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# FROM THE BACKWARD KOLMOGOROV PDE ON THE WASSERSTEIN SPACE TO PROPAGATION OF CHAOS FOR MCKEAN-VLASOV SDES

PAUL-ERIC CHAUDRU DE RAYNAL AND NOUFEL FRIKHA

ABSTRACT. In this article, we provide some new quantitative estimates for propagation of chaos of non-linear stochastic differential equations (SDEs) in the sense of McKean-Vlasov. We obtain explicit error estimates, at the level of the trajectories, at the level of the semi-group and at the level of the densities, for the mean-field approximation by systems of interacting particles under mild regularity assumptions on the coefficients. A first order expansion for the difference between the densities of one particle and its mean-field limit is also established. Our analysis relies on the well-posedness of classical solutions to the backward Kolmogorov partial differential equations (PDEs) defined on the strip  $[0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$ ,  $\mathcal{P}_2(\mathbb{R}^d)$  being the Wasserstein space, that is, the space of probability measures on  $\mathbb{R}^d$  with a finite second-order moment and also on the existence and uniqueness of a fundamental solution for the related parabolic linear operator here stated on  $[0, T] \times \mathcal{P}_2(\mathbb{R}^d)$ .

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## 1. INTRODUCTION

In this work, we are interested in some non-linear stochastic differential equations (SDEs for short) in the sense of McKean-Vlasov with dynamics:

$$(1.1) \quad X_t^\xi = \xi + \int_0^t b(s, X_s^\xi, [X_s^\xi]) ds + \int_0^t \sigma(s, X_s^\xi, [X_s^\xi]) dW_s, \quad [\xi] \in \mathcal{P}_2(\mathbb{R}^d),$$

where  $\xi$  is an  $\mathbb{R}^d$ -valued random variable which is independent of the  $q$ -dimensional Brownian motion  $W = (W^1, \dots, W^q)$  and with coefficients  $b : \mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}^d$  and  $\sigma : \mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}^{d \times q}$ ,  $[\theta]$  denoting the law of the random variable  $\theta$  and its approximation by the associated system of  $N$  particles  $\{(X_t^i)_{t \in [0, T]}, 1 \leq i \leq N\}$  interacting through its empirical measure

$$(1.2) \quad X_t^i = \xi^i + \int_0^t b(s, X_s^i, \mu_s^N) ds + \int_0^t \sigma(s, X_s^i, \mu_s^N) dW_s^i, \quad \mu_t^N := \frac{1}{N} \sum_{i=1}^N \delta_{X_t^i}, \quad i = 1, \dots, N,$$

where  $\{(\xi^i, (W_t^i)_{t \in [0, T]}), 1 \leq i \leq N\}$  are  $N$  i.i.d. copies of  $(\xi, W)$ . The connection between the two above systems of SDEs comes from fact that the dynamics (1.1) describes the limiting behaviour of an individual particle in (1.2) when the size of the population  $N$  grows to infinity as stated by the so-called propagation of chaos phenomenon, originally studied by McKean [McK67] and then investigated by Sznitman [Szn91]. Roughly speaking, it is expected that the dynamics of  $k$  particles among  $N$ , say  $(X^1, \dots, X^k)$ ,  $k$  being a fixed integer, consists of  $k$  independent copies  $(\bar{X}^1, \dots, \bar{X}^k)$  of a process following the law of the unique solution to the limiting equation (1.1) as  $N$  goes to infinity. Since the original works of Kac [Kac56] in kinetic theory and of McKean [McK66] in non-linear parabolic partial differential

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equations (PDEs for short), theoretical and numerical aspects of McKean-Vlasov SDEs have been an active research area in several directions during the last decades such as the well-posedness of the related martingale problem, the propagation of chaos and other limit theorems, probabilistic representations to non-linear parabolic PDEs and their numerical approximation schemes. We refer to Tanaka [Tan78], Funaki [Fun84], Oelschläger [Oel84], Gärtner [Gär88], [Szn91], Mishura and Veretenikov [MV18], Chaudru de Raynal [CdR19], Lacker [Lac18] for a small sample among others.

Here, our main objective consists in revisiting and rigorously justify the mean-field approximation of (1.1) by its system of particles (1.2) under mild assumptions on the coefficients. Our analysis strongly relies on the smoothing properties of the mean-field SDE under the assumption that  $a = \sigma\sigma^*$  is uniformly elliptic. We achieve this goal by bringing to light some new quantitative estimates of propagation of chaos for the system of particles (1.2) at three different levels. Namely, we prove the  $L^2(\mathbb{P})$ -convergence of the trajectories of  $(X_t^i)_{t \in [0, T]}$  to its McKean-Vlasov limit dynamics. We also establish an explicit error estimate and a first order expansion for the difference between the transition densities of one particle and its limit. Eventually, we provide some convergence rate for the difference between the flow of empirical measures  $(\mu_t^N)_{t \in [0, T]}$  of the system of particles and its limit given by the flow of probability measures  $(\mu_t)_{t \in [0, T]}$  associated to the dynamics (1.1) when they both act on some irregular map of  $\mathcal{P}_2(\mathbb{R}^d)$ .

A natural question to be addressed before investigating the convergence problem for the system of particles (1.2) is the well-posedness in the weak or strong sense of its mean-field limit (1.1). This problem has been intensively investigated under various settings by many authors. We refer e.g. to [Gär88], [Szn91], Jourdain [Jou97], and more recently, Li and Min [LM16], [MV18] and Hammersley et al. [HvS18] for a short sample.

In our recent contribution [CdRF18], we revisited the problem of the unique solvability by tackling the corresponding formulation of the martingale problem under mild regularity assumptions on the coefficients  $b$  and  $a$ , namely:  $a = \sigma\sigma^*$  is uniformly elliptic,  $b$  is bounded, measurable and Lipschitz in  $\mu$  with respect to the total variation metric,  $a$  is bounded,  $\eta$ -Hölder continuous in space and admits a bounded and  $\eta$ -Hölder continuous linear functional derivative. Under an additional regularity assumption, namely if  $(x, \mu) \mapsto b(t, x, \mu)$ ,  $a(t, x, \mu)$  are uniformly Hölder continuous with respect to the space variable  $x$  and admit two bounded and uniformly Hölder continuous linear functional derivatives on  $\mathcal{P}_2(\mathbb{R}^d)$ , it then turns out that the transition density  $p(\mu, s, t, z)$  of the SDE (1.1) at time  $t$  starting from the initial distribution  $\mu$  at time  $s$  exists and is smooth with respect to the variables  $s$  and  $\mu$ , the derivatives in the measure argument being understood for a stronger notion of differentiation, namely in the sense of Lions. More precisely, the map  $(s, \mu) \mapsto p(\mu, s, t, z) \in \mathcal{C}^{1,2}([0, t] \times \mathcal{P}_2(\mathbb{R}^d))$  (see Section 2.1 for a definition of this space). The previous regularity properties of the density finally allows to establish the existence and uniqueness of classical solutions for a class of linear parabolic PDEs on the Wasserstein space, namely

$$(1.3) \quad \begin{cases} (\partial_t + \mathcal{L}_t)U(t, x, \mu) = f(t, x, \mu) & \text{for } (t, x, \mu) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d), \\ U(T, x, \mu) = h(x, \mu) & \text{for } (x, \mu) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d), \end{cases}$$

where the source term  $f : \mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$  and the terminal condition  $h : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$  are some given functions and the operator  $\mathcal{L}_t$  is defined by

$$(1.4) \quad \begin{aligned} \mathcal{L}_t g(x, \mu) &= \sum_{i=1}^d b_i(t, x, \mu) \partial_{x_i} g(x, \mu) + \frac{1}{2} \sum_{i,j=1}^d a_{i,j}(t, x, \mu) \partial_{x_i x_j}^2 g(x, \mu) \\ &+ \int \left\{ \sum_{i=1}^d b_i(t, z, \mu) [\partial_{\mu} g(x, \mu)(z)]_i + \frac{1}{2} \sum_{i,j=1}^d a_{i,j}(t, z, \mu) \partial_{z_i} [\partial_{\mu} g(x, \mu)(z)]_j \right\} \mu(dz) \end{aligned}$$

and acts on sufficiently smooth test functions  $g : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$ . The aforementioned well-posedness and smoothing property results for the dynamics (1.1) and the PDE (1.3) allow us to investigate in turn the convergence problem of the particle system (1.2) at the three levels mentioned above within the same framework.

The former convergence problem of the trajectories has been thoroughly investigated under the standard framework of Lipschitz continuous coefficients  $b$  and  $\sigma$  over  $\mathbb{R}^d \times \mathcal{P}_p(\mathbb{R}^d)$ ,  $\mathcal{P}_p(\mathbb{R}^d)$  being the space of probability measures with finite moment of order  $p$  equipped with the Wasserstein distance  $W_p$ , by using the very effective and now well-known coupling argument between the solution of the system of particles (1.2) and  $N$  independent copies of the unique strong solution of the nonlinear SDE (1.1) taken with

the same input  $(\xi^i, W^i)_{1 \leq i \leq N}$ . We refer to [Szn91], Léonard [L86], Méléard [M96] for a presentation of this argument and also to Jourdain and Méléard [JM98] and Malrieu [Mal03] for some extensions to non-linear SDEs with coefficients depending locally on its density and to granular media equations respectively.

It actually turns out to be a challenging question to go beyond the aforementioned framework by weakening the Lipschitz regularity assumption on the coefficients. Let us however mention the recent work of Holding [Hol16] in which some quantitative propagation of chaos estimates for systems of interacting particles with a constant diffusion coefficient and a drift coefficient with an Hölder continuous interacting kernel of first order type are established. Therein, an error bound for the Wasserstein distance of order 1 between the empirical measure  $(\mu_t^N)_{t \in [0, T]}$  of the system of particles and its mean-field limit is obtained with a convergence rate depending on the Hölder exponent of the interacting kernel.

Our first contribution is a general rate of convergence for the  $L^2(\mathbb{P})$ -error on the trajectories of the solution of the system of particles (1.2) and  $N$  independent copies of its mean-field dynamics (1.1) as well as for the Wasserstein distance of order 2 between  $\mu_t^N$  and its corresponding limit. The main novelty here compared to the aforementioned references is that we make the approach as systematic as possible by connecting the above convergence problem to the well-posedness and the regularity properties of the solution  $U$  of the backward Kolmogorov PDE (1.3) with source term  $f(t, x, \mu) = b(t, x, \mu)$  and terminal condition  $U(T, x, \mu) \equiv h \equiv 0$ . This strategy is reminiscent of Zvonkin's method for solving SDEs driven by a bounded and measurable drift [Zvo74]. Indeed, testing the solution  $U$  on the dynamics of the system of particles notably allows to remove the drift from the convergence analysis and to achieve the expected convergence rate of the framework of Lipschitz coefficients but with weaker conditions on the drift coefficient, namely the drift is assumed to be bounded, Hölder continuous in space and with two bounded and Hölder continuous linear functional derivatives in the measure argument.

Our second contribution is an error bound together with a first order expansion for the difference between the densities of the one-dimensional marginal of the system of particles and its corresponding limit. Here again, the technique of proof is based on the well-posedness of the backward Kolmogorov PDE here stated on the strip  $[0, T] \times \mathcal{P}_2(\mathbb{R}^d)$  for which we introduce and study a notion of fundamental solution. The natural candidate for being the unique classical solution is the transition density of the McKean-Vlasov SDE (1.1) with initial distribution  $\mu$  at time  $s$ , namely the map  $[0, t] \times \mathcal{P}_2(\mathbb{R}^d) \ni (s, \mu) \mapsto p(\mu, s, t, z)$ ,  $(t, z) \in (0, T] \times \mathbb{R}^d$  being fixed. By taking advantage of its regularity properties, the key idea then consists in testing the fundamental solution along the empirical measure  $(\mu_s^N)_{s \in [0, t]}$  of the system of particles. On the one hand, the proxies  $\{[0, t] \ni s \mapsto p(\mu_s^N, s, t, z), N \geq 1\}$  should get closer and closer in average to the (constant) map  $s \mapsto p(\mu_s, s, t, z) = p(\mu, 0, t, z)$  up to a remainder term that vanishes as  $N$  goes to infinity. On the other hand, as  $(s, \mu) \mapsto p(\mu, s, t, z)$  is the fundamental solution of the backward Kolmogorov PDE and by symmetry of the dynamics (1.2),  $s \mapsto p(\mu_s^N, s, t, z)$  converges weakly to the one-dimensional marginal density function of the system of particles as  $s$  goes to  $t$ . Combining these two facts yields our results.

Our third contribution consists in an analysis of a weak form of propagation of chaos. Inspired by Remark 5.110 in [CD18], we quantify the distance between the empirical measure of the particle system and the law of the solution of the equation both acting on a large class of functions of  $\mathcal{P}_2(\mathbb{R}^d)$ . We provide an explicit error estimate for the difference between the semigroup generated by the mean field system and its approximation by the system of particles. The key tool to prove such result is very closed to the one developed to handle the previous estimates on the densities. Namely, it first consists in investigating the regularity properties of the solution to the Cauchy problem related to the backward Kolmogorov PDE stated on the strip  $[0, T] \times \mathcal{P}_2(\mathbb{R}^d)$ , without source term and with a terminal condition  $h : \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$  admitting two bounded and Hölder continuous linear functional derivatives and then to test such a solution along the empirical measure and the limiting law. Although we refrain to go further in that direction, we are convinced that repeating the previous strategy in order to obtain a first order expansion for the difference between the densities would lead to a first order expansion for the semigroups.

Taking benefit of the well-posedness of classical solutions to the backward Kolmogorov PDE on the Wasserstein space to prove the aforementioned quantitative estimates of propagation of chaos for the system of particles thus plays a central role in our analysis. Let us however mention that the strategy developed here is clearly reminiscent of the point of view taken by Cardaliaguet & al. [CDLL15], Mischler and Mouhot [MM13] and by Mischler, Mouhot and Wennberg in their subsequent work [MMW15]. In [CDLL15], the authors study the convergence problem, as  $N \uparrow \infty$ , of the  $N$ -Nash system consisting of a system of  $N$  coupled Hamilton-Jacobi equations. The limit equation is no longer a linear backward

Kolmogorov equation but a non-linear PDE of second order type also stated on the space of probability measures, the so-called master equation of mean-field games. The strategy developed by the authors to establish their estimates of the rate of convergence consists exactly in testing the solution of the master equation as an approximate solution to the  $N$ -Nash system. Obviously, the very nature of our approach is the same, except that, in our case we work with a linear PDE and its fundamental solution under mild regularity conditions while in [CDLL15], the PDE is non-linear but has smooth coefficients. The point of view expressed to establish propagation of chaos estimates for systems of particles undergoing collisions in [MM13] and for mean-field systems undergoing jumps and/or diffusions in the subsequent work [MMW15] is also very close to ours. One of the main difference being that in [MM13] the quantitative estimates are uniform in time while ours are established on a finite time horizon. Moreover, in [MMW15], the authors directly compares the semigroup generated by the system of particles and the *lifted* one, that is, the one generated by the mean-field limit both acting on symmetric functions on  $(\mathbb{R}^d)^N$  while in our case we work at the level of the densities. An error bound of order  $N^{-1/2}$  for the total variation distance between  $k$  particles and  $k$  independent copies of the mean-field limit for non-linear SDEs with a constant diffusion coefficient and a drift with general and singular interacting kernel of first order type has been established in Jabin and Wang [JW18]. We also refer to the book of Kolokolstov [Kol10] and to the work by Kolokolstov, Troeva and Yang [KTY14] for a point of view based on measure-valued Markov processes and some quantitative estimates for mean-field games approximation. Let us finally mention the recent work of Chassagneux, Szpruch and Tse [CST19] where an expansion for the difference  $\mathbb{E}[h(\mu_t^N)] - h(\mu_t)$ ,  $t \in [0, T]$ , is established by exploiting the well-posedness and the regularity properties of the backward Kolmogorov PDE (1.3) (with  $f \equiv 0$ ) stated on  $[0, T] \times \mathcal{P}_2(\mathbb{R}^d)$  in the spirit of Buckdhan & al. [BLPR17], under the assumptions that  $h$ ,  $b$  and  $\sigma$  are smooth functions of the variables  $x$  and  $\mu$ .

The article is organized as follows. The basic notions of differentiation on the Wasserstein space with an emphasis on the smoothing properties of McKean-Vlasov SDEs that will play a key role in our analysis are presented in Section 2. The general set-up together with the assumptions and the main results are described in Section 3. The proof of the existence and uniqueness of the fundamental solution of the backward Kolmogorov PDE on the Wasserstein space together with some additional regularity properties of the transition SDE associated to (1.1) are addressed in Section 4. The propagation of chaos estimates are established in Section 5. The proof of some useful but auxiliary technical results are given in Appendix.

**Notations:** In the following we will denote by  $C$  and  $K$  some generic positive constants that may depend on the coefficients  $b$  and  $\sigma$ . We reserve the notation  $c$  for constants depending on  $|\sigma|_\infty$ ,  $\lambda$  (see assumption **(HE)** in Section 3) and possibly on  $N$  but not on the time horizon  $T$ . Moreover, the value of both  $C$ ,  $K$  or  $c$  may change from line to line.

We will denote by  $\mathcal{P}(\mathbb{R}^d)$  the space of probability measures on  $\mathbb{R}^d$  and by  $\mathcal{P}_2(\mathbb{R}^d) \subset \mathcal{P}(\mathbb{R}^d)$  the space of probability measures with finite second moment. For  $\mu \in \mathcal{P}(\mathbb{R}^d)$  and  $q > 0$ , we set  $M_q(\mu) := (\int_{\mathbb{R}^d} |x|^q \mu(dx))^{1/q}$  if  $\int_{\mathbb{R}^d} |x|^q \mu(dx) < +\infty$  and  $M_q(\mu) = +\infty$  otherwise.

For a positive variance-covariance matrix  $\Sigma$ , the function  $y \mapsto g(\Sigma, y)$  stands for the  $d$ -dimensional Gaussian kernel with  $\Sigma$  as covariance matrix  $g(\Sigma, x) = (2\pi)^{-\frac{d}{2}} (\det \Sigma)^{-\frac{1}{2}} \exp(-\frac{1}{2} \langle \Sigma^{-1} x, x \rangle)$ . We also define the first and second order Hermite polynomials:  $H_1^i(\Sigma, x) := -(\Sigma^{-1} x)_i$  and  $H_2^{i,j}(\Sigma, x) := (\Sigma^{-1} x)_i (\Sigma^{-1} x)_j - (\Sigma^{-1})_{i,j}$ ,  $1 \leq i, j \leq d$  which are related to the previous Gaussian density as follows  $\partial_{x_i} g(\Sigma, x) = H_1^i(\Sigma, x) g(\Sigma, x)$ ,  $\partial_{x_i, x_j}^2 g(\Sigma, x) = H_2^{i,j}(\Sigma, x) g(\Sigma, x)$ . We will sometimes use the following relations:  $\partial_\Sigma g(\Sigma, x) = -\frac{1}{2} (H_2.g)(\Sigma, x)$ ,  $\partial_\Sigma^2 g(\Sigma, x) = \frac{1}{4} (H_4.g)(\Sigma, x)$  where  $H_2(\Sigma, x) = (H_2^{i,j}(\Sigma, x))_{1 \leq i, j \leq d}$  and  $H_4(\Sigma, x) = (H_4^{i,j,k,l}(\Sigma, x))_{1 \leq i, j, k, l \leq d}$  satisfies  $\partial_{x_i, x_j, x_k, x_l}^4 g(\Sigma, x) = H_4^{i,j,k,l}(\Sigma, x) g(\Sigma, x)$ . Also, when  $\Sigma = cI_d$ , for some positive constant  $c$ , the latter notation is simplified to  $g(c, x) := (1/(2\pi c))^{d/2} \exp(-|x|^2/(2c))$ .

One of the key inequality that will be used intensively in this work is the following: for any  $p, q > 0$  and  $x \in \mathbb{R}$ ,  $|x|^p e^{-qx^2} \leq (p/(2qe))^{p/2}$ . As a direct consequence, we obtain the *space-time inequality*,

$$(1.5) \quad \forall p, c > 0, \quad |x|^p g(ct, x) \leq Ct^{p/2} g(c't, x)$$

which in turn gives the *standard Gaussian estimates* for the first and second order derivatives of Gaussian density, namely for all  $c > 0$

$$(1.6) \quad |H_1^i(ct, x)|g(ct, x) \leq \frac{C}{t^{\frac{1}{2}}} g(c't, x), \quad |H_2^{i,j}(ct, x)|g(ct, x) \leq \frac{C}{t} g(c't, x), \quad |H_4^{i,j,k,l}(ct, x)|g(ct, x) \leq \frac{C}{t^2} g(c't, x)$$

for some positive constants  $C, c'$ . Since we will employ it quite frequently, we will often omit to mention it explicitly at some places. We finally define the Mittag-Leffler function  $E_{\alpha,\beta}(z) := \sum_{n \geq 0} z^n / \Gamma(\alpha n + \beta)$ ,  $z \in \mathbb{R}$ ,  $\alpha, \beta > 0$ .

## 2. PRELIMINARIES: DIFFERENTIATION ON THE WASSERSTEIN SPACE AND SMOOTHING PROPERTIES

**2.1. Differentiation on the Wasserstein space.** In this section, we briefly present the regularity notions we will use when working with mappings defined on  $\mathcal{P}_2(\mathbb{R}^d)$ . We refer the reader to Lions' seminal lectures [Lio14], to Cardaliaguet's lectures notes [Car13], to the recent work Cardaliaguet et al. [CDLL15] or to Chapter 5 of Carmona and Delarue's monograph [CD18] for a more complete and detailed exposition. The space  $\mathcal{P}_2(\mathbb{R}^d)$  is equipped with the 2-Wasserstein metric

$$W_2(\mu, \nu) = \inf_{\pi \in \mathcal{P}(\mu, \nu)} \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 \pi(dx, dy) \right)^{\frac{1}{2}}$$

where, for given  $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ ,  $\mathcal{P}(\mu, \nu)$  denotes the set of measures on  $\mathbb{R}^d \times \mathbb{R}^d$  with marginals  $\mu$  and  $\nu$ .

Following our recent work [CdRF18], we will employ two notions of differentiation of a continuous map  $U$  defined on  $\mathcal{P}_2(\mathbb{R}^d)$ . The first one, called the *linear functional derivative* and denoted by  $\frac{\delta U}{\delta m}$ , will play an important role in our linearization procedure to strengthen the regularity properties of the transition densities of the McKean-Vlasov SDE (1.1) and its corresponding decoupling field.

**Definition 2.1.** The continuous map  $U : \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$  is said to have a continuous linear functional derivative if there exists a continuous function  $\frac{\delta U}{\delta m} : \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \rightarrow \mathbb{R}$  such that  $y \mapsto \frac{\delta U}{\delta m}(m)(y)$  has at most quadratic growth in  $y$ , uniformly in  $m$  for  $m \in \mathcal{K}$ ,  $\mathcal{K}$  being any compact subset of  $\mathcal{P}_2(\mathbb{R}^d)$  and such that for any  $m, m' \in \mathcal{P}_2(\mathbb{R}^d)$ ,

$$\lim_{\varepsilon \downarrow 0} \frac{U((1 - \varepsilon)m + \varepsilon m') - U(m)}{\varepsilon} = \int_{\mathbb{R}^d} \frac{\delta U}{\delta m}(m)(y) d(m' - m)(y).$$

The map  $y \mapsto \frac{\delta U}{\delta m}(m)(y)$  being defined up to an additive constant, we will follow the usual normalization convention  $\int_{\mathbb{R}^d} \frac{\delta U}{\delta m}(m)(y) dm(y) = 0$ .

From the above definition, we observe that

$$\forall m, m' \in \mathcal{P}_2(\mathbb{R}^d), \quad U(m') - U(m) = \int_0^1 \int_{\mathbb{R}^d} \frac{\delta U}{\delta m}((1 - \lambda)m + \lambda m')(y) d(m' - m)(y) d\lambda.$$

It is then readily seen that if  $(m, y) \mapsto \frac{\delta U}{\delta m}(m)(y)$  is bounded, then one has

$$\forall m, m' \in \mathcal{P}_2(\mathbb{R}^d), \quad |U(m) - U(m')| \leq \sup_{m'' \in \mathcal{P}_2(\mathbb{R}^d)} \left\| \frac{\delta U}{\delta m}(m'')(\cdot) \right\|_{\infty} d_{TV}(m, m')$$

where  $d_{TV}$  is the total variation metric so that  $U$  is Lipschitz continuous with respect to this distance. If,  $y \mapsto \frac{\delta U}{\delta m}(m)(y)$  is Lipschitz continuous, with a Lipschitz modulus bounded uniformly with respect to the variable  $m$ , then from the Monge-Kantorovich duality principle

$$\forall m, m' \in \mathcal{P}_2(\mathbb{R}^d), \quad |U(m) - U(m')| \leq \sup_{m''} \left\| \partial_y \left[ \frac{\delta U}{\delta m}(m'')(\cdot) \right] \right\|_{\infty} W_1(m, m').$$

We will also work with higher order derivatives. This is naturally defined by induction as follows.

**Definition 2.2.** Let  $p \geq 1$ . The continuous map  $U : \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$  is said to have a continuous linear functional derivative at order  $p$  if there exists a continuous function  $\frac{\delta^p U}{\delta m^p} : \mathcal{P}_2(\mathbb{R}^d) \times (\mathbb{R}^d)^{p-1} \times \mathbb{R}^d \rightarrow \mathbb{R}$  such that  $\mathbb{R}^d \ni y_p \mapsto \frac{\delta^p U}{\delta m^p}(m)(\mathbf{y}_{p-1}, y_p)$  has at most quadratic growth in  $y_p$ , uniformly in  $(m, \mathbf{y}_{p-1})$  for  $(m, \mathbf{y}_{p-1}) \in \mathcal{K}$ ,  $\mathcal{K}$  being any compact subset of  $\mathcal{P}_2(\mathbb{R}^d) \times (\mathbb{R}^d)^{p-1}$  and such that for any  $m, m' \in \mathcal{P}_2(\mathbb{R}^d)$  and for any  $\mathbf{y}_{p-1} \in (\mathbb{R}^d)^{p-1}$

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{-1} \left( \frac{\delta^{p-1}}{\delta m^{p-1}} U((1 - \varepsilon)m + \varepsilon m')(\mathbf{y}_{p-1}) - \frac{\delta^{p-1}}{\delta m^{p-1}} U(m)(\mathbf{y}_{p-1}) \right) = \int_{\mathbb{R}^d} \frac{\delta^p U}{\delta m^p}(m)(\mathbf{y}_p) d(m' - m)(y_p)$$

with the notation  $\mathbf{y}_p := (\mathbf{y}_{p-1}, y_p)$  and the convention  $\frac{\delta^0}{\delta m} U(m) \equiv U(m)$ . We again follow the usual normalization convention  $\int_{\mathbb{R}^d} \frac{\delta^p U}{\delta m^p}(m)(\mathbf{y}_p) dm(y_p) = 0$

Again, for more details on the above notion of derivative, we refer to Chapter 5 of [CD18].

We now briefly present the second notion of derivatives as originally introduced by Lions [Lio14]. The basic strategy consists in considering the canonical lift of the real-valued function  $U : \mathcal{P}_2(\mathbb{R}^d) \ni \mu \mapsto U(\mu)$  into a function  $\mathcal{U} : \mathbb{L}_2 \ni Z \mapsto \mathcal{U}(Z) = U([Z]) \in \mathbb{R}$ ,  $(\Omega, \mathcal{F}, \mathbb{P})$  standing for an atomless probability space,

with  $\Omega$  a Polish space,  $\mathcal{F}$  its Borel  $\sigma$ -algebra,  $\mathbb{L}_2 := \mathbb{L}_2(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{R}^d)$  standing for the space of  $\mathbb{R}^d$ -valued random variables defined on  $\Omega$  with finite second moment and  $Z$  being a random variable with law  $\mu$ . The function  $U$  is then said to be differentiable at  $\mu \in \mathcal{P}_2(\mathbb{R}^d)$  if its canonical lift  $\mathcal{U}$  is Fréchet differentiable at some point  $Z$  such that  $[Z] = \mu$ . We denote by  $DU$  its gradient. The Riezs representation theorem then allows to identify  $DU$  as an element of  $\mathbb{L}^2$ . It turns out that  $DU$  is a random variable which is  $\sigma(Z)$ -measurable and given by a function  $DU(\mu)(\cdot)$  from  $\mathbb{R}^d$  to  $\mathbb{R}^d$ , which depends on the law  $\mu$  of  $Z$  and satisfying  $DU(\mu)(\cdot) \in \mathbb{L}^2(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \mu; \mathbb{R}^d)$ . As in [CdRF18], we adopt the notation  $\partial_\mu U(\mu)(\cdot)$  in order to emphasize that we are taking the derivative of the map  $U$  with respect to its measure argument. The  $L$ -derivative of  $U$  at  $\mu$  is the map  $\partial_\mu U(\mu)(\cdot) : \mathbb{R}^d \ni v \mapsto \partial_\mu U(\mu)(v) \in \mathbb{R}^d$ , satisfying  $DU = \partial_\mu U(\mu)(Z)$ .

It is important to note that this representation holds irrespectively of the choice of the original probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . We will restrict our considerations to functions which are  $\mathcal{C}^1$ , that is, functions for which the associated canonical lift is  $\mathcal{C}^1$  on  $\mathbb{L}^2$  and for which there exists a continuous version of the mapping  $\mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \ni (\mu, v) \mapsto \partial_\mu U(\mu)(v) \in \mathbb{R}^d$ . It then appears that this version is unique. We straightforwardly extend the above discussion to  $\mathbb{R}^d$ -valued or  $\mathbb{R}^{d \times d}$ -valued maps  $U$  defined on  $\mathcal{P}_2(\mathbb{R}^d)$ , component by component.

In order to establish the existence and uniqueness of a fundamental solution of the Kolmogorov PDE on the Wasserstein space as well as our quantitative estimates for the mean-field approximation by systems of particles, we will employ at several places a chain rule formula for  $(U(t, Y_t, [X_t]))_{t \geq 0}$ , where  $(X_t)_{t \geq 0}$  and  $(Y_t)_{t \geq 0}$  are two Itô processes defined for sake of simplicity on the same probability space  $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$  assumed to be equipped with a right-continuous and complete filtration  $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ . Their dynamics are given by

$$(2.1) \quad X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s, \quad X_0 \in \mathbb{L}_2,$$

$$(2.2) \quad Y_t = Y_0 + \int_0^t \eta_s ds + \int_0^t \gamma_s dW_s$$

where  $W = (W_t)_{t \geq 0}$  is an  $\mathbb{F}$ -adapted  $d$ -dimensional Brownian,  $(b_t)_{t \geq 0}$ ,  $(\eta_t)_{t \geq 0}$ ,  $(\sigma_t)_{t \geq 0}$  and  $(\gamma_t)_{t \geq 0}$  are  $\mathbb{F}$ -progressively measurable processes, with values in  $\mathbb{R}^d$ ,  $\mathbb{R}^d$ ,  $\mathbb{R}^{d \times d}$  and  $\mathbb{R}^{d \times q}$  respectively, satisfying the following conditions

$$(2.3) \quad \forall T > 0, \quad \mathbb{E} \left[ \int_0^T (|b_t|^2 + |\sigma_t|^4) dt \right] < \infty \text{ and } \mathbb{P} \left( \int_0^T (|\eta_t| + |\gamma_t|^2) dt < +\infty \right) = 1.$$

We now introduce two classes of functions we will work with throughout the paper.

**Definition 2.3.** (The spaces  $\mathcal{C}^{p,2,2}([0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$  and  $\mathcal{C}_f^{p,2,2}([0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$ , for  $p = 0, 1$ ) Let  $T > 0$  and  $p \in \{0, 1\}$ .

The continuous function  $U : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$  is in  $\mathcal{C}^{p,2,2}([0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$  if the following conditions hold:

- (i) For any  $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ , the mapping  $[0, T] \times \mathbb{R}^d \ni (t, x) \mapsto U(t, x, \mu)$  is in  $\mathcal{C}^{p,2}([0, T] \times \mathbb{R}^d)$  and the functions  $[0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \ni (t, x, \mu) \mapsto \partial_t^p U(t, x, \mu)$ ,  $\partial_x U(t, x, \mu)$ ,  $\partial_x^2 U(t, x, \mu)$  are continuous.
- (ii) For any  $(t, x) \in [0, T] \times \mathbb{R}^d$ , the mapping  $\mathcal{P}_2(\mathbb{R}^d) \ni \mu \mapsto U(t, x, \mu)$  is continuously  $L$ -differentiable and for any  $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ , we can find a version of the mapping  $\mathbb{R}^d \ni v \mapsto \partial_\mu U(t, x, \mu)(v)$  such that the mapping  $[0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \ni (t, x, \mu, v) \mapsto \partial_\mu U(t, x, \mu)(v)$  is locally bounded and is continuous at any  $(t, x, \mu, v)$  such that  $v \in \text{Supp}(\mu)$ .
- (iii) For the version of  $\partial_\mu U$  mentioned above and for any  $(t, x, \mu)$  in  $[0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$ , the mapping  $\mathbb{R}^d \ni v \mapsto \partial_\mu U(t, x, \mu)(v)$  is continuously differentiable and its derivative  $\partial_v[\partial_\mu U(t, x, \mu)](v) \in \mathbb{R}^{d \times d}$  is jointly continuous in  $(t, x, \mu, v)$  at any point  $(t, x, \mu, v)$  such that  $v \in \text{Supp}(\mu)$ .

The continuous function  $U : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$  is in  $\mathcal{C}_f^{p,2,2}([0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$  if  $U \in \mathcal{C}^{p,2,2}([0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$  in the above sense and the following additional condition holds:

- (iv) For each  $v \in \mathbb{R}^d$ , the version  $\mathcal{P}_2(\mathbb{R}^d) \ni \mu \mapsto \partial_\mu U(t, x, \mu)(v)$  discussed in (ii) is  $L$ -differentiable (component by component) with a derivative given by  $(\mu, v, v') \mapsto \partial_\mu^2 U(t, x, \mu)(v)(v') \in \mathbb{R}^{d \times d}$  such that for any  $\mu \in \mathcal{P}_2(\mathbb{R}^d)$  and  $X \in \mathbb{L}_2$  with  $[X] = \mu$ , the  $\mathbb{R}^{d \times d}$ -valued random variable  $\partial_\mu^2 U(t, x, \mu)(v)(X)$  gives the Fréchet derivative of the map  $\mathbb{L}_2 \ni X' \mapsto \partial_\mu^2 U(t, x, [X'])(v)$  for every  $v \in \mathbb{R}^d$ . Denoting  $\partial_\mu^2 U(t, x, \mu)(v)(v')$  by  $\partial_\mu^2 U(t, x, \mu)(v, v')$ , the map  $[0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times (\mathbb{R}^d)^2 \ni (t, x, \mu, v, v') \mapsto \partial_\mu^2 U(t, x, \mu)(v, v')$  is also assumed to be continuous for the product topology.

*Remark 2.4.* We will also consider the spaces  $\mathcal{C}^{1,p}([0, T] \times \mathcal{P}_2(\mathbb{R}^d))$  for  $p = 1, 2$  and  $\mathcal{C}_f^{1,2}([0, T] \times \mathcal{P}_2(\mathbb{R}^d))$ , where we adequately remove the space variable in the Definition 2.3. We will say that  $U \in \mathcal{C}^{1,1}([0, T] \times$

$\mathcal{P}_2(\mathbb{R}^d)$ ) if  $U$  is continuous,  $t \mapsto U(t, \mu) \in \mathcal{C}^1([0, T])$  for any  $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ ,  $(t, \mu) \mapsto \partial_t U(t, \mu)$  being continuous and if for any  $t \in [0, T]$ ,  $\mu \mapsto U(t, \mu)$  is continuously  $L$ -differentiable such that we can find a version of  $v \mapsto \partial_\mu U(t, \mu)(v)$  satisfying:  $(t, \mu, v) \mapsto \partial_\mu U(t, \mu)(v)$  is locally bounded and continuous at any  $(t, \mu, v)$  satisfying  $v \in \text{Supp}(\mu)$ .

We will say that  $U \in \mathcal{C}_f^{1,2}([0, T] \times \mathcal{P}_2(\mathbb{R}^d))$  if  $U \in \mathcal{C}^{1,2}([0, T] \times \mathcal{P}_2(\mathbb{R}^d))$  and for the version of  $\partial_\mu U$  previously considered, for any  $(t, v) \in [0, T] \times \mathbb{R}^d$ , the mapping  $\mathcal{P}_2(\mathbb{R}^d) \ni \mu \mapsto \partial_\mu U(t, \mu)(v)$  is  $L$ -differentiable with a derivative given by  $(t, \mu, v, v') \mapsto \partial_\mu U(t, \mu)(v, v') \in \mathbb{R}^{d \times d}$  such that for any  $\mu \in \mathcal{P}_2(\mathbb{R}^d)$  and  $X \in \mathbb{L}_2$  with  $[X] = \mu$ ,  $\partial_\mu U(t, \mu)(v, X)$  gives the Fréchet derivative of the map  $\mathbb{L}_2 \ni X' \mapsto \partial_\mu U(t, [X'])(v)$  for every  $(t, v) \in [0, T] \times \mathbb{R}^d$ . Moreover, the map  $[0, T] \times \mathcal{P}_2(\mathbb{R}^d) \times (\mathbb{R}^d)^2 \ni (t, \mu, v, v') \mapsto \partial_\mu^2 U(t, \mu)(v, v')$  is assumed to be continuous for the product topology.

**Notations:** We will use the following notations throughout the paper. For a smooth map  $U : \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$  and for  $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ ,  $v, v' \in \mathbb{R}^d$

$$\begin{aligned} \partial_v[\partial_\mu U(\mu)(v)] &= (\partial_{v_j}[\partial_\mu U(\mu)]_i(v))_{1 \leq i, j \leq d} \\ \partial_\mu^2 U(\mu)(v, v') &= ([\partial_\mu[\partial_\mu U(\mu)]_i(v)]_j(v'))_{1 \leq i, j \leq d}. \end{aligned}$$

With the above definitions and notations, we can now provide the chain rule formula on the Wasserstein space that will be play a central role in our analysis.

**Proposition 2.1** ([CD18], Proposition 5.102). Let  $X$  and  $Y$  be two Itô processes, with respective dynamics (2.1) and (2.2), satisfying (2.3). Assume that  $U \in \mathcal{C}^{1,2,2}([0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$  in the sense of Definition 2.3 such that for any compact set  $\mathcal{K} \subset \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$ ,

$$(2.4) \quad \sup_{(t, x, \mu) \in [0, T] \times \mathcal{K}} \left\{ \int_{\mathbb{R}^d} |\partial_\mu U(t, x, \mu)(v)|^2 \mu(dv) + \int_{\mathbb{R}^d} |\partial_v[\partial_\mu U(t, x, \mu)](v)|^2 \mu(dv) \right\} < \infty.$$

Then,  $\mathbb{P}$ -a.s.,  $\forall t \in [0, T]$ , one has

$$(2.5) \quad \begin{aligned} U(t, Y_t, [X_t]) &= U(0, Y_0, [X_0]) + \int_0^t \partial_x U(s, Y_s, [X_s]) \cdot \gamma_s dW_s \\ &+ \int_0^t \left\{ \partial_s U(s, Y_s, [X_s]) + \partial_x U(s, Y_s, [X_s]) \cdot \eta_s + \frac{1}{2} \text{Tr}(\partial_x^2 U(s, Y_s, [X_s]) \gamma_s \gamma_s^T) \right\} ds \\ &+ \int_0^t \left\{ \tilde{\mathbb{E}}[\partial_\mu U(s, Y_s, [X_s])(\tilde{X}_s) \cdot \tilde{b}_s] + \frac{1}{2} \tilde{\mathbb{E}}[\text{Tr}(\partial_v[\partial_\mu U(s, Y_s, [X_s])](\tilde{X}_s) \tilde{a}_s)] \right\} ds \end{aligned}$$

where the Itô process  $(\tilde{X}_t, \tilde{b}_t, \tilde{\sigma}_t)_{0 \leq t \leq T}$  is a copy of the original process  $(X_t, b_t, \sigma_t)_{0 \leq t \leq T}$  defined on a copy  $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$  of the original probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ .

