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# Hausdorff Dimension of the Record Set of a Fractional Brownian Motion

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## Abstract

We prove that the Hausdorff dimension of the record set of a fractional Brownian motion with Hurst parameter  $H$  equals  $H$ .

## 1 Introduction

The statistics of records has been studied in both the physics and mathematics literature, see for example [14, 13, 11, 12, 18, 10, 5]. The record set (denoted  $\text{Rec}$ ) of a random process  $X_t$  is the set of times  $s$  at which  $X_s = \max_{[0,s]} X_t$ . One of the most basic properties of this set is the number of records occurring during a certain time interval. This problem has been well studied for discrete processes such as sequences of i.i.d. random variables [5, 12] or random walks on  $\mathbb{R}$  [4, 11]. However, when considering continuous processes (e.g., the Brownian motion) the question is ill defined. Indeed, an interval will typically contain either zero or infinitely many records. In these cases, a natural way to quantify the size of the record set is to evaluate its Hausdorff dimension. For the Brownian motion, it is shown in [15] that this dimension is  $\frac{1}{2}$ .

The fractional Brownian motion (fBm) is a continuous Gaussian process  $X_t$ , depending on a parameter  $H \in (0, 1)$  called the Hurst index. It has expected value 0 and covariances given by

$$\langle X_t X_s \rangle = \frac{1}{2} \left( |t|^{2H} + |s|^{2H} - |t - s|^{2H} \right).$$

The fBm is scale-invariant, namely  $(a^{-H} X_{at})_{t \geq 0}$  has the same law as  $(X_t)_{t \geq 0}$  for all  $a > 0$ . We emphasize that, even though in general this process could be defined also for  $H = 1$ , we will only consider here  $H$  strictly smaller than 1.

The fractal properties of fBm have been studied extensively (see [1, 19]). In this paper, we show that the Hausdorff dimension of its record set is  $H$ .

## 2 Heuristics

To find the dimension of the record set, first fix a small  $\varepsilon > 0$  and divide the time interval  $[0, 1]$  in  $N_\varepsilon = \frac{1}{\varepsilon}$  small boxes, each of diameter  $\varepsilon$ . We will be interested in finding the number  $M_\varepsilon$  of boxes in which a record has

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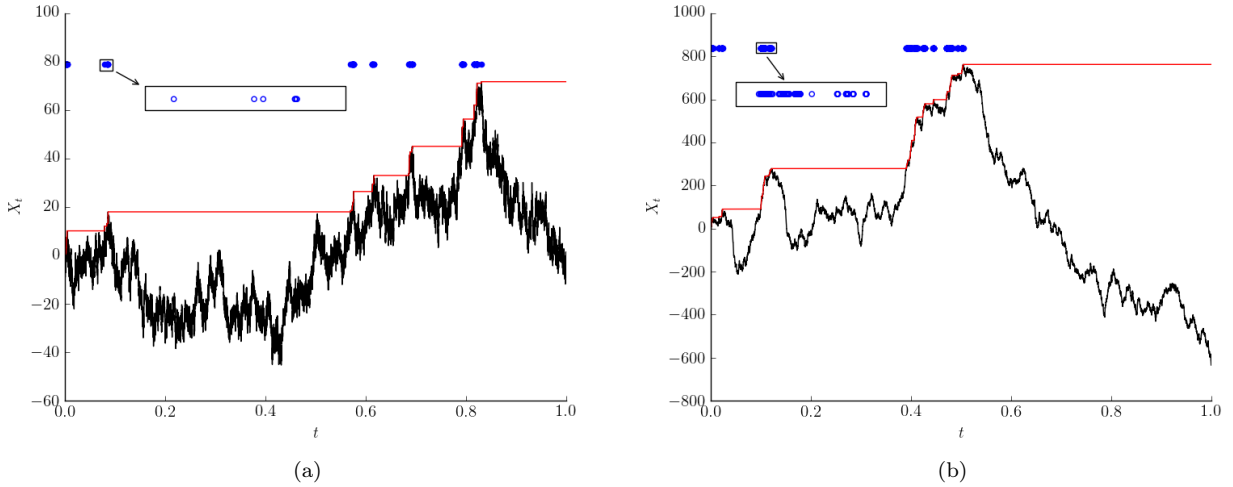


