boundary, a trajectory that arrives to a non-regular point on the boundary, one says that it is not defined (or it stops). In the context of tiling billiards, in the same way as for geodesic flows on flat surfaces (and this is not a mere coincidence!), one can correctly define (possibly branching) singular trajectories.

A piece-wise linear simple path that passes through at least one vertex of a tiling is a **singular tiling billiard trajectory**, if the refraction law holds for all the pieces of such trajectory (even those that enter a vertex). We call a segment of a singular trajectory that connects a vertex with some point on the edge of the same tile a **separatrix segment**. If a singular trajectory comes back to a vertex it started in, we call this trajectory a **separatrix loop**.

To any singular trajectory, one associates a finite number (maybe 0) of singular trajectories passing by the same vertex v , via folding. Indeed, a singular trajectory folds into a chord that intersects the set $\mathcal{F}(V)$. In this case one can look at the intersection of this chord with six folded triangles surrounding the vertex v . The trajectories starting in such way are obviously all singular.

We say that the union of *all* separatrices passing by a fixed vertex in a tiling that are subsets in $\mathcal{F}^{-1}(l)$ for the same chord $l \subset \mathcal{C}$ is a **flower**. We call each of the separatrix loops (or separatrix rays) in this flower a **petal**. So for any segment in the folding passing by a point $\mathcal{F}(v)$ for some $v \in V$ we define a **flower** (the union of petals). We call $v \in V$ a **pistil** of a corresponding flower. A flower is **bounded** if all of its petals are bounded (and hence, periodic) singular trajectories.

As we see in the following, since the symbolic dynamics of any other trajectory can be described in terms of the dynamics of some singular trajectories on which it can be contracted.

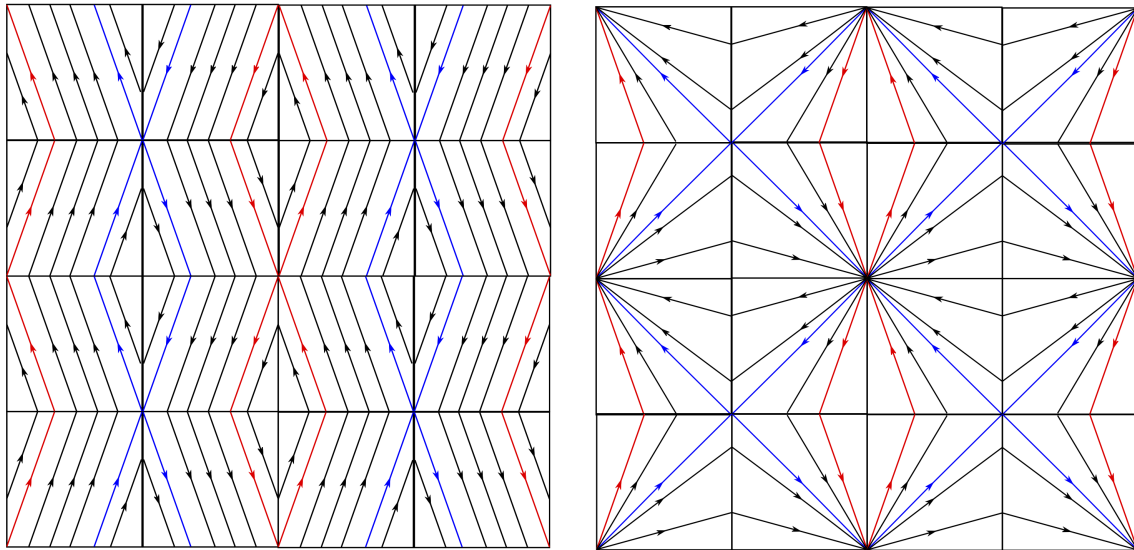


FIGURE 6. Parallel and radial foliations in a square tiling. On the left the parallel foliation \mathcal{P}_ν for some $\nu \in \mathbb{P}^1(\mathbf{R})$ and on the right the radial foliation $\mathcal{R}_v, v \in V$. By blue and red on both pictures we mark the separatrices. Every flower in both foliations consists of either 0 or 2 petals. Red trajectories are contained in both \mathcal{P}_ν and \mathcal{R}_v . Square tiling represents an example of periodicity of parallel and radial foliation: of course, it is not at all the case for a general triangle or cyclic quadrilateral tiling.

4.2. Parallel and radial tiling billiard foliations. We call a foliation a **tiling billiard foliation** if it is an oriented foliation of a plane with a tiling and all of its leaves (singular and not) are the tiling billiard trajectories. We define two foliations that come as preimages of two sheaves of lines on the plane containing the origami.

Take a triangle (quadrilateral) tiling, fixe some base tile τ_0 and an associated folding map \mathcal{F} , with an origami $\mathcal{O} = \mathcal{F}(\mathbf{R}^2)$. Then take some $\nu \in \mathbb{P}^1(\mathbf{R})$ and consider a foliation of the plane by parallel lines with a common direction ν . One considers the intersections of the leaves of this foliation with the origami \mathcal{O} . Then, by unfolding these intersections back to the plane with a tiling one obtains a foliation which we call the **parallel tiling billiard foliation** or simply, a **parallel foliation** \mathcal{P}_ν (or simply, \mathcal{P}).

Now, take some $v \in V$ and a point $p = \mathcal{F}(v) \in \mathcal{C}$. Consider all the lines passing by p . This foliation slices up the origami \mathcal{O} . By unfolding these slices back to the plane with a tiling one obtains a foliation which we call the **radial foliation** \mathcal{R}_v (or simply, \mathcal{R}) centered at the point $v \in V$.

Example. On Figure 6 we give the simplest cases of periodic and parallel foliations for the square tiling. In this special case the orientation on the leaves of the foliation is defined in the same way as on the trajectories (all of them singular in this case) is the same as that on the leaves of the foliation. One can note that two of these foliations have some common (singular) leaves. Hence these two foliations can be used to study these singular leaves, at the same time. This idea is a key idea in the proof of the Tree conjecture for triangle tiling billiards.

Lemma 2. *The following holds for the parallel and radial foliations in triangle and cyclic quadrilateral tilings.*

1. For any $v \in V$ and any $\nu \in \mathbb{P}^1(\mathbf{R})$ the foliations \mathcal{P}_ν and \mathcal{R}_v are well defined.

2. The set of singularities of \mathcal{P} as well as \mathcal{R} coincides with the set V of vertices of the tiles.
3. Each of the leaves of any of the foliations \mathcal{P} and \mathcal{R} is a tiling billiard trajectory.
4. For any $\nu \in \mathbb{P}^1(\mathbf{R})$ and any $v \in V$, there exists a finite number of singular leaves in \mathcal{P}_ν passing by v (at most one by each tile containing v). Conversely, two separatrices in \mathcal{P}_ν passing by v fold into the same line by folding.
5. Take any (possibly singular, not necessarily closed) trajectory δ . Then δ can be included in a unique parallel foliation \mathcal{P}^δ . If under folding δ folds into a chord passing by some element of $\mathcal{F}(V)$, then δ can be included in at least one (and at most two) radial foliations \mathcal{R}^δ .
6. The leaves of a foliation \mathcal{P} (foliation \mathcal{R}) foliate the invariant subsets of the tiling billiard map, e.g. the interior of any closed trajectory δ is foliated by the leaves of \mathcal{P}^δ (or, if possible, \mathcal{R}^δ).
7. The foliations \mathcal{P}_ν and \mathcal{R}_ν are orientable. For any $v \in V$ and $\nu \in \mathbb{P}^1(\mathbf{R})$ the orientations of the leaves of the foliations \mathcal{P}_ν and \mathcal{R}_ν coincide with the orientations of corresponding tiling billiard trajectories.

Proof. All of this follows obviously from the folding, see Lemma 1, united with the points 1.-3. of Theorem 1 as well as Theorem 4. □

4.3. Local behavior of separatrices of triangle tiling billiards. Radial foliations centered at singular points come of use for periodic triangle tilings. In the following, the radial foliation is used in order to understand the combinatorics of local behavior of separatrix segments in the triangle tilings.

Proposition 1. *Take any vertex $v \in V$ of a periodic triangle tiling. Consider the union of all singular trajectories of the foliation $\mathcal{P}_\nu, \nu \in \mathbb{P}^1(\mathbf{R})$, passing by v . Denote the number of separatrix segments containing v by s .*

Then s is a finite and even number. Moreover, in restriction to the union of six triangular tiles containing v , up to a possible change of orientation of all trajectories, there are exactly five possible combinatorial behaviors of separatrix segments. represented on Figure 7. For $s = 0, 4, 6$ there is only one possible combinatorial behavior and for $s = 2$ there are two of them.