We conclude this subsection by enlightening the connection between the  $L$ -derivative of a map  $U : \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$  and the standard partial derivatives of its empirical projection  $U^N : (\mathbb{R}^d)^N \rightarrow \mathbb{R}$ ,  $N$  being a positive integer, defined by

$$(2.6) \quad U^N : (\mathbb{R}^d)^N \ni (x_1, \dots, x_N) \mapsto U\left(\frac{1}{N} \sum_{i=1}^N \delta_{x_i}\right).$$

We refer to Propositions 5.35 and 5.91 of [CD18] for a proof of the following result.

**Proposition 2.2** (Connection between  $L$ -derivatives and empirical projection). If  $U$  is  $\mathcal{C}_f^2(\mathcal{P}_2(\mathbb{R}^d))$  (fully  $\mathcal{C}^2$ ) then its empirical projection  $U^N$  is two times differentiable on  $(\mathbb{R}^d)^N$  and, for all  $x_1, \dots, x_N \in (\mathbb{R}^d)^N$ , for all  $(i, j) \in \{1, \dots, N\}^2$

$$\partial_{x_i} U^N(x_1, \dots, x_N) = \frac{1}{N} \partial_\mu U\left(\frac{1}{N} \sum_{\ell=1}^N \delta_{x_\ell}\right)(x_i)$$

and

$$\partial_{x_i, x_j} U^N(x_1, \dots, x_N) = \frac{1}{N} \partial_v \partial_\mu U\left(\frac{1}{N} \sum_{\ell=1}^N \delta_{x_\ell}\right)(x_i) \delta_{i,j} + \frac{1}{N^2} \partial_\mu^2 U\left(\frac{1}{N} \sum_{\ell=1}^N \delta_{x_\ell}\right)(x_i, x_j)$$

with the notation  $\delta_{i,j} = 1$  if  $i = j$  and  $\delta_{i,j} = 0$  otherwise.



**2.2. Regularization properties by smooth flow of probability measures.** In order to establish quantitative estimates for propagation of chaos for the mean field approximation of the dynamics (1.1) by its system of particles (1.2), an additional study of the regularity properties of the transition density of the McKean-Vlasov SDE is required. We build on our previous work [CdRF18] which highlights the key feature to investigate the smoothing properties of the transition density in the uniformly elliptic framework. Namely, our analysis is mainly based on how a map defined on  $\mathcal{P}_2(\mathbb{R}^d)$  admitting only flat derivatives can be regularized in *the intrinsic sense* by a smooth flow of probability measures. Assuming that the coefficients  $b$  and  $a$  admit bounded and Hölder continuous linear functional derivatives at order 2, it turns out that the density of the unique solution of a McKean-Vlasov SDE with dynamics (1.1) achieves better regularity with respect to its measure argument and is *partially*  $\mathcal{C}^2$ . Clearly, this phenomenon has to be understood as a smoothing property of McKean-Vlasov SDEs in a uniformly elliptic setting. We refer to Section 2.2 in [CdRF18] for a detailed introduction and discussion of this regularization property.

We here want to go one step further by analyzing the *full*  $\mathcal{C}^2$  regularity of the density. The following result will play an important role in our study.

**Proposition 2.3.** Assume that the continuous map  $h : \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$  admits two bounded continuous linear functional derivative. Consider a map  $(t, x, \mu) \mapsto p(\mu, t, T, x, z) \in \mathcal{C}_f^{1,2,2}([0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$ , for some prescribed  $T > 0$ ,  $z \mapsto p(\mu, t, T, x, z)$  being a density function, such that  $z \mapsto (p(\mu, t, T, \cdot, z) \# \mu) \in \mathcal{P}_2(\mathbb{R}^d)$ , locally uniformly with respect to  $(t, \mu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)$ , i.e. uniformly in  $(t, \mu) \in \mathcal{K}$ ,  $\mathcal{K}$  being any compact subset of  $[0, T] \times \mathcal{P}_2(\mathbb{R}^d)$ , such that the mappings  $\mathbb{R}^d \ni x \mapsto \int_{\mathbb{R}^d} |\partial_\mu^2 p(\mu, t, T, x, z)(v, v')| dz$ ,  $\int_{\mathbb{R}^d} |\partial_v^n [\partial_\mu p(\mu, t, T, x, z)](v)| dz$ ,  $\int_{\mathbb{R}^d} |\partial_t p(\mu, t, T, x, z)| dz$ ,  $n = 0, 1$ , are at most of quadratic growth, uniformly in  $(t, \mu, v, v')$  in compact subsets of  $[0, T] \times \mathcal{P}_2(\mathbb{R}^d) \times (\mathbb{R}^d)^2$  and such that for any compact set  $\mathcal{K}' \subset [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \times (\mathbb{R}^d)^3$ , for any  $n = 0, 1$ ,

$$(2.7) \quad \int \sup_{(t, \mu, x, v, v') \in \mathcal{K}'} \left\{ |\partial_t^n p(\mu, t, T, x, z)| + |\partial_x^{1+n} p(\mu, t, T, x, z)| \right. \\ \left. + |\partial_v^n [\partial_\mu p(\mu, t, T, x, z)](v)| + |\partial_\mu^2 p(\mu, t, T, x, z)(v, v')| \right\} dz < \infty.$$

Let  $\Theta(t, \mu) : [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \ni (t, \mu) \mapsto \Theta(t, \mu)(dz) = (p(\mu, t, T, \cdot, z) \# \mu)(dz) = \int_{\mathbb{R}^d} p(\mu, t, T, x, z) \mu(dx) dz$ . Then, one has:

- the map  $[0, T] \times \mathcal{P}_2(\mathbb{R}^d) \ni (t, \mu) \mapsto h(\Theta(t, \mu)) \in \mathcal{C}_f^{1,2}([0, T] \times \mathcal{P}_2(\mathbb{R}^d))$ ,
- the Lions and time derivatives satisfy for  $n = 0, 1$ :

$$(2.8) \quad \partial_v^n [\partial_\mu h(\Theta(t, \mu))](v) = \partial_v^n \left[ \partial_\nu \left[ \int \int \frac{\delta}{\delta m} h(\Theta(t, \mu))(z) p(\nu, t, T, x, z) dz \nu(dx) \right]_{|\nu=\mu} \right] (v) \\ = \int_{\mathbb{R}^d} \left[ \frac{\delta}{\delta m} h(\Theta(t, \mu))(z) - \frac{\delta}{\delta m} h(\Theta(t, \mu))(v) \right] \partial_x^{1+n} p(\mu, t, T, v, z) dz \\ + \int_{(\mathbb{R}^d)^2} \left[ \frac{\delta}{\delta m} h(\Theta(t, \mu))(z) - \frac{\delta}{\delta m} h(\Theta(t, \mu))(x) \right] \partial_v^n [\partial_\mu p(\mu, t, T, x, z)](v) dz \mu(dx),$$

$$(2.9) \quad \partial_t h(\Theta(t, \mu)) = \partial_s \left[ \int_{(\mathbb{R}^d)^2} \frac{\delta}{\delta m} h(\Theta(t, \mu))(z) p(\mu, s, T, x, z) dz \mu(dx) \right]_{s=t} \\ = \int_{(\mathbb{R}^d)^2} \left[ \frac{\delta}{\delta m} h(\Theta(t, \mu))(z) - \frac{\delta}{\delta m} h(\Theta(t, \mu))(x) \right] \partial_t p(\mu, t, T, x, z) dz \mu(dx)$$

and

$$\begin{aligned}
 \partial_\mu^2 h(\Theta(t, \mu))(v, v') &= \partial_\mu \left[ \int_{\mathbb{R}^d} \frac{\delta}{\delta m} h(\Theta(t, \mu))(z) \partial_v p(\mu, t, T, v, z) dz \right] (v') \\
 &\quad + \partial_\mu \left[ \int_{(\mathbb{R}^d)^2} \frac{\delta}{\delta m} h(\Theta(t, \mu))(z) \partial_\mu p(\mu, t, T, x, z)(v) dz \mu(dx) \right] (v') \\
 &= \int_{\mathbb{R}^d} \partial_v p(\mu, t, T, v, z) \otimes \partial_\mu \left[ \frac{\delta}{\delta m} h(\Theta(t, \mu))(z) - \frac{\delta}{\delta m} h(\Theta(t, \mu))(v) \right] (v') dz \\
 &\quad + \int_{\mathbb{R}^d} \left[ \frac{\delta}{\delta m} h(\Theta(t, \mu))(z) - \frac{\delta}{\delta m} h(\Theta(t, \mu))(v) \right] \partial_\mu \partial_v p(\mu, t, T, v, z)(v') dz \\
 (2.10) \quad &\quad + \int_{(\mathbb{R}^d)^2} \partial_\mu p(\mu, t, T, x, z)(v) \otimes \partial_\mu \left[ \frac{\delta}{\delta m} h(\Theta(t, \mu))(z) - \frac{\delta}{\delta m} h(\Theta(t, \mu))(x) \right] (v') dz \mu(dx) \\
 &\quad + \int_{(\mathbb{R}^d)^2} \left[ \frac{\delta}{\delta m} h(\Theta(t, \mu))(z) - \frac{\delta}{\delta m} h(\Theta(t, \mu))(x) \right] \partial_\mu^2 p(\mu, t, T, x, z)(v, v') dz \mu(dx) \\
 &\quad + \int_{\mathbb{R}^d} \left[ \frac{\delta}{\delta m} h(\Theta(t, \mu))(z) - \frac{\delta}{\delta m} h(\Theta(t, \mu))(v') \right] \partial_{v'} \partial_\mu p(\mu, t, T, v', z)(v) dz
 \end{aligned}$$

with the notations:  $\partial_\mu \partial_v p(\mu, t, T, v, z)(v') = ([\partial_\mu \partial_{v_i} p(\mu, t, T, v, z)]_j(v'))_{1 \leq i, j \leq d}$  and  $\partial_{v'} \partial_\mu p(\mu, t, T, v', z)(v) = (\partial_{v'_i} [\partial_\mu p(\mu, t, T, v', z)]_j(v))_{1 \leq i, j \leq d}$ .

*Proof.* From [CdRF18], we already know that  $(t, \mu) \mapsto h(\Theta(t, \mu)) \in \mathcal{C}^{1,2}([0, T] \times \mathbb{R}^d)$  and that (2.8) as well as (2.9) hold. It thus remains to prove that for any  $(t, v) \in [0, T] \times \mathbb{R}^d$ , the map  $\mu \mapsto \partial_\mu h(\Theta(t, \mu))(v)$  is  $\mathcal{C}^1(\mathcal{P}_2(\mathbb{R}^d))$  and that for any  $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ , we can find a version of  $v' \mapsto \partial_\mu^2 h(\Theta(t, \mu))(v, v')$  satisfying (2.10) and such that the mapping  $(t, \mu, v, v') \mapsto \partial_\mu^2 h(\Theta(t, \mu))(v, v')$  is locally bounded and is continuous at any  $(t, \mu, v, v')$  such that  $v, v' \in \text{Supp}(\mu)$ .

From (2.8) with  $n = 0$

$$\begin{aligned}
 \partial_\mu h(\Theta(t, \mu))(v) &= \int_{\mathbb{R}^d} \left[ \frac{\delta}{\delta m} h(\Theta(t, \mu))(z) - \frac{\delta}{\delta m} h(\Theta(t, \mu))(v) \right] \partial_v p(\mu, t, T, v, z) dz \\
 &\quad + \int_{(\mathbb{R}^d)^2} \left[ \frac{\delta}{\delta m} h(\Theta(t, \mu))(z) - \frac{\delta}{\delta m} h(\Theta(t, \mu))(x) \right] \partial_\mu p(\mu, t, T, x, z)(v) dz \mu(dx).
 \end{aligned}$$

Observe now that the maps  $\mu \mapsto \frac{\delta}{\delta m} h(\Theta(t, \mu))(v)$ ,  $\partial_v p(\mu, t, T, v, z)$ ,  $\partial_\mu p(\mu, t, T, x, z)(v)$  are continuously  $L$ -differentiable for any  $(t, x, z, v) \in [0, T] \times (\mathbb{R}^d)^3$ . Moreover, from (2.8) with  $n = 0$  applied to the map  $\delta h / \delta m$  instead of  $h$ , we deduce that  $(t, \mu, v, v') \mapsto \partial_\mu \frac{\delta}{\delta m} h(\Theta(t, \mu))(v, v')$  is continuous and locally bounded. Hence, the integrability conditions allow to differentiate under the integral sign in the above identity. We thus deduce the  $L$ -differentiability of  $\mu \mapsto \partial_\mu h(\Theta(t, \mu))(v)$  and (2.10) follows. Finally, we remark that each integrand appearing in the five integrals of the right-hand side of (2.10) are continuous with respect to the variables  $t, \mu, v, v'$ . The integrability conditions then allow to deduce the global continuity of each term for  $(t, \mu, v, v') \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \times (\mathbb{R}^d)^2$ .

□

An explicit expression of  $\partial_\mu^2 h(\Theta(t, \mu))(v, v')$  can be derived by plugging (2.8) with  $n = 0$  for the map  $\frac{\delta}{\delta m} h(\Theta(t, \mu))$  into (2.10). We obtain

$$\begin{aligned}
& \partial_\mu^2 h(\Theta(t, \mu))(v, v') \\
&= \int_{(\mathbb{R}^d)^2} \left\{ \frac{\delta^2 h}{\delta m^2}(\Theta(t, \mu))(z, z') - \frac{\delta^2 h}{\delta m^2}(\Theta(t, \mu))(v, v') \right\} \partial_v p(\mu, t, T, v, z) \otimes \partial_{v'} p(\mu, t, T, v', z') dz dz' \\
&+ \int_{(\mathbb{R}^d)^3} \left\{ \frac{\delta^2 h}{\delta m^2}(\Theta(t, \mu))(z, z') - \frac{\delta^2 h}{\delta m^2}(\Theta(t, \mu))(v, x) \right\} \partial_v p(\mu, t, T, v, z) \otimes \partial_\mu p(\mu, t, T, x, z')(v') dz dz' d\mu(x) \\
&+ \int_{\mathbb{R}^d} \left\{ \frac{\delta h}{\delta m}(\Theta(t, \mu))(z) - \frac{\delta h}{\delta m}(\Theta(t, \mu))(v) \right\} \partial_\mu \partial_v p(\mu, t, T, v, z)(v') dz \\
(2.11) \quad &+ \int_{\mathbb{R}^d} \left\{ \frac{\delta h}{\delta m}(\Theta(t, \mu))(z) - \frac{\delta h}{\delta m}(\Theta(t, \mu))(v') \right\} \partial_{v'} \partial_\mu p(\mu, t, T, v', z)(v) dz \\
&+ \int_{(\mathbb{R}^d)^3} \left\{ \frac{\delta^2 h}{\delta m^2}(\Theta(t, \mu))(z, z') - \frac{\delta^2 h}{\delta m^2}(\Theta(t, \mu))(x, v') \right\} \partial_\mu p(\mu, t, T, x, z)(v) \otimes \partial_{v'} p(\mu, t, T, v', z') dz dz' d\mu(x) \\
&+ \int_{(\mathbb{R}^d)^4} \left\{ \frac{\delta^2 h}{\delta m^2}(\Theta(t, \mu))(z, z') - \frac{\delta^2 h}{\delta m^2}(\Theta(t, \mu))(x, x') \right\} \partial_\mu p(\mu, t, T, x, z)(v) \otimes \partial_\mu p(\mu, t, T, x', z')(v') dz dz' d\mu(x') d\mu(x) \\
&+ \int_{(\mathbb{R}^d)^2} \left\{ \frac{\delta h}{\delta m}(\Theta(t, \mu))(z) - \frac{\delta h}{\delta m}(\Theta(t, \mu))(x) \right\} \partial_\mu^2 p(\mu, t, T, x, z)(v, v') dz d\mu(x).
\end{aligned}$$

The three relations (2.8), (2.9) and (2.11) play a central role for the analysis of the regularity properties of the transition density related to the dynamics (1.1). Indeed, under the additional assumption that the maps  $y \mapsto \frac{\delta h}{\delta m}(m)(y)$ ,  $(y, y') \mapsto \frac{\delta^2 h}{\delta m^2}(m)(y, y')$  are Hölder continuous and if  $(t, \mu, x) \mapsto p(\mu, t, T, x, z)$  as well as its derivatives satisfy suitable Gaussian-type bounds, they allow thanks to the space-time inequality (1.5) to match the diagonal regime of the underlying heat kernel and to benefit from the so-called smoothing property of Gaussian kernels. These key observation will be used repeatedly in the proofs of Lemma 3.1 and Proposition 3.1.

### 3. OVERVIEW, ASSUMPTIONS AND MAIN RESULTS

**3.1. Well-posedness of (1.1), existence and regularity of its transition density.** Let us give a few practical reminders of our previous work [CdRF18] concerning the well-posedness of (1.1), the existence and regularity properties of its transition density. We first provide some assumptions on the coefficients.

**(HR)** (i) The maps  $\mathbb{R}^d \ni x \mapsto b(t, x, m) \in \mathbb{R}^d$ ,  $a(t, x, m) \in \mathbb{R}^{d \times d}$  are bounded and uniformly  $\eta$ -Hölder continuous for some  $\eta \in (0, 1]$ ,

$$\sup_{t \geq 0, x \neq y, m \in \mathcal{P}_2(\mathbb{R}^d)} \frac{|a(t, x, m) - a(t, y, m)|}{|x - y|^\eta} < \infty, \quad \sup_{t \geq 0, x \neq y, m \in \mathcal{P}_2(\mathbb{R}^d)} \frac{|b(t, x, m) - b(t, y, m)|}{|x - y|^\eta} < \infty.$$

(ii) For any  $(i, j) \in \{1, \dots, d\}^2$ , for any  $(t, x, y) \in \mathbb{R}_+ \times (\mathbb{R}^d)^2$ , the map  $m \mapsto a_{i,j}(t, x, m)$  has a bounded continuous linear functional derivative, such that  $(x, y) \mapsto \frac{\delta}{\delta m} a_{i,j}(t, x, m)(y)$  is a bounded and  $\eta$ -Hölder continuous function, for some  $\eta \in (0, 1]$  uniformly with respect to the other variables. The map  $m \mapsto \frac{\delta}{\delta m} a_{i,j}(t, x, m)(y)$  has a bounded continuous linear functional derivative, such that  $(x, y') \mapsto \frac{\delta^2}{\delta m^2} a_{i,j}(t, x, m)(y, y')$  is  $\eta$ -Hölder continuous uniformly with respect to the other variables.

(iii) For any  $i \in \{1, \dots, d\}$ , for any  $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$ , the map  $m \mapsto b_i(t, x, m)$  has a bounded continuous linear functional derivative, such that  $y \mapsto \frac{\delta}{\delta m} b_i(t, x, m)(y)$  is  $\eta$ -Hölder continuous uniformly with respect to the other variables. Moreover, for any  $i \in \{1, \dots, d\}$ , for any  $(t, x, y) \in \mathbb{R}_+ \times (\mathbb{R}^d)^2$ , the map  $m \mapsto \frac{\delta}{\delta m} b_i(t, x, m)(y)$  has a bounded continuous linear functional derivative, such that  $y' \mapsto \frac{\delta^2}{\delta m^2} b_i(t, x, m)(y, y')$  is  $\eta$ -Hölder continuous uniformly with respect to the other variables.

**(HE)** The diffusion coefficient is uniformly elliptic, that is, there exists  $\lambda \geq 1$  such that for every  $(t, m) \in [0, \infty) \times \mathcal{P}_2(\mathbb{R}^d)$  and  $(x, \xi) \in (\mathbb{R}^d)^2$ ,  $\lambda^{-1} |\xi|^2 \leq \langle a(t, x, m) \xi, \xi \rangle \leq \lambda |\xi|^2$  where  $a(t, x, m) = (\sigma \sigma^*)(t, x, m)$ .

Under **(HR)** and **(HE)**, the martingale problem associated with (1.1) is well-posed for any initial distribution  $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ . In particular, weak existence and uniqueness in law holds for the SDE (1.1). We refer to [CdRF18] for a proof using a fixed point argument in a suitable complete metric space under weaker assumptions, notably concerning the regularity in the measure argument. The law of the process  $(X_t^{s,\xi})_{t \geq s}$  given by the unique solution to the SDE (1.1) starting from the initial distribution  $\mu = [\xi]$  at time  $s$  thus only depends upon  $\xi$  through its law  $\mu$ . Given  $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ , it thus makes sense to consider  $([X_t^{s,\xi}])_{t \geq s}$  as a function of the initial distribution  $\mu$  (and of the time variable  $s$ ) without specifying the choice of the lifted random variable  $\xi$  that has  $\mu$  as distribution. We then introduce, for any  $x \in \mathbb{R}^d$ , the following *decoupled stochastic flow* associated to the SDE (1.1)

$$(3.1) \quad X_t^{s,x,\mu} = x + \int_s^t b(r, X_r^{s,x,\mu}, [X_r^{s,\xi}]) dr + \int_s^t \sigma(r, X_r^{s,x,\mu}, [X_r^{s,\xi}]) dW_r.$$

We note that the previous equation is not a McKean-Vlasov SDE since the law appearing in the coefficients is not  $[X_r^{s,x,\mu}]$  but rather  $[X_r^{s,\xi}]$ , that is, the law of the solution to the SDE (1.1) (starting at time  $s$  from the initial distribution  $\mu$ ) at time  $r$ . Under **(HR)**(i) and **(HE)**, the time-inhomogeneous martingale problem associated to the SDE (3.1) is well-posed, see e.g. Stroock and Varadhan [SV79]. In particular, weak existence and uniqueness in law holds for the SDE (3.1).

Moreover, from Friedman [Fri64], it follows that the transition density of the SDE (3.1) exists<sup>1</sup>. In particular, the random variable  $X_t^{s,x,\mu}$  has a density that we denote by  $z \mapsto p(\mu, s, t, x, z)$  which admits a representation in infinite series by means of the parametrix method that we now briefly describe. We refer the reader to [Fri64] for a more complete exposition of the method. Let us first consider the approximation process  $(\widehat{X}_{t_2}^{t_1,x,\mu})_{t_2 \geq t_1}$  obtained from the dynamics (3.1) by removing the drift and freezing the diffusion coefficient in space at a fixed point  $y \in \mathbb{R}^d$

$$(3.2) \quad \widehat{X}_{t_2}^{t_1,x,\mu} = x + \int_{t_1}^{t_2} \sigma(r, y, [X_r^{s,\xi}]) dW_r.$$

The above dynamics is a simple Gaussian process with transition density

$$\widehat{p}^y(\mu, s, t_1, t_2, x, z) := g \left( \int_{t_1}^{t_2} a(r, y, [X_r^{s,\xi}]) dr, z - x \right).$$

To make the notation simpler, we will write  $\widehat{p}(\mu, s, t_1, t_2, x, y) := \widehat{p}^y(\mu, s, t_1, t_2, x, y)$  and  $\widehat{p}^y(\mu, s, t_2, x, z) = \widehat{p}^y(\mu, s, s, t_2, x, z)$ . Note importantly that the variable  $y$  acts twice since it appears as a terminal point where the density is evaluated and also as the point where the diffusion coefficient is frozen. Note also that in what follows we need to separate between the starting time  $t_1$  of the approximation process and the starting time  $s$  of the original McKean-Vlasov dynamics. We also introduce the infinitesimal generators associated to the dynamics (3.1) and (3.2)

$$\begin{aligned} L_{s,t}f(\mu, t, x) &= \sum_{i=1}^d b_i(t, x, [X_t^{s,\xi}]) \partial_{x_i} f(\mu, t, x) + \frac{1}{2} \sum_{i,j=1}^d a_{i,j}(t, x, [X_t^{s,\xi}]) \partial_{x_i x_j}^2 f(\mu, t, x), \\ \widehat{L}_{s,t}f(\mu, t, x) &= \frac{1}{2} \sum_{i,j=1}^d a_{i,j}(t, y, [X_t^{s,\xi}]) \partial_{x_i x_j}^2 f(\mu, t, x) \end{aligned}$$

and define the parametrix kernel  $\mathcal{H}$  for  $(\mu, r, x, y) \in \mathcal{P}_2(\mathbb{R}^d) \times [s, t] \times (\mathbb{R}^d)^2$

$$\begin{aligned} \mathcal{H}(\mu, s, r, t, x, y) &:= (L_{s,r} - \widehat{L}_{s,r}) \widehat{p}(\mu, s, r, t, x, y) \\ &= \sum_{i=1}^d b_i(r, x, [X_r^{s,\xi}]) \partial_{x_i} \widehat{p}(\mu, s, r, t, x, y) \\ &\quad + \frac{1}{2} \sum_{i,j=1}^d (a_{i,j}(r, x, [X_r^{s,\xi}]) - a_{i,j}(r, y, [X_r^{s,\xi}])) \partial_{x_i x_j}^2 \widehat{p}(\mu, s, r, t, x, y). \end{aligned}$$

<sup>1</sup>In [Fri64], it is proved that if  $x \mapsto \bar{b}(r, x) = b(r, x, [X_r^{s,\xi}])$  is bounded and Hölder-continuous then the fundamental solution associated to the infinitesimal generator of (3.1) exists and is unique by means of the parametrix method. However, existence of the transition density as well as weak existence and weak uniqueness can be derived under the sole assumption that the drift is bounded and measurable and the diffusion matrix is uniformly elliptic and Hölder continuous with respect to the space variable uniformly in time.

Now we define the following space-time convolution operator

$$(f \otimes g)(\mu, s, r, t, x, y) := \int_r^t \int_{\mathbb{R}^d} f(\mu, s, r, v, x, z) g(\mu, s, v, t, z, y) dz dv$$

and to simplify the notation we will write  $(f \otimes g)(\mu, s, t, x, y) := (f \otimes g)(\mu, s, s, t, x, y)$ ,  $\mathcal{H}(\mu, s, t, x, z) = \mathcal{H}(\mu, s, s, t, x, z)$  and proceed similarly for other maps. We also define  $f \otimes \mathcal{H}^{(k)} = (f \otimes \mathcal{H}^{(k-1)}) \otimes \mathcal{H}$  for  $k \geq 1$  with the convention that  $f \otimes \mathcal{H}^{(0)} \equiv f$ . With these notations, the following parametrix expansion in infinite series of the transition  $p(\mu, s, t, x, z)$  holds. Let  $T > 0$ . For any  $0 \leq s < t \leq T$  and any  $(\mu, x, y) \in \mathcal{P}_2(\mathbb{R}^d) \times (\mathbb{R}^d)^2$

$$(3.3) \quad p(\mu, s, t, x, y) = \sum_{k \geq 0} (\widehat{p} \otimes \mathcal{H}^{(k)})(\mu, s, t, x, y).$$

Moreover, the above infinite series converge absolutely and uniformly for  $(\mu, x, y) \in \mathcal{P}_2(\mathbb{R}^d) \times (\mathbb{R}^d)^2$  and satisfies the following Gaussian upper-bound: for any  $0 \leq s < t \leq T$  and any  $(\mu, x, y) \in \mathcal{P}_2(\mathbb{R}^d) \times (\mathbb{R}^d)^2$

$$(3.4) \quad |\partial_x^n p(\mu, s, t, x, y)| \leq \frac{C}{(t-s)^{\frac{n}{2}}} g(c(t-s), y-x), \quad n = 0, 1, 2$$

and

$$(3.5)$$

$$\forall \beta \in [0, \eta), \quad |\partial_x^2 p(\mu, s, t, x_1, y) - \partial_x^2 p(\mu, s, t, x_2, y)| \leq C \frac{|x_1 - x_2|^\beta}{(t-s)^{1+\frac{\beta}{2}}} \left[ g(c(t-s), y-x_1) + g(c(t-s), y-x_2) \right]$$

where  $C := C(T, \lambda, \eta)$  and  $c := c(\lambda)$  are two positive constants.

A similar representation in infinite series is also valid for the density of the random variable  $X_t^{s, \mu}$ , denoted by  $z \mapsto p(\mu, s, t, z)$ . Under our assumption, it satisfies the following key relation

$$(3.6) \quad p(\mu, s, t, z) = \int_{\mathbb{R}^d} p(\mu, s, t, x, z) \mu(dx).$$

The representation in infinite series of  $p(\mu, s, t, z)$  is thus obtained by integrating  $x \mapsto p(\mu, s, t, x, z)$  against the initial distribution  $\mu$ . The following result is established in [CdRF18].

**Theorem 3.1.** Assume that **(HE)** and **(HR)** hold. Then, the mapping  $[0, t) \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \ni (s, x, \mu) \mapsto p(\mu, s, t, x, z)$  is in  $C^{1,2,2}([0, t) \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$ . Moreover, for any fixed  $T > 0$ , there exist two positive constants  $C := C(b, a, [\delta b / \delta m], [\delta a / \delta m], T, \eta)$ ,  $c := c(\lambda)$ , such that for any  $(\mu, s, x, x', z, v, v') \in \mathcal{P}_2(\mathbb{R}^d) \times [0, t) \times (\mathbb{R}^d)^5$  and any  $0 \leq s < t \leq T$

$$(3.7) \quad |\partial_v^n [\partial_\mu p(\mu, s, t, x, z)](v)| \leq \frac{C}{(t-s)^{\frac{1+n-\eta}{2}}} g(c(t-s), z-x), \quad n = 0, 1,$$

$$(3.8) \quad |\partial_s p(\mu, s, t, x, z)| \leq \frac{C}{t-s} g(c(t-s), z-x),$$

$$(3.9) \quad \begin{aligned} & |\partial_v^n [\partial_\mu p(\mu, s, t, x, z)](v) - \partial_v^n [\partial_\mu p(\mu, s, t, x', z)](v)| \\ & \leq C \frac{|x - x'|^\beta}{(t-s)^{\frac{1+n+\beta-\eta}{2}}} \{g(c(t-s), z-x) + g(c(t-s), z-x')\}, \end{aligned}$$

where  $\beta \in [0, 1]$  for  $n = 0$  and  $\beta \in [0, \eta)$  for  $n = 1$ ,

$$(3.10)$$

$$\forall \beta \in [0, \eta), \quad |\partial_v [\partial_\mu p(\mu, s, t, x, z)](v) - \partial_v [\partial_\mu p(\mu, s, t, x, z)](v')| \leq C \frac{|v - v'|^\beta}{(t-s)^{1+\frac{\beta-\eta}{2}}} g(c(t-s), z-x),$$

There exist positive constants  $C := C(\mathbf{(HE)}, \mathbf{(HR)}, T)$ ,  $c := c(\lambda)$ , such that for any  $(\mu, \mu', s, x, z, v) \in (\mathcal{P}_2(\mathbb{R}^d))^2 \times [0, t) \times (\mathbb{R}^d)^3$ ,

$$(3.11) \quad |\partial_x^n p(\mu, s, t, x, z) - \partial_x^n p(\mu', s, t, x, z)](v)| \leq C \frac{W_2^\beta(\mu, \mu')}{(t-s)^{\frac{n+\beta-\eta}{2}}} g(c(t-s), z-x),$$

where  $\beta \in [0, 1]$  for  $n = 0, 1$  and  $\beta \in [0, \eta)$  for  $n = 2$ ,

$$(3.12) \quad |\partial_v^n [\partial_\mu p(\mu, s, t, x, z)](v) - \partial_v^n [\partial_\mu p(\mu', s, t, x, z)](v)| \leq C \frac{W_2^\beta(\mu, \mu')}{(t-s)^{\frac{1+n+\beta-\eta}{2}}} g(c(t-s), z-x)$$

where  $\beta \in [0, 1]$  for  $n = 0$  and  $\beta \in [0, \eta)$  for  $n = 1$  and for all  $(s_1, s_2) \in [0, t)$

$$(3.13) \quad \begin{aligned} & |\partial_x^n p(\mu, s_1, t, x, z) - \partial_x^n p(\mu, s_2, t, x, z)| \\ & \leq C \left\{ \frac{|s_1 - s_2|^\beta}{(t - s_1)^{\frac{\eta}{2} + \beta}} g(c(t - s_1), z - x) + \frac{|s_1 - s_2|^\beta}{(t - s_2)^{\frac{\eta}{2} + \beta}} g(c(t - s_2), z - x) \right\}, \end{aligned}$$

where  $\beta \in [0, 1]$  for  $n = 0$ ,  $\beta \in [0, \frac{1+\eta}{2})$  for  $n = 1$  and  $\beta \in [0, \frac{\eta}{2})$  for  $n = 2$  and

$$(3.14) \quad \begin{aligned} & |\partial_v^n [\partial_\mu p(\mu, s_1, t, x, z)](v) - \partial_v^n [\partial_\mu p(\mu, s_2, t, x, z)](v)| \\ & \leq C \left\{ \frac{|s_1 - s_2|^\beta}{(t - s_1)^{\frac{1+\eta-n}{2} + \beta}} g(c(t - s_1), z - x) + \frac{|s_1 - s_2|^\beta}{(t - s_2)^{\frac{1+\eta-n}{2} + \beta}} g(c(t - s_2), z - x) \right\}, \end{aligned}$$

where  $\beta \in [0, (1 + \eta)/2)$  for  $n = 0$  and  $\beta \in [0, \eta/2)$  for  $n = 1$ .

**3.2. Additional regularity of the transition density.** Our approach forces us to investigate additional regularity properties of the map  $(s, \mu) \mapsto p(\mu, s, t, z)$ . To be more specific, our aim is to establish that  $(s, \mu) \mapsto p(\mu, s, t, z)$  belongs to  $\mathcal{C}_f^{1,2}([0, t) \times \mathcal{P}_2(\mathbb{R}^d))$  under mild regularity assumptions on the coefficients. In view of the relation (3.6), it suffices to investigate the regularity of  $(s, x, \mu) \mapsto p(\mu, s, t, x, z)$ .

We start with the study of the Hölder regularity of the map  $\mathcal{P}_2(\mathbb{R}^d) \ni \mu \mapsto \partial_x [\partial_\mu p(\mu, s, t, x, z)](y) = \partial_\mu \partial_x p(\mu, s, t, x, z)(y)$ . The proof of the following lemma follows from similar arguments as those employed to obtain Theorem 3.1 in [CdRF18] and is thus postponed to the Appendix, Section 6.1.

**Lemma 3.1.** Under **(HE)** and **(HR)**, for all  $(\mu, s < t, z, y) \in \mathcal{P}_2(\mathbb{R}^d) \times [0, T]^2 \times (\mathbb{R}^d)^2$  the map  $\mathbb{R}^d \ni x \mapsto \partial_\mu p(\mu, s, t, x, z)(y)$  is continuously differentiable and there exist positive constants  $C := C((\mathbf{HE}), (\mathbf{HR}), T, \eta)$ ,  $c := c(\lambda)$ , such that for all  $(s, x, z, y) \in [0, t) \times (\mathbb{R}^d)^3$  and  $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ , one has

$$|\partial_x [\partial_\mu p(\mu, s, t, x, z)](y)| \leq \frac{C}{(t - s)^{1 - \frac{\eta}{2}}} g(c(t - s), z - x),$$

and for all  $\mu, \mu' \in \mathcal{P}_2(\mathbb{R}^d)$

$$|\partial_x [\partial_\mu p(\mu, s, t, x, z)](y) - \partial_x [\partial_{\mu'} p(\mu', s, t, x, z)](y)| \leq C \frac{W_2^\beta(\mu, \mu')}{(t - s)^{1 + \frac{\beta - \eta}{2}}} g(c(t - s), z - x),$$

for any  $\beta \in [0, \eta)$ .

We will also need the existence of the second derivative  $\partial_\mu^2 p(\mu, s, t, x, z)(v)$  together with the Hölder regularity of  $\mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \times (\mathbb{R}^d)^2 \ni (\mu, x, v) \mapsto \partial_\mu^2 p(\mu, s, t, x, z)(v)$  and some sharp Gaussian type estimates. In this regard, we introduce the following additional assumption on the coefficients:

**(HR<sub>+</sub>)** The coefficients  $b$  and  $\sigma$  satisfy **(HR)**. Moreover, for any  $(i, j) \in \{1, \dots, d\}^2$ , for any  $(t, x, y, y') \in \mathbb{R}_+ \times (\mathbb{R}^d)^3$ , the maps  $m \mapsto \frac{\delta^2}{\delta m^2} a_{i,j}(t, x, m)(y, y')$ ,  $\frac{\delta^2}{\delta m^2} b_i(t, x, m)(y, y')$  admit a bounded continuous linear functional derivative, such that  $(x, y'') \mapsto \frac{\delta^3}{\delta m^3} a_{i,j}(t, x, m)(y, y', y'')$  and  $y'' \mapsto \frac{\delta^3}{\delta m^3} b_i(t, x, m)(y, y', y'')$  are  $\eta$ -Hölder continuous uniformly with respect to the other variables.

The following result establishes the  $\mathcal{C}_f^{1,2,2}([0, t) \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$  regularity of the map  $(s, x, \mu) \mapsto p(\mu, s, t, x, z)$  which is a key step toward our quantitative estimates for propagation of chaos. Its proof is postponed to Section 4.2.

**Proposition 3.1.** Assume that **(HE)** and **(HR<sub>+</sub>)** hold. Then, the map  $(s, x, \mu) \mapsto p(\mu, s, t, x, z) \in \mathcal{C}_f^{1,2,2}([0, t) \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$ . In particular, for any fixed  $(s, x) \in [0, t) \times \mathbb{R}^d$ , the map  $\mu \mapsto p(\mu, s, t, x, z)$  is fully  $\mathcal{C}^2$ . Moreover, for any fixed  $T > 0$ , there exist two positive constants  $C := C((\mathbf{HE}), (\mathbf{HR}), T)$ ,  $c := c(\lambda)$ , such that for any  $(\mu, s, x, x', z, v, v') \in \mathcal{P}_2(\mathbb{R}^d) \times [0, t) \times (\mathbb{R}^d)^5$  and any  $t \in (0, T]$

$$(3.15) \quad |\partial_\mu^2 p(\mu, s, t, x, z)(v, v')| \leq \frac{C}{(t - s)^{1 - \frac{\eta}{2}}} g(c(t - s), z - x)$$

and

$$(3.16) \quad \begin{aligned} & |\partial_\mu^2 p(\mu, s, t, x, z)(v) - \partial_\mu^2 p(\mu', s, t, x', z)(v')| \\ & \leq \frac{K}{(t - s)^{1 + \frac{\beta - \eta}{2}}} [W_2^\beta(\mu, \mu') + |x - x'|^\beta + |v - v'|^\beta] \left\{ g(c(t - s), z - x) + g(c(t - s), z - x') \right\}, \end{aligned}$$

for any  $\beta \in [0, \eta)$  and for some positive constant  $K := K((\mathbf{HE}), (\mathbf{HR}_+), T)$ .

**3.3. Fundamental solution of the backward Kolmogorov PDE on the Wasserstein space.** A key feature of our analysis of convergence rate in the propagation of chaos phenomenon is to bring to light a connection between the transition density functions of the system of particles (1.2) and of its mean-field limit (1.1) by means of the notion of fundamental solution of the parabolic backward Kolmogorov PDE defined on the strip  $[0, T] \times \mathcal{P}_2(\mathbb{R}^d)$  that we now present.