Figure 1.1: Two simulations of fBm, (a) with Hurst index  $\frac{1}{3}$ , and (b) with index  $\frac{2}{3}$ . The fBm path is in black. In red, for every time  $t$ , is the maximum up to time  $t$ . The blue points represent the record set. Zooming in one of the blue clusters, one sees the fractal nature of this set. For generating the fBm we used the algorithm of [9], for a system of size  $2^{25}$ .

occurred. To do so, we first compute the probability to find a record during the time interval  $[(n-1)\varepsilon, n\varepsilon]$ . Stated differently, this is the probability that the maximum of  $X_t$  in  $[0, n\varepsilon]$  is attained in the time interval  $[(n-1)\varepsilon, n\varepsilon]$ . By time reversal symmetry this is the same as the probability to attain the maximum of  $X_t$  during  $[0, n\varepsilon]$  in  $[0, \varepsilon]$ . Since the maximum during the time interval  $[0, \varepsilon]$  scales like  $\varepsilon^H$ , following [8], we claim that this probability is controlled by the probability that  $\max_{[0, n\varepsilon]} X_t$  is of order  $\varepsilon^H$ . That probability, as shown in [6], scales like  $(\varepsilon^H)^{\frac{1-H}{H}} = \varepsilon^{1-H}$ . Summing up the argument so far, we get:

$$\begin{aligned}
 \mathbb{P}[\text{Rec} \cap [(n-1)\varepsilon, n\varepsilon] \neq \emptyset] &= \mathbb{P}\left[\max_{[0, n\varepsilon]} X_t \text{ is attained during } [(n-1)\varepsilon, n\varepsilon]\right] \\
 &= \mathbb{P}\left[\max_{[0, n\varepsilon]} X_t \text{ is attained during } [0, \varepsilon]\right] \\
 &\approx \mathbb{P}\left[\max_{[0, n\varepsilon]} X_t \text{ is of order } \varepsilon^H\right] \\
 &\approx \varepsilon^{1-H}.
 \end{aligned}$$

Thus, the expected number of boxes containing a record scales as  $M_\varepsilon \approx N_\varepsilon \varepsilon^{1-H} = \varepsilon^{-H}$ , suggesting a fractal dimension  $H$  of the record set. This scaling is verified numerically in figure 2.1, as well as in [3].

### 3 Notation and Presentation of the Result

We start by presenting some notations and definitions that will be used throughout the proof. In order to give the definition of the Hausdorff dimension we first define the  $\alpha$ -value of a covering:

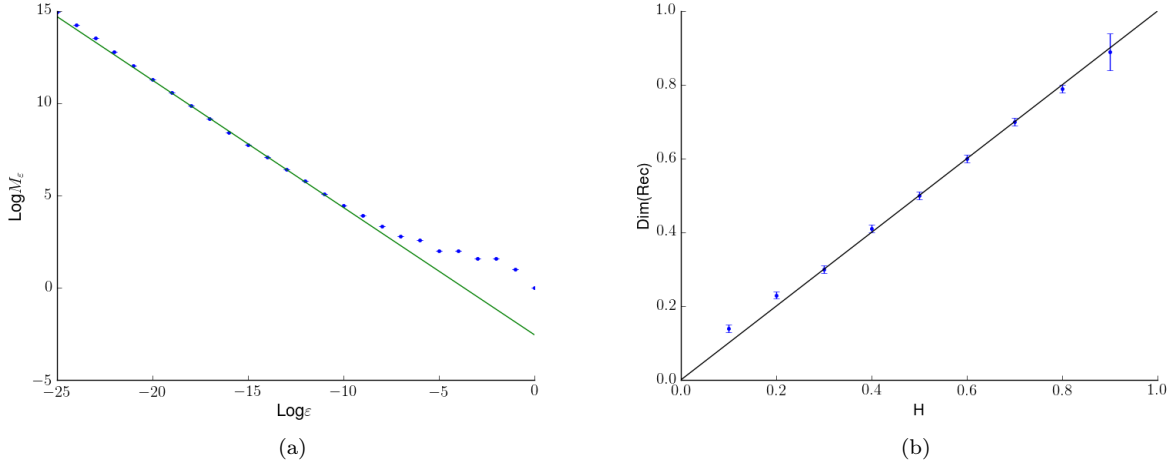


Figure 2.1: In (a) we see a graph showing the relation between  $\log_2 M_\varepsilon$  and  $\log_2 \varepsilon$  for a fBm sample with  $H = \frac{3}{4}$ . The relation is linear with slope  $-0.690$  given by linear regression. In (b), we have extracted this slope for different values of  $H$  (shown in blue), averaging over 100 samples. The black line is  $\text{Dim}(\text{Rec}) = H$ . To generate an fBm we used the algorithm of [9], for a system of size  $2^{25}$ .