Let us be precise what we mean by combinatorial behavior. This means that for any separatrix segment configuration in a foliation \mathcal{P} , the configurations presented on Figure 7 give all the possibilities modulo isometries of the plane. What is important is the number of separatrix segments and their *relative* positions.

Proof. Finiteness of s follows from 4. in Lemma 2. The parity of s follows from the fact that the foliation \mathcal{P}_ν is oriented.

The separatrices passing by v are leaves of both \mathcal{R}_ν and \mathcal{P}_ν , see Figure 8. Then, the radial foliation \mathcal{R}_ν has a very simple form in restriction to six triangles containing v . Indeed, it consists of all segments going into (or out from) the vertex v and the directions of the leaves alternate from one tile to its neighbor. This, and the orientability of \mathcal{R}_ν (see point 7. of Lemma 2) concludes the proof. □

Note. Moreover, the list given in Proposition 1 is realizable: one can find examples of foliations \mathcal{P}_ν (by choosing the forms of tiles and the directions ν) realizing all of the combinatorial behaviors. Although, the first case for $s = 2$ on Figure 7 is not realizable by separatrix loops. This follows from the Proposition 3 to come in the following.

The analogous to Proposition 1 statement can be proven for quadrilateral tilings.

5. TREE CONJECTURE FOR TRIANGLE TILING BILLIARDS.

In this Section, we prove the Tree conjecture for triangle tiling billiards.

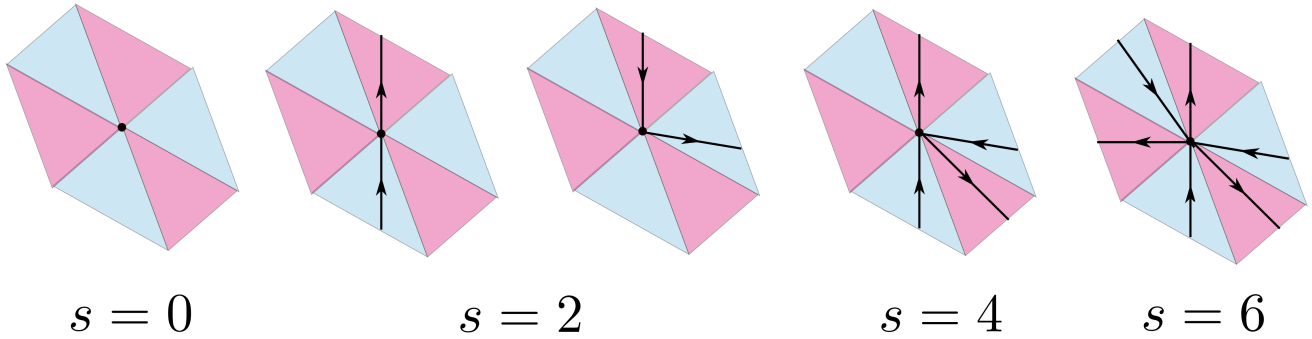


FIGURE 7. All possible local combinatorial behaviors of separatrices in $\mathcal{P}_\nu \cap \mathcal{R}_\nu$, $v \in V$, $\nu \in \mathbb{P}^1(\mathbf{R})$. The angles and exact placement of the segments is deduced from the vector v . For $s = 0$ the only behavior is trivial, and in the neighbourhood of a singular point the foliation \mathcal{P}_ν hence consists of 6-periodic trajectories. For $s = 2$ two behaviors are possible. For $s = 4, 6$ only one local behavior is possible.

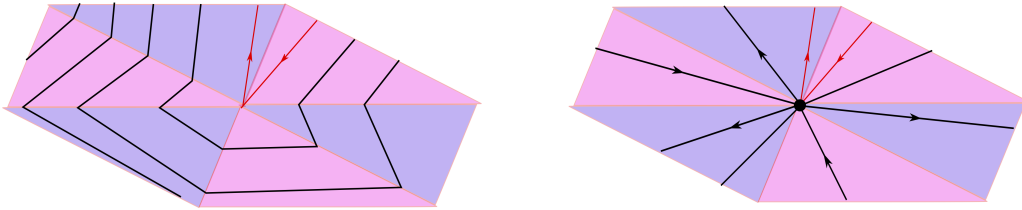


FIGURE 8. A restriction of the periodic foliation \mathcal{P}_ν (on the left) and a radial foliation \mathcal{R}_ν (on the right) to the six tiles around the vertex v . In red are given their common separatrix segments. The leaves of radial foliation alternate orientation from tile to tile.

5.1. Reducing the Tree conjecture to the Flower conjecture. *Tree conjecture* is a statement about the symbolic behavior of (possibly, very large) periodic trajectories. In this paragraph we reduce it to the *Flower conjecture* which is a statement about the topology of separatrix loops of one vertex $v \in V$.

Let us introduce some notations. We say that two tiles are **neighbouring** if they share an edge, and we say that two tiles are **opposite in a vertex** v if they both pass by v and are centrally symmetric to each other with respect to it. For any tile τ_0 passing through a fixed vertex $v \in V$, we denote by τ_0^v its opposite tile (with respect to this vertex). Obviously, $(\tau_0^v)^v = \tau_0$. Sometimes we say that the tiles are **opposite**, if the name of a vertex is clear from the context.

We say that the **Flower conjecture holds** for a tiling if for any vertex v in this tiling the following two conditions hold. First, any separatrix loop passing by v passes by two *neighboring* tiles containing v . And moreover, an edge between this tiles is contained in the interior of the finite domain bounded by the loop. Second, any two separatrix loops in a vertex belonging to the same parallel foliation \mathcal{P} have the *same* index with respect to infinity. In other words, the domains that these separatrix loops bound are disjoint.

The structure of our proof of the *Tree conjecture* is to reduce it to the *Flower Conjecture* and then to prove the second.

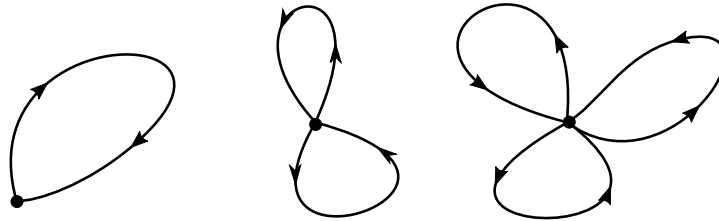


FIGURE 9. Possible non-trivial local behavior of bounded flowers for triangle tiling billiards.

Theorem 5. *The Flower Conjecture holds for periodic triangle tilings.*

The Theorem 5 that we prove in the following, together with the Proposition 1 give four possible forms of bounded flowers belonging to the foliation \mathcal{P} : with zero, one, two or three petals, see Figure 9. Of course, a singular point with no petals is also considered a flower, even though in real life such flowers are a little bit sad.

Note. The second condition of the Flower conjecture can be reformulated in terms of scissor cuts. Indeed, if one folds a tiling into an origami and then one cuts along some line (which is not a separatrix), the plane will fall into an infinite number of connected pieces. The second statement of the Flower conjecture is equivalent to saying that in this case none of the finite pieces is an annulus (and all of them are disks).

Proposition 2. *For a triangle or cyclic quadrilateral tiling, the Flower conjecture is equivalent to the Tree conjecture.*

Proof. First, if the Flower conjecture fails for some flower γ in a vertex v then one finds a periodic trajectory in the parallel foliation \mathcal{P}^γ (by small perturbation of separatrix loops) which contours at least one triangle. Indeed, one should take a periodic trajectory δ that approaches from outside a biggest petal (or union of petals) of the flower. If there exists a petal of a flower that contours a triangle (passes only by its vertex), then δ contours this triangle. If not, i.e. if two petals have different orientations (which means that one of the petals eats up another one), the second petal contains a triangle and hence a trajectory δ as well.⁴

Now let us prove that the Flower conjecture implies the Tree conjecture. Take some periodic trajectory δ and let us prove that it doesn't contour any tile given that the Flower conjecture holds. By 6. in Lemma 2 we include δ in the foliation \mathcal{P}^δ . Then the domain bounded by δ is foliated by a family of trajectories in \mathcal{P}^δ and we consider all (finite number) singular trajectories among this family. These separatrices define the symbolic behavior of δ .