Let us consider the following linear differential operator

$$(3.17) \quad \mathcal{L}_t U(\mu) = \int_{\mathbb{R}^d} \left\{ \sum_{i=1}^d b_i(t, v, \mu) [\partial_\mu U(\mu)(v)]_i + \frac{1}{2} \sum_{i,j} a_{i,j}(t, v, \mu) \partial_{v_i} [\partial_\mu U(\mu)(v)]_j \right\} \mu(dv), \quad t \in [0, T]$$

acting on a smooth real-valued function  $U$  defined on  $\mathcal{P}_2(\mathbb{R}^d)$ . The parabolic backward Kolmogorov PDE defined on the strip  $[0, T] \times \mathcal{P}_2(\mathbb{R}^d)$  is given by

$$(3.18) \quad \begin{cases} (\partial_t + \mathcal{L}_t)U(t, \mu) = 0 & \text{for } (t, \mu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d), \\ U(T, \mu) = h(\mu) & \text{for } \mu \in \mathcal{P}_2(\mathbb{R}^d). \end{cases}$$

Let us underline that under mild assumptions on the functions  $h$ ,  $b$  and  $a$ , the above PDE admits a unique classical solution given by  $U(t, \mu) = h([X_T^{t,\xi}])$ ,  $(X_s^{t,\xi})_{s \in [t, T]}$  being the unique weak solution to the SDE (1.1) starting from the initial distribution  $[\xi] = \mu$  at time  $t$ . We refer to [CdRF18] for irregular terminal condition  $h$  and coefficients  $b$  and  $a$ , in the uniformly elliptic setting. We also refer to [CM17] when the terminal condition  $h$  is irregular by means of Malliavin's calculus, in the uniformly elliptic setting. We finally mention the recent work [BLPR17] for the case of smooth functions  $h$ ,  $b$  and  $a$ . Let us now introduce the notion of fundamental solution related to (3.18).

**Definition 3.2.** A fundamental solution of  $(\partial_s + \mathcal{L}_s) = 0$  in  $[0, T] \times \mathcal{P}_2(\mathbb{R}^d)$  is a map  $[0, t] \times \mathcal{P}_2(\mathbb{R}^d) \ni (s, \mu) \mapsto p(\mu, s, t, z)$  defined for all  $(t, z) \in (0, T] \times \mathbb{R}^d$  satisfying the two following conditions:

- (i) For every fixed  $(t, z) \in (0, T] \times \mathbb{R}^d$ , the map  $[0, t] \times \mathcal{P}_2(\mathbb{R}^d) \ni (s, \mu) \mapsto p(\mu, s, t, z)$  belongs to  $\mathcal{C}^{1,2}([0, t] \times \mathcal{P}_2(\mathbb{R}^d))$  and satisfies the equation

$$(3.19) \quad (\partial_s + \mathcal{L}_s)p(\mu, s, t, z) = 0 \quad \text{on } [0, t] \times \mathcal{P}_2(\mathbb{R}^d).$$

- (ii) For every real-valued continuous function  $f$  defined on  $\mathbb{R}^d$  with at most quadratic growth, for any  $\mu \in \mathcal{P}_2(\mathbb{R}^d)$

$$(3.20) \quad \lim_{s \uparrow t} \int_{\mathbb{R}^d} f(z) p(\mu, s, t, z) dz = \int_{\mathbb{R}^d} f(z) \mu(dz).$$

When there is no possible confusion, we will write  $\lim_{s \uparrow t} p(\mu, s, t, z) = \delta_z(\cdot) \star \mu$  (“ $\star$ ” denoting the usual convolution operator), instead of (3.20).

**Theorem 3.3.** Assume that **(HE)** and **(HR)** hold. Let  $(t, z) \in (0, T] \times \mathbb{R}^d$ . The map  $[0, t] \times \mathcal{P}_2(\mathbb{R}^d) \ni (s, \mu) \mapsto p(\mu, s, t, z)$  defined by (3.6) and (3.3) is a fundamental solution of  $\partial_s + \mathcal{L}_s = 0$ .

Moreover, it is the unique solution among the class of fundamental solutions satisfying (2.4),  $T$  being replaced by any  $t' \in [0, t)$  and the terminal condition (3.20) locally uniformly in  $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ , that is, uniformly in  $\mu \in \mathcal{K}$ ,  $\mathcal{K}$  being any compact set of  $\mathcal{P}_2(\mathbb{R}^d)$ .

**3.4. Propagation of chaos for the system of particles (1.2).** Our primary objective is to study the propagation of chaos for the system of particles (1.2) by quantifying in an appropriate sense its distance from its mean field limit (1.1).

Let us first emphasize that under **(HE)** and **(HR)**, the system of particles with dynamics (1.2) is well posed in the weak sense. Indeed, for any  $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$  and  $\mathbf{x}, \mathbf{y} \in (\mathbb{R}^d)^N$ :

$$b_i(t, x, m_{\mathbf{x}}^N) - b_i(t, x, m_{\mathbf{y}}^N) = \int_0^1 \int_{\mathbb{R}^d} \frac{\delta}{\delta m} b_i(t, x, m^{\lambda, N})(z) (m_{\mathbf{x}}^N - m_{\mathbf{y}}^N)(dz) d\lambda$$

where we used the notations  $m_{\mathbf{x}}^N := N^{-1} \sum_{i=1}^N \delta_{x_i}$  and  $m^{\lambda, N} := \lambda m_{\mathbf{x}}^N + (1 - \lambda) m_{\mathbf{y}}^N$ . From the uniform  $\eta$ -Hölder regularity of  $z \mapsto \frac{\delta}{\delta m} b_i(t, x, m)(z)$ , it is thus seen readily

$$|b_i(t, x, m_{\mathbf{x}}^N) - b_i(t, x, m_{\mathbf{y}}^N)| \leq \sup_{t \in \mathbb{R}_+, x \in \mathbb{R}^d, \mu \in \mathcal{P}_2(\mathbb{R}^d)} \left[ b_i(t, x, \mu)(\cdot) \right]_H |\mathbf{x} - \mathbf{y}|^\eta$$

and the same inequality holds with the map  $a_{i,j}$  instead of  $b_i$ . As a consequence, the maps  $\mathbb{R}^d \times (\mathbb{R}^d)^N \ni (x, \mathbf{x}) \mapsto b(t, x, m_{\mathbf{x}}^N)$ ,  $a(t, x, m_{\mathbf{x}}^N)$  are bounded and  $\eta$ -Hölder continuous uniformly in time so that the martingale problem related to (1.2) is well posed, see e.g. [SV79]. In particular, weak existence and uniqueness holds for the SDE (1.2).

From [Fri64], the  $N \times d$  dimensional random variable  $\mathbf{X}_t = (X_t^1, \dots, X_t^N)$  given by the unique weak solution to (1.2) taken at time  $t$  starting from the  $N$ -fold product measure  $\mu^N$  admits a density function  $(\mathbb{R}^d)^N \ni \mathbf{z} \mapsto \mathbf{p}^N(\mu, 0, t, \mathbf{z})$ ,  $\mathbf{z} = (z_1, \dots, z_N)$ , with respect to the Lebesgue measure on  $(\mathbb{R}^d)^N$ . For any fixed  $i$  in  $\{1, \dots, N\}$ , we denote by  $p^{i,N}$  the density of the  $i^{\text{th}}$  particle obtained by integrating the joint density of the particles  $\mathbf{z} \mapsto \mathbf{p}^N(\mu, 0, t, \mathbf{z})$  over  $z_j$  for  $j \neq i$ . By weak uniqueness of the SDE (1.2) and exchangeability in law of the i.i.d. initial conditions  $(\xi^i)_{1 \leq i \leq N}$ , the one-dimensional marginal distributions of the random variable  $\mathbf{X}_t^N$  are equal. In particular, one has  $p^{i,N} \equiv p^{1,N}$  for any  $i \in \{1, \dots, N\}$ . Moreover, for any fixed time  $T > 0$ , there exists two constants  $\mathbf{C} := \mathbf{C}(T, a, b, N) > 1$ ,  $\mathbf{c} := \mathbf{c}(\lambda, N) > 0$  such that for any  $(t, \mu, \mathbf{z}) \in (0, T] \times \mathcal{P}_2(\mathbb{R}^d) \times (\mathbb{R}^d)^N$  the following two sided Gaussian estimate holds

$$(3.21) \quad \mathbf{C}^{-1} \int_{(\mathbb{R}^d)^N} g(\mathbf{c}^{-1}t, \mathbf{z} - \mathbf{x}) \mu^N(d\mathbf{x}) \leq \mathbf{p}^N(\mu, 0, t, \mathbf{z}) \leq \mathbf{C} \int_{(\mathbb{R}^d)^N} g(ct, \mathbf{z} - \mathbf{x}) \mu^N(d\mathbf{x}).$$

*Remark 3.4.* It is also readily seen that a similar bound hold for  $p^{1,N}$  instead of  $\mathbf{p}^N$  but with constant  $\mathbf{C}$ ,  $\mathbf{c}$  that depend on  $N$ . As a by product of our result, we will establish below a Gaussian upper-bound with two constants  $C$ ,  $c$  that do not depend on  $N$ . To the best of our knowledge, this result is new.

The first propagation of chaos estimate is an error bound of order  $N^{-1}$  for the difference  $(p^{1,N} - p)(\mu, 0, t, z)$  under **(HR)** and **(HE)**. We then establish a first order expansion for this difference under **(HR<sub>+</sub>)** together with an explicit control of the remainder term under additional assumptions. The proof of the following result is postponed to Section 5.

**Theorem 3.5.** Assume that **(HE)** and **(HR)** hold. Then, for all  $(t, \mu, z) \in (0, T] \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d$ ,

$$(3.22) \quad |(p^{1,N} - p)(\mu, 0, t, z)| \leq \frac{K}{N} \left\{ \frac{1}{t^{\frac{1-\eta}{2}}} \int_{\mathbb{R}^d} g(ct, z - x) |x| \mu(dx) + \frac{1}{t^{1-\frac{\eta}{2}}} \int_{\mathbb{R}^d} g(ct, z - x) \mu(dx) \right\}$$

where  $T \mapsto K := K(\mathbf{(HR)}, \mathbf{(HE)}, T, M_2(\mu))$  is a positive non-decreasing function and  $c := c(\lambda)$  is a positive constant.

Under the additional assumption **(HR<sub>+</sub>)**, for all  $(t, \mu, z) \in (0, T] \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d$ , the following first order expansion holds

$$(3.23) \quad \begin{aligned} \forall \alpha \in [0, \eta), \quad (p^{1,N} - p)(\mu, 0, t, z) &= \frac{1}{N} \mathbb{E} \left[ \frac{\delta}{\delta m} p(\mu, 0, t, \xi^1, z)(\xi^1) - \frac{\delta}{\delta m} p(\mu, 0, t, \xi^1, z)(\tilde{\xi}) \right] \\ &+ \frac{1}{2N} \mathbb{E} \left[ \frac{\delta^2}{\delta m^2} p(\mu, 0, t, \xi^1, z)(\tilde{\xi}, \tilde{\xi}) - \frac{\delta^2}{\delta m^2} p(\mu, 0, t, \xi^1, z)(\tilde{\xi}, \xi^2) \right] \\ &+ \frac{1}{N} \int_0^t \mathbb{E}[\mathcal{A}_s p(\mu_s, s, t, z)] ds + \frac{1}{N} \mathcal{R}_N(\mu, 0, t, z) \end{aligned}$$

where  $\tilde{\xi}$  is an  $\mathbb{R}^d$ -valued random variable independent of  $(\xi^i)_{1 \leq i \leq N}$  with law  $\mu$  and  $\mathcal{A}_s$  is the differential operator on  $\mathcal{P}_2(\mathbb{R}^d)$  defined by

$$\mathcal{A}_s U(\mu) = \frac{1}{2} \int_{\mathbb{R}^d} \text{Tr}(a(s, v, \mu) \partial_\mu^2 U(\mu)(v, v)) \mu(dv).$$

Moreover, under the assumption of Theorem 3.8, the following estimate on the remainder term holds:

$$\forall \beta \in [0, \eta), \quad \int_{\mathbb{R}^d} |\mathcal{R}_N(\mu, 0, t, z)| |z|^2 dz \leq \frac{K}{t^{1+\frac{\beta-\eta}{2}}} \varepsilon_N^{\frac{\beta}{2}}$$

for some positive constant  $K := K(\mathbf{(HE)}, \mathbf{(HR}_+), T, M_4(\mu))$  and where  $\varepsilon_N$  is defined by (3.29).

Inspired by the previous result as well as Remark 5.110 in [CD18], we now provide a kind of weak propagation of chaos estimate as well as an error estimate for the difference between the semigroup generated by the system of particles (1.2) and the semigroup associated to its mean-field limit both living on  $\mathcal{P}_2(\mathbb{R}^d)$ . Below, for all  $t \geq 0$ , we denote by  $\mu_t$  the law of  $X_t$ .

**Theorem 3.6.** Assume that **(HR)** and **(HE)** hold. Let  $\mathcal{C}_{b,\alpha}^2(\mathcal{P}_2(\mathbb{R}^d), \mathbb{R})$  be the set of functions  $\phi : \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$  that admit two bounded continuous linear functional derivatives with supremum norm less than 1 and with  $\alpha$ -Hölder continuous linear functional derivatives,  $\alpha$  in  $(0, 1]$ . Then there exists a positive constant  $C := C(T, \mathbf{(HE)}, \mathbf{(HR)}, \alpha)$  ( $T \mapsto C(\mathbf{(HE)}, \mathbf{(HR)}, T, \alpha)$  being non-decreasing) such that for all



$\phi \in \mathcal{C}_{b,\alpha}^2(\mathcal{P}_2(\mathbb{R}^d), \mathbb{R})$  we have

$$(3.24) \quad |\mathbb{E}[\phi(\mu_T^N)] - \phi(\mu_T)| \leq \frac{C}{T^{1-\frac{\alpha}{2}}} \frac{1}{N};$$

$$(3.25) \quad \mathbb{E}[|\phi(\mu_T^N) - \phi(\mu_T)|] \leq \frac{C}{T^{\frac{1-\alpha}{2}}} \begin{cases} N^{-1/2} & \text{if } d < 2, \\ N^{-1/2} \log(1+N) & \text{if } d = 2, \\ N^{-1/d} & \text{if } d > 2. \end{cases}$$

*Remark 3.7.* • Note that when  $\alpha = 1$ , we have that

$$\{\phi : \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}, \phi(\mu) = \int \varphi d\mu, \varphi \text{ is } 1 - \text{Lipschitz}\} \subset \mathcal{C}_{b,1}^2(\mathcal{P}_2(\mathbb{R}^d), \mathbb{R})$$

so that (3.25) implies the convergence in Wasserstein-1 distance by the Kantorovitch-Rubinstein duality theorem.

- From the proof of (3.24), it will be apparent that one could also obtain a first order expansion at the level of the semigroup, that is, an ad hoc version of (3.23). However, we refrain from going further in this direction here.

Our last objective is to prove that the system of particles (1.2) converges in the strong sense to the solution of the McKean-Vlasov SDE (1.1) by extending the classical result of propagation of chaos on the trajectories of the particles to our framework. As in the standard case, we shall quantify the convergence rate of propagation of chaos through a coupling argument with an auxiliary system of particles.

Under the additional assumption that  $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \ni (x, \mu) \mapsto \sigma(t, x, \mu)$  is Lipschitz continuous uniformly in time, from [Ver80], strong uniqueness holds for the system of particles (1.2) and from [CdRF18] the same conclusion holds for its mean-field limit (1.1). Hence, strong well-posedness for both SDEs follows from the Yamada-Watanabe theorem.

In the above framework, we thus choose a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  as well as  $N$  independent  $q$ -Brownian motion  $(W^i)_{1 \leq i \leq N}$  on it. We also assume that the probability space carries the i.i.d. sequence of  $\mathbb{R}^d$ -valued and  $\mathcal{F}_0$ -measurable random variables  $(\xi^i)_{1 \leq i \leq N}$  with common law  $\mu$  satisfying  $M_2(\mu) < \infty$ .

For any  $i \in \{1, \dots, N\}$ , we then introduce the process  $\bar{X}^i = (\bar{X}_t^i)_{0 \leq t \leq T}$  given by the unique strong solution to the McKean-Vlasov SDE (1.1) but with the input  $(\xi^i, W^i)_{1 \leq i \leq N}$  instead of  $(\xi, W)$

$$(3.26) \quad \bar{X}_t^i = \xi^i + \int_0^t b(s, \bar{X}_s^i, [\bar{X}_s^i]) ds + \int_0^t \sigma(s, \bar{X}_s^i, [\bar{X}_s^i]) dW_s^i, \quad i = 1, \dots, N.$$

By weak uniqueness for the SDE (1.1), the two processes  $\bar{X}^i$  and  $X$  have the same law, in particular  $[\bar{X}_t^i] = [X_t] = \mu_t$ , for any  $t \in [0, T]$  and for any  $i \in \{1, \dots, N\}$ . Our last result quantifies the propagation of chaos at level of the trajectories. Its proof is postponed to Section 5.

**Theorem 3.8.** Assume that for some  $q > 4$ ,  $M_q(\mu) < +\infty$ . Assume that the coefficients  $b$  and  $a$  are continuous, satisfy **(HR)** and **(HE)** and that the map  $\mathbb{R}^d \ni (x, \mu) \mapsto \sigma(t, x, \mu)$  is Lipschitz continuous, uniformly in time, with modulus  $[\sigma]_L$ . Then, there exists a positive constant  $C := C(\mathbf{(HE)}, \mathbf{(HR)}, T, [\sigma]_L, M_q(\mu))$  such that

$$(3.27) \quad \sup_{0 \leq t \leq T} \mathbb{E}[W_2^2(\mu_t, \mu_t^N)] + \max_{i=1, \dots, N} \sup_{0 \leq t \leq T} \mathbb{E}[|X_t^i - \bar{X}_t^i|^2] \leq C \varepsilon_N$$

and

$$(3.28) \quad \mathbb{E}\left[\sup_{0 \leq t \leq T} W_2^2(\mu_t, \mu_t^N)\right] + \max_{i=1, \dots, N} \mathbb{E}\left[\sup_{0 \leq t \leq T} |X_t^i - \bar{X}_t^i|^2\right] \leq C \sqrt{\varepsilon_N}$$

with

$$(3.29) \quad \varepsilon_N := \begin{cases} N^{-1/2} & \text{if } d < 4, \\ N^{-1/2} \log(1+N) & \text{if } d = 4, \\ N^{-2/d} & \text{if } d > 4. \end{cases}$$

*Remark 3.9.* The Zvonkin's transform applied in our framework shows that the rate of convergence provided in (3.28) is actually ruled by the quantity  $\mathbb{E}[\sup_{0 \leq t \leq T} W_2^2(\mu_t, \bar{\mu}_t^N)]$  where  $\bar{\mu}_t^N := \frac{1}{N} \sum_{i=1}^N \delta_{\bar{X}_t^i}$  which is in turn known to be of order  $\sqrt{\varepsilon_N}$ , see e.g. Briand et al. [BCCH19]. This last estimate could be improved under stronger integrability assumption on the initial distribution  $\mu$ . We also mention the fact that one could achieve a convergence rate of order  $\varepsilon_N$  under the additional assumption that the map  $\mu \mapsto b(t, x, \mu)$  is Lipschitz continuous uniformly with respect to the variables  $t$  and  $x$  but we do not engage into further reflections in that direction.

## 4. THE BACKWARD KOLMOGOROV EQUATION

This section is dedicated to the proofs of Theorem 3.3 and Proposition 3.1. Hence, we assume that **(HE)** and **(HR)** are in force in the subsection 4.1 and that **(HE)** and **(HR<sub>+</sub>)** are in force in the subsection 4.2.

**4.1. Proof of Theorem 3.3.** We fix  $T > 0$  and  $z \in \mathbb{R}^d$ . From (3.6) and (3.1), we already know that the map  $(s, \mu) \mapsto p(\mu, s, t, z)$  is in  $\mathcal{C}^{1,2}([0, t] \times \mathcal{P}_2(\mathbb{R}^d))$  with derivatives  $\partial_s p(\mu, s, t, z) = \int_{\mathbb{R}^d} \partial_s p(\mu, s, t, x, z) \mu(dx)$  and

$$(4.1) \quad \partial_v^n [\partial_\mu p(\mu, s, t, z)](v) = \partial_v^{1+n} p(\mu, s, t, v, z) + \int_{\mathbb{R}^d} \partial_v^n [\partial_\mu p(\mu, s, t, x, z)](v) \mu(dx), \quad n = 0, 1.$$

We now prove that it satisfies (3.19).

From the Markov property satisfied by the SDE (1.1), stemming from the well-posedness of the related martingale problem, see Theorem 3.4 in [CdRF18], the following relation is satisfied for all  $0 < h < s$

$$p(\mu, s - h, t, z) = p([X_s^{s-h, \xi}], s, t, z).$$

From the relation (4.1) and the estimates (3.7), we deduce that the condition (2.4) of the chain rule formula of Proposition 2.1 (with respect to the measure variable only) is satisfied so that

$$p([X_s^{s-h, \xi}], s, t, z) = p(\mu, s, t, z) + \int_{s-h}^s \mathcal{L}_r p([X_r^{s-h, \xi}], s, t, z) dr$$

which in turn yields

$$\frac{1}{h} (p(\mu, s - h, t, x, z) - p(\mu, s, t, x, z)) = \frac{1}{h} \int_{s-h}^s \mathcal{L}_r p([X_r^{s-h, \xi}], s, t, z) dr.$$

Letting  $h \downarrow 0$ , from the boundedness and the continuity of the coefficients as well as the continuity of the maps  $(\mu, v) \mapsto \partial_\mu p(\mu, s, t, z)(v)$ ,  $\partial_v [\partial_\mu p(\mu, s, t, z)](v)$  and the differentiability of  $[0, t] \ni s \mapsto p(\mu, s, t, x, z)$ , we get that  $(s, \mu) \mapsto p(\mu, s, t, z)$  satisfies (3.19).

We then consider a real-valued continuous function  $f$  defined on  $\mathbb{R}^d$  with at most quadratic growth. From (3.3), one gets

$$(4.2) \quad p(\mu, s, t, x, z) = \widehat{p}(\mu, s, t, x, z) + \mathcal{R}(\mu, s, t, x, z), \quad \mathcal{R}(\mu, s, t, x, z) := \sum_{k \geq 1} (\widehat{p} \otimes \mathcal{H}^{(k)})(\mu, s, t, x, z)$$

together with the following estimate

$$(4.3) \quad |\mathcal{R}(\mu, s, t, x, z)| \leq K(t-s)^{\frac{\eta}{2}} g(c(t-s), z-x)$$

for some positive constant  $K := K(T, a, b, \eta)$ . From the mean-value theorem, the  $\eta$ -Hölder continuity of  $x \mapsto a(t, x, \mu)$  and the space-time inequality (1.5), one has

$$(4.4) \quad |(\widehat{p}^z - \widehat{p}^x)(\mu, s, t, x, z)| \leq K|z-x|^\eta g(c(t-s), z-x) \leq K(t-s)^{\frac{\eta}{2}} g(c(t-s), z-x).$$

Hence, the key relation (3.6) yields

$$\begin{aligned} \int_{\mathbb{R}^d} f(z) p(\mu, s, t, z) dz - \int_{\mathbb{R}^d} f(x) \mu(dx) &= \int_{(\mathbb{R}^d)^2} [f(z) - f(x)] \widehat{p}^x(\mu, s, t, x, z) dz \mu(dx) \\ &\quad + \int_{(\mathbb{R}^d)^2} f(z) [\widehat{p}^z(\mu, s, t, x, z) - \widehat{p}^x(\mu, s, t, x, z)] dz \mu(dx) \\ &\quad + \int_{(\mathbb{R}^d)^2} f(z) \mathcal{R}(\mu, s, t, x, z) dz \mu(dx). \end{aligned}$$

Thanks to (4.3), (4.4), for any compact set  $\mathcal{K} \subset \mathcal{P}_2(\mathbb{R}^d)$ , we derive

$$\begin{aligned} \sup_{\mu \in \mathcal{K}} \left| \int_{(\mathbb{R}^d)^2} f(z) [\widehat{p}^z(\mu, s, t, x, z) - \widehat{p}^x(\mu, s, t, x, z)] dz \mu(dx) \right| \\ + \sup_{\mu \in \mathcal{K}} \left| \int_{(\mathbb{R}^d)^2} f(z) \mathcal{R}(\mu, s, t, x, z) dz \mu(dx) \right| \leq K(t-s)^{\frac{\eta}{2}} (1 + \sup_{\mu \in \mathcal{K}} M_2(\mu)). \end{aligned}$$

The uniform continuity of the map  $[0, t] \times \mathcal{K} \ni (s, \mu) \mapsto \int_{(\mathbb{R}^d)^2} [f(z) - f(x)] \widehat{p}^x(\mu, s, t, x, z) dz \mu(dx) = \int_{(\mathbb{R}^d)^2} [f(x + \Sigma_{s,t}^{1/2} z) - f(x)] e^{-\frac{|z|^2}{2}} (2\pi)^{-\frac{d}{2}} dz \mu(dx)$ , where  $\Sigma_{s,t}^{1/2}$  is the unique principal square root of the positive-semidefinite matrix  $\int_s^t a(r, x, [X_r^{s, \xi}]) dr$ , implies the equicontinuity of the family of maps

$\left\{ \int_{(\mathbb{R}^d)^2} [f(z) - f(x)] \widehat{p}^x(\mu, s, t, x, z) dz \mu(dx), \mu \in \mathcal{K} \right\}$  and the quadratic growth of  $f$  implies its boundedness. We thus deduce

$$\limsup_{s \uparrow t} \sup_{\mu \in \mathcal{K}} \left| \int_{(\mathbb{R}^d)^2} [f(z) - f(x)] \widehat{p}^x(\mu, s, t, x, z) dz \mu(dx) \right| = 0.$$

In order to get the uniqueness result, let us consider any solution  $(s, \mu) \mapsto q(\mu, s, t, z)$  to the backward Kolmogorov equation (3.19) satisfying (2.4) on any interval  $[0, t']$ , with  $t' < t$ , and (3.20) locally uniformly in  $\mu$ . We apply the chain rule formula of Proposition 2.1 to  $\{q([X_r^{s, \xi}], r, t, z), s \leq r < t\}$  and use the fact that  $(\partial_s + \mathcal{L}_s)q(\mu, s, t, z) = 0$ , for any  $(s, \mu) \in [0, t) \times \mathcal{P}_2(\mathbb{R}^d)$  to get

$$\forall r \in [s, t), \quad q([X_r^{s, \xi}], r, t, z) = q(\mu, s, t, z).$$

In order to pass to the limit as  $r \uparrow t$  in the previous relation, we remark that from (3.3) and (3.6),  $\lim_{r \uparrow t} W_2([X_r^{s, \xi}], [X_t^{s, \xi}]) = 0$  and using the local uniform convergence in  $\mu$  of  $r \mapsto \int_{\mathbb{R}^d} f(z) q(\mu, r, t, z) dz$  towards  $\int_{\mathbb{R}^d} f(x) \mu(dx)$  as  $r \uparrow t$ , we obtain

$$\int_{\mathbb{R}^d} f(z) q(\mu, s, t, z) dz = \lim_{r \uparrow t} \int_{\mathbb{R}^d} f(z) q([X_r^{s, \xi}], r, t, z) dz = \int_{\mathbb{R}^d} f(z) [X_t^{s, \xi}](dz) = \int_{\mathbb{R}^d} f(z) p(\mu, s, t, z) dz.$$

for any continuous function  $f$  with at most quadratic growth. We thus deduce that  $q(\mu, s, t, z) = p(\mu, s, t, z)$  which completes the proof of Theorem 3.3.

**4.2. Proof of Proposition 3.1.** The proof of Proposition 3.1 relies on similar arguments as those employed to prove Theorem 3.1 in [CdRF18]. To be more specific, our strategy is based on an approximation argument of the transition density  $p(\mu, s, t, x, z)$  by a Picard iteration scheme and sharp uniform estimates on its derivatives from which we can extract a uniformly convergent subsequence by using Arzela-Ascoli's theorem.

*Step 1: Construction of an approximation sequence and related estimates*

For a given initial condition  $(s, \mu) \in \mathbb{R}_+ \times \mathcal{P}_2(\mathbb{R}^d)$  and a probability measure  $\nu \in \mathcal{P}_2(\mathbb{R}^d)$ ,  $\nu \neq \mu$ , we let  $\mathbb{P}^{(0)} = (\mathbb{P}^{(0)}(t))_{t \geq s}$  be the probability measure on  $\mathcal{C}([s, \infty), \mathbb{R}^d)$ , endowed with its canonical filtration, satisfying  $\mathbb{P}^{(0)}(t) = \nu$ ,  $t \geq s$ . Let us consider the following recursive sequence of probability measures  $\{\mathbb{P}^{(m)}; m \geq 0\}$ , with time marginals  $(\mathbb{P}^{(m)}(t))_{t \geq s}$ , where,  $\mathbb{P}^{(m)}$  being given,  $\mathbb{P}^{(m+1)}$  is the unique solution to the following martingale problem

- (i)  $\mathbb{P}^{(m+1)}(y(r) \in \Gamma; 0 \leq r \leq s) = \mu(\Gamma)$ , for all  $\Gamma \in \mathcal{B}(\mathbb{R}^d)$ .
- (ii) For all  $f \in \mathcal{C}_b^2(\mathbb{R}^d)$ ,

$$f(y_t) - f(y_s) - \int_s^t \left\{ \sum_{i=1}^d b_i(r, y_r, \mathbb{P}^{(m)}(r)) \partial_i f(y_r) + \sum_{i,j=1}^d \frac{1}{2} a_{i,j}(r, y_r, \mathbb{P}^{(m)}(r)) \partial_{i,j}^2 f(y_r) \right\} dr$$

is a continuous square-integrable martingale under  $\mathbb{P}^{(m+1)}$ .

Note that, under the considered assumptions, the well-posedness of the above standard martingale problem follows from classical results, see e.g. [SV79]. In particular, there exists a unique weak solution to the SDE with dynamics

$$(4.5) \quad X_t^{s, \xi, (m+1)} = \xi + \int_s^t b(r, X_r^{s, \xi, (m+1)}, [X_r^{s, \xi, (m)}]) dr + \int_s^t \sigma(r, X_r^{s, \xi, (m+1)}, [X_r^{s, \xi, (m)}]) dW_r.$$

We will also work with the decoupled stochastic flow or characteristics given by the unique weak solution to the SDE with dynamics

$$(4.6) \quad X_t^{s, x, \mu, (m+1)} = x + \int_s^t b(r, X_r^{s, x, \mu, (m+1)}, [X_r^{s, \xi, (m)}]) dr + \int_s^t \sigma(r, X_r^{s, x, \mu, (m+1)}, [X_r^{s, \xi, (m)}]) dW_r.$$

We point out that the notation  $X_t^{s, x, \mu, (m+1)}$  makes sense since by weak uniqueness of solution to the SDE (4.5), the law  $[X_t^{s, \xi, (m)}]$  only depends on the initial condition  $\xi$  through its law  $\mu$ .

From [Fri64], for any  $m \geq 0$ , the two random variables  $X_t^{s, \xi, (m)}$  and  $X_t^{s, x, \mu, (m)}$  admit a density respectively denoted by  $p_m(\mu, s, t, z)$  and  $p_m(\mu, s, t, x, z)$ . Moreover, the following relation is satisfied

$$(4.7) \quad \forall z \in \mathbb{R}^d, \quad p_m(\mu, s, t, z) = \int p_m(\mu, s, t, x, z) \mu(dx)$$

where for all  $m \geq 1$

$$(4.8) \quad p_m(\mu, s, t, x, z) = \sum_{k \geq 0} (\widehat{p}_m \otimes \mathcal{H}_m^{(k)})(\mu, s, t, x, z),$$

with

$$\begin{aligned} \widehat{p}_m(\mu, s, r, t, x, z) &= g \left( \int_r^t a(v, z, [X_v^{s, \xi, (m-1)}]) dv, z - x \right), \\ \mathcal{H}_m(\mu, s, r, t, x, z) &= \left\{ - \sum_{i=1}^d b_i(r, x, [X_r^{s, \xi, (m-1)}]) H_1^i \left( \int_r^t a(v, z, [X_v^{s, \xi, (m-1)}]) dv, z - x \right) \right. \\ &\quad \left. + \frac{1}{2} \left( a_{i,j}(r, x, [X_r^{s, \xi, (m-1)}]) - a_{i,j}(r, z, [X_r^{s, \xi, (m-1)}]) \right) \right. \\ &\quad \left. \times H_2^{i,j} \left( \int_r^t a(v, z, [X_v^{s, \xi, (m-1)}]) dv, z - x \right) \right\} \widehat{p}_m(\mu, s, r, t, x, z) \end{aligned}$$

and  $\mathcal{H}_m^{(k+1)}(\mu, s, t, x, z) = (\mathcal{H}_m^{(k)} \otimes \mathcal{H}_m)(\mu, s, t, x, z)$ ,  $\mathcal{H}_m^{(0)} = I_d$ , with the convention that  $[X_t^{s, \xi, (0)}] = \mathbb{P}^{(0)}(t) = \nu$ ,  $t \geq 0$ . In what follows, we will often make use of the following estimates: there exist constant  $c := c(\lambda) > 1$ ,  $C := C(T, a, b, \eta) > 0$ , such that for all  $0 \leq s < t \leq T$ , for all integer  $k$ , one has

$$(4.9) \quad \forall (x, z) \in (\mathbb{R}^d)^2, \quad |\widehat{p}_m \otimes \mathcal{H}_m^{(k)}(\mu, s, t, x, z)| \leq C^k (t-s)^{k \frac{\eta}{2}} \prod_{i=1}^k B \left( 1 + \frac{(i-1)\eta}{2}, \frac{\eta}{2} \right) g(c(t-s), z-x)$$

where  $B(k, \ell) = \int_0^1 (1-v)^{-1+k} v^{-1+\ell} dv$  stands for the Beta function. As a consequence, from the asymptotics of the Beta function, the series (4.8) converge absolutely and uniformly for  $(\mu, x, z) \in \mathcal{P}_2(\mathbb{R}^d) \times (\mathbb{R}^d)^2$  and satisfies: for all  $m \geq 1$ , for any  $0 \leq s < t \leq T$  and any  $(\mu, x, z) \in \mathcal{P}_2(\mathbb{R}^d) \times (\mathbb{R}^d)^2$

$$(4.10) \quad |\partial_x^n p_m(\mu, s, t, x, z)| \leq C (t-s)^{-\frac{n}{2}} g(c(t-s), z-x), \quad n = 0, 1, 2,$$

where  $C := C(T, b, a, \eta)$  and  $c := c(\lambda)$  are two positive constants and for all  $(x, x') \in (\mathbb{R}^d)^2$ , for any  $m \geq 1$ ,

$$(4.11) \quad \begin{aligned} &|\partial_x^n p_m(\mu, s, r, x, z) - \partial_x^n p_m(\mu, s, r, x', z)| \\ &\leq C \frac{|x - x'|^\beta}{(r-s)^{\frac{n+\beta}{2}}} \{g(c(r-s), z-x) + g(c(r-s), z-x')\}, \quad n = 0, 1, 2, \end{aligned}$$

where  $\beta \in [0, 1]$  if  $n = 0, 1$  and  $\beta \in [0, \eta]$  if  $n = 2$ . We refer to Friedman [Fri64] for a proof of the above estimate.

Denote by  $\Phi_m(\mu, s, r, t, x_1, x_2)$  the solution to the Volterra integral equation

$$(4.12) \quad \Phi_m(\mu, s, r, t, x_1, x_2) = \mathcal{H}_m(\mu, s, r, t, x_1, x_2) + (\mathcal{H}_m \otimes \Phi_m)(\mu, s, r, t, x_1, x_2).$$

From the space-time inequality (1.5), it is easily seen that the singular kernel  $\mathcal{H}_m(\mu, s, r, t, x_1, x_2)$  induces an integrable singularity in time in the above space-time convolution so that the unique solution to the above equation is given by the (uniform) convergent series

$$(4.13) \quad \Phi_m(\mu, s, r, t, x_1, x_2) = \sum_{k \geq 1} \mathcal{H}_m^{(k)}(\mu, s, r, t, x_1, x_2)$$

and (4.8) now writes

$$(4.14) \quad p_m(\mu, s, t, x, z) = \widehat{p}_m(\mu, s, t, x, z) + \int_s^t \int_{\mathbb{R}^d} \widehat{p}_m(\mu, s, r, x, y) \Phi_m(\mu, s, r, t, y, z) dy dr.$$

Finally, from Theorem 7, Chapter 1 in [Fri64], for any  $m \geq 1$ , the map  $x \mapsto \Phi_m(\mu, s, r, t, x, z)$  is Hölder-continuous. More precisely, for any  $\beta \in [0, \eta]$ , there exist two positive constants  $C := C(T, a, b, \eta, \lambda)$ ,  $c(\lambda)$ , thus do not depending on  $m$ , such that

$$(4.15) \quad \begin{aligned} &|\Phi_m(\mu, s, r, t, x, z) - \Phi_m(\mu, s, r, t, y, z)| \\ &\leq C \frac{|x - y|^\beta}{(t-r)^{1+\frac{\beta-\eta}{2}}} \{g(c(t-r), z-x) + g(c(t-r), z-y)\}. \end{aligned}$$

We also recall from [CdRF18] the following properties. Let  $T > 0$ . For any fixed  $(t, z) \in (0, T] \times \mathbb{R}^d$ , for all  $m \geq 1$ :

- The mapping  $[0, t] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \ni (s, x, \mu) \mapsto p_m(\mu, s, t, x, z)$  is in  $\mathcal{C}^{1,2,2}([0, t] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$ .