**Definition 1.** Let  $E$  a metric space,  $\mathfrak{E} = \{E_1, E_2, \dots\}$  a covering of  $E$ , and  $\alpha \geq 0$ . Then the  $\alpha$ -value of  $\mathfrak{E}$  is

$$S_\alpha(\mathfrak{E}) := \sum_{i=1}^{\infty} \text{Diam}(E_i)^\alpha.$$

We can now define the Hausdorff dimension of a set.

**Definition 2.** Let  $E$  a metric space, and for every  $\alpha \geq 0$  consider the  $\alpha$ -Hausdorff measure of  $E$ :

$$\mathcal{H}_\alpha(E) := \liminf_{\delta \downarrow 0} \{S_\alpha(\mathfrak{E}), \mathfrak{E} \text{ a covering of } E \text{ and } \text{Diam}(E_i) \leq \delta\}.$$

Then the Hausdorff dimension of  $E$  is

$$\text{Dim}(E) = \inf \{\alpha \geq 0, \mathcal{H}_\alpha(E) < \infty\} = \sup \{\alpha \geq 0, \mathcal{H}_\alpha(E) = \infty\}.$$

Recall the definition of the record set

$$\text{Rec} = \left\{ t \geq 0, X_t = \max_{s \in [0, t]} X_s \right\}.$$

The main result we present here is:

**Theorem 1.**

$$\text{Dim}(\text{Rec}) = H \text{ a.s.} \quad (3.1)$$

Finally, we will prove the following corollary, describing the scaling for the fBm equivalent of the third arcsine law (for results in the physics literature beyond the asymptotics see [17]):

**Corollary 1.** For all  $\delta > 0$  there exists  $\varepsilon_0 > 0$ , such that for all positive  $\varepsilon < \varepsilon_0$ ,

$$\varepsilon^{1-H+\delta} \leq \mathbb{P} \left( \text{Argmax}_{[0,1]} X_t \in [0, \varepsilon] \right) \leq \varepsilon^{1-H-\delta}. \quad (3.2)$$

## 4 Proof of the Result

We will follow the proof from [15], in which the Hausdorff dimension of the record set is found for the (non-fractional) Brownian motion. The main difference comes from the non-Markovian behavior of the fBm, a difficulty that we will control with Lemma 1 below.

First, we will get a lower bound on the record set dimension using Lemma 4.21 of [15]:

**Proposition 1.** *Let  $f : [0, 1] \rightarrow \mathbb{R}$ ,  $\alpha$ -Hölder continuous, whose maximum is not attained at 0. Then the Hausdorff dimension of its record set is greater or equal to  $\alpha$ .*

For the other bound, we will use the following result, essentially proven in [15]:

**Proposition 2.** *Let  $A \subset [0, 1]$  be a random set and  $\vartheta > 0$  such that for all  $b > 0$ , there exists  $C_b > 0$  and a sequence of positive numbers  $\varepsilon_k$ , converging to zero and satisfying*

$$\forall a \geq b, \mathbb{P}[A \cap [a, a + \varepsilon_k] \neq \emptyset] \leq C_b \varepsilon_k^{1-\vartheta}. \quad (4.1)$$

Then, almost surely,

$$\text{Dim}(A) \leq \vartheta. \quad (4.2)$$

For completeness, we present here the proof:

*Proof.* Let  $b > 0$ , we will show that  $\text{Dim}(A \cap [b, 1]) \leq \vartheta$  using assumption 4.1. The result will then follow by the countable stability of the Hausdorff dimension.

In order to get an upper bound on the dimension, it is enough to find a family of coverings of  $[b, 1] \cap A$  with diameter going to zero such that the  $\vartheta$ -value of each covering is finite. To construct such a covering, consider, for  $k \in \mathbb{N}$ ,

$$N_k = \sup_{j \in \mathbb{N}} \{b + j\varepsilon_k \leq 1\}.$$

Denote, for  $j = 0 \dots N_k - 1$ ,  $I_j := [j\varepsilon_k, (j+1)\varepsilon_k]$  and consider the collection of intervals:

$$\mathcal{J}_k = \{I_j, I_j \cap A \neq \emptyset\} \cup \{[b + N_k\varepsilon_k, 1]\}.$$