Let us contract δ inside in order to obtain a separatrix loop (or a union of loops) with a singularity in some vertex v . If the trajectory contracts to a vertex, it has period 6 and a tree it contours is a point. Otherwise, we can assume that all of the resulting trajectories in a union of loops are singular in only one point v . If it is not true, one of them is a separatrix connecting two vertices of a tiling. But then a corresponding folded chord l connects $\mathcal{F}(v)$ with $\mathcal{F}(v')$ for some $v, v' \in V, v \neq v'$. In this case we perturb the initial direction of δ (without changing its symbolic dynamics, see 3. in Theorems 1 and 4), and restart the process.

This means we obtain a flower (with m petals, where $m \in \{0, 1, 2, 3\}$ for triangle tilings and $m \in \{0, 1, 2\}$ for quadrilateral tilings) that is contained in the foliation \mathcal{P}^δ . Now we take each of the petals we obtained, and approach them by periodic trajectories from the inside. We see that

⁴Our reader can easily find a trajectory δ for all obstructions on Figure 10. Analogously, this is true for quadrilateral tilings.

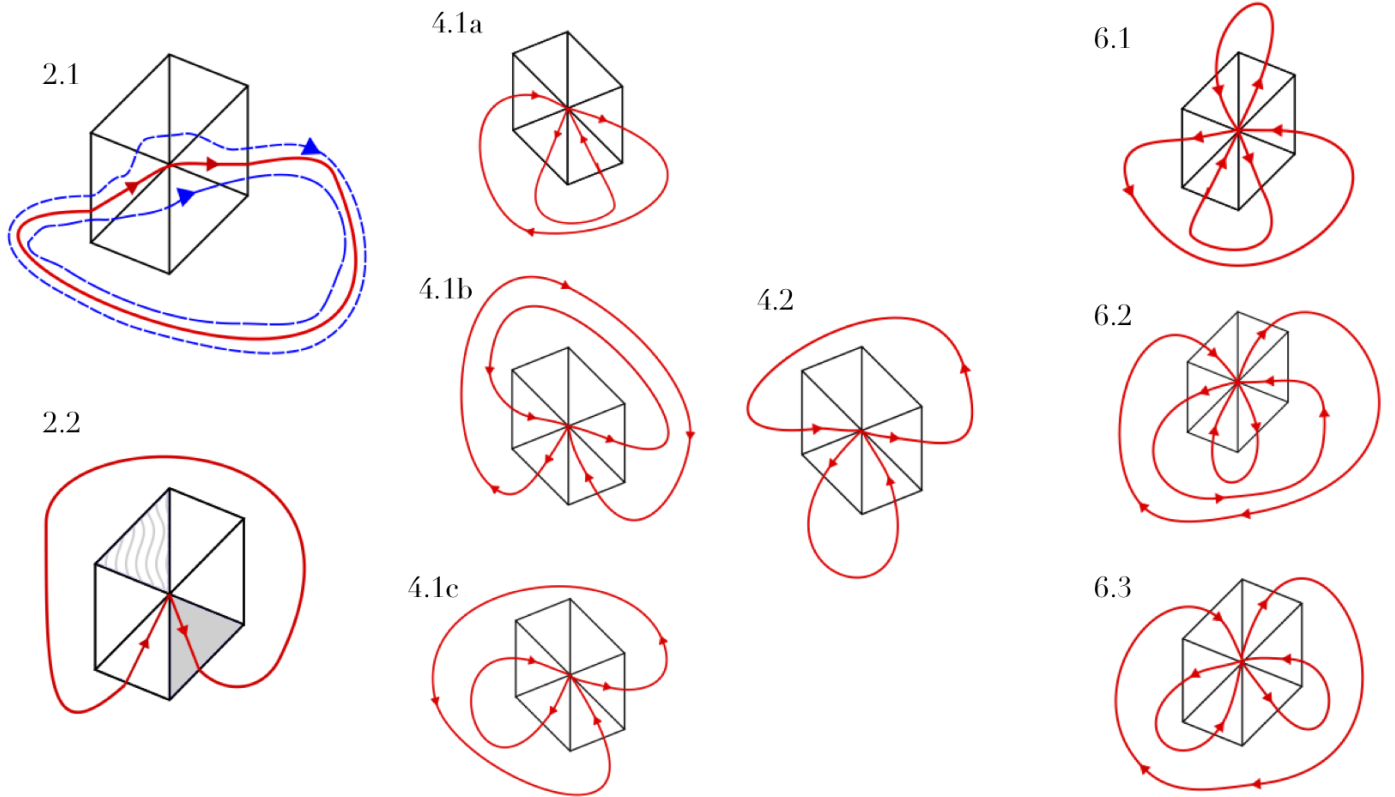


FIGURE 10. *A list of topological obstructions for the Flower conjecture for triangle tiling billiards.* This is the main Figure of this Section. It represents all of the cases of flower configurations that we want to exclude in the proof of the Flower conjecture. This Figure is presented in three columns (from left to right) in a way that the number of petals grows from 2 to 6. Red trajectories are separatrix loops that close themselves up in a way that contradicts the Flower conjecture. They can do it in three different ways: either the petals of one flower have different orientations; the separatrix in \mathcal{R}^v doesn't pass by neighboring tiles in v ; or it does, but its interior doesn't contain an edge between these tiles. This Figure carries topological information, i.e. the way the trajectories are placed with respect to each other globally as well as the local combinatorics discussed in Proposition 1.

a subgraph of Λ contoured by γ is a union of graphs for periodic trajectories in each of the petals united with m edges passing through v inside each of the petal.⁵

In this way, we reduce the Tree Conjecture for the initial trajectory for the Tree Conjecture for a trajectory of a smaller period. In this process, for each periodic trajectory inside a petal, the number of vertices inside it diminishes at least by one with respect to an initial trajectory. At some moment the process stops and hence a graph inside is a tree, by recurrence on the period of a trajectory. \square

5.2. Obstructions to the Flower conjecture. Based on Proposition 2, it suffices to prove Theorem 5 in order to prove Conjecture 1.

⁵Note that also on this step the symbolic dynamics of the initial periodic trajectory is defined by the dynamics of the periodic trajectories approaching the petals from inside and the disposition of the petals.

Proposition 1 gives a finite list of possible local behaviors of separatrices in some vertex $v \in V$. Then, by a simple combinatorial verification we see that the only cases of global behavior of bounded flowers belong to a following list of seven cases that are all illustrated on Figure 10.

A list of behaviors of bounded flowers in some vertex v that contradict the Flower conjecture.

- S2.1** There are two separatrix segments ($s = 2$) and the only petal γ passes by a pair τ_0, τ_0^v of opposite in v triangles.
- S2.2** There are two separatrix segments and the only petal γ passes by neighbouring triangles but the edge between two crossed triangles is not contained in the interior of the domain bounded by a trajectory.
- S4.1** There are four separatrix segments ($s = 4$) and two petals have different indices as curves. For **4.1a** and **4.1b**, the petals are formed by one separatrix γ_1 which crosses two opposite tiles and another separatrix γ_2 which crosses two neighbouring tiles. The two cases occur when the finite domain bounded by γ_1 (or γ_2) contains the domain bounded by γ_2 (or γ_1). The case **4.1c** concerns the situation when both separatrix loops pass by two neighbouring triangles but have different indices.
- S4.2** There are four separatrix segments and the two separatrix loops γ_1, γ_2 have the same index as curves but one of the loops γ_1 passes by opposite triangles.
- S6.1** There are six separatrix segments ($s = 6$) and hence three separatrix loops $\gamma_1, \gamma_2, \gamma_3$. One of the loops γ_3 passes by opposite triangles (and hence, bounds a domain containing γ_2).
- S6.2** There are six separatrix segments. Moreover, one of the separatrix loops γ_1 passes by neighbouring tiles, a separatrix loop γ_2 passes by opposite triangles and has a different orientation from that of γ_1 . Finally, a separatrix loop γ_3 passes by two left neighbouring triangles and has different orientation from that of γ_2 . In such a way, the three domains bounded by the loops are contained one in each other in a sequence.
- S6.3** There are six separatrix segments. All of the separatrix loops pass by neighbouring triangles. Two of these loops γ_1, γ_2 have the same index, and another one γ_3 has an opposite index.

Note. This list is given modulo a possible change of all of the orientations of the leaves to the inverse orientations. Without loss of generality, we fix the orientations as shown on Figure 10.

Our goal is to prove that all of the cases listed above are not realized by trajectories of triangle tiling billiards. In the following we exclude the cases in the list one by one. Before passing to the direct exclusion of cases, we present our main tools.

First of all, we use in a crucial way both of the foliations \mathcal{P} and \mathcal{R} that can be defined for a very large class of tilings.