- There exist two positive constants  $C := C(T, b, a, \frac{\delta b}{\delta m}, \frac{\delta a}{\delta m}, \eta)$ ,  $c := c(\lambda)$ , thus do not depending on  $m$ , such that for any  $(\mu, s, x, x', z, v, v') \in \mathcal{P}_2(\mathbb{R}^d) \times [0, t) \times (\mathbb{R}^d)^5$ ,

$$(4.16) \quad |\partial_v^n [\partial_\mu p_m(\mu, s, t, x, z)](v)| \leq \frac{C}{(t-s)^{\frac{1+n-\eta}{2}}} g(c(t-s), z-x), \quad n = 0, 1,$$

$$(4.17) \quad |\partial_s p_m(\mu, s, t, x, z)| \leq \frac{C}{t-s} g(c(t-s), z-x),$$

$$(4.18) \quad \begin{aligned} & |\partial_v^n [\partial_\mu p_m(\mu, s, t, x, z)](v) - \partial_v^n [\partial_\mu p_m(\mu, s, t, x', z)](v)| \\ & \leq C \frac{|x-x'|^\beta}{(t-s)^{\frac{1+n+\beta-\eta}{2}}} \{g(c(t-s), z-x) + g(c(t-s), z-x')\}, \end{aligned}$$

where  $\beta \in [0, 1]$  for  $n = 0$  and  $\beta \in [0, \eta]$  for  $n = 1$ ,

$$(4.19) \quad \begin{aligned} & \forall \beta \in [0, \eta), \quad |\partial_v [\partial_\mu p_m(\mu, s, t, x, z)](v) - \partial_v [\partial_\mu p_m(\mu, s, t, x, z)](v')| \\ & \leq C \frac{|v-v'|^\beta}{(t-s)^{1+\frac{\beta-\eta}{2}}} g(c(t-s), z-x). \end{aligned}$$

- There exist  $C := C((\mathbf{HE}), (\mathbf{HR}), T) > 0$ ,  $c := c(\lambda) > 0$ , thus do not depending on  $m$ , such that for any  $(\mu, \mu', s, x, z, v, v') \in (\mathcal{P}_2(\mathbb{R}^d))^2 \times [0, t) \times (\mathbb{R}^d)^4$ ,

$$(4.20) \quad |\partial_x^n p_m(\mu, s, t, x, z) - \partial_x^n p_m(\mu', s, t, x, z)| \leq C \frac{W_2^\beta(\mu, \mu')}{(t-s)^{\frac{n+\beta-\eta}{2}}} g(c(t-s), z-x),$$

where  $\beta \in [0, 1]$  for  $n = 0, 1$  and  $\beta \in [0, \eta]$  for  $n = 2$ ,

$$(4.21) \quad \begin{aligned} & |\partial_v^n [\partial_\mu p_m(\mu, s, t, x, z)](v) - \partial_v^n [\partial_\mu p_m(\mu', s, t, x, z)](v)| \\ & \leq C \frac{W_2^\beta(\mu, \mu')}{(t-s)^{\frac{1+n+\beta-\eta}{2}}} g(c(t-s), z-x), \end{aligned}$$

where  $\beta \in [0, 1]$  for  $n = 0$  and  $\beta \in [0, \eta]$  for  $n = 1$ , and for all  $(s_1, s_2) \in [0, t)^2$ ,

$$(4.22) \quad \begin{aligned} & |\partial_x^n p_m(\mu, s_1, t, x, z) - \partial_x^n p_m(\mu, s_2, t, x, z)| \\ & \leq C \left\{ \frac{|s_1 - s_2|^\beta}{(t-s_1)^{\frac{n}{2}+\beta}} g(c(t-s_1), z-x) + \frac{|s_1 - s_2|^\beta}{(t-s_2)^{\frac{n}{2}+\beta}} g(c(t-s_2), z-x) \right\}, \end{aligned}$$

where  $\beta \in [0, 1]$  for  $n = 0$ ,  $\beta \in [0, \frac{1+\eta}{2}]$  for  $n = 1$  and  $\beta \in [0, \frac{\eta}{2}]$  for  $n = 2$  and

$$(4.23) \quad \begin{aligned} & |\partial_v^n [\partial_\mu p_m(\mu, s_1, t, x, z)](v) - \partial_v^n [\partial_\mu p_m(\mu, s_2, t, x, z)](v)| \\ & \leq C \left\{ \frac{|s_1 - s_2|^\beta}{(t-s_1)^{\frac{1+n-\eta}{2}+\beta}} g(c(t-s_1), z-x) + \frac{|s_1 - s_2|^\beta}{(t-s_2)^{\frac{1+n-\eta}{2}+\beta}} g(c(t-s_2), z-x) \right\}, \end{aligned}$$

where  $\beta \in [0, \frac{\eta}{2})$ .

With the above notations and properties at hand, we can now state the following key proposition whose proof is postponed to the subsection 6.2 of the Appendix.

**Proposition 4.1.** Let  $T > 0$ . Assume that **(HE)** and **(HR)** hold. Then, for any fixed  $(t, z) \in (0, T] \times \mathbb{R}^d$ , for all  $m \geq 1$ , the following properties hold:

- The mapping  $[0, t) \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \ni (s, x, \mu) \mapsto p_m(\mu, s, t, x, z)$  is in  $\mathcal{C}_f^{1,2,2}([0, t) \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$ .
- There exist two positive constants  $C := C((\mathbf{HE}), (\mathbf{HR}), T)$ ,  $c := c(\lambda)$ , thus do not depending on  $m$ , such that for any  $(\mu, s, x, x', z, v) \in \mathcal{P}_2(\mathbb{R}^d) \times [0, t) \times (\mathbb{R}^d)^5$ ,

$$(4.24) \quad |\partial_\mu^2 p_m(\mu, s, t, x, z)(v)| \leq \frac{C}{(t-s)^{1-\frac{\eta}{2}}} g(c(t-s), z-x).$$

- If additionally **(HR<sub>+</sub>)** is satisfied, then there exist  $C := C((\mathbf{HE}), (\mathbf{HR}_+), T)$ ,  $c := c(\lambda) > 0$ , thus do not depending on  $m$ , such that for any  $(\mu, s, x, x', z, v) \in \mathcal{P}_2(\mathbb{R}^d) \times [0, t) \times (\mathbb{R}^d)^5$ ,

$$(4.25) \quad |\partial_\mu^2 p_m(\mu, s, t, x, z)(v) - \partial_\mu^2 p_m(\mu', s, t, x, z)(v)| \leq C \frac{W_2^\beta(\mu, \mu')}{(t-s)^{1+\frac{\beta-\eta}{2}}} g(c(t-s), z-x),$$

for all  $\beta \in [0, \eta)$ ,

$$(4.26) \quad \begin{aligned} & |\partial_\mu^2 p_m(\mu, s, t, x, z)(v) - \partial_\mu^2 p_m(\mu, s, t, x', z)(v')| \\ & \leq C \frac{[|x-x'|^\beta + |v-v'|^\beta]}{(t-s)^{1+\frac{\beta-\eta}{2}}} \{g(c(t-s), z-x) + g(c(t-s), z-x')\}, \end{aligned}$$

for all  $\beta \in [0, \eta)$ , for all  $x, x' \in \mathbb{R}^d$ , for all  $v, v' \in (\mathbb{R}^d)^2$  and

$$(4.27) \quad \begin{aligned} & |\partial_\mu^2 p_m(\mu, s_1, t, x, z)(v_1, v_2) - \partial_\mu^2 p_m(\mu, s_2, t, x, z)(v_1, v_2)| \\ & \leq C \left\{ \frac{|s_1 - s_2|^\beta}{(t-s_1)^{1-\frac{\eta}{2}+\beta}} g(c(t-s_1), z-x) + \frac{|s_1 - s_2|^\beta}{(t-s_2)^{1-\frac{\eta}{2}+\beta}} g(c(t-s_2), z-x) \right\}, \end{aligned}$$

where  $\beta \in [0, \eta/2)$ .

The proof of the above result is postponed to the Appendix, subsection 6.2.

*Step 2: Extraction of a convergent subsequence*

Our next step now is to extract from the sequences  $\{\mathbb{L}^2 \ni \xi \mapsto \tilde{p}_m(\xi, s, t, x, z), m \geq 0\}$  (the lifting of  $\mu \mapsto p_m(\mu, s, t, x, z)$ ),  $\{\mathbb{L}^2 \ni \xi \mapsto \partial_\mu p_m(\mu, s, t, x, z)(\xi), m \geq 0\}$ ,  $\{[0, t] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times (\mathbb{R}^d)^2 \ni (s, x, \mu, v) \mapsto \partial_\mu^2 p_m(\mu, s, t, x, z)(v), m \geq 0\}$  the corresponding subsequences which converge locally uniformly using the Arzelà-Ascoli theorem.

Since the coefficients  $b_i, a_{i,j}$  are bounded and the initial condition  $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ , the sequence  $(\mathbb{P}^{(m)})_{m \geq 0}$  constructed in step 1 is tight. Relabelling the indices if necessary, we may assert that  $(\mathbb{P}^{(m)})_{m \geq 0}$  converges weakly to a probability measure  $\mathbb{P}^\infty$ . From standard arguments that we omit (passing to the limit in the characterisation of the martingale problem solved by  $\mathbb{P}^{(m)}$ ) we deduce that  $\mathbb{P}^\infty$  is the probability measure  $\mathbb{P}$  induced by the unique weak solution to the McKean-Vlasov SDE (1.1). As a consequence, every convergent subsequence converges to the same limit  $\mathbb{P}$  and so does the original sequence  $(\mathbb{P}^{(m)})_{m \geq 1}$ .

By Lebesgue's dominated convergence theorem, for any fixed  $t > 0$  and  $z \in \mathbb{R}^d$ , using (4.9), one may pass to the limit as  $m \uparrow \infty$  in the parametrix infinite series (4.8) and thus deduce that the sequence of functions  $\{\mathcal{K} \ni (s, x, \mu) \mapsto p_m(\mu, s, t, x, z), m \geq 1\}$ ,  $\mathcal{K}$  being a compact set of  $[0, t] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$ , converges to  $(s, x, \mu) \mapsto p(\mu, s, t, x, z)$ , for any fixed  $(s, x, \mu)$  given by the infinite series (3.3). Moreover, it is clearly uniformly bounded and from (4.16), (4.17) and (4.10),  $C, c$  being two positive constants independent of  $m$ , it is equicontinuous. Relabelling the indices if necessary, from the Arzelà-Ascoli theorem, we may assert that it converges uniformly. Hence,  $[0, t] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \ni (s, x, \mu) \mapsto p(\mu, s, t, x, z)$  is continuous.

For any  $\mu \in \mathcal{P}_2(\mathbb{R}^d)$  and any integer  $m$ , the mapping  $(s, x) \mapsto p_m(\mu, s, t, x, z)$  is in  $\mathcal{C}^{0,2}([0, t] \times \mathbb{R}^d)$ . Moreover, from the estimates (4.20), (4.22) and (4.11) (for  $n = 1, 2$ ), the sequence of functions  $\mathcal{K} \ni (s, x, \mu) \mapsto \partial_x p_m(\mu, s, t, x, z), \partial_x^2 p_m(\mu, s, t, x, z)$ , are uniformly bounded and equicontinuous. Hence, from Arzelà-Ascoli's theorem, we may assert that  $(s, x) \mapsto p(\mu, s, t, x, z) \in \mathcal{C}^{0,2}([0, t] \times \mathbb{R}^d)$  and that the mappings  $[0, t] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \ni (s, x, \mu) \mapsto \partial_x p(\mu, s, t, x, z), \partial_x^2 p(\mu, s, t, x, z)$  are continuous.

From (4.16), (4.24) and (4.25),  $\{L^2(\Omega, \mathcal{A}, \mathbb{P}) \supset B(0, R) \ni \xi \mapsto D\tilde{p}_m(\xi, s, t, x, z) = \partial_\mu p_m(\mu, s, t, x, z)(\xi), m \geq 1\}$  and  $\{L^2(\Omega, \mathcal{A}, \mathbb{P}) \supset (B(0, R))^2 \ni (\xi, \xi') \mapsto D^2\tilde{p}_m(\xi, \xi', s, t, x, z) = \partial_\mu^2 p_m(\mu, s, t, x, z)(\xi, \xi'), m \geq 1\}$  are uniformly bounded and equicontinuous for any  $R > 0$ . Relabelling the indices if necessary, from the Arzelà-Ascoli theorem, we may assert that it converges uniformly. Hence,  $L^2(\Omega, \mathcal{A}, \mathbb{P}) \supset B(0, R) \ni \xi \mapsto \tilde{p}(\xi, s, t, x, z)$  are two times continuously differentiable. As a consequence,  $\mathcal{P}_2(\mathbb{R}^d) \ni \mu \mapsto p(\mu, s, t, x, z)$  is two times continuously  $L$ -differentiable. From the estimates (4.18), (4.21) and (4.23) (the three for  $n = 0$ ) and (4.16) (for  $n = 1$ ) on the one hand and (4.24), (4.25), (4.26), (4.27) on the other hand, the same conclusion holds for the two sequences  $\mathcal{K} \ni (s, x, \mu, v) \mapsto \partial_\mu p_m(\mu, s, t, x, z)(v), \mathcal{K}' \ni (s, x, \mu, v) \mapsto \partial_\mu^2 p_m(\mu, s, t, x, z)(v), m \geq 1$ ,  $\mathcal{K}, \mathcal{K}'$  being compact sets of  $[0, t] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d$  and  $[0, t] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times (\mathbb{R}^d)^2$  respectively, that is, they are uniformly bounded and equicontinuous so that the map  $[0, t] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \ni (s, x, \mu, v) \mapsto \partial_\mu^2 p(\mu, s, t, x, z)(v), [0, t] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times (\mathbb{R}^d)^2 \ni (s, x, \mu, v) \mapsto \partial_\mu p(\mu, s, t, x, z)(v)$  are continuous.

From the estimates (4.16) and (4.19) (both for  $n = 1$ ), the sequence  $\mathbb{R}^d \supset B(0, R) \ni v \mapsto \partial_v[\partial_\mu p_m(\mu, s, t, x, z)](v), m \geq 1$ , is bounded and equicontinuous. We thus conclude that  $\mathbb{R}^d \ni v \mapsto \partial_\mu p(\mu, s, t, x, z)(v)$  is continuously differentiable.

The continuity of the map  $[0, t) \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \ni (s, x, \mu, v) \mapsto \partial_v[\partial_\mu p(\mu, s, t, x, z)](v)$  can be deduced from the uniform convergence of the sequence of continuous mappings  $\mathcal{K} \ni (s, x, \mu, v) \mapsto \partial_v[\partial_\mu p_m(\mu, s, t, x, z)](v)$ ,  $m \geq 1$ ,  $\mathcal{K}$  being a compact set of  $[0, t) \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d$ , along a subsequence, obtained from the estimates (4.18), (4.19), (4.21) and (4.23) for  $n = 1$  combined with the Arzelà-Ascoli theorem. The estimate (3.15) then follows by passing to the limit in the corresponding upper-bounds proved in the first step.

*Step 3:  $\mathcal{C}_f^{1,2,2}([0, t) \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$  regularity and related estimates.*

Let us now prove that  $(s, x, \mu) \mapsto p(\mu, s, t, x, z)$  is in  $\mathcal{C}_f^{1,2,2}([0, t) \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$ . We follow the same lines of reasonings as those employed in [CdRF18]. From the Markov property satisfied by the SDE (1.1), stemming from the well-posedness of the related martingale problem, the following relation is satisfied for all  $h > 0$

$$p(\mu, s - h, t, x, z) = \mathbb{E}[p([X_s^{s-h, \xi}], s, t, X_s^{s-h, x, \mu}, z)].$$

Combining estimates (3.7) and (3.4) (for  $n = 1$ ) with the chain rule formula of Proposition 2.1 (with respect to the space and measure variables only) we obtain

$$\mathbb{E}[p([X_s^{s-h, \xi}], s, t, X_s^{s-h, x, \mu}, z)] = p(\mu, s, t, x, z) + \mathbb{E} \left[ \int_{s-h}^s \mathcal{L}_r p([X_r^{s-h, \xi}], s, t, X_r^{s-h, x, \mu}, z) dr \right]$$

where the operator  $\mathcal{L}_r$  is given by (1.4).

Hence, one has

$$\frac{1}{h}(p(\mu, s - h, t, x, z) - p(\mu, s, t, x, z)) = \frac{1}{h} \mathbb{E} \left[ \int_{s-h}^s \mathcal{L}_r p([X_r^{s-h, \xi}], s, t, X_r^{s-h, x, \mu}, z) dr \right]$$

so that, letting  $h \downarrow 0$ , from the boundedness and continuity of the coefficients as well as the continuity of the maps  $(\mu, x) \mapsto p(\mu, s, t, x, z)$ ,  $\partial_x^{1+n} p(\mu, s, t, x, z)$ ,  $\partial_v^n [\partial_\mu p(\mu, s, t, x, z)]$ , for  $n = 0, 1$ , we deduce that  $[0, t) \ni s \mapsto p(\mu, s, t, x, z)$  is left differentiable. Still from the continuity of the coefficients and of the map  $(s, x, \mu) \mapsto \mathcal{L}_s p(\mu, s, t, x, z)$ , we then conclude that it is differentiable in time on the interval  $[0, t)$  with a time derivative satisfying

$$\partial_s p(\mu, s, t, x, z) = -\mathcal{L}_s p(\mu, s, t, x, z) \quad \text{on } [0, t) \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d).$$

## 5. PROPAGATION OF CHAOS

This section is devoted to the proof of Theorems 3.5 and 3.8. As already mentioned, our propagation of chaos results crucially relies on the regularity properties provided by the Theorems 3.1, 3.3 and the Proposition 3.1.

**5.1. Proof of Theorem 3.5.** The strategy consists in testing the fundamental solution  $p(\mu, s, t, z)$  to the backward Kolmogorov PDE (3.19) stated on the Wasserstein space as an approximate solution to the one-dimensional marginal density of the  $N$ -dimensional particle systems. The natural candidate for being an approximate solution is

$$p(\mu_s^N, s, t, z), \quad s \in [0, t), z \in \mathbb{R}^d, \quad \text{with} \quad \mu_s^N := \frac{1}{N} \sum_{i=1}^N \delta_{X_s^i}$$

where the particles  $\{(X_t^i)_{t \in [0, T]}, 1 \leq i \leq N\}$  are given by the unique weak solution to the system particles with dynamics given by (1.2).

*First step: proof of the error bound (3.22)*

Under **(HE)** and **(HR)**, we again consider the sequence of  $\mathcal{C}_f^{1,2,2}([0, t) \times \mathcal{P}_2(\mathbb{R}^d))$  maps  $\{[0, t) \times \mathcal{P}_2(\mathbb{R}^d) \ni (s, \mu) \mapsto p_m(\mu, s, t, z)\}$ , constructed in Section 4.2. Then, Proposition 2.2 implies that the empirical projection function defined by

$$[0, t) \times (\mathbb{R}^d)^N \ni (s, (x_1, \dots, x_N)) \mapsto p_m\left(\frac{1}{N} \sum_{i=1}^N \delta_{x_i}, s, t, z\right)$$

belongs to the space  $C^{1,2}([0, t] \times (\mathbb{R}^d)^N)$  so that, from the standard Itô's formula

$$\begin{aligned}
(5.1) \quad p_m(\mu_s^N, s, t, z) &= p_m(\mu_0^N, 0, t, z) + \int_0^s (\partial_r + \mathcal{L}_r)p_m(\mu_r^N, r, t, z) dr \\
&+ \int_0^s \frac{1}{N} \sum_{i=1}^N \partial_\mu p_m(\mu_r^N, r, t, z)(X_r^i) \cdot (\sigma(r, X_r^i, \mu_r^N) dW_r^i) \\
&+ \int_0^s \frac{1}{2N^2} \sum_{i=1}^N \text{Tr} \left( a(r, X_r^i, \mu_r^N) \partial_\mu^2 p_m(\mu_r^N, r, t, z)(X_r^i, X_r^i) \right) dr.
\end{aligned}$$

From the relation

$$(5.2) \quad \partial_\mu p_m(\mu, r, t, z)(v) = \partial_v p_m(\mu, r, t, v, z) + \int_{\mathbb{R}^d} \partial_\mu p_m(\mu, r, t, x, z)(v) \mu(dx)$$

and the estimates (4.10) and (4.16), we get  $|\partial_\mu p_m(\mu_r^N, r, t, z)(v)| \leq K := K(t-s, b, a, \frac{\delta}{\delta m} b, \frac{\delta}{\delta m} a, \eta)$ , for any  $r \in [0, s]$ , so that the local martingale appearing in the right-hand side of (5.1) is a true martingale. Taking expectation in both sides, we obtain

$$\begin{aligned}
(5.3) \quad &\mathbb{E} \left[ p_m(\mu_s^N, s, t, z) - p_m(\mu, 0, t, z) - \int_0^s (\partial_r + \mathcal{L}_r)p_m(\mu_r^N, r, t, z) dr \right] \\
&= \mathbb{E} [p_m(\mu_0^N, 0, t, z) - p_m(\mu, 0, t, z)] + \int_0^s \frac{1}{2N} \mathbb{E} \left[ \text{Tr} \left( a(r, X_r^1, \mu_r^N) \partial_\mu^2 p_m(\mu_r^N, r, t, z)(X_r^1, X_r^1) \right) \right] dr.
\end{aligned}$$

From the relation

$$\begin{aligned}
(5.4) \quad \partial_\mu^2 p_m(\mu, r, t, z)(v, v') &= \partial_v [\partial_\mu p_m(\mu, r, t, v, z)(v')] + \partial_{v'} [\partial_\mu p_m(\mu, r, t, v', z)(v)] \\
&+ \int_{\mathbb{R}^d} \partial_\mu^2 p_m(\mu, r, t, x, z)(v, v') \mu(dx)
\end{aligned}$$

and the estimates (6.21) and (4.24), we get the following upper-bound

$$|\partial_\mu^2 p_m(\mu, r, t, z)(v, v')| \leq \frac{K}{(t-r)^{1-\frac{\eta}{2}}} \left\{ g(c(t-r), z-v) + g(c(t-r), z-v') + \int_{\mathbb{R}^d} g(c(t-r), z-x) \mu(dx) \right\}$$

where we importantly emphasize that the positive constants  $K$  and  $c$  do not depend on  $m$ . Hence, using the boundedness of  $a$  and the previous estimate, we thus derive the following bound on the last term in the r.h.s. of (5.3)

$$(5.5) \quad \left| \mathbb{E} \left[ \text{Tr} \left( a(r, X_r^1, \mu_r^N) \partial_\mu^2 p_m(\mu_r^N, r, t, z)(X_r^1, X_r^1) \right) \right] \right| \leq \frac{K}{(t-r)^{1-\frac{\eta}{2}}} \int_{\mathbb{R}^d} g(c(t-r), z-y) p^{1,N}(\mu, 0, r, y) dy$$

which, plugged into (5.3), in turn implies

$$\begin{aligned}
(5.6) \quad &\left| \mathbb{E} \left[ p_m(\mu_s^N, s, t, z) - \int_0^s (\partial_r + \mathcal{L}_r)p_m(\mu_r^N, r, t, z) dr \right] \right| \\
&\leq K \left\{ \int_{\mathbb{R}^d} g(ct, z-x) \mu(dx) + \frac{1}{N} \int_0^s \frac{1}{(t-r)^{1-\frac{\eta}{2}}} \int_{\mathbb{R}^d} g(c(t-r), z-y) p^{1,N}(\mu, 0, r, y) dy dr \right\}.
\end{aligned}$$

We now pass to the limit as  $m \uparrow \infty$  in the previous inequality, along a convergent subsequence. Noting that  $(p_m, \partial_r p_m, \mathcal{L}_r p_m)(\mu, r, t, z)$  converges to  $(p, \partial_r p, \mathcal{L}_r p)(\mu, r, t, z)$  for any fixed  $\mu, r, t, z$ , the estimates (4.10), (4.17), (4.16) as well as the relations (5.2) and (5.4) and the dominated convergence theorem yield

$$\begin{aligned}
(5.7) \quad \lim_{m \uparrow \infty} \mathbb{E} \left[ p_m(\mu_s^N, s, t, z) - \int_0^s (\partial_r + \mathcal{L}_r)p_m(\mu_r^N, r, t, z) dr \right] &= \mathbb{E} \left[ p(\mu_s^N, s, t, z) - \int_0^s (\partial_r + \mathcal{L}_r)p(\mu_r^N, r, t, z) dr \right] \\
&= \mathbb{E} \left[ p(\mu_s^N, s, t, z) \right]
\end{aligned}$$

where we used the important fact that  $(\partial_r + \mathcal{L}_r)p(\mu, r, t, z) = 0$  for any  $(r, \mu) \in [0, t] \times \mathcal{P}_2(\mathbb{R}^d)$  for the last equality.

In particular, the above argument combined with (5.6) yields

$$(5.8) \quad \mathbb{E} \left[ p(\mu_s^N, s, t, z) \right] \leq K \left\{ \int_{\mathbb{R}^d} g(ct, z-x) \mu(dx) + \frac{1}{N} \int_0^s \frac{1}{(t-r)^{1-\frac{\eta}{2}}} \int_{\mathbb{R}^d} g(c(t-r), z-y) p^{1,N}(\mu, 0, r, y) dy dr \right\}.$$



We now pass to the limit as  $s \uparrow t$  in the previous inequality. We first note that

$$\mathbb{E}[p(\mu_s^N, s, t, z)] = \mathbb{E}\left[\int_{\mathbb{R}^d} p(\mu_s^N, s, t, x, z)\mu_s^N(dx)\right] = \mathbb{E}\left[p(\mu_s^N, s, t, X_s^1, z)\right] = \int_{(\mathbb{R}^d)^N} p(m_{\mathbf{x}}^N, s, t, x_1, z)\mathbf{p}^N(\mu, 0, s, \mathbf{x}) d\mathbf{x}$$

where we used the notations  $\mathbf{x} = (x_1, \dots, x_N)$ ,  $d\mathbf{x} = dx_1 \cdots dx_N$ ,  $m_{\mathbf{x}}^N = \frac{1}{N} \sum_{i=1}^N \delta_{x_i}$  and denoted by  $\mathbf{x} \mapsto \mathbf{p}^N(\mu, 0, s, \mathbf{x})$  the density function of the  $N$ -tuple  $\mathbf{X}_s = (X_s^1, \dots, X_s^N)$  given by the unique weak solution to the particle system at time  $s$  starting at time 0 from the  $N$ -fold product measure  $\mu^{\otimes N}$ . We then make use again of the decomposition (4.2) and the computations that appear shortly after, namely

$$p(m_{\mathbf{x}}^N, s, t, x_1, z) = \widehat{p}^z(m_{\mathbf{x}}^N, s, t, x_1, z) + \mathcal{R}(m_{\mathbf{x}}^N, s, t, x_1, z)$$

with

$$|\mathcal{R}(m_{\mathbf{x}}^N, s, t, x_1, z)| \leq C(t-s)^{\frac{\eta}{2}} g(c(t-s), z-x_1).$$

Denoting  $\tilde{\mathbf{x}} = (z, x_2, \dots, x_N)$ ,  $\xi, \xi'$  two random variables with  $[\xi] = m_{\mathbf{x}}^N$ ,  $[\xi'] = m_{\tilde{\mathbf{x}}}^N$ , from (6.5) with  $\beta = \eta$

$$|a_{i,j}(r, z, [X_r^{s,\xi}]) - a_{i,j}(r, z, [X_r^{s,\xi'}])| \leq CW_2^\eta(m_{\mathbf{x}}^N, m_{\tilde{\mathbf{x}}}^N) \leq C|z-x|^\eta$$

which in turn by the mean-value theorem and the space-time inequality (1.5) yield

$$|\widehat{p}^z(m_{\mathbf{x}}^N, s, t, x_1, z) - \widehat{p}^z(m_{\tilde{\mathbf{x}}}^N, s, t, x_1, z)| \leq C(t-s)^{\frac{\eta}{2}} g(c(t-s), z-x_1).$$

From the previous computations and (3.21),

$$\int_{(\mathbb{R}^d)^N} p(m_{\mathbf{x}}^N, s, t, x_1, z) p_N(\mu, 0, s, \mathbf{x}) d\mathbf{x} = \int_{(\mathbb{R}^d)^N} \widehat{p}^z(m_{\mathbf{x}}^N, s, t, x_1, z) p_N(\mu, 0, s, \mathbf{x}) d\mathbf{x} + \mathcal{O}((t-s)^{\frac{\eta}{2}}) \int g(ct, z-x)\mu(dx)$$

We finally perform the change of variable  $x_1 = \Sigma_{s,t}^{1/2} y_1 + z$ , where  $\Sigma_{s,t}^{1/2}$  is the unique principal square root of the positive definite matrix  $\int_s^t a(r, z, [X_r^{s,\xi'}]) dr$ ,  $[\xi'] = m_{\tilde{\mathbf{x}}}^N$ , in the integral appearing in the right-hand side of the previous equality and then let  $s \uparrow t$ , by dominated convergence

$$\lim_{s \uparrow t} \int_{(\mathbb{R}^d)^N} p(m_{\mathbf{x}}^N, s, t, x_1, z) p_N(\mu, 0, s, \mathbf{x}) d\mathbf{x} = \lim_{s \uparrow t} \int_{(\mathbb{R}^d)^N} \widehat{p}^z(m_{\tilde{\mathbf{x}}}^N, s, t, x_1, z) p_N(\mu, 0, s, \mathbf{x}) d\mathbf{x} = p^{1,N}(\mu, 0, t, z).$$

Passing to the limit as  $s \uparrow t$  in (5.8), we thus obtain

$$(5.9) \quad \lim_{s \uparrow t} \mathbb{E}\left[p(\mu_s^N, s, t, z)\right] = p^{1,N}(\mu, 0, t, z) \leq K \left\{ \int_{\mathbb{R}^d} g(ct, z-x)\mu(dx) + \frac{1}{N} \int_0^t \frac{1}{(t-r)^{1-\frac{\eta}{2}}} \int_{\mathbb{R}^d} g(c(t-r), z-y) p^{1,N}(\mu, 0, r, y) dy dr \right\}.$$

The previous inequality can be iterated and by an induction argument that we omit, we deduce

$$(5.10) \quad p^{1,N}(\mu, 0, t, z) \leq K \int_{\mathbb{R}^d} g(ct, z-x)\mu(dx)$$

where we importantly note that the two positive constants  $K$  and  $c$  does not depend on  $N$ .

We now come back to (5.3) and prove an appropriate estimate for the quantity  $|\mathbb{E}[p(\mu_0^N, 0, t, z)] - p(\mu, 0, t, z)|$ .

By exchangeability in law of  $(\xi^i)_{1 \leq i \leq N}$  and the mean-value theorem

$$(5.11) \quad \begin{aligned} & \mathbb{E}[p_m(\mu_0^N, 0, t, z) - p_m(\mu, 0, t, z)] \\ &= \mathbb{E}[p_m(\mu_0^N, 0, t, \xi^1, z) - p_m(\mu, 0, t, \xi^1, z)] \\ &= \int_0^1 \int_{\mathbb{R}^d} \mathbb{E}\left[\frac{\delta}{\delta m} p_m(\mu_0^{\lambda_1, N}, 0, t, \xi^1, z)(y) (\mu_0^N - \mu)(dy)\right] d\lambda_1 \\ &= \frac{1}{N} \int_0^1 \mathbb{E}\left[\frac{\delta}{\delta m} p_m(\mu_0^{\lambda_1, N}, 0, t, \xi^1, z)(\xi^1) - \frac{\delta}{\delta m} p_m(\mu_0^{\lambda_1, N}, 0, t, \xi^1, z)(\tilde{\xi})\right] d\lambda_1 \\ &\quad + \frac{N-1}{N} \int_0^1 \mathbb{E}\left[\frac{\delta}{\delta m} p_m(\mu_0^{\lambda_1, N}, 0, t, \xi^1, z)(\xi^2) - \frac{\delta}{\delta m} p_m(\mu_0^{\lambda_1, N}, 0, t, \xi^1, z)(\tilde{\xi})\right] d\lambda_1 \\ &= \frac{1}{N} \int_{[0,1]^2} \mathbb{E}\left[\partial_\mu p_m(\mu_0^{\lambda_1, N}, 0, t, \xi^1, z)(\lambda_2 \xi^1 + (1-\lambda_2)\tilde{\xi}) \cdot (\xi^1 - \tilde{\xi})\right] d\lambda_1 d\lambda_2 \\ &\quad + \frac{N-1}{N} \int_{[0,1]^2} \mathbb{E}\left[\partial_\mu p_m(\mu_0^{\lambda_1, N}, 0, t, \xi^1, z)(\lambda_2 \xi^2) \cdot \xi^2 - \partial_\mu p_m(\mu_0^{\lambda_1, N}, 0, t, \xi^1, z)(\lambda_2 \tilde{\xi}) \cdot \tilde{\xi}\right] d\lambda_1 d\lambda_2 \end{aligned}$$

with the notation  $\mu_0^{\lambda_1, N} := \lambda_1 \mu_0^N + (1 - \lambda_1) \mu$  and where  $\tilde{\xi}$  is a random variable independent of the sequence  $(\xi^i)_{1 \leq i \leq N}$  with law  $\mu$ . We now introduce the measure  $\tilde{\mu}_0^{\lambda_1, N} := \lambda_1 \tilde{\mu}_0^N + (1 - \lambda_1) \mu$  with  $\tilde{\mu}_0^N := \mu_0^N + \frac{1}{N}(\delta_{\tilde{\xi}^1} - \delta_{\xi^2})$  and notice that

$$\mathbb{E} \left[ \partial_\mu p_m(\mu_0^{\lambda_1, N}, 0, t, \xi^1, z)(\lambda_2 \xi^2) \cdot \xi^2 \right] = \mathbb{E} \left[ \partial_\mu p_m(\tilde{\mu}_0^{\lambda_1, N}, 0, t, \xi^1, z)(\lambda_2 \tilde{\xi}) \cdot \tilde{\xi} \right].$$

Hence, again by the mean-value theorem

$$\begin{aligned} & \mathbb{E} \left[ \partial_\mu p_m(\mu_0^{\lambda_1, N}, 0, t, \xi^1, z)(\lambda_2 \xi^2) \cdot \xi^2 - \partial_\mu p_m(\mu_0^{\lambda_1, N}, 0, t, \xi^1, z)(\lambda_2 \tilde{\xi}) \cdot \tilde{\xi} \right] \\ &= \frac{\lambda_1}{N} \int_0^1 \mathbb{E} \left[ \frac{\delta}{\delta m} \partial_\mu p_m(\tilde{\mu}_0^{\lambda_1, \lambda_3, N}, 0, t, \xi^1, z)(\lambda_2 \tilde{\xi}, \tilde{\xi}) \cdot \tilde{\xi} - \frac{\delta}{\delta m} \partial_\mu p_m(\tilde{\mu}_0^{\lambda_1, \lambda_3, N}, 0, t, \xi^1, z)(\lambda_2 \tilde{\xi}, \xi^2) \cdot \tilde{\xi} \right] d\lambda_3 \\ &= \frac{\lambda_1}{N} \int_{[0,1]^2} \mathbb{E} \left[ \tilde{\xi} \cdot \partial_\mu^2 p_m(\tilde{\mu}_0^{\lambda_1, \lambda_3, N}, 0, t, \xi^1, z)(\lambda_2 \tilde{\xi}, \lambda_4 \tilde{\xi}) \cdot \tilde{\xi} - \xi^2 \cdot \partial_\mu^2 p_m(\tilde{\mu}_0^{\lambda_1, \lambda_3, N}, 0, t, \xi^1, z)(\lambda_2 \tilde{\xi}, \lambda_4 \xi^2) \cdot \tilde{\xi} \right] d\lambda_3 d\lambda_4 \end{aligned}$$

with the notation  $\mu_0^{\lambda_1, \lambda_3, N} := \lambda_3 \tilde{\mu}_0^{\lambda_1, N} + (1 - \lambda_3) \mu_0^{\lambda_1, N}$ .

Therefore,

$$\begin{aligned} & \mathbb{E} [p_m(\mu_0^N, 0, t, z) - p_m(\mu, 0, t, z)] \\ &= \frac{1}{N} \int_{[0,1]^2} \mathbb{E} \left[ \partial_\mu p_m(\mu_0^{\lambda_1, N}, 0, t, \xi^1, z)(\lambda_2 \xi^1 + (1 - \lambda_2) \tilde{\xi}) \cdot (\xi^1 - \tilde{\xi}) \right] d\lambda_1 d\lambda_2 \\ &+ \frac{(N-1)}{N^2} \int_{[0,1]^4} \lambda_1 \mathbb{E} \left[ \tilde{\xi} \cdot \partial_\mu^2 p_m(\tilde{\mu}_0^{\lambda_1, \lambda_3, N}, 0, t, \xi^1, z)(\lambda_2 \tilde{\xi}, \lambda_4 \tilde{\xi}) \cdot \tilde{\xi} \right. \\ &\quad \left. - \xi^2 \cdot \partial_\mu^2 p_m(\tilde{\mu}_0^{\lambda_1, \lambda_3, N}, 0, t, \xi^1, z)(\lambda_2 \tilde{\xi}, \lambda_4 \xi^2) \cdot \tilde{\xi} \right] d\lambda_1 d\lambda_2 d\lambda_3 d\lambda_4 \end{aligned}$$

so that by (4.16) and (4.24)

(5.12)

$$|\mathbb{E} [p_m(\mu_0^N, 0, t, z) - p_m(\mu, 0, t, z)]| \leq \frac{K}{N} \left\{ \frac{1}{t^{\frac{1-\eta}{2}}} \int_{\mathbb{R}^d} g(ct, z-x) |x| \mu(dx) + \frac{1}{t^{1-\frac{\eta}{2}}} \int_{\mathbb{R}^d} g(ct, z-x) \mu(dx) \right\}.$$

From the previous estimate, (5.7), (5.9), (5.5) and (5.3), we thus deduce

$$\begin{aligned} |(p^{1, N} - p)(\mu, 0, t, z)| &\leq \frac{K}{N} \left\{ \frac{1}{t^{\frac{1-\eta}{2}}} \int_{\mathbb{R}^d} g(ct, z-x) |x| \mu(dx) + \frac{1}{t^{1-\frac{\eta}{2}}} \int_{\mathbb{R}^d} g(ct, z-x) \mu(dx) \right. \\ &\quad \left. + \frac{1}{N} \int_0^t \frac{1}{(t-r)^{1-\frac{\eta}{2}}} \int_{\mathbb{R}^d} g(c(t-r), z-y) p^{1, N}(\mu, 0, r, y) dy dr \right\} \end{aligned}$$

which in turn by (5.10) yields

$$|(p^{1, N} - p)(\mu, 0, t, z)| \leq \frac{K}{N} \left\{ \frac{1}{t^{\frac{1-\eta}{2}}} \int_{\mathbb{R}^d} g(ct, z-x) |x| \mu(dx) + \frac{1}{t^{1-\frac{\eta}{2}}} \int_{\mathbb{R}^d} g(ct, z-x) \mu(dx) \right\}.$$

This concludes the proof (3.22).