Let  $j \in \{0, \dots, N_k - 1\}$ . Taking  $a = b + j\varepsilon_k$  in the assumption 4.1, we have

$$\mathbb{P}[A \cap I_j \neq \emptyset] \leq C_b \varepsilon_k^{1-\vartheta}. \quad (4.3)$$

Therefore the covering  $\mathcal{J}_k$  of  $A \cap [b, 1]$  has an expected  $\vartheta$ -value of

$$\mathbb{E}[\mathcal{S}_\vartheta(\mathcal{J}_k)] = \sum_{j=0}^{N_k-1} \mathbb{P}[A \cap I_j \neq \emptyset] \varepsilon_k^\vartheta + (1 - b - N_k\varepsilon_k)^\vartheta \leq 2C_b.$$

Using Fatou's lemma, we have

$$\mathbb{E} \left[ \liminf_{k \rightarrow \infty} \mathcal{S}_\vartheta(\mathcal{J}_k) \right] \leq \liminf_{k \rightarrow \infty} \mathbb{E}[\mathcal{S}_\vartheta(\mathcal{J}_k)] \leq 2C_b. \quad (4.4)$$

Hence, the liminf is almost surely finite. In particular, there exists a family of coverings whose diameter is going to zero with bounded  $\vartheta$ -value, and we can conclude that almost surely

$$\text{Dim}(A \cap [b, 1]) \leq \vartheta. \quad (4.5)$$

□

To use this proposition, we will need the following lemma:

**Lemma 1.** *For all  $\delta > 0$  and  $b > 0$ , there exists a constant  $C(b, \delta, H) > 0$ , such that, for small enough  $\varepsilon > 0$ ,*

$$\forall a \geq b, \quad \mathbb{P}[\text{Rec} \cap [a, a + \varepsilon] \neq \emptyset] \leq C \varepsilon^{1-H-\delta}. \quad (4.6)$$

*Proof.* We begin by introducing two inequalities concerning the supremum of  $X$ :

(i) For all  $\delta' > 0$ , there exists a constant  $M = M(\delta, H) > 0$ , such that, for small enough  $u > 0$ :

$$\mathbb{P}\left[\sup_{0 \leq t \leq 1} X_t \leq u\right] \leq M u^{\frac{1-H-\delta}{H}}. \quad (4.7)$$

(ii) There exists a constant  $M' = M'(H) > 0$ , such that, for large enough  $v$ , we have

$$\mathbb{P}\left[\sup_{0 \leq t \leq 1} X_t > v\right] \leq M' v^{1/H} \Psi(v), \quad (4.8)$$

where  $\Psi(v) = \mathbb{P}(N > v)$  for  $N$  a standard normal random variable.

The first inequality is a weak consequence of corollary 2 in [6]. The statement of the second inequality can be found in Theorem D.4. of [16] (see also [2]).

Let  $\delta, \varepsilon$  and  $\theta$  be three positive real numbers, with  $\theta$  satisfying  $\theta < H$ . By time reversal symmetry, the process  $t \mapsto \tilde{X}_t = X_{a+\varepsilon-t} - X_{a+\varepsilon}$  is again a fractional Brownian motion starting at 0 with Hurst index  $H$ . Hence,

$$\mathbb{P}[\text{Rec} \cap [a, a + \varepsilon] \neq \emptyset] = \mathbb{P}\left[\sup_{[0, a+\varepsilon]} \tilde{X}_t = \sup_{[a, a+\varepsilon]} \tilde{X}_t\right] = \mathbb{P}\left[\sup_{[0, a+\varepsilon]} X_t = \sup_{[0, \varepsilon]} X_t\right].$$

Decomposing this last term into the two terms

$$A_\varepsilon = \mathbb{P}\left[\sup_{[0, a+\varepsilon]} X_t \leq \varepsilon^{H-\theta}, \sup_{[0, a+\varepsilon]} X_t = \sup_{[0, \varepsilon]} X_t\right],$$

$$B_\varepsilon = \mathbb{P}\left[\sup_{[0, a+\varepsilon]} X_t > \varepsilon^{H-\theta}, \sup_{[0, a+\varepsilon]} X_t = \sup_{[0, \varepsilon]} X_t\right],$$

and using the scaling invariance of  $X$ , we get that

$$A_\varepsilon \leq \mathbb{P}\left[\sup_{[0, a+\varepsilon]} \frac{X_t}{(a+\varepsilon)^H} \leq \frac{\varepsilon^{H-\theta}}{(a+\varepsilon)^H}\right] = \mathbb{P}\left[\sup_{[0, 1]} X_t \leq \frac{\varepsilon^{H-\theta}}{(a+\varepsilon)^H}\right]. \quad (4.9)$$