Then, we use two tools which are proper to a periodic triangle tiling. First, the periodic symbolic words are the *squares* of some symbolic words, see 5. in Theorem 1. This *square property* is a very strong property following from the relation of triangle tiling billiards to interval exchange transformations with flips. Second, we use the *symmetry* of the radial foliation \mathcal{R} for triangle tiling billiards centered at a singularity. Both of these tools are very strongly related to the special features of this particular periodic triangle tiling and hence, we believe, can be hard (but maybe, not impossible?...) to apply directly in more general contexts.

In order to show these tools in action, we first show that the cases **S2.1** and **S2.2** are not realized by separatrices of triangle tiling billiards, in much detail, and then treat all the other cases with the same methods.

5.3. Introduction of main tools and exclusion of obstructions for $s = 2$. Every triangle tiling billiard trajectory has a corresponding symbolic code in the alphabet $\mathcal{A}_\Delta = \{ab, ba, ac, ca, bc, cb\}$

consisting of two-letter symbols, where the letters correspond to the codes of the edges a, b, c of the tiles. We define its symbolic code as follows: it is a sequence of *two-letter* symbols corresponding to the edges crossed by a trajectory. Each two-letter symbol in a code starts by the last letter of a preceding symbol and finishes by the first letter of the following word.

Note. We consider the symbolic codes of *periodic trajectories* as cyclic words, i.e. we consider that the following two periodic words are equal:

$$w_0 \dots w_n = w_k w_{k+1} \dots w_n w_0 \dots w_{k-1}, k \in \{1, \dots, n\}, w_j \in \mathcal{A}_\Delta.$$

By point 5. of Theorem 1, any symbolic code is a *square* of some word in alphabet $\mathcal{A}_\Delta = \{ab, ba, bc, cb, ca, ac\}$. For example, a simplest 6-periodic orbit has a following cyclic periodic code: $(ab\ bc\ ca)^2 = (bc\ ca\ ab)^2$.

We simplify a symbolic code in \mathcal{A}_Δ by a code with two symbols: we consider a simplified **sign alphabet** $\mathcal{S}_\Delta = \{-, +\}$. In order to do this, we replace any of the symbols ab, bc or ca by a symbol $+$ and any of the symbols ba, cb and ac by a symbol $-$. See more on this simplified coding in paragraph 6.3.

A word on the notation. We denote by the letters γ with (or without) indices the separatrix loops and by δ with (or without) indices periodic trajectories approaching these loops.

Moreover, we identify the trajectories with their symbolic orbits. We denote by γ an oriented trajectory (as a curve on the plane) and a cyclic periodic word in the alphabet \mathcal{A}_Δ (or in \mathcal{S}_Δ) which is a symbolic code of this trajectory.

In order to exclude the case **S2.1**, one uses symbolic dynamics.

Proposition 3. *A configuration **S2.1** is never realized by a separatrix loop of a triangle tiling billiard.*

Proof. Suppose that a configuration **S2.1** is realized by some separatrix loop γ in the vertex v of a triangle tiling billiard, see Figure 11. There are no other separatrices passing through v contained in the parallel foliation \mathcal{P}^γ corresponding to γ .

In this case, we perturb γ in the foliation \mathcal{P}^γ in order to obtain two periodic trajectories δ_{in} and δ_{out} in the neighbourhood of γ . We can suppose that outside six triangles around v , δ_{in} and δ_{out} have the same symbolic code as γ . Then, one can write the cyclic symbolic words in the alphabet \mathcal{S}_Δ , corresponding to δ_{in} and δ_{out} . There exists word $S \in \mathcal{S}_\Delta^{\mathbb{N}}$ of even length such that:

$$\begin{aligned} \delta_{\text{in}} &= + - - + S, \\ \delta_{\text{out}} &= - + + - S. \end{aligned}$$

We split S into a concatenation of two words $S = s\bar{s}$ of equal length and use the fact that δ_{in} and δ_{out} have symbolic codes which are squares of some words in \mathcal{A}_Δ (and hence, in \mathcal{S}_Δ). This gives

$$\begin{aligned} \delta_{\text{in}} &= - + s\bar{s} + - \\ \delta_{\text{out}} &= + - s\bar{s} - +. \end{aligned}$$

Then we have that simultaneously holds $- + s = \bar{s} + -$ and $+ - s = \bar{s} - +$ for $s, \bar{s} \neq \emptyset$. Hence the word s finishes by $+$ and $-$ at the same time, which is a contradiction. \square

In order to exclude **S2.2**, one uses a symmetry argument for the radial foliation \mathcal{R}_v .

Proposition 4. *A configuration **S2.2** is never realized by a separatrix loop of a triangle tiling billiard.*

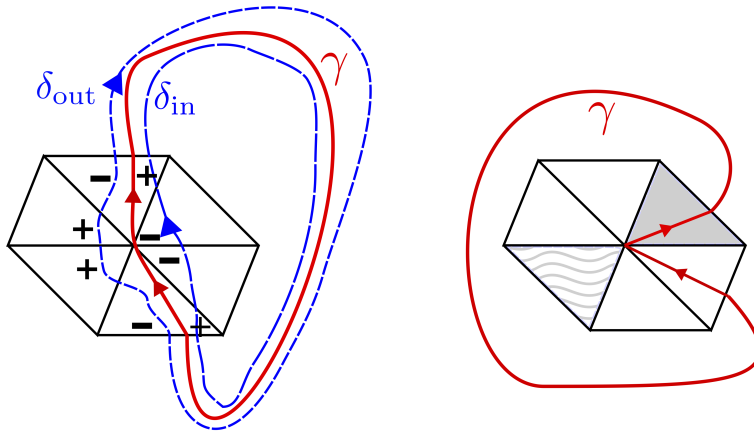


FIGURE 11. Two obstructions **S2.1** and **S2.2** and illustration for the proofs of Propositions 3 and 4. On the left, the case **S2.1**, a petal γ and two periodic trajectories approaching it in a parallel foliation \mathcal{P}^γ : a trajectory δ_{in} is contained inside a domain bounded by γ and a trajectory δ_{out} is contained outside this domain. The symbolic codes of δ_{in} and δ_{out} on the patch of six triangles surrounding v are correspondingly $+ - - +$ and $- + + -$. On the right, the case **S2.2**, a hungry triangle (with a separatrix loop γ passing through it) eats up its opposite triangle.

Note that for the case **S2.2** there exists a tile τ_0 such that the following property holds. There exists a separatrix loop γ such that its segment is contained in τ_0 and such that the opposite tile τ_0^v is contained in the interior of the loop γ . In this case, we say that the triangle τ_0 is a **hungry triangle** and that it eats up τ_0^v . We call a flower (not necessarily bounded) in some \mathcal{P}^γ a **hungry flower** if there exists a petal in this flower passing by a hungry triangle. This property is shared by configurations **2.2**, **4.1c**, **6.2** and **6.3**. In order to prove Proposition 4, we prove a more general statement that will exclude all the cases that we have just mentioned.

Proposition 5. *A configuration of separatrices forming a hungry flower is never realized by tiling billiard trajectories.*

Proof. Triangles τ_0 and τ_0^v fold into two triangles, symmetric under reflection with respect to the diameter d of the circle \mathcal{C} that passes by $\mathcal{F}(v)$, see Figure 12. A separatrix that eats up a triangle, folds in some segment in the circle \mathcal{C} . Then, one considers a leaf of the radial foliation \mathcal{R}_v , which is *symmetric* to the leaf corresponding to an initial separatrix with respect to the diameter d . Then, for any trajectory passing by the vertex v there exists a symmetric trajectory in the folding. One obtains a new separatrix loop that now passes by τ_0^v in such a way that now makes it a hungry triangle, and τ_0^v eats up τ_0 in its turn, by symmetry. On the unfolded situation on the plane, the corresponding separatrices have to intersect, which is impossible since they belong to the same radial foliation \mathcal{R}_v . This is a contradiction. \square

Note. Our proof shows an even stronger statement - the radial foliation \mathcal{R}_v has central symmetry on the plane with respect to the vertex v .

Corollary 1. *Configuration **2.2**, **4.1c**, **6.2** and **6.3** are never realized by separatrix loops of a triangle tiling billiard.*

The possible obstruction that are left to exclude are **4.1a**, **4.1b**, **4.2**, **6.1**. They are treated analogously to **S2.1** in the next paragraph.