*Second step: proof of first order expansion (3.23).*

We now assume that **(HR<sub>+</sub>)** holds and establish the first order expansion (3.23). The proof follows similar lines of reasonings except that we replace the approximation sequence  $(p_m(\mu, 0, t, z))_{m \geq 1}$  by its limit  $p(\mu, 0, t, z)$ . In particular, similarly to (5.3), one has

$$(5.13) \quad \mathbb{E} [p(\mu_s^N, s, t, z) - p(\mu, 0, t, z)] = \mathbb{E} [p(\mu_0^N, 0, t, z) - p(\mu, 0, t, z)] + \int_0^s \frac{1}{2N} \mathbb{E} \left[ \text{Tr} \left( a(r, X_r^1, \mu_r^N) \partial_\mu^2 p(\mu_r^N, r, t, z)(X_r^1, X_r^1) \right) \right] dr.$$

We then pass to the limit as  $s \uparrow t$  in the previous identity using similar arguments as those previously employed. Hence,

$$\begin{aligned} (p^{1, N} - p)(\mu, 0, t, z) &= \mathbb{E} [p(\mu_0^N, 0, t, z) - p(\mu, 0, t, z)] + \int_0^t \frac{1}{2N} \mathbb{E} \left[ \text{Tr} \left( a(r, X_r^1, \mu_r^N) \partial_\mu^2 p(\mu_r^N, r, t, z)(X_r^1, X_r^1) \right) \right] dr \\ &= \mathbb{E} [p(\mu_0^N, 0, t, z) - p(\mu, 0, t, z)] + \int_0^t \frac{1}{2N} \mathbb{E} [\mathcal{A}_s p(\mu_s, s, t, z)] ds + \frac{1}{N} \mathcal{R}_1^N(\mu, 0, t, z) \end{aligned}$$

with

$$\mathcal{R}_1^N(\mu, 0, t, z) := \frac{1}{2} \int_0^t \mathbb{E} \left[ \text{Tr} \left( a(r, X_r^1, \mu_r^N) \partial_\mu^2 p(\mu_r^N, r, t, z)(X_r^1, X_r^1) \right) - \text{Tr} \left( a(r, X_r, \mu_r) \partial_\mu^2 p(\mu_r, r, t, z)(X_r, X_r) \right) \right] dr.$$

Moreover, still with the notations introduced above, one has

$$\begin{aligned}
& \mathbb{E}[p(\mu_0^N, 0, t, z) - p(\mu, 0, t, z)] \\
&= \frac{1}{N} \int_0^1 \mathbb{E} \left[ \frac{\delta}{\delta m} p(\mu_0^{\lambda_1, N}, 0, t, \xi^1, z)(\xi^1) - \frac{\delta}{\delta m} p(\mu_0^{\lambda_1, N}, 0, t, \xi^1, z)(\tilde{\xi}) \right] d\lambda_1 \\
&\quad + \frac{N-1}{N} \int_0^1 \mathbb{E} \left[ \frac{\delta}{\delta m} p(\mu_0^{\lambda_1, N}, 0, t, \xi^1, z)(\xi^2) - \frac{\delta}{\delta m} p(\mu_0^{\lambda_1, N}, 0, t, \xi^1, z)(\tilde{\xi}) \right] d\lambda_1 \\
&= \frac{1}{N} \int_0^1 \mathbb{E} \left[ \frac{\delta}{\delta m} p(\mu_0^{\lambda_1, N}, 0, t, \xi^1, z)(\xi^1) - \frac{\delta}{\delta m} p(\mu_0^{\lambda_1, N}, 0, t, \xi^1, z)(\tilde{\xi}) \right] d\lambda_1 \\
&\quad + \frac{N-1}{N^2} \int_{[0,1]^2} \mathbb{E} \left[ \frac{\delta^2}{\delta m^2} p(\tilde{\mu}_0^{\lambda_1, \lambda_2, N}, 0, t, \xi^1, z)(\tilde{\xi}, \tilde{\xi}) - \frac{\delta^2}{\delta m^2} p(\tilde{\mu}_0^{\lambda_1, \lambda_2, N}, 0, t, \xi^1, z)(\tilde{\xi}, \xi^2) \right] \lambda_1 d\lambda_1 d\lambda_2 \\
&= \frac{1}{N} \mathbb{E} \left[ \frac{\delta}{\delta m} p(\mu, 0, t, \xi^1, z)(\xi^1) - \frac{\delta}{\delta m} p(\mu, 0, t, \xi^1, z)(\tilde{\xi}) \right] \\
&\quad + \frac{1}{2N} \mathbb{E} \left[ \frac{\delta^2}{\delta m^2} p(\mu, 0, t, \xi^1, z)(\tilde{\xi}, \tilde{\xi}) - \frac{\delta^2}{\delta m^2} p(\mu, 0, t, \xi^1, z)(\tilde{\xi}, \xi^2) \right] \\
&\quad + \frac{1}{N} \mathcal{R}_2^N(\mu, 0, t, z)
\end{aligned}$$

with

$$\begin{aligned}
\mathcal{R}_2^N(\mu, 0, t, z) &:= \int_0^1 \left( \mathbb{E} \left[ \frac{\delta}{\delta m} p(\mu_0^{\lambda_1, N}, 0, t, \xi^1, z)(\xi^1) - \frac{\delta}{\delta m} p(\mu, 0, t, \xi^1, z)(\xi^1) \right] \right. \\
&\quad \left. - \mathbb{E} \left[ \frac{\delta}{\delta m} p(\mu_0^{\lambda_1, N}, 0, t, \xi^1, z)(\tilde{\xi}) - \frac{\delta}{\delta m} p(\mu, 0, t, \xi^1, z)(\tilde{\xi}) \right] \right) d\lambda_1 \\
&\quad + \int_{[0,1]^2} \left( \mathbb{E} \left[ \frac{\delta^2}{\delta m^2} p(\tilde{\mu}_0^{\lambda_1, \lambda_2, N}, 0, t, \xi^1, z)(\tilde{\xi}, \tilde{\xi}) - \frac{\delta^2}{\delta m^2} p(\mu, 0, t, \xi^1, z)(\tilde{\xi}, \tilde{\xi}) \right] \right. \\
&\quad \left. - \mathbb{E} \left[ \frac{\delta^2}{\delta m^2} p(\tilde{\mu}_0^{\lambda_1, \lambda_2, N}, 0, t, \xi^1, z)(\tilde{\xi}, \xi^2) - \frac{\delta^2}{\delta m^2} p(\mu, 0, t, \xi^1, z)(\tilde{\xi}, \xi^2) \right] \right) \lambda_1 d\lambda_1 d\lambda_2 \\
&\quad - \frac{1}{N} \int_{[0,1]^2} \mathbb{E} \left[ \frac{\delta^2}{\delta m^2} p(\tilde{\mu}_0^{\lambda_1, \lambda_2, N}, 0, t, \xi^1, z)(\tilde{\xi}, \tilde{\xi}) - \frac{\delta^2}{\delta m^2} p(\tilde{\mu}_0^{\lambda_1, \lambda_2, N}, 0, t, \xi^1, z)(\tilde{\xi}, \xi^2) \right] \lambda_1 d\lambda_1 d\lambda_2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
(p^{1,N} - p)(\mu, 0, t, z) &= \mathbb{E}[p(\mu_0^N, 0, t, z) - p(\mu, 0, t, z)] + \int_0^t \frac{1}{2N} \mathbb{E} \left[ \text{Tr} \left( a(r, X_r^1, \mu_r^N) \partial_\mu^2 p(\mu_r^N, r, t, z)(X_r^1, X_r^1) \right) \right] dr \\
&= \frac{1}{N} \mathbb{E} \left[ \frac{\delta}{\delta m} p(\mu, 0, t, \xi^1, z)(\xi^1) - \frac{\delta}{\delta m} p(\mu, 0, t, \xi^1, z)(\tilde{\xi}) \right] \\
&\quad + \frac{1}{2N} \mathbb{E} \left[ \frac{\delta^2}{\delta m^2} p(\mu, 0, t, \xi^1, z)(\tilde{\xi}, \tilde{\xi}) - \frac{\delta^2}{\delta m^2} p(\mu, 0, t, \xi^1, z)(\tilde{\xi}, \xi^2) \right] + \int_0^t \frac{1}{2N} \mathbb{E}[\mathcal{A}_s p(\mu_s, s, t, z)] ds \\
&\quad + \frac{1}{N} \mathcal{R}^N(\mu, 0, t, z)
\end{aligned}$$

where  $\mathcal{R}^N(\mu, 0, t, z) := \mathcal{R}_1^N(\mu, 0, t, z) + \mathcal{R}_2^N(\mu, 0, t, z)$ . The last step of the proof consists in establishing some estimates on both remainder terms under the assumptions of Theorem 3.8.

*Last step: estimate of the remainder*

We now assume that  $(\mathbf{HR}_+)$  holds and that the assumptions of Theorem 3.8 are in force. Introducing the coupling dynamics  $\bar{X}^1$ , we write

$$\mathcal{R}_1^N(\mu, 0, t, z) = \frac{1}{2} \int_0^t \mathbb{E} \left[ \text{Tr} \left( a(r, X_r^1, \mu_r^N) \partial_\mu^2 p(\mu_r^N, r, t, z)(X_r^1, X_r^1) \right) - \text{Tr} \left( a(r, \bar{X}_r^1, \mu_r) \partial_\mu^2 p(\mu_r, r, t, z)(\bar{X}_r^1, \bar{X}_r^1) \right) \right] dr$$

and decompose the integrand appearing in right-hand side as the sum of the three following terms  $\mathcal{R}_1^{N,1}(\mu, r, t, z)$ ,  $\mathcal{R}_1^{N,2}(\mu, r, t, z)$  and  $\mathcal{R}_1^{N,3}(\mu, r, t, z)$  defined by

$$\begin{aligned}
\mathcal{R}_1^{N,1}(\mu, r, t, z) &:= \mathbb{E} \left[ \text{Tr} \left( [a(r, X_r^1, \mu_r^N) - a(r, \bar{X}_r^1, \mu_r)] \partial_\mu^2 p(\mu_r^N, r, t, z)(X_r^1, X_r^1) \right) \right], \\
\mathcal{R}_1^{N,2}(\mu, r, t, z) &:= \mathbb{E} \left[ \text{Tr} \left( a(r, \bar{X}_r^1, \mu_r) [\partial_\mu^2 p(\mu_r^N, r, t, z)(X_r^1, X_r^1) - \partial_\mu^2 p(\mu_r, r, t, z)(X_r^1, X_r^1)] \right) \right], \\
\mathcal{R}_1^{N,3}(\mu, r, t, z) &:= \mathbb{E} \left[ \text{Tr} \left( a(r, \bar{X}_r^1, \mu_r) [\partial_\mu^2 p(\mu_r, r, t, z)(X_r^1, X_r^1) - \partial_\mu^2 p(\mu_r, r, t, z)(\bar{X}_r^1, \bar{X}_r^1)] \right) \right].
\end{aligned}$$

From (5.4) (with  $p$  instead of  $p_m$ ) and the estimates (3.15) and (3.7) (with  $n = 1$ ), we first obtain

$$(5.14) \quad \begin{aligned} |\partial_{\mu}^2 p(\mu_r^N, r, t, z)(X_r^1, X_r^1)| &\leq \frac{K}{(t-r)^{1-\frac{\eta}{2}}} \left\{ g(c(t-r), z - X_r^1) + \int g(c(t-r), z - x) \mu_r^N(dx) \right\} \\ &= \frac{K}{(t-r)^{1-\frac{\eta}{2}}} \left\{ g(c(t-r), z - X_r^1) + \frac{1}{N} \sum_{i=1}^N g(c(t-r), z - X_r^i) \right\}. \end{aligned}$$

This estimate will be used in the sequel. The uniform Lipschitz regularity of the map  $(x, \mu) \mapsto a(t, x, \mu)$  then gives

$$(5.15) \quad |a(r, X_r^1, \mu_r^N) - a(r, \bar{X}_r^1, \mu_r)| \leq K[|X_r^1 - \bar{X}_r^1| + W_2(\mu_r^N, \mu_r)].$$

Combining the two previous estimates with the Fubini theorem and the Cauchy-Schwarz inequality,

$$\int_{\mathbb{R}^d} |\mathcal{R}_1^{N,1}(\mu, 0, r, z)||z|^2 dz \leq \frac{K}{(t-r)^{1-\frac{\eta}{2}}} \left\{ \mathbb{E}[|X_r^1 - \bar{X}_r^1|^2]^{1/2} + \mathbb{E}[W_2^2(\mu_r^N, \mu_r)]^{1/2} \right\} \mathbb{E}[|X_r^1|^4]^{1/2} \leq \frac{K}{(t-r)^{1-\frac{\eta}{2}}} \sqrt{\varepsilon_N}.$$

We now deal with  $\mathcal{R}_1^{N,2}(\mu, r, t, z)$ . From (5.4) with  $p$  instead of  $p_m$  and the estimates (3.1), (3.16), for any  $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$  and for any coupling  $\pi$  between  $\mu$  and  $\nu$ , we get

$$\begin{aligned} |\partial_{\mu}^2 p(\mu, r, t, z)(v, v) - \partial_{\mu}^2 p(\nu, r, t, z)(v, v)| &\leq K|\partial_x \partial_{\mu} p(\mu, r, t, v, z)(v) - \partial_x \partial_{\mu} p(\nu, r, t, v, z)(v)| \\ &\quad + K \left| \int_{(\mathbb{R}^d)^2} [\partial_{\mu}^2 p(\mu, r, t, x, z)(v, v) - \partial_{\mu}^2 p(\mu, r, t, y, z)(v, v)] \pi(dx, dy) \right| \\ &\quad + K \int_{\mathbb{R}^d} |\partial_{\mu}^2 p(\mu, r, t, x, z)(v, v) - \partial_{\mu}^2 p(\nu, r, t, x, z)(v, v)| \nu(dx) \\ &\leq K \frac{W_2^{\beta}(\mu, \nu)}{(t-r)^{1+\frac{\beta-\eta}{2}}} \left\{ g(c(t-r), z - v) + \int_{\mathbb{R}^d} g(c(t-r), z - x) \nu(dx) \right\} \\ &\quad + \frac{K}{(t-r)^{1+\frac{\beta-\eta}{2}}} \int_{(\mathbb{R}^d)^2} |x - y|^{\beta} \{g(c(t-r), z - x) + g(c(t-r), z - y)\} \pi(dx, dy) \end{aligned}$$

which directly yields

$$\begin{aligned} \int_{\mathbb{R}^d} |z|^2 |\partial_{\mu}^2 p(\mu, r, t, z)(v, v) - \partial_{\mu}^2 p(\nu, r, t, z)(v, v)| dz &\leq K \frac{W_2^{\beta}(\mu, \nu)}{(t-r)^{1+\frac{\beta-\eta}{2}}} (1 + |v|^2) \\ &\quad + \frac{K}{(t-r)^{1+\frac{\beta-\eta}{2}}} \int_{(\mathbb{R}^d)^2} |x - y|^{\beta} (1 + |x|^2 + |y|^2) \pi(dx, dy) \end{aligned}$$

so that using Cauchy-Schwarz's inequality and then taking the infimum over  $\pi$  finally implies

$$\int_{\mathbb{R}^d} |z|^2 |\partial_{\mu}^2 p(\mu, r, t, z)(v, v) - \partial_{\mu}^2 p(\nu, r, t, z)(v, v)| dz \leq K \frac{W_2^{\beta}(\mu, \nu)}{(t-r)^{1+\frac{\beta-\eta}{2}}} (1 + |v|^2 + M_4^2(\mu) + M_4^2(\nu)).$$

From the preceding estimate, the Fubini theorem and the fact that  $\mathbb{E}[M_4^4(\mu_r^N)] \leq KM_4^4(\mu_0)$  and  $M_4(\mu_r) \leq KM_4(\mu_0)$ , we thus get

$$\int_{\mathbb{R}^d} |z|^2 \mathbb{E}[|\partial_{\mu}^2 p(\mu_r^N, r, t, z)(X_r^1, X_r^1) - \partial_{\mu}^2 p(\mu_r, r, t, z)(X_r^1, X_r^1)|] dz \leq K \frac{\mathbb{E}[W_2^2(\mu_r^N, \mu_r)]^{\frac{\beta}{2}}}{(t-r)^{1+\frac{\beta-\eta}{2}}}$$

so that

$$\int_{\mathbb{R}^d} |\mathcal{R}_1^{N,2}(\mu, r, t, z)||z|^2 dz \leq \frac{K}{(t-r)^{1+\frac{\beta-\eta}{2}}} \varepsilon_N^{\frac{\beta}{2}}.$$

From (3.16), (3.9) and (3.10), for any  $(v_1, v_2) \in \mathbb{R}^d$ , we obtain

$$\begin{aligned} |\partial_{\mu}^2 p(\mu, r, t, z)(v_1, v_1) - \partial_{\mu}^2 p(\mu, r, t, z)(v_2, v_2)| &\leq \frac{K}{(t-r)^{1+\frac{\beta-\eta}{2}}} |v_1 - v_2|^{\beta} \{g(c(t-r), z - v_1) + g(c(t-r), z - v_2)\} \\ &\quad + \frac{K}{(t-r)^{1+\frac{\beta-\eta}{2}}} |v_1 - v_2|^{\beta} \int_{\mathbb{R}^d} g(c(t-r), z - x) \mu(dx). \end{aligned}$$

Hence,

$$\int_{\mathbb{R}^d} |\mathcal{R}_1^{N,3}(\mu, r, t, z)||z|^2 dz \leq \frac{K}{(t-r)^{1+\frac{\beta-\eta}{2}}} \mathbb{E}[|X_r^1 - \bar{X}_r^1|^{\beta} (1 + |X_r^1|^2 + |\bar{X}_r^1|^2)] \leq \frac{K}{(t-r)^{1+\frac{\beta-\eta}{2}}} \varepsilon_N^{\frac{\beta}{2}}$$

Gathering the previous estimates, we finally obtain

$$\forall \beta \in [0, \eta), \quad \int_{\mathbb{R}^d} |\mathcal{R}_1^N(\mu, 0, t, z)| |z|^2 dz \leq K t^{\frac{\eta-\beta}{2}} \varepsilon_N^{\frac{\beta}{2}}.$$

We then make use of the decomposition  $\mathcal{R}_2^N(\mu, 0, t, z) =: \mathcal{R}_2^{N,1}(\mu, 0, t, z) + \mathcal{R}_2^{N,2}(\mu, 0, t, z) + \mathcal{R}_2^{N,3}(\mu, 0, t, z)$ . We write

$$\frac{\delta}{\delta m} p(\mu_0^{\lambda_1, N}, 0, t, \xi^1, z)(\xi^1) - \frac{\delta}{\delta m} p(\mu, 0, t, \xi^1, z)(\xi^1) = \int_0^1 \left[ \partial_\mu p(\mu_0^{\lambda_1, N}, 0, t, \xi^1, z)(\lambda_2 \xi^1) - \partial_\mu p(\mu, 0, t, \xi^1, z)(\lambda_2 \xi^1) \right] \cdot \xi_1 d\lambda_2$$

so that, from (3.12) with  $n = 0, \beta = 1$  and noting that  $W_2(\mu_0^{\lambda_1, N}, \mu) = \lambda_1 W_2(\mu_0^N, \mu)$

$$\left| \frac{\delta}{\delta m} p(\mu_0^{\lambda_1, N}, 0, t, \xi^1, z)(\xi^1) - \frac{\delta}{\delta m} p(\mu, 0, t, \xi^1, z)(\xi^1) \right| \leq K \frac{W_2(\mu_0^N, \mu)}{t^{1-\frac{\eta}{2}}} g(ct, z - \xi_1) |\xi_1|.$$

Similarly,

$$\left| \frac{\delta}{\delta m} p(\mu_0^{\lambda_1, N}, 0, t, \xi^1, z)(\tilde{\xi}) - \frac{\delta}{\delta m} p(\mu, 0, t, \xi^1, z)(\tilde{\xi}) \right| \leq K \frac{W_2(\mu_0^N, \mu)}{t^{1-\frac{\eta}{2}}} g(ct, z - \xi_1) |\tilde{\xi}|.$$

Gathering the two previous estimates and using the fact that  $\tilde{\xi}$  is independent of  $\mu_0^N$ , we obtain

$$|\mathcal{R}_2^{N,1}(\mu, 0, t, z)| \leq \frac{K}{t^{1-\frac{\eta}{2}}} \mathbb{E} \left[ W_2(\mu_0^N, \mu) g(ct, z - \xi_1) (1 + |\xi_1|) \right]$$

and by Cauchy-Schwarz's inequality

$$\int_{\mathbb{R}^d} |\mathcal{R}_2^{N,1}(\mu, 0, t, z)| |z|^2 dz \leq \frac{K}{t^{1-\frac{\eta}{2}}} \mathbb{E} \left[ W_2(\mu_0^N, \mu) (1 + |\xi_1|^3) \right] \leq \frac{K}{t^{1-\frac{\eta}{2}}} \mathbb{E} [W_2^4(\mu_0^N, \mu)]^{\frac{1}{4}} = \frac{K}{t^{1-\frac{\eta}{2}}} \varepsilon_N^{\frac{1}{2}}$$

where we used the bound  $\mathbb{E}[W_2^4(\mu_0^N, \mu)]^{\frac{1}{4}} \leq K \varepsilon_N^{\frac{1}{2}}$  stemming, after some standard computations, from the concentration inequality established in Theorem 2 by Fournier and Guillin [FG15].

From similar arguments, using (3.16) and the fact that  $W_2(\tilde{\mu}_0^{\lambda_1, \lambda_2, N}, \mu) \leq W_2(\mu_0^N, \mu)$

$$\left| \mathbb{E} \left[ \frac{\delta^2}{\delta m^2} p(\tilde{\mu}_0^{\lambda_1, \lambda_2, N}, 0, t, \xi^1, z)(\tilde{\xi}, \tilde{\xi}) - \frac{\delta^2}{\delta m^2} p(\mu, 0, t, \xi^1, z)(\tilde{\xi}, \tilde{\xi}) \right] \right| \leq \frac{K}{t^{1+\frac{\beta-\eta}{2}}} \mathbb{E} \left[ W_2^\beta(\mu_0^N, \mu) g(ct, z - \xi_1) \right]$$

and

$$\left| \mathbb{E} \left[ \frac{\delta^2}{\delta m^2} p(\tilde{\mu}_0^{\lambda_1, \lambda_2, N}, 0, t, \xi^1, z)(\tilde{\xi}, \xi^2) - \frac{\delta^2}{\delta m^2} p(\mu, 0, t, \xi^1, z)(\tilde{\xi}, \xi^2) \right] \right| \leq \frac{K}{t^{1+\frac{\beta-\eta}{2}}} \mathbb{E} [W_2^\beta(\mu_0^N, \mu) g(ct, z - \xi_1)]$$

so that

$$|\mathcal{R}_2^{N,2}(\mu, 0, t, z)| \leq \frac{K}{t^{1+\frac{\beta-\eta}{2}}} \mathbb{E} [W_2^\beta(\mu_0^N, \mu) g(ct, z - \xi_1)]$$

and

$$\int_{\mathbb{R}^d} |\mathcal{R}_2^{N,2}(\mu, 0, t, z)| |z|^2 dz \leq \frac{K}{t^{1+\frac{\beta-\eta}{2}}} \varepsilon_N^{\frac{\beta}{2}}$$

for any  $\beta \in [0, \eta)$ . For the last term, from (3.15), we directly get

$$|\mathcal{R}_2^{N,3}(\mu, 0, t, z)| \leq \frac{K}{t^{1-\frac{\eta}{2}}} N^{-1} \mathbb{E} [g(ct, z - \xi_1)]$$

so that

$$\int_{\mathbb{R}^d} |\mathcal{R}_2^{N,3}(\mu, 0, t, z)| |z|^2 dz \leq \frac{K}{t^{1-\frac{\eta}{2}}} N^{-1}.$$

Gathering the previous estimates, we thus conclude

$$\forall \beta \in [0, \eta), \quad \int_{\mathbb{R}^d} |\mathcal{R}_2^N(\mu, 0, t, z)| |z|^2 dz \leq \frac{K}{t^{1+\frac{\beta-\eta}{2}}} \varepsilon_N^\beta.$$

**5.2. Proof of Theorem 3.6.** For a fixed function  $\phi$  in  $\mathcal{C}_{b,\alpha}^2(\mathcal{P}_2(\mathbb{R}^d), \mathbb{R})$ , we consider the following PDE on the Wasserstein space

$$(5.16) \quad (\partial_t + \mathcal{L}_t)U = 0, \quad U(T, \cdot) = \phi(\cdot),$$

where the operator  $\mathcal{L}_t$  is given by (3.17). From Theorem 3.3 in [CdRF18], under **(HE)** and **(HR)**, there exists a unique solution  $U \in \mathcal{C}^{1,2}([0, T] \times \mathcal{P}_2(\mathbb{R}^d))$  to the above PDE (5.16) given by

$$\forall(t, \mu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d), \quad U(t, \mu) = \phi([X_T^{t,\mu}]).$$

Again, we will work with an approximation sequence  $(U^{(m)})_{m \geq 0}$  of  $U$  defined by

$$(5.17) \quad \forall(t, \mu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d), \quad U^{(m)}(t, \mu) = \phi([X_T^{t,\mu,(m)}]),$$

where  $X^{t,\mu,(m)}$  is the unique weak solution to the SDE (4.5) starting from the initial distribution  $\mu$  at time  $t$ . From Proposition 2.3 (see also eq. (2.11)) and from the estimates (4.16) to (4.23) and (4.24), we derive that for each positive integer  $m$ ,  $U^{(m)} \in \mathcal{C}_f^{1,2}([0, T] \times \mathcal{P}_2(\mathbb{R}^d))$  and with bounded derivatives (uniformly in  $m$ ) satisfying the following estimates: there exists  $C := C(\mathbf{(HE)}, \mathbf{(HR)}, T) > 0$  such that for all  $(t, \mu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)$  and  $v, v'$  in  $\mathbb{R}^d$ :

$$(5.18) \quad \partial_v^n \partial_\mu U^{(m)}(t, \mu)(v) \leq C(T-t)^{-\frac{n+1-\alpha}{2}}, \quad n = 0, 1, \quad \partial_\mu^2 U^{(m)}(t, \mu)(v, v') \leq C(T-t)^{-1+\frac{\alpha}{2}}.$$

Note carefully that the time singularities appearing in the previous bounds on the first and second Lions' derivatives of  $U^{(m)}$  are integrable over  $[0, T]$ . Moreover, from the estimates (4.16) to (4.23) and the expressions of  $\partial_v^n \partial_\mu U^{(m)}(t, \mu)(v)$  stemming from Proposition 2.3, relabelling the indices if necessary, one may assert that the sequence  $(U^{(m)}(t, \mu), \mathcal{L}_t U^{(m)}(t, \mu))_{m \geq 0}$  converges towards  $(U(t, \mu), \mathcal{L}_t U(t, \mu))$  for every  $(t, \mu)$  in  $[0, T] \times \mathcal{P}_2(\mathbb{R}^d)$ .

On the one hand, from the standard Itô's formula, we have

$$\begin{aligned} U^{(m)}(t, \mu_t^N) &= U^{(m)}(0, \mu_0^N) + \frac{1}{N} \sum_{i=1}^N \int_0^t \sigma(s, X_s^i, \mu_s) \partial_\mu U^{(m)}(s, X_s^i, \mu_s^N)(X_s^i) dW_s^i \\ &\quad + \frac{1}{2N^2} \sum_{i=1}^N \int_0^t \text{Tr} \left[ a(s, X_s^i, \mu_s^N) \partial_\mu^2 U^{(m)}(s, X_s^i, \mu_s^N)(X_s^i, X_s^i) \right] ds \\ &\quad + \int_0^t (\partial_s + \mathcal{L}_s) U^{(m)}(s, \mu_s^N) ds. \end{aligned}$$

On the other hand, from the Markov property stemming from the well-posedness of the martingale problem related to (1.1)

$$U(t, [X_t^{s,\xi}]) = \phi([X_T^{t,X_t^{s,\xi}}]) = \phi([X_T^{s,\xi}]) = U(s, \mu),$$

for all  $s$  in  $[0, t]$  and especially for  $s = 0$ . Hence

$$\begin{aligned} U^{(m)}(t, \mu_t^N) - U(t, \mu_t) &= (U^{(m)}(0, \mu_0^N) - U(0, \mu)) + \frac{1}{N} \sum_{i=1}^N \int_0^t \sigma(s, X_s^i, \mu_s) \partial_\mu U^{(m)}(s, \mu_s^N)(X_s^i) dW_s^i \\ (5.19) \quad &\quad + \frac{1}{2N^2} \sum_{i=1}^N \int_0^t \text{Tr} \left[ a(s, X_s^i, \mu_s^N) \partial_\mu^2 U^{(m)}(s, \mu_s^N)(X_s^i, X_s^i) \right] ds \\ &\quad + \int_0^t (\partial_s + \mathcal{L}_s) U^{(m)}(s, \mu_s^N) ds. \end{aligned}$$

Using the Burkholder-Davis-Gundy inequality, the estimates (5.18), we get

$$\mathbb{E}[|U^{(m)}(t, \mu_t^N) - U(t, \mu_t)|] \leq \mathbb{E}[|U^{(m)}(0, \mu) - U(0, \mu_0^N)|] + \frac{C}{N} + \int_0^t \mathbb{E}[|(\partial_s + \mathcal{L}_s) U^{(m)}(s, \mu_s^N)|] ds$$

where  $T \mapsto C := C(\mathbf{(HE)}, \mathbf{(HR)}, T)$  is a non-decreasing positive function. Hence, sending  $m \uparrow \infty$  together with dominated convergence theorem and the fact that  $U$  solves (5.16) and then choosing  $t = T$

$$\mathbb{E}[|U(T, \mu_T) - U(T, \mu_T^N)|] \leq \mathbb{E}[|U(0, \mu) - U(0, \mu_0^N)|] + \frac{C}{N} \leq \frac{C}{T^{1-\frac{\alpha}{2}}} \mathbb{E}[W_1(\mu, \mu_0^N)] + \frac{C}{N},$$

where we used the fact that, for any  $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ , the flat derivative  $y \mapsto [\delta U / \delta m](0, \mu)(y)$  is a Lipschitz function uniformly in  $\mu$  with a Lipschitz modulus bounded by  $\sup_{\mu \in \mathcal{P}_2(\mathbb{R}^d)} |\partial_\mu U(0, \mu)(\cdot)|_\infty$  together with

(5.18). The conclusion of point (ii) then follows from Theorem 1 of [FG15] using the fact that the initial distribution  $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ .

In order to obtain the point (i), we first take the expectation in (5.19). Doing so we get rid of the martingale part therein and hence obtain

$$(5.20) \quad \left| U(t, \mu_t) - \mathbb{E} \left[ U^{(m)}(t, \mu_t^N) \right] \right| \leq \left| U(0, \mu) - \mathbb{E} \left[ U^{(m)}(0, \mu_0^N) \right] \right| + \frac{C}{N} + \int_0^t \mathbb{E} \left[ \left| (\partial_s + \mathcal{L}_s) U^{(m)}(s, \mu_s^N) \right| \right] ds.$$

It remains to establish an error bound for the quantity  $\mathbb{E}[U^{(m)}(0, \mu_0^N)] - U^{(m)}(0, \mu)$ . We follow similar lines of reasonings as those employed in (5.11). One may also refer to [CST19] for a similar argument. We briefly repeat the proof here for sake of completeness. From the mean-value theorem and the exchangeability in law of  $(\xi^i)_{1 \leq i \leq N}$

$$\begin{aligned} & \mathbb{E}[U^{(m)}(0, \mu_0^N) - U^{(m)}(0, \mu)] \\ &= \int_0^1 \int_{\mathbb{R}^d} \mathbb{E} \left[ \frac{\delta}{\delta m} U^{(m)}(0, \mu_0^{\lambda_1, N})(y) (\mu_0^N - \mu)(dy) \right] d\lambda_1 \\ &= \int_0^1 \mathbb{E} \left[ \frac{\delta}{\delta m} U^{(m)}(0, \mu_0^{\lambda_1, N})(\xi^1) - \frac{\delta}{\delta m} U^{(m)}(0, \mu_0^{\lambda_1, N})(\tilde{\xi}) \right] d\lambda_1 \\ &= \int_0^1 \mathbb{E} \left[ \frac{\delta}{\delta m} U^{(m)}(0, \tilde{\mu}_0^{\lambda_1, N})(\tilde{\xi}) - \frac{\delta}{\delta m} U^{(m)}(0, \mu_0^{\lambda_1, N})(\tilde{\xi}) \right] d\lambda_1 \\ &= \frac{1}{N} \int_{[0,1]^2} \lambda_1 \mathbb{E} \left[ \frac{\delta^2}{\delta m^2} U^{(m)}(0, \tilde{\mu}_0^{\lambda_1, \lambda_2, N})(\tilde{\xi}, \tilde{\xi}) - \frac{\delta^2}{\delta m^2} U^{(m)}(0, \tilde{\mu}_0^{\lambda_1, \lambda_2, N})(\tilde{\xi}, \xi^1) \right] d\lambda_1 d\lambda_2. \end{aligned}$$

where we used the notations:  $\mu_0^{\lambda_1, N} := \lambda_1 \mu_0^N + (1 - \lambda_1) \mu$ ,  $\tilde{\mu}_0^{\lambda_1, N} := \lambda_1 \tilde{\mu}_0^N + (1 - \lambda_1) \mu$ ,  $\tilde{\mu}_0^N := \mu_0^N + \frac{1}{N}(\delta_{\tilde{\xi}} - \delta_{\xi^1})$ ,  $\tilde{\mu}_0^{\lambda_1, \lambda_2, N} := \lambda_2 \tilde{\mu}_0^{\lambda_1, N} + (1 - \lambda_2) \mu_0^{\lambda_1, N}$ ,  $\tilde{\xi}$  being a random variable independent of  $(\xi^i)_{1 \leq i \leq N}$  with law  $\mu$ . The previous identity together with (5.18) finally yields

$$\left| U^{(m)}(0, \mu_0) - \mathbb{E} \left[ U^{(m)}(0, \mu_0^N) \right] \right| \leq T^{-1 + \frac{\alpha}{2}} \frac{K}{N}$$

for some positive constant  $K$  which is independent of  $m$ . Plugging the previous bound in (5.20), then letting  $m \uparrow \infty$  and finally choosing  $t = T$  allows to conclude.

**5.3. Proof of Theorem 3.8.** As already mentioned in the introduction, the strategy relies on Zvonkin's transform. To do so, we introduce the following PDE

$$(5.21) \quad (\partial_t + \mathcal{L}_t)U(t, x, \mu) = b(t, x, \mu), \quad U(T, \cdot, \cdot) = 0,$$

where the operator  $(\mathcal{L}_t)_{t \geq 0}$  is given by (1.4). From Theorem 3.3 in [CdRF18], there exists a unique solution  $U \in \mathcal{C}^{1,2,2}([0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$  to the above PDE (5.21) given by

$$\forall (t, x, \mu) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d), \quad U(t, x, \mu) = \mathbb{E} \left[ \int_t^T b(s, X_s^{t,x,\mu}, [X_s^{t,\mu}]) ds \right].$$

Following the lines of Proposition 6.1 in [CdRF18] we readily obtain that  $U$  satisfies the following estimates: there exists a positive constant  $C := C((\mathbf{HE}), (\mathbf{HR}), T, \eta)$  such that for all  $(t, x, \mu) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$

$$(5.22) \quad |\partial_v^n \partial_\mu U(t, x, \mu)(v)| + |\partial_x^{n+1} U(t, x, \mu)| \leq C(T - t)^{\frac{1-n+\eta}{2}}, \quad n = 0, 1.$$

From (6.11), the map  $x \mapsto \partial_\mu U(t, x, \mu)(v)$  is continuously differentiable for all  $(t, \mu, v)$  in  $[0, T] \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d$ . Moreover, from (6.12), its derivative satisfies

$$(5.23) \quad \partial_x \partial_\mu U(t, x, \mu)(v) \leq C(T - t)^{\frac{\eta}{2}}.$$

In order to apply Zvonkin's transform, we need to investigate the quantity  $\partial_\mu^2 U$ , which, under the sole assumptions **(HE)** and **(HR)**, is out of reach with our procedure. We thus work with a suitable approximate version  $U^{(m)}$  of  $U$  and estimate such a quantity uniformly with respect to the mollifying procedure. To do so, we come back to the approximation procedure described at the *step 1* of the section 4.2. For each  $m > 0$ , we are hence given the unique weak solution  $X^{t,\mu,(m)}$  and  $X^{t,x,\mu,(m)}$  of the SDEs

(4.5) and (4.6). Having such processes at hand, we then define for each positive integer  $m$  the map

$$(5.24) \quad \begin{aligned} U^{(m)} &: [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \ni (t, x, \mu) \mapsto U^{(m)}(t, x, \mu) \\ &:= \mathbb{E} \left[ \int_t^T b(s, X_s^{t,x,\mu,(m)}, [X_s^{t,\mu,(m)}]) ds \right] = \int_{\mathbb{R}^d} \int_t^T b(s, y, [X_s^{t,\mu,(m)}]) p_m(\mu, t, s, x, y) ds dy \end{aligned}$$

where, according to our notations,  $p_m(\mu, t, s, x, \cdot)$  denotes the density function of the r.v.  $X_s^{t,x,\mu,(m)}$ . The key point is that from Proposition 4.1 such a sequence of maps  $(U^{(m)})_{m>0}$  satisfies under **(HE)** and **(HR)**:

- (i)  $U^{(m)}$  is in  $\mathcal{C}_f^{1,2,2}([0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$ ,
- (ii) There exists  $K := K(\mathbf{(HE)}, \mathbf{(HR)}, T) > 0$  such that for all positive integer  $m$ , for all  $(t, x, \mu, v, v')$  in  $(0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times (\mathbb{R}^d)^2$

$$\begin{aligned} |\partial_\mu U^{(m)}(t, x, \mu)(v)| &\leq K(T-t)^{\frac{1+\eta}{2}}, \quad |\partial_\mu^2 U^{(m)}(t, x, \mu)(v, v')| \leq K(T-t)^{\frac{\eta}{2}}, \\ |\partial_t U^{(m)}(t, x, \mu)(v)| &\leq K, \quad |\partial_x^n U^{(m)}(t, x, \mu)| \leq K(T-t)^{\frac{2-n+\eta}{2}} \end{aligned}$$

for  $n = 0, 1, 2$ .

$$(iii) \quad \forall (t, x, \mu) \in (0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d), \quad \lim_{m \uparrow \infty} U^{(m)}(t, x, \mu) = U(t, x, \mu),$$

$$(iv) \quad \forall (t, x, \mu) \in (0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d), \quad \lim_{m \uparrow \infty} (\partial_t + \mathcal{L}_t)U^{(m)}(t, x, \mu) = (\partial_t + \mathcal{L}_t)U(t, x, \mu) = b(t, x, \mu),$$

along a subsequence.