Therefore, for small enough  $\varepsilon$ , we can apply inequality 4.7 with a positive parameter  $\delta' < 1 - H$ :

$$A_\varepsilon \leq M \frac{\varepsilon^{1-H-\delta'}}{(a+\varepsilon)^{1-H-\delta'}} \varepsilon^{-\theta(1-H-\delta')/H} \leq C_1 \varepsilon^{1-H-\delta},$$

where we now fix  $\theta$  and  $\delta'$  sufficiently small, chosen to verify

$$\delta = -\delta' - \theta \frac{1-H-\delta'}{H},$$

and where, recalling that  $b \leq a$ ,  $C_1$  is defined as:

$$C_1(b, \delta, H) = M(\delta, H) b^{-1+H+\delta'}.$$

We then bound  $B_\varepsilon$ , using again the scaling invariance and applying 4.8 for  $\varepsilon$  small enough:

$$\begin{aligned} B_\varepsilon &= \mathbb{P} \left[ \sup_{[0,\varepsilon]} X_t > \varepsilon^{H-\theta}, \sup_{[0,a+\varepsilon]} X_t = \sup_{[0,\varepsilon]} X_t \right] \leq \mathbb{P} \left[ \sup_{[0,1]} X_t > \varepsilon^{-\theta} \right] \\ &\leq M'(H) \varepsilon^{-\theta/H} \Psi(\varepsilon^{-\theta}) \\ &\leq C_2 \varepsilon^{1-H-\delta}, \end{aligned}$$

with  $C_2 = C_2(\delta, H)$  and where the last inequality is a consequence of the rapid decay of  $\Psi$  as  $\varepsilon$  tends to 0. Summing the bounds over  $A_\varepsilon$  and  $B_\varepsilon$  concludes the proof of the lemma.  $\square$

Putting everything together, we are ready to prove Theorem 1.

*Proof of Theorem 1.* We first prove the lower bound using Proposition 1. Indeed, the sample-paths of the fractional Brownian motion are almost surely  $\alpha$ -Hölder continuous for any  $\alpha < H$  (see Theorem 3.1 of [7]). Hence, we get that

$$\text{Dim}(\text{Rec}) \geq \alpha$$

for any  $\alpha < H$ . The lower bound follows letting  $\alpha$  go to  $H$ .

In order to get the upper bound, we use Proposition 2 combined with Lemma 1 to find:

$$\text{Dim}(\text{Rec}) \leq H + \delta$$

for all  $\delta > 0$ . The upper bound follows letting  $\delta$  go to zero.  $\square$

We can now give a proof of Corollary 1:

*Proof.* By the time reversibility property of the fractal Brownian motion, we can see that (cf. proof of Lemma 1):

$$\mathbb{P} \left[ \text{Argmax}_{[0,1]} X_t \in [0, \varepsilon] \right] = \mathbb{P} [\text{Rec} \cap [1 - \varepsilon, 1] \neq \emptyset].$$

Therefore, the upper bound in the inequality 3.2 is a direct consequence of Lemma 1 (taking  $\frac{\delta}{2}$  in order to absorb the constant  $C$ ).

For the lower bound, we need to show that for all  $\delta > 0$ , there exists  $\varepsilon_0 > 0$  such that

$$\forall \varepsilon < \varepsilon_0, \quad \varepsilon^{1-H+\delta} \leq \mathbb{P} [\text{Rec} \cap [1 - \varepsilon, 1] \neq \emptyset]. \quad (4.10)$$

Reasoning by contradiction, let  $\delta > 0$  and  $(\varepsilon_k)_{k \geq 0}$  such that  $\varepsilon_k \rightarrow 0$  and

$$\mathbb{P} [\text{Rec} \cap [1 - \varepsilon_k, 1] \neq \emptyset] < \varepsilon_k^{1-H+\delta}. \quad (4.11)$$