5.4. Exclusion of remaining cases and finalisation of the proof. We now exclude the remaining cases.

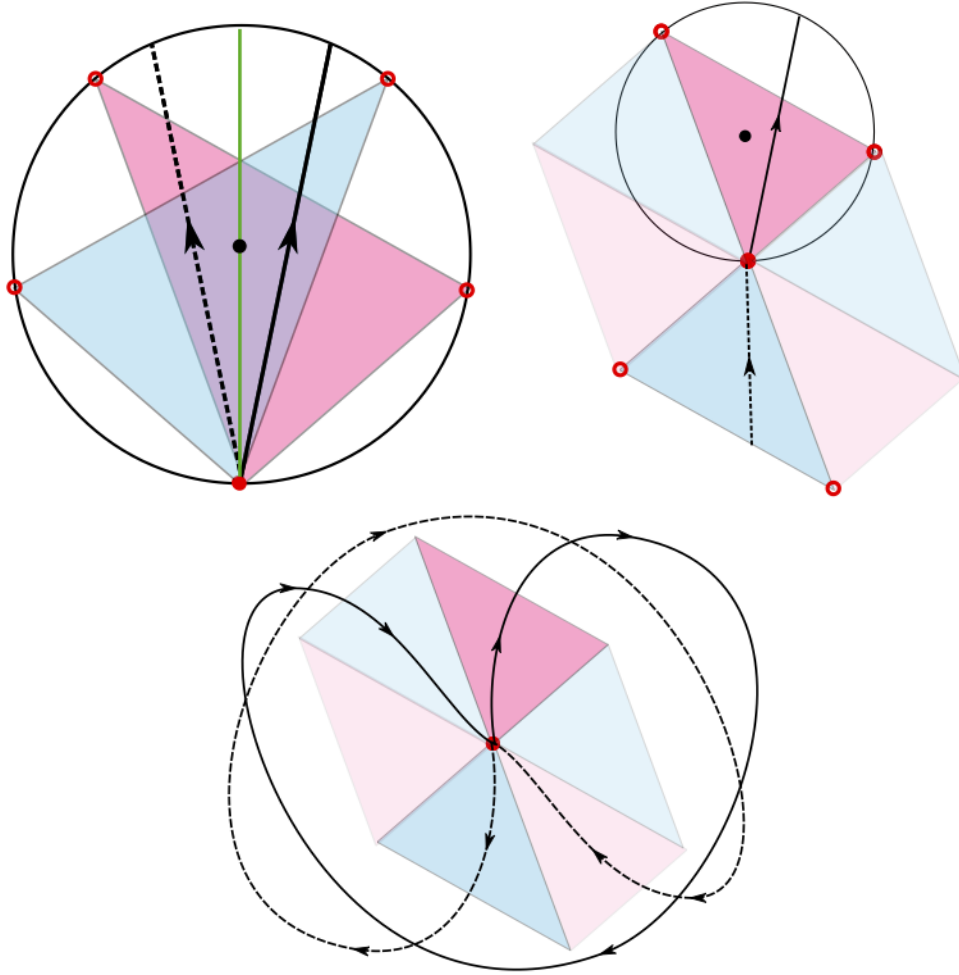


FIGURE 12. *Symmetry of \mathcal{R} and illustration to the proof of Proposition 5.* First row: Folded triangles $\mathcal{F}(\tau_0)$ and $\mathcal{F}(\tau_0^v)$ are symmetric to each other with respect to a diameter of \mathcal{C} passing through $\mathcal{F}(v)$, as shown on the left. Hence to any singular trajectory through v starting in τ_0 one associates a symmetric trajectory finishing in τ_0^v , as shown on the right. In the second row one can see that by unfolding these trajectories, if one of them has a form as **S2.2**, one obtains intersections of leaves in radial foliation \mathcal{R}^v and a contradiction.

Proposition 6. *Configurations **S4.1a** and **S4.1b** are never realized by a separatrix loop of a triangle tiling billiard.*

Proof. Consider the case **S4.1a**. Denote $\gamma_{\text{in}} := \gamma_2$ and $\gamma_{\text{out}} := \gamma_1$. Approach γ_{in} by a periodic non-singular trajectory δ_1 (from inside), and the trajectory γ_{out} by a periodic non-singular trajectory δ_2 (from outside). One can also choose a non-singular periodic trajectory δ , close to the union $\gamma_{\text{out}} \cup \gamma_{\text{in}}$ in between two loops. This can be done in a way that all of the trajectories $\gamma_{\text{in}}, \gamma_{\text{out}}, \delta, \delta_1, \delta_2$ belong to the same parallel foliation $\mathcal{P} = \mathcal{P}^{\gamma_1} = \mathcal{P}^{\gamma_2}$. For some words w and u in the alphabet \mathcal{A}_Δ , we have

$$\begin{aligned}\delta_1 &= (w - -)^2, \\ \delta_2 &= (u - + + -)^2, \\ \delta &= + + w - - w + + u - + + - u.\end{aligned}$$

But δ has to be also a square. Hence, from the considerations of length, we have the equality:

$$-w + +u - + = + - u + +w -$$

which is impossible since $- \neq +$ correspond to different symbolic codings.

The argument for the case **S4.1b** is the same, just put $\gamma_{\text{in}} := \gamma_1$ and $\gamma_{\text{out}} := \gamma_2$. \square

Proposition 7. *Configuration S4.2 is never realized by a separatrix loop of a triangle tiling billiard.*

Proof. Define three periodic trajectories δ_1, δ_2 and δ belonging to the same parallel foliation \mathcal{P} as the separatrices γ_1, γ_2 . First, δ_1 and δ_2 are two periodic trajectories approaching each of γ_1 and γ_2 from the inside. We take these periodic trajectories close enough to the loops that they have the same symbolic codes as separatrix loops. Finally, δ is a periodic trajectory close to $\gamma_1 \cup \gamma_2$.

Then, there exist words $s, \bar{s}, w \in \mathcal{A}_\Delta^{\mathbb{N}}$ such that $|s| = |\bar{s}|$ and

$$\begin{aligned} \delta_1 &= - + + - s\bar{s}, \\ \delta_2 &= (- - w)^2, \\ \delta &= + + w - -w + +s\bar{s}. \end{aligned}$$

This implies, by length considerations:

$$\begin{aligned} \bar{s} - + &= + - s, \\ -w + +s &= \bar{s} + +w - . \end{aligned}$$

This implies that the word s finishes by $-$ and $+$, contradiction. \square

Proposition 8. *Configuration S6.1 is never realized by a separatrix loop of a triangle tiling billiard.*

Proof. As shown on the Figure 10, in this case we have three separatrix loops γ_1, γ_2 and γ_3 such that the domain bounded by the loop γ_3 contains the domain bounded by γ_2 . Then one chooses periodic trajectories $\delta_j, j = 1, 2, 3, 4$ approaching these loops in such a way that δ_1 (δ_2) is a periodic trajectory approaching γ_1 (γ_2) from inside. A trajectory δ_3 approaches the union $\gamma_1 \cup \gamma_3$ from outside, and a trajectory δ_4 approaches the union $\gamma_2 \cup \gamma_3$ in the space between these two.

Then there exist the words $w, v, x \in \mathcal{A}_\Delta^{\mathbb{N}}$ such that

$$\begin{aligned} \delta_1 &= (w - -)^2, \\ \delta_2 &= (v + +)^2, \\ \delta_3 &= + + w - -w + +x, \\ \delta_4 &= - - v + +v - -x. \end{aligned}$$

Since both δ_3 and δ_4 are squares, one can split the word $x = U\bar{U}$ in two words of equal length, $|\bar{U}| = |U|$. Hence the length considerations for δ_3 and δ_4 give two equations :

$$\begin{aligned} -w + +U &= \bar{U} + +w -, \\ \bar{U} - -v - &= +v - -U. \end{aligned}$$

Hence the word \bar{U} has to start from $+$ and $-$ at the same time, a contradiction. \square

The Flower Conjecture now follows.

Proof. Take any vertex and all the separatrix loops in it belonging to some fixed parallel foliation \mathcal{P} . One can suppose that any of these loops doesn't pass through any other vertex in a tiling than its base v . Indeed, one can perturb the angles of a triangle in order to destroy this very special configuration without changing the combinatorics of the behaviour of the separatrices (this can be seen on the folding since connection of vertices is equivalent to the fact that a corresponding chord in the circle connects to vertices on the origami). Then, an obtained flower has to satisfy the Flower conjecture since all other cases have been excluded in Propositions 3–8. \square

By Proposition 2, this finishes the proof of the Tree conjecture for triangle tiling billiards.