Indeed, the first point follows from the fact that for any positive integer  $m$  the map  $(t, x, \mu) \mapsto p_m(\mu, t, s, x, y)$  belongs to  $\mathcal{C}_f^{1,2,2}([0, s] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$  together with Proposition 2.3 applied to the map  $\mu \mapsto b(t, x, \mu)$ . The second point is a consequence of the estimates (4.16), (4.17), (4.24) and (4.10) as well as Proposition 4.1 which allow to derive the following estimates: there exists  $C := C(\mathbf{(HE)}, \mathbf{(HR)}, T) > 0$  such that

$$|\partial_v^n \partial_\mu b(s, y, [X_s^{t,\mu,(m)}])| \leq C(t-s)^{\frac{-1-n+\eta}{2}}, \quad n = 0, 1, \quad |\partial_\mu^2 b(s, y, [X_s^{t,\mu,(m)}])| \leq C(t-s)^{-1+\frac{\eta}{2}}$$

and

$$|\partial_t b(s, y, [X_s^{t,\mu,(m)}])| \leq C(t-s)^{-1+\frac{\eta}{2}}.$$

The third point follows from the fact that  $\lim_m W_2([X_s^{t,\mu,(m)}], [X_s^{t,\mu}]) = 0$ ,  $\mathcal{P}_2(\mathbb{R}^d) \ni \mu \mapsto b(t, x, \mu)$  is continuous and that the sequence  $(p_m(\mu, t, s, x, y))_{m \geq 0}$  converges to  $p(\mu, s, t, x, y)$  for any fixed  $\mu, x, y$  and  $t < s$  combined with dominated convergence theorem. Eventually, we can deduce from the above estimates, from Proposition 2.3 (see also eq. (2.11) in the proof) and from estimates (4.16) to (4.23) that one can extract converging subsequences of  $(\partial_t U^{(m)}(t, x, \mu), \mathcal{L}_t U^{(m)}(t, x, \mu))_{m \geq 0}$  to  $\partial_t U(t, x, \mu)$  and  $\mathcal{L}_t U(t, x, \mu)$ . Hence, relabelling the indices if necessary, one may assert that

$$(5.25) \quad \forall (t, x, \mu) \in (0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d), \quad \lim_{m \uparrow \infty} (\partial_t + \mathcal{L}_t)U^{(m)}(t, x, \mu) = (\partial_t + \mathcal{L}_t)U(t, x, \mu) = b(t, x, \mu).$$

We are now ready to complete the proof of Theorem 3.8. From the chain rule formula of Proposition 2.1 we have

$$(5.26) \quad \begin{aligned} \bar{X}_t^i - U^{(m)}(t, \bar{X}_t^i, \mu_t) \\ = \xi^i - U^{(m)}(0, \xi^i, \mu_0) - \int_0^t \left( \left[ \partial_x U^{(m)} - \mathbf{1} \right] \sigma \right) (s, \bar{X}_s^i, \mu_s) dW_s^i - \int_0^t [(\partial_s + \mathcal{L}_s)U^{(m)} - b](s, \bar{X}_s^i, \mu_s) ds \end{aligned}$$

and from classical Itô's formula

$$(5.27) \quad \begin{aligned} X_t^i - U^{(m)}(t, X_t^i, \mu_t^N) \\ = \xi^i - U^{(m)}(0, \xi^i, \mu_0^N) - \int_0^t \left( \left[ \partial_x U^{(m)} - \mathbf{1} \right] \sigma \right) (s, X_s^i, \mu_s^N) dW_s^i - \int_0^t [(\partial_s + \mathcal{L}_s)U^{(m)} - b](s, X_s^i, \mu_s^N) ds \\ + \frac{1}{2N^2} \sum_{j=1}^N \int_0^t \text{Tr} \left[ a(s, X_s^i, \mu_s^N) \partial_\mu^2 U^{(m)}(s, X_s^i, \mu_s^N) (X_s^j, X_s^j) \right] ds \\ + \frac{1}{N} \sum_{j=1}^N \int_0^t \partial_\mu U^{(m)}(s, X_s^i, \mu_s^N) (X_s^j) dW_s^j. \end{aligned}$$



Taking the difference between (5.26) and (5.27) yields

$$\begin{aligned}
(5.28) \quad \bar{X}_t^i - X_t^i &= U^{(m)}(0, \xi^i, \mu_0^N) - U^{(m)}(0, \xi^i, \mu_0) + U^{(m)}(t, \bar{X}_t^i, \mu_t) - U^{(m)}(t, X_t^i, \mu_t^N) \\
&\quad - \frac{1}{2N^2} \sum_{j=1}^N \int_0^t \text{Tr} \left[ \sigma \sigma^*(s, X_s^i, \mu_s^N) \partial_\mu^2 U^{(m)}(s, X_s^i, \mu_s^N)(X_s^j, X_s^j) \right] ds \\
&\quad - \int_0^t \left( \left[ \partial_x U^{(m)} - \mathbf{1} \right] \sigma \right) (s, \bar{X}_s^i, \mu_s) - \left( \left[ \partial_x U^{(m)} - \mathbf{1} \right] \sigma \right) (s, X_s^i, \mu_s^N) dW_s^i \\
&\quad - \frac{1}{N} \sum_{j=1}^N \int_0^t \partial_\mu U^{(m)}(s, X_s^i, \mu_s^N)(X_s^j) dW_s^j \\
&\quad - \int_0^t [(\partial_s + \mathcal{L}_s)U^{(m)} - b](s, X_s^i, \mu_s^N) ds + \int_0^t [(\partial_s + \mathcal{L}_s)U^{(m)} - b](s, \bar{X}_s^i, \mu_s) ds.
\end{aligned}$$

On the one hand, from the estimates (ii) and the boundedness of  $\sigma$  it follows that the terms in the third and fourth lines appearing in the right-hand side of the above identity are true martingales and that the term in the second line of the above r.h.s. is a  $\mathcal{O}(N^{-1})$ . On the other hand, we have from the point (iv) together with the estimates (ii) and the dominated convergence theorem that, along a subsequence, for any  $0 \leq t < T$

$$\begin{aligned}
(5.29) \quad \lim_{m \uparrow \infty} &\left\{ \max_{1 \leq i \leq N} \mathbb{E} \left[ \left( \int_0^t |[(\partial_s + \mathcal{L}_s)U^{(m)} - b](s, X_s^i, \mu_s^N)| ds \right)^2 \right] \right. \\
&\quad \left. + \max_{1 \leq i \leq N} \mathbb{E} \left[ \left( \int_0^t |[(\partial_s + \mathcal{L}_s)U^{(m)} - b](s, \bar{X}_s^i, \mu_s)| ds \right)^2 \right] \right\} = 0.
\end{aligned}$$

Therefore, coming back to (5.28), taking the square of the norm, summing over  $i$ , using then Burkholder-Davis-Gundy's inequality and finally letting  $m \uparrow \infty$  (along the considered subsequence) give

$$\begin{aligned}
\mathbb{E} \left[ \frac{1}{N} \sum_{i=1}^N |\bar{X}_t^i - X_t^i|^2 \right] &\leq C \left\{ \mathbb{E} [|U(0, \xi^1, \mu_0^N) - U(0, \xi^1, \mu_0)|^2] + \frac{1}{N} \sum_{i=1}^N \mathbb{E} [|U(t, X_t^i, \mu_t) - U(t, \bar{X}_t^i, \mu_t^N)|^2] \right. \\
&\quad \left. + \frac{1}{N} \sum_{i=1}^N \int_0^t \mathbb{E} \left[ |([\partial_x U - \mathbf{1}] \sigma)(s, \bar{X}_s^i, \mu_s) - ([\partial_x U - \mathbf{1}] \sigma)(s, X_s^i, \mu_s^N)|^2 \right] ds + \frac{C}{N} \right\}.
\end{aligned}$$

Using (5.22), (5.23) and the uniform Lipschitz regularity of  $(x, \mu) \mapsto \sigma(t, x, \mu)$ , we deduce that there exists  $C_T := C((\mathbf{HE}), (\mathbf{HR}), T) > 0$  satisfying  $C_T \downarrow 0$  when  $T \downarrow 0$  such that

$$\begin{aligned}
\mathbb{E} \left[ \frac{1}{N} \sum_{i=1}^N |\bar{X}_t^i - X_t^i|^2 \right] &\leq C_T \left\{ \mathbb{E} [W_2^2(\mu_0, \mu_0^N)] + \frac{1}{N} \sum_{i=1}^N \mathbb{E} [|\bar{X}_t^i - X_t^i|^2] + \mathbb{E} [W_2^2(\mu_t, \mu_t^N)] \right. \\
&\quad \left. + \frac{1}{N} \sum_{i=1}^N \int_0^t (\mathbb{E} [|\bar{X}_s^i - X_s^i|^2] + \mathbb{E} [W_2^2(\mu_s, \mu_s^N)]) ds \right\} + \frac{C}{N}.
\end{aligned}$$

We now introduce  $\bar{\mu}_t^N := N^{-1} \sum_{i=1}^N \delta_{\bar{X}_t^i}$ ,  $t \in [0, T]$ , the empirical measure associated with the i.i.d. r.v.  $(\bar{X}_t^i)_{1 \leq i \leq N}$ . Noticing now that for all  $t \in [0, T]$ ,

$$(5.30) \quad W_2^2(\mu_t, \mu_t^N) \leq 2W_2^2(\mu_t, \bar{\mu}_t^N) + 2W_2^2(\bar{\mu}_t^N, \mu_t^N) \leq 2W_2^2(\mu_t, \bar{\mu}_t^N) + \frac{2}{N} \sum_{i=1}^N |X_t^i - \bar{X}_t^i|^2$$

choosing  $T$  small enough<sup>2</sup> so that  $C_T \leq 1/4$  and using Gronwall's lemma lead to

$$\frac{1}{N} \sum_{i=1}^N \mathbb{E} [|\bar{X}_t^i - X_t^i|^2] \leq C_T \left\{ \mathbb{E} [W_2^2(\mu_0, \mu_0^N)] + \sup_{0 \leq t \leq T} \mathbb{E} [W_2^2(\mu_t, \bar{\mu}_t^N)] + \int_0^T \mathbb{E} [W_2^2(\mu_s, \bar{\mu}_s^N)] ds \right\} + \frac{C}{N}.$$

Finally, the strong well-posedness of the SDEs (3.26) and (1.2) together with the exchangeability of  $(\xi^i, W^i)_{1 \leq i \leq N}$  imply that the random variables  $(\bar{X}^i, X^i)_{1 \leq i \leq N}$  are i.i.d. so that  $N^{-1} \sum_{i=1}^N \mathbb{E} [|\bar{X}_t^i - X_t^i|^2] = \mathbb{E} [|\bar{X}_t^1 - X_t^1|^2]$ . Hence,

$$\sup_{0 \leq t \leq T} \mathbb{E} [|\bar{X}_t^i - X_t^i|^2] \leq C_T \left\{ \mathbb{E} [W_2^2(\mu_0, \bar{\mu}_0^N)] + \sup_{0 \leq t \leq T} \mathbb{E} [W_2^2(\mu_t, \bar{\mu}_t^N)] \right\} + \frac{C}{N} \leq C \varepsilon_N$$

<sup>2</sup>There exists  $T = T((\mathbf{HE}), (\mathbf{HR})) > 0$  such that for all  $T \leq \mathcal{T}$  we have  $C_T \leq 1/4$ .

where we used Theorem 1 in Fournier and Guillin [FG15] for the last inequality. One may then extend the above estimate to an arbitrary finite time horizon  $T$  by considering a partition of the time interval  $[0, T]$  with a sufficiently small time mesh and repeating the above argument. Taking expectation in (5.30), one then concludes that a similar estimate holds for the quantity

$$\sup_{0 \leq t \leq T} \mathbb{E}[W_2^2(\mu_t, \mu_t^N)].$$

Finally, coming back to (5.28), one can apply similar lines of reasonings but taking first the square of the norm, then the supremum in time and obtain, thanks to the above estimate, for  $T$  small enough

$$\max_{1 \leq i \leq N} \mathbb{E} \left[ \sup_{0 \leq t \leq T} |X_t^i - \bar{X}_t^i|^2 \right] \leq C_T \left\{ \mathbb{E} [W_2^2(\mu_0, \bar{\mu}_0^N)] + \mathbb{E} \left[ \sup_{0 \leq t \leq T} W_2^2(\mu_t, \bar{\mu}_t^N) \right] + \int_0^T \mathbb{E} [W_2^2(\mu_s, \bar{\mu}_s^N)] ds \right\} + \frac{C}{N}.$$

The first and third terms appearing in the right-hand side of the above inequality are handled using Theorem 1 in [FG15]. The second term provides the rate of convergence and require the following lemma borrowed from [BCCH19].

**Lemma 5.1.** Let  $\{Y^i\}_{1 \leq i \leq N}$  be an i.i.d. sequence of copies of a process  $Y$  satisfying  $\sup_{t \in [0,1]} \mathbb{E}|Y_t|^q < +\infty$ , for some  $q > 4$  and for some  $p > 2$ :

$$(5.31) \quad \begin{aligned} \mathbb{E}[|Y_s - Y_r|^p | Y_s - Y_t|^p] &\leq C|t - r|^2, \quad \text{for } 0 \leq r < s < t \leq 1; \\ \mathbb{E}[|Y_t - Y_s|^p] &\leq C|t - s|, \quad \text{for } 0 \leq s \leq t \leq 1; \\ \mathbb{E}[|Y_t - Y_s|^2] &\leq C|t - s|, \quad \text{for } 0 \leq s \leq t \leq 1. \end{aligned}$$

Then, introducing the notations  $\nu_s := [Y_s]$  and  $\bar{\nu}_s^N := N^{-1} \sum_{i=1}^N \delta_{Y_s^i}$ , there exists  $C > 0$  such that

$$(5.32) \quad \mathbb{E} \left[ \sup_{0 \leq s \leq T} W_2^2(\bar{\nu}_s^N, \nu_s) \right] \leq C\sqrt{\varepsilon_N}.$$

We thus derive

$$\max_{1 \leq i \leq N} \mathbb{E} \left[ \sup_{0 \leq t \leq T} |X_t^i - \bar{X}_t^i|^2 \right] \leq C\sqrt{\varepsilon_N}.$$

for some positive constant  $C := C((\mathbf{HE}), (\mathbf{HR}), T, M_q(\mu))$ . Taking first the supremum in time and then expectation in (5.30), one then concludes that a similar estimate holds for the quantity

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} W_2^2(\mu_t, \mu_t^N) \right].$$

## 6. APPENDIX

We here collect the proofs of some technical results used throughout the article.

**6.1. Proof of Lemma 3.1.** We here freely use the notations as well as the results of [CdRF18]. Since the arguments and the computations are quite similar to those employed in [CdRF18], we will deliberately be short on some technical details. For a map  $h : \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$ , we use the notation  $\Delta_{\mu, \mu'} h(\mu) = h(\mu) - h(\mu')$  for  $\mu, \mu' \in \mathcal{P}_2(\mathbb{R}^d)$ . First, from [CdRF18], the following representation holds:

$$(6.1) \quad \partial_\mu p(\mu, s, t, x, z)(y) = \sum_{k \geq 0} \left( [\partial_\mu \hat{p} + p \otimes \partial_\mu \mathcal{H}] \otimes \mathcal{H}^{(k)} \right) (\mu, s, t, x, z)(y)$$

with

$$(6.2) \quad \partial_\mu \hat{p}^{y'}(\mu, s, r, t, x, z)(y) = -\frac{1}{2}(H_2.g) \left( \int_r^t a(v, y', [X_v^{s,\xi}]) dv, z - x \right) \cdot \int_r^t \partial_\mu a(v, y', [X_v^{s,\xi}]) (y) dv,$$

$$(6.3) \quad |\partial_\mu \mathcal{H}(\mu, s, r, t, x, z)(y)| \leq K \left( \frac{1}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{\frac{1}{2}}} \wedge \frac{1}{(t-r)(r-s)^{\frac{1-\eta}{2}}} \right) g(c(t-r), z-x)$$

which yield the absolute convergence of the infinite series (6.1). Moreover, the following estimates hold:

$$(6.4) \quad |\partial_\mu a(r, y', [X_r^{s,\xi}]) (y)| \leq \frac{K}{(r-s)^{\frac{1-\eta}{2}}},$$

$$(6.5) \quad |\Delta_{\mu, \mu'} a_{i,j}(r, y', [X_r^{s,\xi}]) (y)| \leq K \frac{W_2^\beta(\mu, \mu')}{(r-s)^{\frac{\beta-\eta}{2}}},$$

and for any  $\beta \in [0, 1]$

(6.6)

$$|\Delta_{\mu, \mu'} \mathcal{H}(\mu, s, r, t, y, z)| \leq K \left\{ \frac{1}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{\frac{\beta}{2}}} \wedge \frac{1}{(t-r)(r-s)^{\frac{\beta-\eta}{2}}} \right\} W_2^\beta(\mu, \mu') g(c(t-r), z-y),$$

$$(6.7) \quad |\Delta_{\mu, \mu'} \partial_\mu b_i(r, y, [X_r^{s, \xi}])(v)| + |\Delta_{\mu, \mu'} \partial_\mu a_{i,j}(r, y, [X_r^{s, \xi}])(v)| \leq K \frac{1}{(r-s)^{\frac{1+\beta-\eta}{2}}} W_2^\beta(\mu, \mu'),$$

$$(6.8) \quad \begin{aligned} & |\Delta_{\mu, \mu'} \partial_\mu \mathcal{H}(\mu, s, r, t, y, z)(v)| \\ & \leq K \left\{ \frac{1}{(t-r)(r-s)^{\frac{1+\beta-\eta}{2}}} \wedge \frac{1}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{\frac{1+\beta}{2}}} \right\} W_2^\beta(\mu, \mu') g(c(t-r), z-y). \end{aligned}$$

From (6.2), (6.4) and the space-time inequality (1.5), one gets

$$(6.9) \quad |\partial_x [\partial_\mu \widehat{p}^y](\mu, s, t, x, z)(y)| \leq \frac{K}{(t-s)^{1-\frac{\eta}{2}}} g(c(t-s), z-x)$$

and combining (6.5), (6.7) with the mean-value theorem

$$(6.10) \quad \forall \beta \in [0, \eta), \quad |\Delta_{\mu, \mu'} \partial_x [\partial_\mu \widehat{p}^y](\mu, s, t, x, z)(y)| \leq K \frac{W_2^\beta(\mu, \mu')}{(t-s)^{1+\frac{\beta-\eta}{2}}} g(c(t-s), z-x).$$

From (6.9) and standard computations, we derive the absolute (and uniform in  $x$ ) convergence of the series  $\sum_{k \geq 0} (\partial_x \partial_\mu \widehat{p} \otimes \mathcal{H}^{(k)})(\mu, s, t, x, z)(y)$  as well as the following Gaussian upper bound

$$\sum_{k \geq 0} |(\partial_x \partial_\mu \widehat{p} \otimes \mathcal{H}^{(k)})(\mu, s, t, x, z)(y)| \leq \frac{K}{(t-s)^{1-\frac{\eta}{2}}} g(c(t-s), z-x).$$

From (3.4) with  $n = 1$  and (6.3), dividing the time convolution into the two disjoint parts  $r \in [s, \frac{t+s}{2}]$  and  $r \in [\frac{t+s}{2}, t]$  in order to balance the time singularity, we derive:

$$|(\partial_x p \otimes \partial_\mu \mathcal{H})(\mu, s, t, x, z)(y)| \leq \frac{K}{(t-s)^{1-\frac{\eta}{2}}} g(c(t-s), z-x)$$

which yields, after some standard computations that we omit, the absolute (and uniform in  $x$ ) convergence of series  $\sum_{k \geq 0} ((\partial_x p \otimes \partial_\mu \mathcal{H}) \otimes \mathcal{H}^{(k)})(\mu, s, t, x, z)(y)$  as well as the following Gaussian upper bound:

$$\sum_{k \geq 0} |((\partial_x p \otimes \partial_\mu \mathcal{H}) \otimes \mathcal{H}^{(k)})(\mu, s, t, x, z)(y)| \leq \frac{K}{(t-s)^{1-\frac{\eta}{2}}} g(c(t-s), z-x).$$

We thus deduce that  $\mathbb{R}^d \ni x \mapsto \partial_\mu p(\mu, s, t, x, z)(y)$  given by (6.1) is continuously differentiable with

$$(6.11) \quad \partial_x [\partial_\mu p(\mu, s, t, x, z)](y) = \sum_{k \geq 0} \left( [\partial_x \partial_\mu \widehat{p} + \partial_x p \otimes \partial_\mu \mathcal{H}] \otimes \mathcal{H}^{(k)} \right)(\mu, s, t, x, z)(y)$$

and satisfies the following estimate

$$(6.12) \quad |\partial_x [\partial_\mu p(\mu, s, t, x, z)](y)| \leq \frac{K}{(t-s)^{1-\frac{\eta}{2}}} g(c(t-s), z-x).$$

From (6.11), the following relation holds:  $\partial_x [\partial_\mu p(\mu, s, t, x, z)](y) = (\partial_x \partial_\mu \widehat{p} + \partial_x p \otimes \partial_\mu \mathcal{H})(\mu, s, t, x, z)(y) + (\partial_x \partial_\mu p \otimes \mathcal{H})(\mu, s, t, x, z)(y)$  which in turn yields

$$\begin{aligned} \Delta_{\mu, \mu'} \partial_x [\partial_\mu p(\mu, s, t, x, z)](y) &= (\Delta_{\mu, \mu'} \partial_x \partial_\mu \widehat{p} + \Delta_{\mu, \mu'} [\partial_x p \otimes \partial_\mu \mathcal{H}])(\mu, s, t, x, z)(y) \\ &\quad + (\partial_x \partial_\mu p \otimes \Delta_{\mu, \mu'} \mathcal{H})(\mu, s, t, x, z)(y) + (\Delta_{\mu, \mu'} \partial_x \partial_\mu p \otimes \mathcal{H})(\mu, s, t, x, z)(y) \end{aligned}$$

and by a direct induction argument

$$\begin{aligned} & \Delta_{\mu, \mu'} \partial_x [\partial_\mu p(\mu, s, t, x, z)](y) \\ &= \sum_{k \geq 0} \left( (\Delta_{\mu, \mu'} \partial_x \partial_\mu \widehat{p} + \Delta_{\mu, \mu'} [\partial_x p \otimes \partial_\mu \mathcal{H}]) + (\partial_x \partial_\mu p \otimes \Delta_{\mu, \mu'} \mathcal{H}) \right) \otimes \mathcal{H}^{(k)}(\mu, s, t, x, z)(y). \end{aligned}$$

We now quantify each term appearing in the above series. First, from (6.10) and standard computations, one has

$$\sum_{k \geq 0} |(\Delta_{\mu, \mu'} \partial_x \partial_{\mu'} \widehat{p} \otimes \mathcal{H}^{(k)})(\mu, s, t, x, z)(y)| \leq K \frac{W_2^\beta(\mu, \mu')}{(t-s)^{1+\frac{\beta-\eta}{2}}} g(c(t-s), z-x).$$

Then, from (6.12) and (6.6), we obtain

$$\forall \beta \in [0, 2\eta), |(\partial_x \partial_{\mu} p \otimes \Delta_{\mu, \mu'} \mathcal{H})(\mu, s, t, x, z)(y)| \leq \frac{K}{(t-s)^{1+\frac{\beta}{2}-\eta}} W_2^\beta(\mu, \mu') g(c(t-s), z-x)$$

which in turn by induction yields

$$\sum_{k \geq 0} |(\partial_x \partial_{\mu} p \otimes \Delta_{\mu, \mu'} \mathcal{H}) \otimes \mathcal{H}^{(k)}(\mu, s, t, x, z)(y)| \leq \frac{K}{(t-s)^{1+\frac{\beta}{2}-\eta}} W_2^\beta(\mu, \mu') g(c(t-s), z-x).$$

From (4.20) and (6.3), we get

$$\forall \beta \in [0, 2\eta), |(\Delta_{\mu, \mu'} \partial_x p \otimes \partial_{\mu} \mathcal{H})(\mu, s, t, x, z)(y)| \leq \frac{K}{(t-s)^{1+\frac{\beta}{2}-\eta}} W_2^\beta(\mu, \mu') g(c(t-s), z-x)$$

and from (3.4) and (6.8)

$$\forall \beta \in [0, \eta), |(\partial_x p \otimes \Delta_{\mu, \mu'} \partial_{\mu} \mathcal{H})(\mu, s, t, x, z)(y)| \leq \frac{K}{(t-s)^{1+\frac{\beta-\eta}{2}}} W_2^\beta(\mu, \mu') g(c(t-s), z-x).$$

The two previous estimates imply

$$\forall \beta \in [0, \eta), \sum_{k \geq 0} |\Delta_{\mu, \mu'} [\partial_x p \otimes \partial_{\mu} \mathcal{H}] \otimes \mathcal{H}^{(k)}(\mu, s, t, x, z)(y)| \leq \frac{K}{(t-s)^{1+\frac{\beta-\eta}{2}}} W_2^\beta(\mu, \mu') g(c(t-s), z-x).$$

Gathering the previous estimates finally allows to conclude

$$\forall \beta \in [0, \eta), |\Delta_{\mu, \mu'} \partial_x [\partial_{\mu} p(\mu, s, t, x, z)](y)| \leq \frac{K}{(t-s)^{1+\frac{\beta-\eta}{2}}} W_2^\beta(\mu, \mu') g(c(t-s), z-x).$$

This completes the proof.

**6.2. Proof of Proposition 4.1.** The proof follows similar lines of reasonings as those used in the proof of Theorem 3.10 in our previous work [CdRF18]. We will thus omit technical details at some places and be short on some standard computations. First remark that we already know from [CdRF18] that  $(s, x, \mu) \mapsto p_m(\mu, s, t, x, z) \in \mathcal{C}^{1,2,2}([0, t] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$  for all  $m \geq 1$ . It thus suffices to prove that  $\mu \mapsto \partial_{\mu} p_m(\mu, s, t, x, z)(v)$  is continuously  $L$ -differentiable with a derivative  $\partial_{\mu}^2 p_m(\mu, s, t, x, z)(v, v')$  being continuous with respect to  $s, x, \mu, v, v'$  for the product topology. We proceed by induction on  $m$ . For  $m = 1$ , observe that  $p_1(\mu, s, t, x, z) = \sum_{k \geq 0} (\widehat{p}_1 \otimes \mathcal{H}_1^{(k)})(\mu, s, t, x, z)$ , where we emphasise from the very definition of our iterative scheme that  $\widehat{p}_1$  and  $\mathcal{H}_1$  do not depend on the law  $\mu$  but only on the initial probability measure  $\mathbb{P}^{(0)}$  of the iterative scheme. Hence,  $\mu \mapsto p_1(\mu, s, t, x, z)$  is two times continuously  $L$ -differentiable with  $\partial_{\mu}^2 p_1(\mu, s, t, x, z)(v) \equiv 0$  and  $(s, x, \mu, v) \mapsto \partial_{\mu}^2 p_1(\mu, s, t, x, z)(v)$  is obviously continuous.

Assuming that the induction hypothesis is valid at step  $m$ , we then remark that if  $(s_n, x_n, \mu_n)_{n \geq 1}$  is a sequence of  $[0, t] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$  satisfying  $\lim_n |s_n - s| = \lim_n |x_n - x| = \lim_n W_2(\mu_n, \mu) = 0$  for some  $(s, x, \mu) \in [0, t] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$ , then, from the continuity of the map  $[0, t] \times \mathcal{P}_2(\mathbb{R}^d) \ni (s, \mu) \mapsto p_m(\mu, s, t, z)$ , (4.7), (4.8) and (4.9), we obtain  $\lim_n W_2([X_t^{s_n, \xi_n, (m)}], [X_t^{s, \xi, (m)}]) = 0$ , where  $[\xi_n] = \mu_n$  and  $[\xi] = \mu$ , so that  $\lim_n a(t, x_n, [X_t^{s_n, \xi_n, (m)}]) = a(t, x, [X_t^{s, \xi, (m)}])$  and  $\lim_n b(t, x_n, [X_t^{s_n, \xi_n, (m)}]) = b(t, x, [X_t^{s, \xi, (m)}])$ . Note that the same conclusion holds for the maps  $\frac{\delta^r}{\delta m^r} a$  and  $\frac{\delta^r}{\delta m^r} b$ , for  $r = 1, 2$ , instead of  $a$  and  $b$ .

We next apply Proposition 2.3 to the maps  $(s, \mu, x) \mapsto p_m(\mu, s, t, x, z) \in \mathcal{C}_f^{1,2,2}([0, t] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$  and  $m \mapsto a_{i,j}(t, x, m)$ ,  $b_i(t, x, m)$ , for  $(i, j) \in \{1, \dots, d\}^2$ . Note that from the estimates (4.10), (4.16), (4.17) and (4.24), the map  $[0, r] \times \mathcal{P}_2(\mathbb{R}^d) \times (s, x, \mu) \mapsto p_m(\mu, s, r, x, z)$  satisfies the integrability conditions of Proposition 2.3. We thus deduce that  $(s, \mu) \mapsto a(r, x, [X_r^{s, \xi, (m)}])$ ,  $b(r, x, [X_r^{s, \xi, (m)}]) \in \mathcal{C}_f^{1,2}([0, r] \times \mathcal{P}_2(\mathbb{R}^d))$  with derivatives satisfying

(6.13)

$$\partial_s a(r, x, [X_r^{s, \xi, (m)}]) = \int_{(\mathbb{R}^d)^2} \left( \frac{\delta}{\delta m} a(r, x, [X_r^{s, \xi, (m)}])(y') - \frac{\delta}{\delta m} a(r, x, [X_r^{s, \xi, (m)}])(x') \right) \partial_s p_m(\mu, s, r, x', y') dy' \mu(dx'),$$

$$\begin{aligned}
\partial_v^n [\partial_\mu a(r, x, [X_r^{s,\xi,(m)}])](v) &= \int \left( \frac{\delta}{\delta m} a(r, x, [X_r^{s,\xi,(m)}])(y') - \frac{\delta}{\delta m} a(r, x, [X_r^{s,\xi,(m)}])(v) \right) \partial_x^{1+n} p_m(\mu, s, r, v, y') dy' \\
(6.14) \quad &+ \int_{(\mathbb{R}^d)^2} \left( \frac{\delta}{\delta m} a(r, x, [X_r^{s,\xi,(m)}])(y') - \frac{\delta}{\delta m} a(r, x, [X_r^{s,\xi,(m)}])(x') \right) \partial_v^n [\partial_\mu p_m(\mu, s, r, x', y')](v) dy' \mu(dx')
\end{aligned}$$

and similarly

$$\begin{aligned}
\partial_s b(r, x, [X_r^{s,\xi,(m)}]) &= \int_{(\mathbb{R}^d)^2} \left( \frac{\delta}{\delta m} b(r, x, [X_r^{s,\xi,(m)}])(y') - \frac{\delta}{\delta m} b(r, x, [X_r^{s,\xi,(m)}])(x') \right) \partial_s p_m(\mu, s, r, x', y') dy' \mu(dx'), \\
(6.15) \quad &
\end{aligned}$$

$$\begin{aligned}
\partial_v^n [\partial_\mu b(r, x, [X_r^{s,\xi,(m)}])](v) &= \int \left( \frac{\delta}{\delta m} b(r, x, [X_r^{s,\xi,(m)}])(y') - \frac{\delta}{\delta m} b(r, x, [X_r^{s,\xi,(m)}])(v) \right) \partial_x^{1+n} p_m(\mu, s, r, v, y') dy' \\
&+ \int_{(\mathbb{R}^d)^2} \left( \frac{\delta}{\delta m} b(r, x, [X_r^{s,\xi,(m)}])(y') - \frac{\delta}{\delta m} b(r, x, [X_r^{s,\xi,(m)}])(x') \right) \partial_v^n [\partial_\mu p_m(\mu, s, r, x', y')](v) dy' \mu(dx').
\end{aligned}$$

As a consequence, the maps  $[0, r] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \ni (s, x, \mu, v) \mapsto \partial_s a(r, x, [X_r^{s,\xi,(m)}])$ ,  $\partial_v^n [\partial_\mu a(r, x, [X_r^{s,\xi,(m)}])](v)$  and  $[0, r] \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \ni (s, \mu, v) \mapsto \partial_s b(r, x, [X_r^{s,\xi,(m)}])$ ,  $\partial_v^n [\partial_\mu b(r, x, [X_r^{s,\xi,(m)}])](v)$  are continuous for  $n = 0, 1$ . From (2.11) and similar arguments, we also derive that  $[0, r] \times \mathcal{P}_2(\mathbb{R}^d) \times (\mathbb{R}^d)^2 \ni (s, \mu, v) \mapsto \partial_\mu^2 b(r, x, [X_r^{s,\xi,(m)}])(v)$ ,  $\partial_\mu^2 a(r, x, [X_r^{s,\xi,(m)}])(v)$  are continuous.

From (6.14), (6.15), the estimates (4.16) and the  $\eta$ -Hölder regularity of  $y \mapsto \frac{\delta}{\delta m} a(t, x, \mu)(y)$ ,  $\frac{\delta}{\delta m} b(t, x, \mu)(y)$ , we obtain

$$(6.16) \quad |\partial_v^n [\partial_\mu a(r, x, [X_r^{s,\xi,(m)}])](v)| + |\partial_v^n [\partial_\mu b(r, x, [X_r^{s,\xi,(m)}])](v)| \leq K(r-s)^{-\frac{1+n-\eta}{2}}$$

for some positive constant  $K := K(T, a, b, \frac{\delta}{\delta m} a, \frac{\delta}{\delta m} b)$  independent of  $m$ . We also know from [CdRF18] that the derivatives  $\partial_y^n [\partial_\mu p_m(\mu, s, t, x, z)](y)$ ,  $n = 0, 1$ , satisfy the following relation

$$(6.17) \quad \partial_y^n [\partial_\mu p_m(\mu, s, t, x, z)](y) = \sum_{k \geq 0} (\partial_y^n [\partial_\mu \hat{p}_m] + p_m \otimes \partial_y^n [\partial_\mu \mathcal{H}_m]) \otimes \mathcal{H}_m^{(k)}(\mu, s, t, x, z)(y)$$

with the following estimates: for all integer  $r$  and  $m \geq 1$ ,  $n = 0, 1$

$$(6.18) \quad |\partial_x^r \partial_y^n [\partial_\mu \hat{p}_m^y(\mu, s, t, x, z)](y)| \leq K(t-s)^{-\frac{1+r+n-\eta}{2}} g(c(t-s), z-x),$$

and

$$(6.19) \quad |\partial_x^r \partial_y^n [\partial_\mu \hat{p}_m^y(\mu, s, r, t, x, z)](y)| \leq K(r-s)^{-\frac{1+r+n-\eta}{2}} g(c(t-r), z-x)$$

$$(6.20)$$

$$|\partial_y^n [\partial_\mu \mathcal{H}_{m+1}(\mu, s, r, t, x, z)](y)| \leq K \left( \frac{1}{(t-r)^{1-\frac{\eta}{2}} (r-s)^{\frac{1+n}{2}}} \wedge \frac{1}{(t-r)(r-s)^{\frac{1+n-\eta}{2}}} \right) g(c(t-r), z-x)$$

for some positive constant  $K := K(T, a, b, \frac{\delta}{\delta m} a, \frac{\delta}{\delta m} b)$  independent of  $m$ . The estimates (6.18) (with  $n = 0$  and  $r = 1$ ), (4.10) (with  $n = 1$ ) and (6.20) (with  $n = 0$ ) together with the relation (6.17) (with  $n = 0$ ) allow to conclude that  $x \mapsto \partial_\mu p_m(\mu, s, t, x, z)(y)$  is continuously differentiable. Moreover, there exists a positive constant  $K := K(T, a, b, \frac{\delta}{\delta m} a, \frac{\delta}{\delta m} b, \eta)$  independent of  $m$ , such that

$$(6.21) \quad |\partial_x \partial_\mu p_m(\mu, s, t, x, z)(y)| \leq K(t-s)^{-1+\frac{\eta}{2}} g(c(t-s), z-x).$$

Since the previous argument is standard, we omit its proof. See also the proof of Lemma 3.1 for similar computations.

In order to obtain some estimates on  $\partial_\mu^2 b(r, x, [X_r^{s,\xi,(m)}])$ ,  $\partial_\mu^2 a(r, x, [X_r^{s,\xi,(m)}])$ , we make use of the relation (2.11) with  $h$  equals to  $b$  or  $a$  (component by component). Then, from the estimates (4.16), (6.21), the uniform  $\eta$ -Hölder regularity of the maps  $v \mapsto \frac{\delta^r}{\delta m^r} b(t, x, \mu)(v)$ ,  $\frac{\delta^r}{\delta m^r} a(t, x, \mu)(v)$ ,  $r = 1, 2$ , and the space-time inequality (1.5), we deduce that the following estimate holds: there exists a positive constant  $K := K((\mathbf{HE}), (\mathbf{HR}), T)$  that do not depend on  $m$  such that

$$\begin{aligned}
(6.22) \quad &|\partial_\mu^2 b(r, x, [X_r^{s,\xi,(m)}])|(y) + |\partial_\mu^2 a(r, x, [X_r^{s,\xi,(m)}])|(y) \\
&\leq K \left\{ (r-s)^{-1+\frac{\eta}{2}} + \int \int (|y' - x'|^\eta \wedge 1) |\partial_\mu^2 p_m(\mu, s, t, x', y')(y)| dy' d\mu(x') \right\}
\end{aligned}$$

so that from the induction hypothesis

$$(6.23) \quad |\partial_\mu^2 b(r, x, [X_r^{s,\xi,(m)}])(v)| + |\partial_\mu^2 a(r, x, [X_r^{s,\xi,(m)}])(v)| \leq K(r-s)^{-1+\frac{\eta}{2}}.$$

By the chain rule, we therefore deduce that  $\mathcal{P}_2(\mathbb{R}^d) \ni \mu \mapsto \mathcal{H}_{m+1}(\mu, s, r, t, x, z)$ ,  $\widehat{p}_{m+1}^{y'}(\mu, s, r, t, x, z)$ ,  $\widehat{p}_{m+1}^{y'}(\mu, s, t, x, z)$  are two times continuously  $L$ -differentiable and that  $\partial_\mu^2 \mathcal{H}_{m+1}(\mu, s, r, t, x, z)(y)$ ,  $\partial_\mu^2 \widehat{p}_{m+1}^{y'}(\mu, s, t, x, z)(y)$ ,  $\partial_\mu^2 \widehat{p}_{m+1}^{y'}(\mu, s, r, t, x, z)(y)$  satisfy for any fixed  $0 \leq s \leq r < t$ ,  $(x, y', z) \in (\mathbb{R}^d)^3$ ,  $y = (y_1, y_2) \in (\mathbb{R}^d)^2$ ,  $\mu \in \mathcal{P}_2(\mathbb{R}^d)$

$$(6.24) \quad \begin{aligned} \partial_\mu^2 \widehat{p}_{m+1}^{y'}(\mu, s, r, t, x, z)(y) &= -\frac{1}{2}(H_2.g) \left( \int_r^t a(v, y', [X_v^{s,\xi,(m)}])(y) dv, z-x \right) \cdot \int_r^t \partial_\mu^2 a(v, y', [X_v^{s,\xi,(m)}])(y) dv \\ &+ \frac{1}{4} \left( \int_r^t \partial_\mu a(v, y', [X_v^{s,\xi,(m)}])(y_2) dv \right)^\dagger (H_4.g) \left( \int_r^t a(v, y', [X_v^{s,\xi,(m)}])(y) dv, z-x \right) \cdot \int_r^t \partial_\mu a(v, y', [X_v^{s,\xi,(m)}])(y_1) dv \end{aligned}$$

so that, by (6.23), (6.16) and (1.6)

$$(6.25) \quad |\partial_\mu^2 \widehat{p}_{m+1}^{y'}(\mu, s, r, t, x, z)(y)| \leq K(r-s)^{-1+\frac{\eta}{2}} g(c(t-r), z-x)$$

and

$$(6.26) \quad |\partial_\mu^2 \widehat{p}_{m+1}^{y'}(\mu, s, t, x, z)(y)| \leq C(t-s)^{-1+\frac{\eta}{2}} g(c(t-s), z-x).$$

Moreover, one has

$$(6.27) \quad \partial_\mu^2 \mathcal{H}_{m+1}(\mu, s, r, t, x, z)(y) =: \text{I}(y) + \text{II}(y) + \text{III}(y),$$

with

$$\begin{aligned} \text{I}(y) &:= \left\{ -\sum_{i=1}^d H_1^i \left( \int_r^t a(v, z, [X_v^{s,\xi,(m)}]) dv, z-x \right) \partial_\mu^2 b_i(r, x, [X_r^{s,\xi,(m)}])(y) \right\} \widehat{p}_{m+1}(\mu, s, r, t, x, z) \\ &+ \left\{ -\sum_{i=1}^d b_i(r, x, [X_r^{s,\xi,(m)}]) \partial_\mu^2 H_1^i \left( \int_r^t a(v, z, [X_v^{s,\xi,(m)}]) dv, z-x \right) (y) \right\} \widehat{p}_{m+1}(\mu, s, r, t, x, z) \\ &=: \text{I}_1(y) + \text{I}_2(y), \\ \text{II}(y) &:= \left\{ \frac{1}{2} \sum_{i,j=1}^d \partial_\mu^2 \left( a_{i,j}(r, x, [X_r^{s,\xi,(m)}]) - a_{i,j}(r, z, [X_r^{s,\xi,(m)}]) \right) (y) \right. \\ &\quad \left. \times H_2^{i,j} \left( \int_r^t a(v, z, [X_v^{s,\xi,(m)}]) dv, z-x \right) \right\} \widehat{p}_{m+1}(\mu, s, r, t, x, z) \\ &+ \left\{ \frac{1}{2} \sum_{i,j=1}^d \left( a_{i,j}(r, x, [X_r^{s,\xi,(m)}]) - a_{i,j}(r, z, [X_r^{s,\xi,(m)}]) \right) \right. \\ &\quad \left. \times \partial_\mu^2 H_2^{i,j} \left( \int_r^t a(v, z, [X_v^{s,\xi,(m)}]) dv, z-x \right) (y) \right\} \widehat{p}_{m+1}(\mu, s, r, t, x, z) \\ &=: \text{II}_1(y) + \text{II}_2(y), \\ \text{III}(y) &:= \left\{ -\sum_{i=1}^d b_i(r, x, [X_r^{s,\xi,(m)}]) H_1^i \left( \int_r^t a(v, z, [X_v^{s,\xi,(m)}]) dv, z-x \right) \right. \\ &\quad \left. + \frac{1}{2} \sum_{i,j=1}^d \left( a_{i,j}(r, x, [X_r^{s,\xi,(m)}]) - a_{i,j}(r, z, [X_r^{s,\xi,(m)}]) \right) H_2^{i,j} \left( \int_r^t a(v, z, [X_v^{s,\xi,(m)}]) dv, z-x \right) \right\} \\ &\quad \times \partial_\mu^2 \widehat{p}_{m+1}(\mu, s, r, t, x, z)(y). \end{aligned}$$

Combining the previous expression with (6.23), (6.25) and using the uniform  $\eta$ -Hölder regularity of  $x \mapsto a_{i,j}(r, x, \mu)$  with the space time inequality (1.5) yield

$$\begin{aligned} |\text{I}| &\leq \frac{K}{(t-r)^{\frac{1}{2}}(r-s)^{1-\frac{\eta}{2}}} g(c(t-r), z-x), \\ |\text{III}| &\leq \frac{K}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{1-\frac{\eta}{2}}} g(c(t-r), z-x). \end{aligned}$$

From the key decomposition (2.11) applied to the map  $\mu \mapsto a_{i,j}(r, x, \mu)$  and  $\Theta(s, \mu) = [X_r^{s,\xi,(m)}]$ , the estimates (4.10), (4.16), (4.24) at step  $m$  and using either the  $\eta$ -Hölder regularity of  $x \mapsto \frac{\delta^2}{\delta m^2} a_{i,j}(t, x, \mu)(y)$  or the  $\eta$ -Hölder regularity of  $y \mapsto \frac{\delta^2}{\delta m^2} a_{i,j}(t, x, \mu)(y)$  for each term appearing in the right-hand side of (2.11), we get

$$\left| \partial_\mu^2 \left( a_{i,j}(r, x, [X_r^{s,\xi,(m)}]) - a_{i,j}(r, z, [X_r^{s,\xi,(m)}]) \right) (y) \right| \leq K \left\{ \frac{|z-x|^\eta}{r-s} \wedge \frac{1}{(r-s)^{1-\frac{\eta}{2}}} \right\}$$

so that, by the space-time inequality (1.5), one finally obtains

$$|\text{III}| \leq K \left( \frac{1}{(t-r)^{1-\frac{\eta}{2}}(r-s)} \wedge \frac{1}{(t-r)(r-s)^{1-\frac{\eta}{2}}} \right) g(c(t-r), z-x).$$

Gathering the three previous estimates on I, II and III, we get

$$(6.28) \quad |\partial_\mu^2 \mathcal{H}_{m+1}(\mu, s, r, t, x, z)(y)| \leq K \left( \frac{1}{(t-r)^{1-\frac{\eta}{2}}(r-s)} \wedge \frac{1}{(t-r)(r-s)^{1-\frac{\eta}{2}}} \right) g(c(t-r), z-x).$$

From (6.26), (6.28) and standard computations (separating notably the time convolution into the two disjoint intervals  $[s, \frac{t+s}{2}]$  and  $[\frac{t+s}{2}, t]$  to balance the singularity generated by (6.28)) yield the absolute and uniform convergence of the two series  $\sum_{k \geq 0} \partial_\mu^2 \widehat{p}_{m+1} \otimes \mathcal{H}_{m+1}^{(k)}(\mu, s, t, x, z)(y)$  and  $\sum_{k \geq 0} (p_{m+1} \otimes \partial_\mu^2 \mathcal{H}_{m+1}) \otimes \mathcal{H}_{m+1}^{(k)}(\mu, s, t, x, z)(y)$  as well as their global continuity with respect to the variables  $s, x, \mu, y$  on  $[0, t) \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times (\mathbb{R}^d)^2$ .