Let  $b > 0$ ,  $a \geq b$ , and  $\varepsilon'_k$  to be chosen later on. Consider the rescaled process  $t \rightarrow Y_t = (a + \varepsilon'_k)^H X_{(a+\varepsilon'_k)t}$ . By scaling invariance,  $Y$  is a fractional Brownian motion of Hurst index  $H$  whose record set  $\text{Rec}(Y)$  on  $[0, 1]$  is the rescaled record set of  $X$ . Hence,

$$\begin{aligned} \mathbb{P} [\text{Rec}(X) \cap [a, a + \varepsilon'_k] \neq \emptyset] &= \mathbb{P} \left[ \text{Rec}(Y) \cap \left[ 1 - \frac{\varepsilon'_k}{a + \varepsilon'_k}, 1 \right] \neq \emptyset \right] \\ &\leq \mathbb{P} \left[ \text{Rec} \cap \left[ 1 - \frac{\varepsilon'_k}{b + \varepsilon'_k}, 1 \right] \neq \emptyset \right]. \end{aligned}$$

Choosing  $\varepsilon'_k = \frac{b \varepsilon_k}{1 - \varepsilon_k}$ , so that  $\frac{\varepsilon'_k}{b + \varepsilon'_k} = \varepsilon_k$ , 4.11 yields:

$$\mathbb{P} [\text{Rec} \cap [a, a + \varepsilon'_k] \neq \emptyset] \leq b^{-1+\delta+H} \varepsilon_k^{1-(H-\delta)}.$$

This is exactly the assumption of Proposition 2, thus

$$\text{Dim}(\text{Rec}) \leq H - \delta,$$

in contradiction with Theorem 1.  $\square$

## 5 Further Questions

There are various topics for further research concerning the record statistics of continuous processes. For example, one may study the duration of the longest record or the waiting time for a first record to occur after some fixed positive time. It could also be interesting to study non-Gaussian or non-stationary processes. Another question would be to extend the study of records to fields of higher dimensions (both in space and in time), given an appropriate order on these spaces.

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## References

- [1] RJ Adler. *The geometry of random fields*. Wiley, London, 1981.
- [2] Robert J Adler and Jonathan E Taylor. *Random fields and geometry*. Springer Science & Business Media, 2009.
- [3] A Aliakbari, P Manshour, and MJ Salehi. Records in fractal stochastic processes. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 27(3):033116, 2017.
- [4] Erik Sparre Andersen. On the fluctuations of sums of random variables. *Mathematica Scandinavica*, pages 263–285, 1954.
- [5] BC Arnold, N Balakrishnan, and HN Nagaraja. Records john wiley and sons. *New York*, 1998.
- [6] Frank Aurzada et al. On the one-sided exit problem for fractional Brownian motion. *Electron. Commun. Probab*, 16:392–404, 2011.
- [7] Laurent Decreusefond et al. Stochastic analysis of the fractional Brownian motion. *Potential analysis*, 10(2):177–214, 1999.
- [8] Mathieu Delorme and Kay Jörg Wiese. Perturbative expansion for the maximum of fractional Brownian motion. *Physical Review E*, 94(1):012134, 2016.
- [9] A. B. Dieker. *Simulation of fractional Brownian motion*. PhD thesis, University of Twente, 2004.
- [10] P. Le Doussal and K.J. Wiese. Driven particle in a random landscape: disorder correlator, avalanche distribution and extreme value statistics of records. *Phys. Rev. E*, 79:051105, 2009.
- [11] William Feller. *An introduction to probability theory and its applications*, volume 2. John Wiley & Sons New York, 1966.
- [12] Ned Glick. Breaking records and breaking boards. *American Mathematical Monthly*, pages 2–26, 1978.
- [13] Claude Godreche, Satya N. Majumdar, and Gregory Schehr. Record statistics of a strongly correlated time series: random walks and lévy flights. *To appear in J. Phys. A. arXiv:1702.00586*, 2017.
- [14] Satya N Majumdar and Robert M Ziff. Universal record statistics of random walks and lévy flights. *Physical review letters*, 101(5):050601, 2008.



- [15] P. Mörters and Y. Peres. *Brownian Motion*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, 2010.
- [16] Vladimir I Piterbarg. *Asymptotic methods in the theory of Gaussian processes and fields*, volume 148. American Mathematical Soc., 2012.
- [17] T. Sadhu, M. Delorme, and K.J. Wiese. Generalized arcsine laws for fractional Brownian motion. *arXiv*, 1706.01675, 2017.
- [18] Gregor Wergen, Miro Bogner, and Joachim Krug. Record statistics for biased random walks, with an application to financial data. *Physical Review E*, 83(5):051109, 2011.
- [19] Yimin Xiao. Recent developments on fractal properties of gaussian random fields. In *Further Developments in Fractals and Related Fields*, pages 255–288. Springer, 2013.

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