Note that our strategy of the proof gives a new proof of a result in [16] which states the special case of the Tree conjecture for obtuse tiles.

Corollary 2 (Theorem 5.7. in [7]). *Any periodic trajectory in a periodic obtuse triangle tiling billiard encloses a tree which is a path.*

Proof. One can easily see, by folding six triangles surrounding a vertex on a circle, that each flower for obtuse triangle tiling has *at most two* petals. \square

6. POSSIBLE DIRECTIONS AND OPEN QUESTIONS.

6.1. Tree conjecture for cyclic quadrilateral tilings: challenges. The theory of tiling billiards in cyclic quadrilateral tilings in many ways analogous to that of triangle tiling billiards as already mentioned in [16], see paragraph 3.1 above. In this paragraph, we discuss the differences and challenges that presents this system, in comparison to triangle tiling billiards.

As already formulated in [16], the Tree Conjecture for cyclic quadrilateral tilings seems to hold as well for any periodic orbit, see Figure 13.

Conjecture 2 (Tree conjecture for cyclic quadrilateral tilings). *Take any periodic closed trajectory δ of a periodic cyclic quadrilateral billiard. A trajectory δ incloses some bounded domain $U \subset \mathbf{R}^2$ in the plane, $\partial U = \delta$. Then the graph $G := U \cap \Lambda$ (as a subgraph of Λ) is a tree. In other words, a trajectory δ passes by all the tiles that intersect U .*

We didn't yet manage to prove this Conjecture. Even though one can prove easily the analogue of Proposition 1, and parallel and radial foliations are well defined, the symbolic behavior of quadrilateral tilings seems to be much more complicated than that of triangle tilings.

The symbolic dynamics of quadrilateral tilings is defined as follows. Denote by $\alpha, \beta, \gamma, \delta$ the angles of a quadrilateral tile and its corresponding sides by a, b, c, d (a being a side between angles α and β etc.). We suppose that any quadrilateral in the tiling is oriented in a way that a counterclockwise tour of its boundary reads $abcd$. Then a **symbolic code** of a curve on a plane with respect to a periodic cyclic quadrilateral tiling is defined as a word in the alphabet $\mathcal{A}_{\square} = \{ab, ac, ad, ba, bc, bd, ca, cb, cd, da, db, dc\}$. For example, a code of a simple closed curve making a tour of a vertex in this tiling in a clock-wise manner reads $ab\ bc\ cd\ da$.

The symbolic coding for quadrilateral tilings doesn't have the properties 4.-5. of Theorem 1 for triangle tilings, and the orbits have periods that do not necessarily belong to the set $\{4n + 2, n \in \mathbf{N}^*\}$. The simplest example being 4-periodic orbits on square tilings. But this is not the only example, and an orbit can have period 20 and be quite assymetric, see Figure 14.

The Flower conjecture and the Tree Conjecture seem to continue to hold even though we didn't manage to prove them by using the tools we have developped here, without a better understanding of symbolic dynamics of trajectories of cyclic quadrilateral tiling billiards.

6.2. Density property. In this paragraph we present a generalization of the Tree Conjecture that can be applied for any (not necessarily periodic) trajectory, a so-called Density property. We show that a trajectory *constructs dynamically* two graphs, and in the case of non-periodic behavior, both of these graphs are trees.

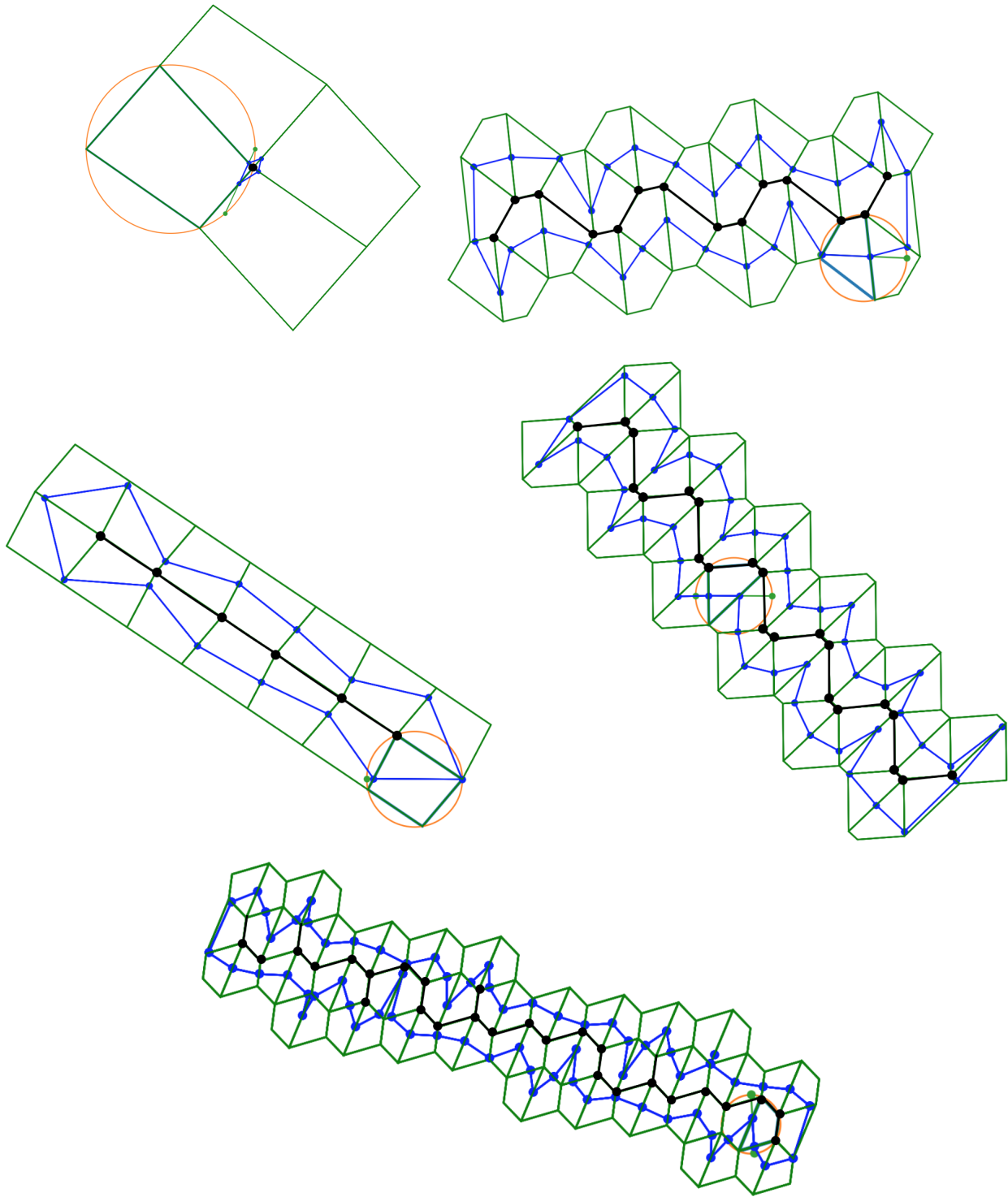


FIGURE 13. Several examples of cyclic quadrilateral tiling billiard trajectories and the corresponding trees. The simplest trajectory is a four-periodic trajectory and the corresponding graph is a simplest tree (one vertex without edges). The trees obtained can be much more complicated and are not necessarily paths as shows the last example.

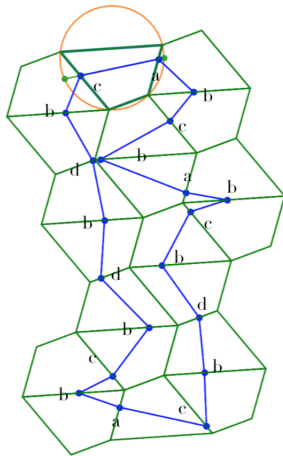


FIGURE 14. An orbit represented here has its symbolic behavior given by a periodic word $w = cabcbabcbdbcabcbdbdb$ and its period is equal to 20. The word w can't be represented as a square of some shorter word. A corresponding arithmetic orbit is not symmetric.

Consider a (not necessarily closed) trajectory δ of a tiling billiard. Let $V(\delta) \subset V$ be a set of vertices such that the trajectory intersects at least one edge with $v \in V(\delta)$ as an extremity. We will color the vertices of this set in two colors, black and white by following a trajectory. We define a map $\mathcal{L} : V(\delta) \rightarrow \{B, W\}$ in order to obtain a decomposition of the set $V(\delta)$ into two complement sets $B(\delta) = \mathcal{L}^{-1}(0)$, $W(\delta) = \mathcal{L}^{-1}(1)$. An algorithm of simultaneous construction of sets $B(\delta)$, $W(\delta)$ goes as follows.