We then formally differentiate two times with respect to the variable  $\mu$  the relation  $p_{m+1}(\mu, s, t, x, z) = \widehat{p}_{m+1}(\mu, s, t, x, z) + (p_{m+1} \otimes \mathcal{H}_{m+1})(\mu, s, t, x, z)$ , we obtain

$$\begin{aligned} \partial_\mu^2 p_{m+1}(\mu, s, t, x, z)(y) &= \partial_\mu^2 \widehat{p}_{m+1}(\mu, s, t, x, z)(y) + (p_{m+1} \otimes \partial_\mu^2 \mathcal{H}_{m+1})(\mu, s, t, x, z)(y) \\ &\quad + (\partial_\mu^2 p_{m+1} \otimes \mathcal{H}_{m+1})(\mu, s, t, x, z)(y) \end{aligned}$$

so that a direct iteration yields the following key relation

$$(6.29) \quad \partial_\mu^2 p_{m+1}(\mu, s, t, x, z)(y) = \sum_{k \geq 0} (\partial_\mu^2 \widehat{p}_{m+1} + p_{m+1} \otimes \partial_\mu^2 \mathcal{H}_{m+1}) \otimes \mathcal{H}_{m+1}^{(k)}(\mu, s, t, x, z)(y).$$

From (6.25), (6.28) and some standard computations, the above series converge absolutely and uniformly on any compact set  $\mathcal{K} \subset [0, t) \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times (\mathbb{R}^d)^2$ . We thus deduce that  $\mu \mapsto p_{m+1}(\mu, s, t, x, z)$  is twice continuously  $L$ -differentiable and that  $[0, t) \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times (\mathbb{R}^d)^2 \ni (s, x, \mu, y) \mapsto \partial_\mu^2 p_{m+1}(\mu, s, t, x, z)(y)$  is continuous.

We now prove the estimates (4.24) to (4.27) at step  $m+1$ . Since their proofs are rather long, technical and relies on similar ideas and arguments as those established in [CdRF18], we will not prove all the announced estimates and will be short on some technical details. In particular, we will deliberately omit the proof of (4.27). We start with (4.24). We introduce the two following quantities for  $n = 0, 1$

$$\begin{aligned} u_m(s, t) &:= \sup_{y \in (\mathbb{R}^d)^2} \int \int (|y' - x'|^\eta \wedge 1) |\partial_\mu^2 p_m(\mu, s, t, x', y')(y)| dy' \mu(dx'), \\ v_m(s, t) &:= \sup_{y \in (\mathbb{R}^d)^2} \int \int |\partial_\mu^2 p_m(\mu, s, t, x', y')(y)| dy' \mu(dx') \end{aligned}$$

and prove by induction on  $m$  the following key inequalities:

$$u_m(s, t) \leq C_m(s, t)(t-s)^{-1+\eta}, \quad v_m(s, t) \leq C_m(s, t)(t-s)^{-1+\frac{\eta}{2}}$$

with  $C_m(s, t) := \sum_{k=1}^m C^k \prod_{i=1}^k B\left(\frac{\eta}{2}, i\frac{\eta}{2}\right) (t-s)^{(k-1)\frac{\eta}{2}}$ ,  $C := C((\mathbf{HE}), (\mathbf{HR}), T)$  being a positive constant independent of  $m$ . This result being straightforward for  $m = 1$ , we assume that it holds at step  $m$ .

We first remark that from (6.24), (6.22), (6.16) (with  $n = 0$ ), there exist positive constants  $K := K(T, \frac{\delta}{\delta m} a, \frac{\delta^2}{\delta m^2} a)$ ,  $c := c(\lambda)$ , which may vary from line to line, such that for all  $m \geq 1$

$$\begin{aligned} & |\partial_\mu^2 \widehat{p}_{m+1}(\mu, s, t, x, z)|(y) \\ & \leq K \left\{ \frac{1}{(t-s)^{1-\frac{\eta}{2}}} + \frac{1}{t-s} \int_s^t \int \int (|y' - x'|^\eta \wedge 1) |\partial_\mu^2 p_m(\mu, s, r, x', y')|(y) dy' \mu(dx') dr \right\} \\ & \quad \times g(c(t-s), z-x) \end{aligned}$$

so that by the induction hypothesis

$$\begin{aligned} & |\partial_\mu^2 \widehat{p}_{m+1}(\mu, s, t, x, z)|(y) \\ & \leq K \left\{ \frac{1}{(t-s)^{1-\frac{\eta}{2}}} + \frac{1}{t-s} \int_s^t \frac{C_m(s, r)}{(r-s)^{1-\eta}} dr \right\} g(c(t-s), z-x) \\ & \leq K \left\{ \frac{1}{(t-s)^{1-\frac{\eta}{2}}} + \frac{1}{(t-s)^{\frac{\eta}{2}}} \int_s^t \frac{C_m(s, r)}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{1-\eta}} dr \right\} g(c(t-s), z-x) \\ (6.30) \quad & \leq K \left\{ \frac{1}{(t-s)^{1-\frac{\eta}{2}}} + \frac{1}{(t-s)^{1-\frac{\eta}{2}}} \sum_{k=1}^m C^k \prod_{i=1}^{k+1} B\left(\frac{\eta}{2}, i\frac{\eta}{2}\right) (t-s)^{k\frac{\eta}{2}} \right\} \\ & \quad \times g(c(t-s), z-x). \end{aligned}$$

Hence, by induction on  $r$ ,

$$\begin{aligned} |(\partial_\mu^2 \widehat{p}_{m+1} \otimes \mathcal{H}_{m+1}^{(r)})(\mu, s, t, x, z)(y)| & \leq K^r \left\{ 1 + \sum_{k=1}^m C^k \prod_{i=1}^{k+1} B\left(\frac{\eta}{2}, i\frac{\eta}{2}\right) (t-s)^{k\frac{\eta}{2}} \right\} \\ & \quad \times (t-s)^{-1+\frac{\eta}{2}+r\frac{\eta}{2}} \prod_{i=1}^r B\left(\frac{\eta}{2}, i\frac{\eta}{2}\right) g(c(t-s), z-x) \end{aligned}$$

which in turn implies

$$\begin{aligned} & \sum_{r \geq 0} |(\partial_\mu^2 \widehat{p}_{m+1} \otimes \mathcal{H}_{m+1}^{(r)})(\mu, s, t, x, z)(y)| \\ & \leq \frac{K}{(t-s)^{1-\frac{\eta}{2}}} \left\{ 1 + \sum_{k=1}^m C^k \prod_{i=1}^{k+1} B\left(\frac{\eta}{2}, i\frac{\eta}{2}\right) (t-s)^{k\frac{\eta}{2}} \right\} g(c(t-s), z-x) \\ (6.31) \quad & \leq \frac{K}{(t-s)^{1-\frac{\eta}{2}}} \left\{ B\left(\frac{\eta}{2}, \frac{\eta}{2}\right) + \sum_{k=1}^m C^k \prod_{i=1}^{k+1} B\left(\frac{\eta}{2}, i\frac{\eta}{2}\right) (t-s)^{k\frac{\eta}{2}} \right\} g(c(t-s), z-x). \end{aligned}$$

We now come back to the decomposition (6.27) of  $\partial_\mu^2 \mathcal{H}_{m+1}(\mu, s, r, t, x, z)(y)$ . From (6.22) and the induction hypothesis, we directly get

$$\begin{aligned} |I_1(y)| & = \left| - \sum_{i=1}^d H_1^i \left( \int_r^t a(v, z, [X_v^{s, \xi, (m)}]) dv, z-x \right) \partial_\mu^2 b_i(r, x, [X_r^{s, \xi, (m)}])(y) \right| \widehat{p}_{m+1}(\mu, r, t, x, z) \\ & \leq K \frac{(1 + (r-s)^{1-\frac{\eta}{2}} u_m(s, r))}{(t-r)^{\frac{1}{2}} (r-s)^{1-\frac{\eta}{2}}} g(c(t-r), z-x) \\ & \leq K \frac{(1 + C_m(s, r)(r-s)^{\frac{\eta}{2}})}{(t-r)^{\frac{1}{2}} (r-s)^{1-\frac{\eta}{2}}} g(c(t-r), z-x). \end{aligned}$$

Next again from (6.16), (6.22), the induction hypothesis and the space-time inequality (1.5)

$$\begin{aligned} |I_2(y)| & = \left| - \sum_{i=1}^d b_i(r, x, [X_r^{s, \xi, (m)}]) \partial_\mu^2 H_1^i \left( \int_r^t a(v, z, [X_v^{s, \xi, (m)}]) dv, z-x \right) (y) \right| \widehat{p}_{m+1}(\mu, r, t, x, z) \\ & \leq K |z-x| \left( \frac{1}{(t-r)^3} \left( \int_r^t \frac{1}{(v-s)^{\frac{1-\eta}{2}}} dv \right)^2 + \frac{1}{(t-r)^2} \int_r^t \frac{(1 + (v-s)^{1-\frac{\eta}{2}} u_m(s, v))}{(v-s)^{1-\frac{\eta}{2}}} dv \right) g(c(t-r), z-x) \\ & \leq K \frac{(1 + (t-r)^{-1} \int_r^t C_m(s, v)(v-s)^{\frac{\eta}{2}} dv)}{(t-r)^{\frac{1}{2}} (r-s)^{1-\frac{\eta}{2}}} g(c(t-r), z-x). \end{aligned}$$



Hence, one concludes

$$\begin{aligned} |\mathbb{I}(y)| &\leq K \frac{(1 + C_m(s, r)(r - s)^{\frac{\eta}{2}} + (t - r)^{-1} \int_r^t C_m(s, v)(v - s)^{\frac{\eta}{2}} dv)}{(t - r)^{\frac{1}{2}}(r - s)^{1 - \frac{\eta}{2}}} g(c(t - r), z - x) \\ &\leq K \left( \frac{1}{(t - r)^{1 - \frac{\eta}{2}}(r - s)} \wedge \frac{1}{(t - r)(r - s)^{1 - \frac{\eta}{2}}} \right) \\ &\quad \times \left( 1 + C_m(s, r)(r - s)^{\frac{\eta}{2}} + (t - r)^{-1} \int_r^t C_m(s, v)(v - s)^{\frac{\eta}{2}} dv \right) g(c(t - r), z - x). \end{aligned}$$

We now estimate  $\mathbb{II}$  which is the tricky part of our computations. From Proposition 2.3,  $\mathbb{II}_1(y)$  can be written as

$$\begin{aligned} \mathbb{II}_1(y) &= \frac{1}{2} \left\{ \sum_{i,j=1}^d H_2^{i,j} \left( \int_r^t a(v, z, [X_v^{s,\xi,(m)}]) dv, z - x \right) \right. \\ &\quad \left. \times \partial_\mu^2 \left[ a_{i,j}(r, x, [X_r^{s,\xi,(m)}]) - a_{i,j}(r, z, [X_r^{s,\xi,(m)}]) \right] (y) \right\} \widehat{p}_m(\mu, r, t, x, z) \\ &=: \left\{ \frac{1}{2} \sum_{i,j=1}^d H_2^{i,j} \left( \int_r^t a(v, z, [X_v^{s,\xi,(m)}]) dv, z - x \right) \times J_{i,j}(y) \right\} \widehat{p}_m(\mu, r, t, x, z) \end{aligned}$$

where, from (2.11),  $J_{i,j}(y) = \sum_{\ell=1}^7 J_{i,j}^\ell(y)$  with

$$\begin{aligned} J_{i,j}^1(y) &:= \int \int \left\{ \frac{\delta^2}{\delta m^2} a_{i,j}(r, x, [X_r^{s,\xi,(m)}])(z', z'') - \frac{\delta^2}{\delta m^2} a_{i,j}(r, x, [X_r^{s,\xi,(m)}])(y) \right. \\ &\quad \left. - \left[ \frac{\delta^2}{\delta m^2} a_{i,j}(r, z, [X_r^{s,\xi,(m)}])(z', z'') - \frac{\delta^2}{\delta m^2} a_{i,j}(r, z, [X_r^{s,\xi,(m)}])(y) \right] \right\} \partial_{y_1} p_m(\mu, s, r, y_1, z') \otimes \partial_{y_2} p_m(\mu, s, r, y_2, z'') dz' dz'', \\ J_{i,j}^2(y) &:= \int \int \int \left\{ \frac{\delta^2}{\delta m^2} a_{i,j}(r, x, [X_r^{s,\xi,(m)}])(z', z'') - \frac{\delta^2}{\delta m^2} a_{i,j}(r, x, [X_r^{s,\xi,(m)}])(y_1, x') \right. \\ &\quad \left. - \left[ \frac{\delta^2}{\delta m^2} a_{i,j}(r, z, [X_r^{s,\xi,(m)}])(z', z'') - \frac{\delta^2}{\delta m^2} a_{i,j}(r, z, [X_r^{s,\xi,(m)}])(y_1, x') \right] \right\} \\ &\quad \cdot \partial_{y_1} p_m(\mu, s, r, y_1, z') \otimes \partial_\mu p_m(\mu, s, r, x', z'')(y_2) dz' dz'' d\mu(x'), \\ J_{i,j}^3(y) &:= \int \left\{ \frac{\delta}{\delta m} a_{i,j}(r, x, [X_r^{s,\xi,(m)}])(z') - \frac{\delta}{\delta m} a_{i,j}(r, x, [X_r^{s,\xi,(m)}])(y_1) \right. \\ &\quad \left. - \left[ \frac{\delta}{\delta m} a_{i,j}(r, z, [X_r^{s,\xi,(m)}])(z') - \frac{\delta}{\delta m} a_{i,j}(r, z, [X_r^{s,\xi,(m)}])(y_1) \right] \right\} \partial_v \partial_\mu p_m(\mu, s, r, y_1, z')(y_2) dz', \\ (6.32) \quad J_{i,j}^4(y) &:= \int \left\{ \frac{\delta}{\delta m} a_{i,j}(r, x, [X_r^{s,\xi,(m)}])(z') - \frac{\delta}{\delta m} a_{i,j}(r, x, [X_r^{s,\xi,(m)}])(y_2) \right. \\ &\quad \left. - \left[ \frac{\delta}{\delta m} a_{i,j}(r, z, [X_r^{s,\xi,(m)}])(z') - \frac{\delta}{\delta m} a_{i,j}(r, z, [X_r^{s,\xi,(m)}])(y_2) \right] \right\} \partial_{y_2} \partial_\mu p_m(\mu, s, r, y_2, z')(y_1) dz', \\ J_{i,j}^5(y) &:= \int \int \int \left\{ \frac{\delta^2}{\delta m^2} a_{i,j}(r, x, [X_r^{s,\xi,(m)}])(z', z'') - \frac{\delta^2}{\delta m^2} a_{i,j}(r, x, [X_r^{s,\xi,(m)}])(x', y_2) \right. \\ &\quad \left. - \left[ \frac{\delta^2}{\delta m^2} a_{i,j}(r, z, [X_r^{s,\xi,(m)}])(z', z'') - \frac{\delta^2}{\delta m^2} a_{i,j}(r, z, [X_r^{s,\xi,(m)}])(x', y_2) \right] \right\} \\ &\quad \partial_\mu p_m(\mu, s, r, x', z')(y_1) \otimes \partial_{y_2} p_m(\mu, s, r, y_2, z'') dz' dz'' d\mu(x'), \\ J_{i,j}^6(y) &:= \int \int \int \int \left\{ \frac{\delta^2}{\delta m^2} a_{i,j}(r, x, [X_r^{s,\xi,(m)}])(z', z'') - \frac{\delta^2}{\delta m^2} a_{i,j}(r, x, [X_r^{s,\xi,(m)}])(x', x'') \right. \\ &\quad \left. - \left[ \frac{\delta^2}{\delta m^2} a_{i,j}(r, z, [X_r^{s,\xi,(m)}])(z', z'') - \frac{\delta^2}{\delta m^2} a_{i,j}(r, z, [X_r^{s,\xi,(m)}])(x', x'') \right] \right\} \\ &\quad \partial_\mu p_m(\mu, s, r, x', z')(y_1) \otimes \partial_\mu p_m(\mu, s, r, x'', z'')(y_2) dz' dz'' d\mu(x'') d\mu(x'), \\ J_{i,j}^7(y) &:= \int \int \left\{ \frac{\delta}{\delta m} a_{i,j}(r, x, [X_r^{s,\xi,(m)}])(z') - \frac{\delta}{\delta m} a_{i,j}(r, x, [X_r^{s,\xi,(m)}])(x') \right. \\ &\quad \left. - \left[ \frac{\delta}{\delta m} a_{i,j}(r, z, [X_r^{s,\xi,(m)}])(z') - \frac{\delta}{\delta m} a_{i,j}(r, z, [X_r^{s,\xi,(m)}])(x') \right] \right\} \partial_\mu^2 p_m(\mu, s, r, x', z')(y_1, y_2) dz' d\mu(x'). \end{aligned}$$

On the one hand, using the induction hypothesis and the fact that  $x \mapsto \frac{\delta}{\delta m} a_{i,j}(r, x, \mu)(y_1)$ ,  $\frac{\delta^2}{\delta m^2} a_{i,j}(r, x, \mu)(y_1, y_2)$  are uniformly  $\eta$ -Hölder continuous, one gets

$$(6.33) \quad \sum_{\ell=1}^6 |J_{i,j}^\ell(y)| \leq K \frac{|z-x|^\eta \wedge 1}{r-s}, \quad |J_{i,j}^7(y)| \leq K(|z-x|^\eta \wedge 1)v_m(s, r).$$

On the other hand, from the uniform  $\eta$ -Hölder regularity of  $\mathbb{R}^d \ni y_1 \mapsto \frac{\delta}{\delta m} a_{i,j}(r, x, \mu)(y_1)$ ,  $(\mathbb{R}^d)^2 \ni y \mapsto \frac{\delta^2}{\delta m^2} a_{i,j}(r, x, \mu)(y)$  and the induction hypothesis, one gets

$$(6.34) \quad \sum_{\ell=1}^6 |J_{i,j}^\ell(y)| \leq \frac{K}{(r-s)^{1-\frac{\eta}{2}}}, \quad |J_{i,j}^7(y)| \leq K u_m(s, r).$$

Consequently, combining the estimates (6.33) and (6.34), we obtain

$$(6.35) \quad |J_{i,j}(y)| \leq K \left\{ \frac{|z-x|^\eta}{r-s} \wedge \frac{1}{(r-s)^{1-\frac{\eta}{2}}} \right\} (1 + (r-s)^{1-\frac{\eta}{2}} u_m(s, r) + (r-s)v_m(s, r)).$$

Hence, from the previous estimate, the space-time inequality (1.5) and the induction hypothesis, we deduce

$$\begin{aligned} |\Pi_1(y)| &\leq \frac{K}{t-r} \left( \frac{|z-x|^\eta}{(r-s)} \wedge \frac{1}{(r-s)^{1-\frac{\eta}{2}}} \right) (1 + C_m(s, r)(r-s)^{\frac{\eta}{2}}) g(c(t-r), z-x) \\ &\leq K \left( \frac{1}{(t-r)^{1-\frac{\eta}{2}}(r-s)} \wedge \frac{1}{(t-r)(r-s)^{1-\frac{\eta}{2}}} \right) (1 + C_m(s, r)(r-s)^{\frac{\eta}{2}}) g(c(t-r), z-x). \end{aligned}$$

We now turn our attention to  $\Pi_2$ . From the very definition of  $H_2^{i,j}$ , (6.22), (6.16), the induction hypothesis and noticing that for any differentiable map  $\mathcal{P}_2(\mathbb{R}^d) \ni \nu \mapsto \Sigma(\nu)$  taking values in the set of positive definite matrix one has  $\partial_\mu(\Sigma^{-1}(\mu))_{i,j} = -(\Sigma^{-1}(\mu)\partial_\mu\Sigma(\mu)\Sigma^{-1}(\mu))_{i,j} = -\sum_{k_1, k_2}(\Sigma^{-1}(\mu))_{i, k_1}\partial_\mu(\Sigma(\mu))_{k_1, k_2}(\Sigma^{-1}(\mu))_{k_2, j}$ , we get

$$(6.36) \quad \begin{aligned} &\left| \partial_\mu^2 \left[ H_2^{i,j} \left( \int_r^t a(v, z, [X_v^{s, \xi, (m)}]) dv, z-x \right) \right] (y) \right| \\ &\leq K \left\{ \left( \frac{|z-x|^2}{(t-r)^3} + \frac{1}{(t-r)^2} \right) \int_r^t \max_{i,j} \left| \partial_\mu^2 [a_{i,j}(v, z, [X_v^{s, \xi, (m)}])](y) \right| dv \right. \\ &\quad \left. + \left( \frac{|z-x|^2}{(t-r)^4} + \frac{1}{(t-r)^3} \right) \left( \int_r^t \max_{i,j} \left| \partial_\mu [a_{i,j}(v, z, [X_v^{s, \xi, (m)}])](y_1) \right| dv \right) \left( \int_r^t \max_{i,j} \left| \partial_\mu [a_{i,j}(v, z, [X_v^{s, \xi, (m)}])](y_2) \right| dv \right) \right\} \\ &\leq K \left( \frac{|z-x|^2}{(t-r)^2} + \frac{1}{t-r} \right) \frac{(1 + (t-r)^{-1} \int_r^t C_m(s, v)(v-s)^{\frac{\eta}{2}} dv)}{(r-s)^{1-\frac{\eta}{2}}} \end{aligned}$$

so that, from the space-time inequality (1.5), we clearly deduce

$$\begin{aligned} |\Pi_2(y)| &\leq K \left( \frac{|z-x|^{2+\eta}}{(t-r)^2} + \frac{|z-x|^\eta}{(t-r)} \right) \frac{(1 + (t-r)^{-1} \int_r^t C_m(s, v)(v-s)^{\frac{\eta}{2}} dv)}{(r-s)^{1-\frac{\eta}{2}}} g(c(t-r), z-x) \\ &\leq \frac{K}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{1-\frac{\eta}{2}}} (1 + (t-r)^{-1} \int_r^t C_m(s, v)(v-s)^{\frac{\eta}{2}} dv) g(c(t-r), z-x) \\ &\leq K \left( \frac{1}{(t-r)^{1-\frac{\eta}{2}}(r-s)} \wedge \frac{1}{(t-r)(r-s)^{1-\frac{\eta}{2}}} \right) (1 + (t-r)^{-1} \int_r^t C_m(s, v)(v-s)^{\frac{\eta}{2}} dv) \\ &\quad \times g(c(t-r), z-x). \end{aligned}$$

Hence, gathering estimates on  $\Pi_1$  and  $\Pi_2$ , we get for all  $y \in (\mathbb{R}^d)^2$

$$|\Pi(y)| \leq K \left( \frac{1}{(t-r)^{1-\frac{\eta}{2}}(r-s)} \wedge \frac{1}{(t-r)(r-s)^{1-\frac{\eta}{2}}} \right) (1 + (t-r)^{-1} \int_r^t C_m(s, v)(v-s)^{\frac{\eta}{2}} dv) \times g(c(t-r), z-x).$$

Finally, from the relation (6.24) with  $r \neq s$ , and the estimates (6.16) and (6.22), we get

$$(6.37) \quad \left| \partial_\mu^2 \widehat{p}_{m+1}(\mu, s, r, t, x, z)(y) \right| \leq \frac{K}{(r-s)^{1-\frac{\eta}{2}}} \left( 1 + (t-r)^{-1} \int_r^t C_m(s, v)(v-s)^{\frac{\eta}{2}} dv \right) g(c(t-r), z-x)$$

so that

$$|\text{III}(y)| \leq K \left( \frac{1}{(t-r)^{1-\frac{\eta}{2}}(r-s)} \wedge \frac{1}{(t-r)(r-s)^{1-\frac{\eta}{2}}} \right) \left( 1 + (t-r)^{-1} \int_r^t C_m(s,v)(v-s)^{\frac{\eta}{2}} dv \right) \times g(c(t-r), z-x).$$

Gathering the previous estimates together, we finally obtain

$$(6.38) \quad |\partial_\mu^2 \mathcal{H}_{m+1}(\mu, s, r, t, x, z)(y)| \leq K \left( \frac{1}{(t-r)^{1-\frac{\eta}{2}}(r-s)} \wedge \frac{1}{(t-r)(r-s)^{1-\frac{\eta}{2}}} \right) \times \left[ 1 + C_m(s,r)(r-s)^{\frac{\eta}{2}} + (t-r)^{-1} \int_r^t C_m(s,v)(v-s)^{\frac{\eta}{2}} dv \right] \times g(c(t-r), z-x).$$

Now, our aim is to establish an upper-bound for the quantity  $p_{m+1} \otimes \partial_\mu^2 \mathcal{H}_{m+1}(\mu, r, t, x, z)(y)$ . The estimate (6.38) allows to balance the singularity in time induced by  $\partial_\mu^2 \mathcal{H}_{m+1}$ . Indeed, assuming first that  $r \in [s, \frac{t+s}{2}]$ , one has  $t-r \geq (t-s)/2$  which directly implies

$$\int |p_{m+1}(\mu, s, r, x, y')| |\partial_\mu^2 \mathcal{H}_{m+1}(\mu, s, r, t, y', z)](y)| dy' \leq \frac{K}{(t-s)(r-s)^{1-\frac{\eta}{2}}} \left( 1 + C_m(s,r)(r-s)^{\frac{\eta}{2}} + (t-s)^{-1} \int_s^t C_m(s,v)(v-s)^{\frac{\eta}{2}} dv \right) \times g(c(t-s), z-x)$$

so that

$$\begin{aligned} & \int_s^{\frac{t+s}{2}} \int |p_{m+1}(\mu, s, r, x, y')| |\partial_\mu^2 \mathcal{H}_{m+1}(\mu, s, r, t, y', z)](y)| dy' dr \\ & \leq K \left\{ \frac{B(1, \frac{\eta}{2})}{(t-s)^{1-\frac{\eta}{2}}} + \frac{1}{t-s} \int_s^t \frac{C_m(s,r)}{(r-s)^{1-\eta}} dr \right\} g(c(t-s), z-x) \\ & \leq K \left\{ \frac{B(\frac{\eta}{2}, \frac{\eta}{2})}{(t-s)^{1-\frac{\eta}{2}}} + \frac{1}{(t-s)^{\frac{\eta}{2}}} \int_s^t \frac{C_m(s,r)}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{1-\eta}} dr \right\} \\ & \quad \times g(c(t-s), z-x) \\ & \leq \frac{K}{(t-s)^{1-\frac{\eta}{2}}} \left\{ B\left(\frac{\eta}{2}, \frac{\eta}{2}\right) + \sum_{k=1}^m C^k \prod_{i=1}^{k+1} B\left(\frac{\eta}{2}, i\frac{\eta}{2}\right) (t-s)^{k\frac{\eta}{2}} \right\} \\ & \quad \times g(c(t-s), z-x). \end{aligned}$$

Then, assuming that  $r \in [\frac{t+s}{2}, t]$ , one has  $r-s \geq (t-s)/2$  so that

$$\int |p_{m+1}(\mu, s, r, x, y')| |\partial_\mu^2 \mathcal{H}_{m+1}(\mu, s, r, t, y', z)](y)| dy' \leq \frac{K}{(t-s)(t-r)^{1-\frac{\eta}{2}}} (1 + C_m(s,r)(r-s)^{\frac{\eta}{2}} + (t-r)^{-1} \int_r^t C_m(s,v)(v-s)^{\frac{\eta}{2}} dv) \times g(c(t-s), z-x)$$

which in turn, by Fubini's theorem, directly yields

$$\begin{aligned} & \int_s^t \int |p_{m+1}(\mu, s, r, x, y')| |\partial_\mu^2 \mathcal{H}_{m+1}(\mu, s, r, t, y', z)](y)| dy' dr \\ & \leq \frac{K}{t-s} \int_s^t \frac{1}{(t-r)^{1-\frac{\eta}{2}}} (1 + C_m(s,r)(r-s)^{\frac{\eta}{2}} + (t-r)^{-1} \int_r^t C_m(s,v)(v-s)^{\frac{\eta}{2}} dv) dr \\ & \quad \times g(c(t-s), z-x) \\ & \leq K \left\{ \frac{B(1, \frac{\eta}{2})}{(t-s)^{1-\frac{\eta}{2}}} + \frac{1}{(t-s)^{\frac{\eta}{2}}} \int_s^t \frac{C_m(s,v)}{(t-v)^{1-\frac{\eta}{2}}(v-s)^{1-\eta}} dv \right\} g(c(t-s), z-x) \\ & \leq \frac{K}{(t-s)^{1-\frac{\eta}{2}}} \left\{ B\left(\frac{\eta}{2}, \frac{\eta}{2}\right) + \sum_{k=1}^m C^k \prod_{i=1}^{k+1} B\left(\frac{\eta}{2}, i\frac{\eta}{2}\right) (t-s)^{k\frac{\eta}{2}} \right\} g(c(t-s), z-x). \end{aligned}$$

Gathering the two previous cases, we clearly obtain

$$(6.39) \quad \begin{aligned} & |p_{m+1} \otimes \partial_\mu^2 \mathcal{H}_{m+1}(\mu, s, t, x, z)(y)| \\ & \leq \frac{K}{(t-s)^{1-\frac{\eta}{2}}} \left( B\left(\frac{\eta}{2}, \frac{\eta}{2}\right) + \sum_{k=1}^m C^k \prod_{i=1}^{k+1} B\left(\frac{\eta}{2}, i\frac{\eta}{2}\right) (t-s)^{k\frac{\eta}{2}} \right) g(c(t-s), z-x) \end{aligned}$$

so that

$$(6.40) \quad \begin{aligned} & \sum_{r \geq 0} |(p_{m+1} \otimes \partial_\mu^2 \mathcal{H}_{m+1}) \otimes \mathcal{H}_{m+1}^{(r)}(\mu, s, t, x, z)(y)| \\ & \leq \frac{K}{(t-s)^{1-\frac{\eta}{2}}} \left\{ B\left(\frac{\eta}{2}, \frac{\eta}{2}\right) + \sum_{k=1}^m C^k \prod_{i=1}^{k+1} B\left(\frac{\eta}{2}, i\frac{\eta}{2}\right) (t-s)^{k\frac{\eta}{2}} \right\} g(c(t-s), z-x). \end{aligned}$$

The estimates (6.31) and (6.40) together with the representation formula (6.29) imply that there exist two constants  $K, c$  (independent of  $m$  and  $C$ ) such that

$$(6.41) \quad \begin{aligned} & |\partial_\mu^2 p_{m+1}(\mu, s, t, x, z)(y)| \\ & \leq \frac{K}{(t-s)^{1-\frac{\eta}{2}}} \left\{ B\left(\frac{\eta}{2}, \frac{\eta}{2}\right) + \sum_{k=1}^m C^k \prod_{i=1}^{k+1} B\left(\frac{\eta}{2}, i\frac{\eta}{2}\right) (t-s)^{k\frac{\eta}{2}} \right\} g(c(t-s), z-x) \end{aligned}$$

so that, by the space-time inequality (1.5)

$$u_{m+1}(s, t) \leq \frac{K}{(t-s)^{1-\eta}} \left\{ B\left(\frac{\eta}{2}, \frac{\eta}{2}\right) + \sum_{k=1}^m C^k \prod_{i=1}^{k+1} B\left(\frac{\eta}{2}, i\frac{\eta}{2}\right) (t-s)^{k\frac{\eta}{2}} \right\}$$

and similarly,

$$v_{m+1}(s, t) \leq \frac{K}{(t-s)^{1-\frac{\eta}{2}}} \left\{ B\left(\frac{\eta}{2}, \frac{\eta}{2}\right) + \sum_{k=1}^m C^k \prod_{i=1}^{k+1} B\left(\frac{\eta}{2}, i\frac{\eta}{2}\right) (t-s)^{k\frac{\eta}{2}} \right\}.$$

Since the constant  $K$  does not depend either on the constant  $C$  appearing in the definition of  $C_m(s, t)$  or  $m$ , one may change  $C$  once for all and derive the induction hypothesis at step  $m+1$  for  $u_{m+1}$  and  $v_{m+1}$ . From (6.41) and the asymptotics of the Beta function, we also conclude that (4.24) holds at step  $m+1$ . This concludes the proof of (4.24) for all  $m \geq 0$ .