First, pick some edge e crossed by a trajectory. Denote its extremities b_0 and w_0 , in any arbitrary order. Add $b_0 \in B(\delta)$, $w_0 \in W(\delta)$. At the next step, one adds one element in one of the sets $B(\delta)$ or $W(\delta)$. We will say that on each step we add $b_{j+1} \in B(\delta)$, $w_{j+1} \in W(\delta)$ but that either $b_j = b_{j+1}$ or $w_j = w_{j+1}$. The elements added satisfy the following relation: the edges of $b_j b_{j+1}$ and $w_j w_{j+1}$ are not crossed by the trajectory and the edges $b_j w_{j+1}$ and $w_j b_{j+1}$ are crossed by the trajectory. Note that some of these edges degenerate into vertices. It may also happen for some $k < j - 1$ that $b_j = b_k$, $k < j - 1$.

Take two graphs $\Gamma_{B(\delta)}$ and $\Gamma_{W(\delta)}$ (subgraphs of Λ) with the sets of vertices $B(\delta)$ and $W(\delta)$ correspondingly.

Then the Conjectures 1 and 2 can be generalized to have the following form:

Conjecture 3 (Density property). *For any trajectory δ , at least one of the graphs $\Gamma_{B(\delta)}$, $\Gamma_{W(\delta)}$ is a tree. A trajectory is periodic if and only if another graph has a unique cycle in it. A trajectory is non-periodic if and only if both of the graphs are cycles.*

The Density property holds for triangle tiling billiards and its proof is analogous to the proof of Theorem 3. We won't make this proof explicit but the only difference with a proof of the Tree Conjecture is that now we have to consider not only bounded flowers but also flowers with some petals that are unbounded. The obstructions will be although presented by bounded petals and are treated with the methods we presented above. Although, the Density property for quadrilateral tilings is an open question, and it seems to hold for simulated trajectories.

The Density property can be reformulated in terms of scissor cuts of an origami.

Reformulated Density property. Consider a periodic (triangle or cyclic quadrilateral) tiling of the plane and fold the plane into an origami contained in a disk. Then, cut along some chord in this disk. Then, the plane falls into an infinite number of connected components and *none* of these components contains a full triangle.

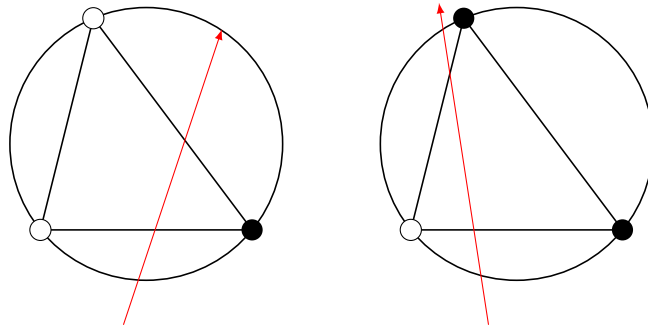


FIGURE 15. Coloring of the set of vertices V in the tiling with respect to a folded trajectory.

A difficulty in proving such a statement directly is that when one makes a cut of an origami, one does not cut out one trajectory but an infinite number of trajectories with the same first integral. We do not exclude a possibility that maybe the Tree Conjecture has a much simpler proof than the one we presented here based only on the idea of folding into a circle. We present here a following statement proved directly by looking at a folding and not with a use of foliations on the plane. Of course, this statement follows obviously from Theorem 3. But we present this statement here in order to show its proof.

Proposition 9. *There is no trajectory δ of triangle tiling billiard that doesn't pass through a triangle Δ but does pass through all of its neighboring triangles (the three triangles $\Delta_a, \Delta_b, \Delta_c$ sharing the sides with Δ).*

Proof. Consider a chord corresponding to a folding of δ into the origami. Color the vertices of the tiling in two colors: black and white, depending on their placement on the circle C , on the left or on the right from this chord. Then all the points in V are colored, and this coloring is compatible with a coloring of $V(\delta)$ defined above (the latter is a subset of the former).

Now, if a trajectory δ as in the assumption exists, then the vertices A, B, C of Δ are colored in the same color and the other three vertices A', B', C' of neighbouring triangles $\Delta_a, \Delta_b, \Delta_c$ are colored in an opposite color. By looking at the system in a circumcircle, one remarks that the image of Δ in the circle lies on one side of the chord corresponding to the trajectory. In this case, its reflection with respect to at least one of its sides also lies on the same side of this chord. This reflection is an image of one of the triangles $\Delta_a, \Delta_b, \Delta_c$, see Figure 15. This forces one of the vertices A', B', C' to have the same color as the vertices of Δ . We obtain a contradiction. \square

6.3. Symbolic dynamics of the family CET_τ^n . This paragraph deals with the following question. Can one formulate the Tree conjecture in the language of symbolic dynamics? Can this give another proof of the Tree conjecture?

Consider a circle $\mathbb{S}^1 = \mathbf{R}/\mathbf{Z}$ of length 1, and the parameters $\tau \in [0, 1)$ and $l_j \in \mathbf{R}_+, j = 1, \dots, n$ such that $\sum_{j=1}^n l_j = 1$. Define $a_0, \dots, a_n \in \mathbb{S}^1$ as $a_0 = 0, a_1 = l_1, \dots, a_j = l_1 + \dots + l_j, \dots, a_{n-1} = 1 - l_n, a_n = a_0$. Define a family of maps $F_\tau^{l_1, \dots, l_n} : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ in a following way. For any $x \in \mathbb{S}^1$ it belongs to a unique interval $[a_{i-1}, a_i]$ for some $i \in \{1, \dots, n\}$, then define

$$F_\tau^{l_1, \dots, l_n}(x) := -x + l_i + \tau \pmod{1}.$$

We call any map from this family a **fully flipped n -interval exchange transformation on the circle with trivial combinatorics**. We denote the set of all such transformations by CET_τ^n , see Figure 16 for illustration.

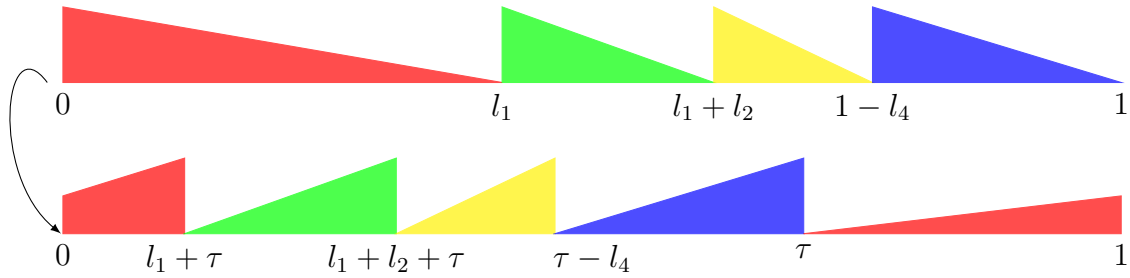


FIGURE 16. This is a pictorial representation of some map $F_\tau^{l_1, \dots, l_4} \in \text{CET}_\tau^4$. The shapes above the intervals are drawn in order to facilitate the understanding behind interval exchange transformations with flips. This representation visualizes the action of the map F on the circle and shows that the beginning of each intervals maps to the end of each interval in the image and vice-versa, see [Figure 9, [7]] for the first use of this representation.

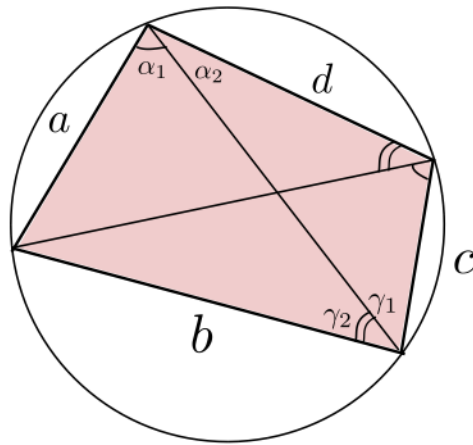


FIGURE 17. Cyclic quadrilateral tile in a tiling and the corresponding angle parameters.

It happens that the behavior of the trajectories of triangle (cyclic quadrilateral) tiling billiards is completely described by the behavior of the maps in the family CET_τ^3 (CET_τ^4) as shown in [7, 16]. The study of symbolic dynamics of a trajectory in a tiling billiard is hence reduced to the study of symbolic dynamics of the map $F \in \text{CET}_\tau^n$ where the symbolic orbit of a point corresponds to the sequence of intervals I_j of continuity of F that this orbit visits. Note that the question of symbolic dynamics in the family CET_τ^n can be studied for any n , independently on the existence of a tiling.

Note. For some cyclic quadrilateral and a trajectory defined by a chord with a parameter τ , a corresponding fully flipped map is a map

$$F_\tau^{\frac{\alpha_1}{\pi}, \frac{\gamma_2}{\pi}, \frac{\gamma_1}{\pi}, \frac{\alpha_2}{\pi}} \in \text{CET}_\tau^4,$$

where $\alpha_1 + \alpha_2 = \alpha$, $\beta = \gamma_1 + \alpha_2$, $\gamma = \gamma_1 + \gamma_2$, $\delta = \alpha_1 + \gamma_2$ and the angles α_1, α_2 (γ_1, γ_2) are the angles in which the diagonal of a tile splits the angle α (and γ , correspondingly), see Figure 17.

We propose here the interpretation of the Tree Conjecture in the terms of the winding number for the family CET_τ^n for any n .

Consider some cyclic polygon \mathcal{P} with n sides. For any n , such a polygon defines a system of **reflections in the circumcircle**, defined in [16]. The dynamics of the system of reflections in the circumcircle depends on an additional parameter $\tau \in \mathbb{S}^1$. Indeed, we consider a circle of length 1 and a chord in it connecting some fixed point (a zero) with a point $\tau \in \mathbb{S}^1$. Denote the sides of \mathcal{P} by

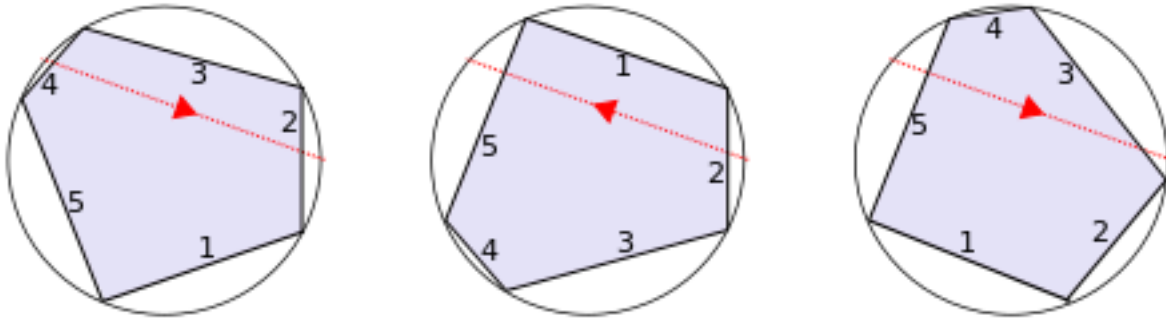


FIGURE 18. Two iterations of the system of reflections in the circumcircle for a pentagon \mathcal{P} . After each iteration, one changes the direction of the chord defined by the parameter τ : hence after two iterations, the chord comes retrieves its initial orientation. The symbolic code of the orbit $(X, F(X), F^2(X))$ is in this case $w = a_4a_2 a_2a_5$. The map $F \in \text{CET}_\tau^5$. We do not mark $F(X)$ on the circle since its placement does not coincide with the vertex A , since the direction of the chord changes for odd iterations of F . For more on this issue, see [7].

reading the boundary of a polygon in a counterclockwise order as a_1, a_2, \dots, a_n . We also consider these sides cyclically, meaning $a_{n+1} := a_1$. Then for any $X \in \mathbb{S}^1$ one can inscribe the polygon \mathcal{P} in the circle in such a way that the vertex $A = a_1 \cap a_n$ is placed exactly in X . The map of reflections in the circumcircle maps a corresponding polygon to a congruent polygon of different orientation sharing one side with \mathcal{P} . A label of the side is defined by looking at the positive intersection with the chord defined by τ , see Figure 18 for an example of the dynamics of system of reflection in the case when \mathcal{P} is a pentagon. The dynamics of this system is equivalent to that of the dynamics of a tiling billiard if a corresponding tiling exists (so only for $n = 3, 4$). For any n , the data (cyclic polygon+chord) defines a map $F \in \text{CET}_\tau^n$. In this paragraph, we discuss the symbolic dynamics of such maps. The following is based on our discussions with Pierre Dehornoy.

Consider an alphabet \mathcal{A} defined as an alphabet of all two-letter words $a_i a_j, i, j = 1, \dots, n$. Then we say that the **winding** is a map $\text{wd} : \mathcal{A} \rightarrow \{0, 1, -1\}$ defined as follows. For any element $w \in \mathcal{A}, w = a_i a_j$, $\text{wd}(w) = 1$ if $j = i + 1$ (cyclically) and $\text{wd}(w) = -1$ if $i = j + 1$. For all other situations, we define $\text{wd}(w) = 0$. Note that winding has already been introduced implicitly when a simplified alphabet \mathcal{S}_Δ was introduced for the proof of the Tree Conjecture.

Any word representing a symbolic code of an orbit of the system of reflections in a circumcircle is a concatenation of symbols in \mathcal{A} . Then one defines its winding as a sum of windings of its subsequent subwords.

Example. If \mathcal{P} is a quadrilateral with sides a_1, a_2, a_3, a_4 , one can consider a (periodic) orbit of a system of reflections in a circumcircle with a symbolic code $w = a_1a_2 a_2a_3 a_3a_4 a_4a_1$ (on the plane it corresponds to a 4-periodic orbit). Then $\text{wd}(w) = 4$.

One defines a **winding of a trajectory** of the system of reflections in the circumcircle as winding of its symbolic code, and a **winding of a periodic trajectory** as a winding of the minimal periodic word.

Let us note that one should redefine what a periodic trajectory is since for the drift-periodic trajectories of tiling billiards in triangles and quadrilaterals correspond to periodic trajectories in reflections in the circumcircle system and we want to exclude this case. For now, we define

periodic trajectories as those trajectories that are stable under the small perturbation of the polygon \mathcal{P} .⁶

Lemma 3. • *A winding of a simple closed curve that doesn't touch the vertices (not necessarily a trajectory) in the triangle (cyclic quadrilateral) tiling is equal to ± 6 (± 4) depending on if it turns clockwise or counter-clockwise.*

- *For any $n \in \mathbf{N}, \geq 3$, the winding of a periodic trajectory in a system of reflections in a circumcircle is well-defined, i.e. doesn't depend on the representation of the period. This winding is equal to $\pm 2n$ if n is odd, and to $\pm n$ if n is even.*

Proof. In order to prove the first statement, consider a vector orthogonal to the curve δ and count the (algebraic) number of turns it makes when it moves along a simple closed curve in the tiling. One can see that this number is exactly $\frac{1}{6}\text{wd}(\delta)$ for the triangle tiling billiard and $\frac{1}{4}\text{wd}(\delta)$ for cyclic quadrilateral tiling billiard.

For the second statement we first note that the winding of a periodic trajectory is well-defined. Second, one can see that the only addition to winding is done by the words that use subsequent letters. For $n > 4$ the corresponding tiling doesn't exist but one still can unfold the trajectory to some broken trajectory in some tiling with self-intersections. When one comes back to the same tile in the system of reflections, one comes back to the same tile on this unfolding as well. \square

Conjecture 4 (Winding Conjecture). *For any map $F \in \text{CET}_\tau^n$, for any periodic points $x, y \in \mathbb{S}^1$ of F , the winding numbers of their symbolic codings are equal. Hence, to every map $F \in \text{CET}_\tau^n$ having a periodic point⁷ one can associate a number $\text{wd}(F)$ corresponding to a winding number of its periodic orbits. This number $\text{wd}(F)$ is equal either to $2n$ or $-2n$ (if n is odd), or n or $-n$ (if n is even).*

In terms of tiling billiards, Winding Conjecture states that periodic orbits with the same τ (or, equivalently, obtained by the same scissor cut) have the same orientation with respect to infinity. From the previous discussion it follows, that the Winding Conjecture is true for $n = 3$. We believe it is true for $n = 4$ and we have no idea for $n > 4$.

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⁶Another way of approaching periodic trajectories is to exclude the cases of polygons with rationally dependent angles.

⁷By the Theorem of Nogueira, almost any map in the family CET_τ^n has a periodic point (and even an interval of periodic points).

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