In order to obtain (4.25) we proceed in a similar manner. To lighten the notations, we introduce the quantities

$$\begin{aligned} u_m(s, t) & := \sup_{(y, \mu, \mu') \in (\mathbb{R}^d)^2 \times (\mathcal{P}_2(\mathbb{R}^d))^2, \mu \neq \mu'} \int \int (|y'' - x'|^\eta \wedge 1) \frac{|\Delta_{\mu, \mu'} \partial_\mu^2 p_m(\mu, s, t, x', y'')(y)|}{W_2^\alpha(\mu, \mu')} dy'' \mu'(dx'), \\ v_m(s, t) & := \sup_{(y, \mu, \mu') \in \mathbb{R}^d \times (\mathcal{P}_2(\mathbb{R}^d))^2, \mu \neq \mu'} \int \int \frac{|\Delta_{\mu, \mu'} \partial_\mu^2 p_m(\mu, s, t, x', y'')(y)|}{W_2^\alpha(\mu, \mu')} dy'' \mu'(dx') \end{aligned}$$

for a fixed  $\alpha \in [0, \eta]$ . Here and below we will use the notation  $\Delta_{\mu, \mu'} f(\mu) = f(\mu) - f(\mu')$  for a map  $\mathcal{P}_2(\mathbb{R}^d) \ni \mu \mapsto f(\mu)$ . We prove by induction the following key inequalities:

$$u_m(s, t) \leq C_m(s, t)(t-s)^{-(1+\frac{\alpha}{2}-\eta)} \quad \text{and} \quad v_m(s, t) \leq C_m(s, t)(t-s)^{-(1+\frac{\alpha-\eta}{2})},$$

with  $C_m(s, t) := \sum_{k=1}^m C^k \prod_{i=1}^k B\left(\frac{\eta}{2}, \frac{\eta-\alpha}{2} + (i-1)\frac{\eta}{2}\right) (t-s)^{(k-1)\frac{\eta}{2}}$ . The result being straightforward for  $m=1$ , we assume that it holds at step  $m$ . We recall from [CdRF18] the following estimates: there exist two positive constant  $K, c$  such that for all  $m \geq 0$  and for all  $\alpha \in [0, 1]$

$$(6.42) \quad |\Delta_{\mu, \mu'} p_m(\mu, s, t, x, z)| \leq K \frac{W_2^\alpha(\mu, \mu')}{(t-s)^{\frac{\alpha-\eta}{2}}} g(c(t-s), z-x),$$

$$(6.43) \quad \begin{aligned} & |\Delta_{\mu, \mu'} \mathcal{H}_{m+1}(\mu, s, r, t, y, z)| \\ & \leq K \left\{ \frac{1}{(t-r)^{1-\frac{\eta}{2}} (r-s)^{\frac{\alpha}{2}}} \wedge \frac{1}{(t-r)(r-s)^{\frac{\alpha-\eta}{2}}} \right\} W_2^\alpha(\mu, \mu') g(c(t-r), z-y) \end{aligned}$$

and, for  $r=s$

$$(6.44) \quad \forall \alpha \in [0, \eta), \quad |\Delta_{\mu, \mu'} \mathcal{H}_{m+1}(\mu, s, t, y, z)| \leq K \frac{W_2^\alpha(\mu, \mu')}{(t-s)^{1+\frac{\alpha-\eta}{2}}} g(c(t-s), z-y).$$

We now claim: there exist two positive constant  $K, c$  such that for all  $m \geq 0$ ,

(6.45)

$$\forall \alpha \in [0, \eta), \quad |\Delta_{\mu, \mu'} \partial_{\mu}^2 \widehat{p}_{m+1}(\mu, s, r, t, x, z)(y)| \leq KW_2^\alpha(\mu, \mu') \left( \left\{ \frac{1}{(t-s)^{1+\frac{\alpha-\eta}{2}}} \mathbf{1}_{\{r=s\}} + \frac{1}{(r-s)^{1+\frac{\alpha-\eta}{2}}} \mathbf{1}_{\{r>s\}} \right\} + \frac{1}{t-r} \int_r^t u_m(s, v) dv \right) g(c(t-r), z-x).$$

In order to prove the previous estimate, we make use of the following decomposition derived from (6.24):

$$\Delta_{\mu, \mu'} \partial_{\mu}^2 \widehat{p}_{m+1}(\mu, s, r, t, x, z)(y) = \text{I}(y) + \text{II}(y) + \text{III}(y) + \text{IV}(y),$$

with

$$\begin{aligned} \text{I}(y) &:= -\frac{1}{2} \Delta_{\mu, \mu'} \left[ (H_2.g) \left( \int_r^t a(v, z, [X_v^{s, \xi, (m)}]) dv, z-x \right) \right] \cdot \int_r^t \partial_{\mu}^2 a(v, z, [X_v^{s, \xi, (m)}])(y) dv, \\ \text{II}(y) &:= -\frac{1}{2} (H_2.g) \left( \int_r^t a(v, z, [X_r^{s, \xi', (m)}]) dr, z-x \right) \cdot \int_r^t \Delta_{\mu, \mu'} \partial_{\mu}^2 a(v, z, [X_v^{s, \xi, (m)}])(y) dv, \\ \text{III}(y) &:= \frac{1}{4} \left( \int_r^t \partial_{\mu} a(v, z, [X_v^{s, \xi, (m)}])(y_2) dv \right)^{\dagger} \Delta_{\mu, \mu'} (H_4.g) \left( \int_r^t a(v, z, [X_v^{s, \xi, (m)}]) dv, z-x \right) \\ &\quad \cdot \int_r^t \partial_{\mu} a(v, z, [X_v^{s, \xi, (m)}])(y_1) dv, \\ \text{IV}(y) &:= \frac{1}{4} \left[ \left( \int_r^t \partial_{\mu} a(v, z, [X_v^{s, \xi, (m)}])(y_2) dv \right)^{\dagger} (H_4.g) \left( \int_r^t a(v, z, [X_v^{s, \xi', (m)}]) dv \right) \int_r^t \partial_{\mu} a(v, z, [X_v^{s, \xi, (m)}])(y_1) dv \right. \\ &\quad \left. - \left( \int_r^t \partial_{\mu} a(v, z, [X_v^{s, \xi', (m)}])(y_2) dv \right)^{\dagger} (H_4.g) \left( \int_r^t a(v, z, [X_v^{s, \xi, (m)}]) dv \right) \int_r^t \partial_{\mu} a(v, z, [X_v^{s, \xi', (m)}])(y_1) dv \right]. \end{aligned}$$

From Proposition 2.3, if  $h$  admits a bounded linear functional derivative  $\frac{\delta h}{\delta m}$  such that  $y \mapsto \frac{\delta}{\delta m} h(m)(y)$  is uniformly  $\eta$ -Hölder continuous, then, for any  $m \geq 1$  and any  $\alpha \in [0, 1]$

$$\begin{aligned} &|h([X_r^{s, \xi, (m)}]) - h([X_r^{s, \xi', (m)}])| \\ &\leq \left| \int_0^1 \int_{(\mathbb{R}^d)^2} \frac{\delta}{\delta m} h(\Theta_{\lambda, r}^{(m)})(y') p_m(\mu, s, r, x', y') dy' (\mu - \mu')(dx') d\lambda \right| \\ (6.46) \quad &+ \left| \int_0^1 \int_{(\mathbb{R}^d)^2} \left( \frac{\delta}{\delta m} h(\Theta_{\lambda, r}^{(m)})(y') - \frac{\delta}{\delta m} h(\Theta_{\lambda, r}^{(m)})(x') \right) \Delta_{\mu, \mu'} p_m(\mu, s, r, x', y') dy' \mu'(dx') d\lambda \right| \\ &\leq K \left\{ \frac{W_2^\alpha(\mu, \mu')}{(r-s)^{\frac{\alpha-\eta}{2}}} + \int_{(\mathbb{R}^d)^2} (|y' - x'|^\eta \wedge 1) |\Delta_{\mu, \mu'} p_m(\mu, s, r, x', y')| dy' \mu'(dx') \right\} \\ &\leq K \frac{W_2^\alpha(\mu, \mu')}{(r-s)^{\frac{\alpha-\eta}{2}}} \end{aligned}$$

where we used (6.42) and the fact that  $x' \mapsto \int_{\mathbb{R}^d} \frac{\delta}{\delta m} h(\Theta_{\lambda, r}^{(m)})(y') p_m(\mu, s, r, x', y') dy'$  is  $\alpha$ -Hölder with a modulus bounded by  $K(r-s)^{\frac{\eta-\alpha}{2}}$  for the last but one inequality.

Hence, from (6.22) and (4.24) (with a constant  $C$  independent of  $m$ ), one concludes that (6.23) holds with a constant  $K$  independent of  $m$  so that from the mean-value theorem and (6.46) applied to the map  $\mu \mapsto a(t, x, \mu)$  there exists a constant  $K := K(T, a, b)$  independent of  $m$  such that

$$\begin{aligned} |\text{I}(y)| &\leq \frac{K}{(t-r)^2} \int_r^t \max_{i,j} |a_{i,j}(v, z, [X_v^{s, \xi, (m)}]) - a_{i,j}(v, z, [X_v^{s, \xi', (m)}])| dv \int_r^t (v-s)^{-1+\frac{\eta}{2}} dv g(c(t-r), z-x) \\ &\leq \frac{K}{(t-r)^2} \int_r^t \frac{W_2^\alpha(\mu, \mu')}{(v-s)^{\frac{\alpha-\eta}{2}}} dv \times [(t-s)^{\frac{\eta}{2}} \mathbf{1}_{\{r=s\}} + (t-r)(r-s)^{-1+\frac{\eta}{2}} \mathbf{1}_{\{r>s\}}] g(c(t-r), z-x) \\ &\leq KW_2^\alpha(\mu, \mu') \left[ \frac{1}{(t-s)^{1+\frac{\alpha}{2}-\eta}} \mathbf{1}_{\{r=s\}} + \frac{1}{(r-s)^{1+\frac{\alpha}{2}-\eta}} \mathbf{1}_{\{r>s\}} \right] g(c(t-r), z-x) \end{aligned}$$

for any  $\alpha \in [0, 1]$ . Thus, from the identity (2.11) and the estimates (6.46) applied to the maps  $\frac{\delta}{\delta m} a_{i,j}$ ,  $\frac{\delta^2}{\delta m^2} a_{i,j}$ , (4.20) and (4.21), skipping some technical details, for all  $\alpha \in [0, \eta]$ , we get

(6.47)

$$\begin{aligned} & |\partial_\mu^2 a(v, z, [X_v^{s,\xi,(m)}])(y) - \partial_\mu^2 a(v, z, [X_v^{s,\xi',(m)}])(y)| \\ & \leq K \left\{ \frac{W_2^\alpha(\mu, \mu')}{(v-s)^{1+\frac{\alpha-\eta}{2}}} + \int_{(\mathbb{R}^d)^2} (|y' - x'|^\eta \wedge 1) |\Delta_{\mu,\mu'} \partial_\mu^2 p_m(\mu, s, v, x', y')(y)| dy' \mu'(dx') \right\}. \end{aligned}$$

We thus obtain

$$\begin{aligned} |\text{II}(y)| & \leq K \left\{ \frac{W_2^\alpha(\mu, \mu')}{(t-s)^{1+\frac{\alpha-\eta}{2}}} \mathbf{1}_{\{r=s\}} + \frac{W_2^\alpha(\mu, \mu')}{(r-s)^{1+\frac{\alpha-\eta}{2}}} \mathbf{1}_{\{r>s\}} \right. \\ & \left. + \frac{1}{t-r} \int_r^t \int_{(\mathbb{R}^d)^2} (|y' - x'|^\eta \wedge 1) |\Delta_{\mu,\mu'} \partial_\mu^2 p(\mu, s, v, x', y')(y)| dy' \mu'(dx') dv \right\} g(c(t-r), z-x). \end{aligned}$$

From (6.16) (with  $n=0$ ) and the mean-value theorem with (6.46) (applied to the map  $a_{i,j}$ ),

$$\begin{aligned} |\text{III}(y)| & \leq K \frac{W_2^\alpha(\mu, \mu')}{(t-r)^3} \int_r^t \frac{1}{(v-s)^{\frac{\alpha-\eta}{2}}} dv \left[ \int_r^t \frac{1}{(v-s)^{\frac{1-\eta}{2}}} dv \right]^2 g(c(t-r), z-x) \\ & \leq K W_2^\alpha(\mu, \mu') \left\{ \frac{1}{(t-s)^{1+\frac{\alpha-3\eta}{2}}} \mathbf{1}_{\{r=s\}} + \frac{1}{(r-s)^{1+\frac{\alpha-3\eta}{2}}} \mathbf{1}_{\{r>s\}} \right\} g(c(t-r), z-x). \end{aligned}$$

From similar arguments as those used to establish (6.47), we get

$$(6.48) \quad \forall \alpha \in [0, 1], \quad |\partial_\mu a(v, z, [X_v^{s,\xi,(m)}])(y) - \partial_\mu a(v, z, [X_v^{s,\xi',(m)}])(y)| \leq K \frac{W_2^\alpha(\mu, \mu')}{(v-s)^{\frac{1+\alpha-\eta}{2}}}$$

which in turn yields

$$\begin{aligned} |\text{IV}(y)| & \leq K \frac{W_2^\alpha(\mu, \mu')}{(t-r)^2} \int_r^t \frac{1}{(v-s)^{\frac{1+\alpha-\eta}{2}}} dv \int_r^t \frac{1}{(v-s)^{\frac{1-\eta}{2}}} dv g(c(t-r), z-x) \\ & \leq K W_2^\alpha(\mu, \mu') \left\{ \frac{1}{(t-s)^{1+\frac{\alpha}{2}-\eta}} \mathbf{1}_{\{r=s\}} + \frac{1}{(r-s)^{1+\frac{\alpha}{2}-\eta}} \mathbf{1}_{\{r>s\}} \right\} g(c(t-r), z-x). \end{aligned}$$

Gathering the previous estimates completes the proof of (6.45). As a consequence, from the induction hypothesis, we directly get

$$\begin{aligned} & |\Delta_{\mu,\mu'} \partial_\mu^2 \widehat{p}_{m+1}(\mu, s, t, x, z)(y)| \\ & \leq K \left\{ \frac{1}{(t-s)^{1+\frac{\alpha-\eta}{2}}} + \frac{1}{t-s} \int_s^t \frac{C_{m,n}(s,r)}{(r-s)^{1+\frac{\alpha}{2}-\eta}} dr \right\} W_2^\alpha(\mu, \mu') g(c(t-s), z-x) \\ & \leq \frac{K}{(t-s)^{1+\frac{\eta-\alpha}{2}}} \left\{ B\left(\frac{\eta}{2}, \frac{\eta-\alpha}{2}\right) + \sum_{k=1}^m C^k \prod_{i=1}^{k+1} B\left(\frac{\eta}{2}, \frac{\eta-\alpha}{2} + (i-1)\frac{\eta}{2}\right) (t-s)^{k\frac{\eta}{2}} \right\} \\ & \quad \times W_2^\alpha(\mu, \mu') g(c(t-s), z-x) \end{aligned}$$

which in turn yields

$$\begin{aligned} & \sum_{r \geq 0} |\Delta_{\mu,\mu'} \partial_\mu^2 \widehat{p}_{m+1} \otimes \mathcal{H}_{m+1}^{(r)}(\mu, s, t, x, z)(v)| \\ (6.49) \quad & \leq K \left\{ B\left(\frac{\eta}{2}, \frac{\eta-\alpha}{2}\right) + \sum_{k=1}^m C^k \prod_{i=1}^{k+1} B\left(\frac{\eta}{2}, \frac{\eta-\alpha}{2} + (i-1)\frac{\eta}{2}\right) (t-s)^{k\frac{\eta}{2}} \right\} \frac{W_2^\alpha(\mu, \mu')}{(t-s)^{1+\frac{\alpha-\eta}{2}}} g(c(t-s), z-x). \end{aligned}$$

From (6.27), we easily obtain the following decomposition

$$\Delta_{\mu,\mu'} \partial_\mu^2 \mathcal{H}_{m+1}(\mu, s, r, t, y, z)(v) = A + B + C + D + E = \sum_{i=1}^3 A_i + B_i + C_i + D_i + E_i$$

with

$$\begin{aligned}
A_1 &:= - \sum_{i=1}^d \Delta_{\mu, \mu'} \partial_{\mu'}^2 b_i(r, y, [X_r^{s, \xi, (m)}]) (v) H_1^i \left( \int_r^t a(v', z, [X_{v'}^{s, \xi, (m)}]) dv', z - y \right) \widehat{p}_{m+1}(\mu, s, r, t, y, z), \\
A_2 &:= - \sum_{i=1}^d \partial_{\mu'}^2 b_i(r, y, [X_r^{s, \xi', (m)}]) (v) \Delta_{\mu, \mu'} H_1^i \left( \int_r^t a(v', z, [X_{v'}^{s, \xi, (m)}]) dv', z - y \right) \widehat{p}_{m+1}(\mu, s, r, t, y, z), \\
A_3 &:= - \sum_{i=1}^d \partial_{\mu'}^2 b_i(r, y, [X_r^{s, \xi', (m)}]) (v) H_1^i \left( \int_r^t a(v', z, [X_{v'}^{s, \xi', (m)}]) dv', z - y \right) \Delta_{\mu, \mu'} \widehat{p}_{m+1}(\mu, s, r, t, y, z), \\
B_1 &:= \frac{1}{2} \sum_{i,j=1}^d \Delta_{\mu, \mu'} [\partial_{\mu'}^2 [a_{i,j}(r, y, [X_r^{s, \xi, (m)}]) - a_{i,j}(r, z, [X_r^{s, \xi, (m)}])] (v) H_2^{i,j} \left( \int_r^t a(v', z, [X_{v'}^{s, \xi, (m)}]) dv', z - y \right) \widehat{p}_{m+1}(\mu, s, r, t, y, z), \\
B_2 &:= \frac{1}{2} \sum_{i,j=1}^d \partial_{\mu'}^2 [a_{i,j}(r, y, [X_r^{s, \xi', (m)}]) - a_{i,j}(r, z, [X_r^{s, \xi', (m)}])] (v) \Delta_{\mu, \mu'} H_2^{i,j} \left( \int_r^t a(v', z, [X_{v'}^{s, \xi, (m)}]) dv', z - y \right) \widehat{p}_{m+1}(\mu, s, r, t, y, z), \\
B_3 &:= \frac{1}{2} \sum_{i,j=1}^d \partial_{\mu'}^2 [a_{i,j}(r, y, [X_r^{s, \xi', (m)}]) - a_{i,j}(r, z, [X_r^{s, \xi', (m)}])] (v) H_2^{i,j} \left( \int_r^t a(v', z, [X_{v'}^{s, \xi', (m)}]) dv', z - y \right) \Delta_{\mu, \mu'} \widehat{p}_{m+1}(\mu, s, r, t, y, z), \\
C_1 &:= - \sum_{i=1}^d \Delta_{\mu, \mu'} b_i(r, y, [X_r^{s, \xi, (m)}]) \partial_{\mu'}^2 H_1^i \left( \int_r^t a(v', z, [X_{v'}^{s, \xi, (m)}]) dv', z - y \right) (v) \widehat{p}_{m+1}(\mu, s, r, t, y, z), \\
C_2 &:= - \sum_{i=1}^d b_i(r, y, [X_r^{s, \xi', (m)}]) \Delta_{\mu, \mu'} \partial_{\mu'}^2 H_1^i \left( \int_r^t a(v', z, [X_{v'}^{s, \xi, (m)}]) dv', z - y \right) (v) \widehat{p}_{m+1}(\mu, s, r, t, y, z), \\
C_3 &:= - \sum_{i=1}^d b_i(r, y, [X_r^{s, \xi', (m)}]) \partial_{\mu'}^2 H_1^i \left( \int_r^t a(v', z, [X_{v'}^{s, \xi', (m)}]) dv', z - y \right) (v) \Delta_{\mu, \mu'} \widehat{p}_{m+1}(\mu, s, r, t, y, z), \\
D_1 &:= \frac{1}{2} \sum_{i,j=1}^d \Delta_{\mu, \mu'} (a_{i,j}(r, y, [X_r^{s, \xi, (m)}]) - a_{i,j}(r, z, [X_r^{s, \xi, (m)}])) \partial_{\mu'}^2 H_2^{i,j} \left( \int_r^t a(v', z, [X_{v'}^{s, \xi, (m)}]) dv', z - y \right) (v) \widehat{p}_{m+1}(\mu, s, r, t, y, z), \\
D_2 &:= \frac{1}{2} \sum_{i,j=1}^d (a_{i,j}(r, y, [X_r^{s, \xi', (m)}]) - a_{i,j}(r, z, [X_r^{s, \xi', (m)}])) \Delta_{\mu, \mu'} \partial_{\mu'}^2 H_2^{i,j} \left( \int_r^t a(v', z, [X_{v'}^{s, \xi, (m)}]) dv', z - y \right) (v) \widehat{p}_{m+1}(\mu, s, r, t, y, z), \\
D_3 &:= \frac{1}{2} \sum_{i,j=1}^d (a_{i,j}(r, y, [X_r^{s, \xi', (m)}]) - a_{i,j}(r, z, [X_r^{s, \xi', (m)}])) \partial_{\mu'}^2 H_2^{i,j} \left( \int_r^t a(v', z, [X_{v'}^{s, \xi', (m)}]) dv', z - y \right) (v) \Delta_{\mu, \mu'} \widehat{p}_{m+1}(\mu, s, r, t, y, z), \\
E_1 &:= - \sum_{i=1}^d \Delta_{\mu, \mu'} [b_i(r, y, [X_r^{s, \xi, (m)}]) H_1^i \left( \int_r^t a(v', z, [X_{v'}^{s, \xi, (m)}]) dv', z - y \right)] \partial_{\mu'}^2 \widehat{p}_{m+1}(\mu, s, r, t, y, z)(v), \\
E_2 &:= \frac{1}{2} \sum_{i,j=1}^d \Delta_{\mu, \mu'} \left[ (a_{i,j}(r, y, [X_r^{s, \xi, (m)}]) - a_{i,j}(r, z, [X_r^{s, \xi, (m)}])) H_2^{i,j} \left( \int_r^t a(v', z, [X_{v'}^{s, \xi, (m)}]) dv', z - y \right) \right] \\
&\quad \times \partial_{\mu'}^2 \widehat{p}_{m+1}(\mu, s, r, t, y, z)(v), \\
E_3 &:= \left\{ - \sum_{i=1}^d b_i(r, y, [X_r^{s, \xi, (m)}]) H_1^i \left( \int_r^t a(v', z, [X_{v'}^{s, \xi, (m)}]) dv', z - y \right) \right. \\
&\quad \left. + \frac{1}{2} \sum_{i,j=1}^d (a_{i,j}(r, y, [X_r^{s, \xi, (m)}]) - a_{i,j}(r, z, [X_r^{s, \xi, (m)}])) H_2^{i,j} \left( \int_r^t a(v', z, [X_{v'}^{s, \xi, (m)}]) dv', z - y \right) \right\} \\
&\quad \times \Delta_{\mu, \mu'} \partial_{\mu'}^2 \widehat{p}_{m+1}(\mu, s, r, t, y, z)(v).
\end{aligned}$$

• **Estimates on A:**

Similarly to (6.47), we get

$$|\Delta_{\mu, \mu'} \partial_{\mu, \mu'}^2 b_i(r, y, [X_r^{s, \xi, (m)}]) (v)| \leq K \left\{ \frac{W_2^\alpha(\mu, \mu')}{(r-s)^{1+\frac{\alpha-\eta}{2}}} + \int_{(\mathbb{R}^d)^2} (|y' - x'|^\eta \wedge 1) |\Delta_{\mu, \mu'} \partial_{\mu}^2 p_m(\mu, s, r, x', y')(y)| dy' \mu'(dx') \right\}$$

so that

$$|A_1| \leq \frac{K}{(t-r)^{\frac{1}{2}}(r-s)^{1+\frac{\alpha-\eta}{2}}} (1 + C_m(s, r)(r-s)^{\frac{\eta}{2}}) W_2^\alpha(\mu, \mu') g(c(t-r), z-y).$$

For  $A_2$ , from the mean-value theorem and (6.46) (applied to the map  $a_{i,j}$ ), for any  $\alpha \in [0, 1]$

$$\begin{aligned} & \left| \Delta_{\mu, \mu'} H_1^i \left( \int_r^t a(v', z, [X_{v'}^{s, \xi, (m)}]) dv', z-y \right) \widehat{p}_{m+1}(\mu, s, r, t, y, z) \right| \\ & \leq \frac{K}{(t-r)^{\frac{3}{2}}} \int_r^t \max_{i,j} |a_{i,j}(v', z, [X_{v'}^{s, \xi, (m)}]) - a_{i,j}(v', z, [X_{v'}^{s, \xi', (m)}])| dv' g(c(t-r), z-y) \\ & \leq K \frac{W_2^\alpha(\mu, \mu')}{(t-r)^{\frac{1}{2}}(r-s)^{\frac{\alpha-\eta}{2}}} g(c(t-r), z-y) \end{aligned}$$

which combined with (6.23) yield

$$\forall \alpha \in [0, 1], \quad |A_2| \leq K \frac{W_2^\alpha(\mu, \mu')}{(t-r)^{\frac{1}{2}}(r-s)^{1+\frac{\alpha}{2}-\eta}} g(c(t-r), z-y).$$

For  $A_3$ , from (6.46) (applied to the map  $a_{i,j}$ ) and the mean value theorem, one similarly gets

$$(6.50) \quad \left| \Delta_{\mu, \mu'} \widehat{p}_{m+1}(\mu, s, r, t, y, z) \right| \leq K \frac{W_2^\alpha(\mu, \mu')}{(r-s)^{\frac{\alpha-\eta}{2}}} g(c(t-r), z-y)$$

which in turn, with (6.23), directly imply

$$\forall \alpha \in [0, 1], \quad |A_3| \leq K \frac{W_2^\alpha(\mu, \mu')}{(t-r)^{\frac{1}{2}}(r-s)^{1+\frac{\alpha}{2}-\eta}} g(c(t-r), z-y).$$

Combining the previous estimates, we finally obtain

$$|A| \leq \frac{K}{(t-r)^{\frac{1}{2}}(r-s)^{1+\frac{\alpha-\eta}{2}}} (1 + C_m(s, r)(r-s)^{\frac{\eta}{2}}) W_2^\alpha(\mu, \mu') g(c(t-r), z-y).$$

#### • Estimates on B:

For  $B_1$ , we employ a similar decomposition as for  $\Delta_{\mu, \mu'} \partial_\mu^2 b_i(r, y, [X_r^{s, \xi, (m)}])(v)$ , namely from (2.11) we write

$$\Delta_{\mu, \mu'} \partial_\mu^2 [a_{i,j}(r, y, [X_r^{s, \xi, (m)}]) - a_{i,j}(r, z, [X_r^{s, \xi, (m)}])](v) = \sum_{\ell=1}^7 T_{i,j}^\ell$$

with

$$\begin{aligned} T_{i,j}^1 &:= \Delta_{\mu, \mu'} \left[ \int_{(\mathbb{R}^d)^2} \left\{ \frac{\delta^2}{\delta m^2} a_{i,j}(r, y, [X_r^{s, \xi, (m)}])(z', z'') - \frac{\delta^2}{\delta m^2} a_{i,j}(r, z, [X_r^{s, \xi, (m)}])(z', z'') \right\} \right. \\ & \quad \left. \cdot \partial_{v_1} p_m(\mu, s, r, v_1, z') \otimes \partial_{v_2} p_m(\mu, s, r, v_2, z'') dz' dz'' \right], \\ T_{i,j}^2 &:= \Delta_{\mu, \mu'} \left[ \int_{(\mathbb{R}^d)^3} \left\{ \frac{\delta^2}{\delta m^2} a_{i,j}(r, y, [X_r^{s, \xi, (m)}])(z', z'') - \frac{\delta^2}{\delta m^2} a_{i,j}(r, z, [X_r^{s, \xi, (m)}])(z', z'') \right\} \right. \\ & \quad \left. \cdot \partial_{v_1} p_m(\mu, s, r, v_1, z') \otimes \partial_\mu p_m(\mu, s, r, x, z'') (v_2) dz' dz'' d\mu(x) \right], \\ T_{i,j}^3 &:= \Delta_{\mu, \mu'} \left[ \int_{\mathbb{R}^d} \left\{ \frac{\delta}{\delta m} a_{i,j}(r, y, [X_r^{s, \xi, (m)}])(z') - \frac{\delta}{\delta m} a_{i,j}(r, z, [X_r^{s, \xi, (m)}])(z') \right\} \partial_{v_1} \partial_\mu p_m(\mu, s, r, v_1, z') (v_2) dz' \right], \\ T_{i,j}^4 &:= \Delta_{\mu, \mu'} \left[ \int_{\mathbb{R}^d} \left\{ \frac{\delta}{\delta m} a_{i,j}(r, y, [X_r^{s, \xi, (m)}])(z') - \frac{\delta}{\delta m} a_{i,j}(r, z, [X_r^{s, \xi, (m)}])(z') \right\} \partial_{v_2} \partial_\mu p_m(\mu, s, r, v_2, z') (v_1) dz' \right], \\ T_{i,j}^5 &:= \Delta_{\mu, \mu'} \left[ \int_{(\mathbb{R}^d)^3} \left\{ \frac{\delta^2}{\delta m^2} a_{i,j}(r, y, [X_r^{s, \xi, (m)}])(z', z'') - \frac{\delta^2}{\delta m^2} a_{i,j}(r, z, [X_r^{s, \xi, (m)}])(z', z'') \right\} \right. \\ & \quad \left. \cdot \partial_\mu p_m(\mu, s, r, x', z') (v_1) \otimes \partial_{v_2} p_m(\mu, s, r, v_2, z'') dz' dz'' d\mu(x') \right], \\ T_{i,j}^6 &:= \Delta_{\mu, \mu'} \left[ \int_{(\mathbb{R}^d)^4} \left\{ \frac{\delta^2}{\delta m^2} a_{i,j}(r, y, [X_r^{s, \xi, (m)}])(z', z'') - \frac{\delta^2}{\delta m^2} a_{i,j}(r, z, [X_r^{s, \xi, (m)}])(z', z'') \right\} \right. \\ & \quad \left. \cdot \partial_\mu p_m(\mu, s, r, x', z') (v_1) \otimes \partial_\mu p_m(\mu, s, r, x'', z') (v_2) dz dz'' d\mu(x'') d\mu(x') \right], \\ T_{i,j}^7 &:= \Delta_{\mu, \mu'} \left[ \int_{(\mathbb{R}^d)^2} \left\{ \frac{\delta}{\delta m} a_{i,j}(r, y, [X_r^{s, \xi, (m)}])(z') - \frac{\delta}{\delta m} a_{i,j}(r, z, [X_r^{s, \xi, (m)}])(z') \right\} \partial_\mu^2 p_m(\mu, s, r, x', z') (v) dz' d\mu(x') \right]. \end{aligned}$$



As previously done, we quantify the contribution of each term in the above decomposition. We remark that if  $h$  is the map  $a_{i,j}$ ,  $\frac{\delta}{\delta m}a_{i,j}$  or  $\frac{\delta^2}{\delta m^2}a_{i,j}$  then by the mean value theorem, one has

$$\begin{aligned} \Delta_{\mu,\mu'}h(r,y,[X_r^{s,\xi,(m)}]) &:= h(r,y,[X_r^{s,\xi,(m)}]) - h(r,y,[X_r^{s,\xi',(m)}]) \\ &= \int_0^1 \partial_\lambda h(r,y,\Theta_{\lambda,r}^{(m)})(z') d\lambda \\ &= \int_0^1 \int_{\mathbb{R}^d} \frac{\delta}{\delta m} h(r,y,\Theta_{r,\lambda})(y')(p_m(\mu,s,r,y') - p_m(\mu',s,r,y')) dy' d\lambda \\ &= \int_0^1 \int_{(\mathbb{R}^d)^2} \frac{\delta}{\delta m} h(r,y,\Theta_{r,\lambda})(y')(p_m(\mu,s,r,x',y') - p_m(\mu',s,r,x',y')) dy' \mu(dx') d\lambda \\ &\quad + \int_0^1 \int_{\mathbb{R}^d} \frac{\delta}{\delta m} h(r,y,\Theta_{r,\lambda})(y') p_m(\mu',s,r,x',y') dy' (\mu - \mu')(dx') d\lambda \end{aligned}$$

where we used the notation  $\Theta_{\lambda,r}^{(m)} := \lambda[X_r^{s,\xi,(m)}] + (1-\lambda)[X_r^{s,\xi',(m)}]$ . On the one hand, the uniform  $\eta$ -Hölder regularity of  $y \mapsto \frac{\delta}{\delta m}h(r,y,\mu)(y')$ , (4.20) and (4.10) yield

$$(6.51) \quad |\Delta_{\mu,\mu'}h(r,y,[X_r^{s,\xi,(m)}]) - \Delta_{\mu,\mu'}h(r,z,[X_r^{s,\xi,(m)}])| \leq K \frac{|y-z|^\eta}{(r-s)^{\frac{\eta}{2}}} W_2(\mu,\mu') \leq K |y-z|^\eta \frac{W_2^\alpha(\mu,\mu')}{(r-s)^{\frac{\alpha}{2}}}$$

if  $W_2(\mu,\mu') \leq (r-s)^{\frac{1}{2}}$ . Hence, from the previous argument and (4.20)

$$|T_{i,j}^1| \leq K |y-z|^\eta \frac{W_2^\alpha(\mu,\mu')}{(r-s)^{1+\frac{\alpha}{2}}}$$

if  $W_2(\mu,\mu') \leq (r-s)^{\frac{1}{2}}$ . The previous estimate directly follows from the uniform  $\eta$ -Hölder regularity of  $y \mapsto \frac{\delta^2}{\delta m^2}a_{i,j}(r,y,[X_r^{s,\xi,(m)}])(z')$  and (4.10) if  $W_2(\mu,\mu') \geq (r-s)^{\frac{1}{2}}$ . On the other hand, from (6.46) applied to the map  $\frac{\delta^2}{\delta m^2}a_{i,j}$  and (4.20), we also get

$$|T_{i,j}^1| \leq K \frac{W_2^\alpha(\mu,\mu')}{(r-s)^{1+\frac{\alpha-\eta}{2}}}.$$

Combining the two previous estimates with the space-time inequality (1.5), we thus deduce

$$(6.52) \quad \begin{aligned} \forall \alpha \in [0,\eta), \quad & \left| T_{i,j}^1 \times H_2^{i,j} \left( \int_r^t a(v',z,[X_{v'}^{s,\xi,(m)}]) dv', z-y \right) \widehat{p}_{m+1}(\mu,s,r,t,y,z) \right| \\ & \leq K \left\{ \frac{1}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{1+\frac{\alpha}{2}}} \wedge \frac{1}{(t-r)(r-s)^{1+\frac{\alpha-\eta}{2}}} \right\} W_2^\alpha(\mu,\mu') g(c(t-r),z-y). \end{aligned}$$

We proceed in a completely analogous way for  $T_{i,j}^2, \dots, T_{i,j}^6$ . Skipping some technical details, we obtain

$$(6.53) \quad \begin{aligned} \forall \alpha \in [0,\eta), \quad & \left| \sum_{\ell=1}^6 T_{i,j}^\ell \times H_2^{i,j} \left( \int_r^t a(v',z,[X_{v'}^{s,\xi,(m)}]) dv', z-y \right) \widehat{p}_{m+1}(\mu,s,r,t,y,z) \right| \\ & \leq K \left\{ \frac{1}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{1+\frac{\alpha}{2}}} \wedge \frac{1}{(t-r)(r-s)^{1+\frac{\alpha-\eta}{2}}} \right\} W_2^\alpha(\mu,\mu') g(c(t-r),z-y). \end{aligned}$$

In order to handle  $T_{i,j}^7$ , we make use of the decomposition  $T_{i,j}^7 := T_{i,j}^{7,1} + T_{i,j}^{7,2} + T_{i,j}^{7,3}$  with

$$\begin{aligned} T_{i,j}^{7,1} &:= \int_{(\mathbb{R}^d)^2} \Delta_{\mu,\mu'} \left[ \frac{\delta}{\delta m} a_{i,j}(r,y,[X_r^{s,\xi,(m)}])(z') - \frac{\delta}{\delta m} a_{i,j}(r,z,[X_r^{s,\xi,(m)}])(z') \right] \partial_\mu^2 p_m(\mu,s,r,x',z')(v) dz' d\mu(x'), \\ T_{i,j}^{7,2} &:= \int_{(\mathbb{R}^d)^2} \left\{ \frac{\delta}{\delta m} a_{i,j}(r,y,[X_r^{s,\xi',(m)}])(z') - \frac{\delta}{\delta m} a_{i,j}(r,z,[X_r^{s,\xi',(m)}])(z') \right\} \Delta_{\mu,\mu'} \partial_\mu^2 p_m(\mu,s,r,x',z')(v) dz' d\mu(x'), \\ T_{i,j}^{7,3} &:= \int_{(\mathbb{R}^d)^2} \left\{ \frac{\delta}{\delta m} a_{i,j}(r,y,[X_r^{s,\xi,(m)}])(z') - \frac{\delta}{\delta m} a_{i,j}(r,z,[X_r^{s,\xi,(m)}])(z') \right\} \partial_\mu^2 p_m(\mu',s,r,x',z')(v) dz' d(\mu-\mu')(x'). \end{aligned}$$

From (4.24), the  $\eta$ -Hölder regularity of  $x \mapsto \frac{\delta}{\delta m}a_{i,j}(r,x,\mu)(z)$  as well as similar arguments as those employed to handle  $T_{i,j}^1$ , one gets

$$|T_{i,j}^{7,1}| \leq K \frac{|z-y|^\eta}{(r-s)^{1+\frac{\alpha-\eta}{2}}} W_2^\alpha(\mu,\mu').$$

Using either the  $\eta$ -Hölder regularity of  $x \mapsto \frac{\delta}{\delta m} a_{i,j}(r, x, \mu)(z)$  or the  $\eta$ -Hölder regularity of  $z \mapsto \frac{\delta}{\delta m} a_{i,j}(r, x, \mu)(z)$ , we get

$$|T_{i,j}^{7,2}| \leq K \{|z - y|^\eta v_m(s, r) \wedge u_m(s, r)\} W_2^\alpha(\mu, \mu').$$

From the representation formula (6.29) and standard arguments that we omit, one obtains the  $\alpha$ -Hölder regularity property of  $x \mapsto \partial_\mu^2 p_m(\mu, s, t, x, z)(y)$  with a modulus independent of  $m$ , namely

$$|\partial_\mu^2 p_m(\mu, s, t, x, z)(y) - \partial_\mu^2 p_m(\mu, s, t, x', z)(y)| \leq K \frac{|x - x'|^\alpha}{(t - s)^{1 + \frac{\alpha - \eta}{2}}} \{g(c(t - s), z - x) + g(c(t - s), z - x')\}$$

which in turn implies that the map

$$x' \mapsto \int_{\mathbb{R}^d} \left\{ \frac{\delta}{\delta m} a_{i,j}(r, y, [X_r^{s, \xi', (m)}])(z') - \frac{\delta}{\delta m} a_{i,j}(r, z, [X_r^{s, \xi', (m)}])(z') \right\} \partial_\mu^2 p_m(\mu', s, r, x', z')(v) dz'$$

is  $\alpha$ -Hölder with a modulus bounded by  $K|z - y|^\eta (r - s)^{-1 + \frac{\eta - \alpha}{2}}$  for any  $\alpha \in [0, \eta)$ . We thus conclude

$$|T_{i,j}^{7,3}| \leq K \frac{|z - y|^\eta}{(r - s)^{1 + \frac{\alpha - \eta}{2}}} W_2^\alpha(\mu, \mu').$$

Gathering the previous estimates and using the space-time inequality (1.5), we get

$$\begin{aligned} \forall \alpha \in [0, \eta), \quad & \left| T_{i,j}^7 H_2^{i,j} \left( \int_r^t a(v', z, [X_{v'}^{s, \xi, (m)}]) dv', z - y \right) \widehat{p}_{m+1}(\mu, s, r, t, y, z) \right| \\ (6.54) \quad & \leq K \left\{ \frac{1}{(t - r)^{1 - \frac{\eta}{2}} (r - s)^{1 + \frac{\alpha - \eta}{2}}} + \frac{v_m(s, r)}{(t - r)^{1 - \frac{\eta}{2}}} \wedge \frac{u_m(s, r)}{t - r} \right\} W_2^\alpha(\mu, \mu') g(c(t - r), z - y). \end{aligned}$$

From the preceding computations, we finally obtain

$$\begin{aligned} |B_1| & \leq K \left\{ \left\{ \frac{1}{(t - r)^{1 - \frac{\eta}{2}} (r - s)^{1 + \frac{\alpha}{2}}} \wedge \frac{1}{(t - r)(r - s)^{1 + \frac{\alpha - \eta}{2}}} \right\} + \left\{ \frac{u_m^n(s, r)}{t - r} \wedge \frac{v_m^n(s, r)}{(t - r)^{1 - \frac{\eta}{2}}} \right\} \right\} W_2^\alpha(\mu, \mu') \\ & \quad \times g(c(t - r), z - y) \\ & \leq K \left\{ \frac{1}{(t - r)(r - s)^{1 + \frac{\alpha - \eta}{2}}} \wedge \frac{1}{(t - r)^{1 - \frac{\eta}{2}} (r - s)^{1 + \frac{\alpha}{2}}} \right\} (1 + C_m(s, r)(r - s)^{\frac{\eta}{2}}) W_2^\alpha(\mu, \mu') \\ & \quad \times g(c(t - r), z - y). \end{aligned}$$

for any  $\alpha \in [0, \eta)$ . For  $B_2$ , from (6.35) (bounding  $(r - s)^{1 - \frac{\eta}{2}} u_m(s, r)$  and  $(r - s) v_m(s, r)$  by a constant  $K$  independent of  $m$ ), one gets

$$(6.55) \quad |\partial_\mu^2 [a_{i,j}(r, y, [X_r^{s, \xi', (m)}]) - a_{i,j}(r, z, [X_r^{s, \xi', (m)}])](v)| \leq K \left\{ \frac{|z - y|^\eta}{(r - s)} \wedge \frac{1}{(r - s)^{1 - \frac{\eta}{2}}} \right\}$$

and, by the mean-value theorem, the space-time inequality (1.5) and (6.46) (applied to the map  $a_{i,j}$ )

$$\begin{aligned} & |\Delta_{\mu, \mu'} H_2^{i,j} \left( \int_r^t a(v', z, [X_{v'}^{s, \xi, (m)}]) dv', z - y \right) \widehat{p}_{m+1}(\mu, r, t, y, z)| \\ (6.56) \quad & \leq \frac{K}{(t - r)^2} \int_r^t \max_{i,j} |a_{i,j}(v', z, [X_{v'}^{s, \xi, (m)}]) - a_{i,j}(v', z, [X_{v'}^{s, \xi', (m)}])| dv' g(c(t - r), z - y) \\ & \leq K \frac{W_2^\alpha(\mu, \mu')}{(t - r)(r - s)^{\frac{\alpha - \eta}{2}}} g(c(t - r), z - y) \end{aligned}$$

so that

$$\begin{aligned} |B_2| & \leq K \left\{ \frac{|z - y|^\eta}{(r - s)} \wedge \frac{1}{(r - s)^{1 - \frac{\eta}{2}}} \right\} \frac{W_2^\alpha(\mu, \mu')}{(t - r)(r - s)^{\frac{\alpha - \eta}{2}}} g(c(t - r), z - y) \\ & \leq K \left\{ \frac{1}{(t - r)(r - s)^{1 + \frac{\alpha}{2} - \eta}} \wedge \frac{1}{(t - r)^{1 - \frac{\eta}{2}} (r - s)^{1 + \frac{\alpha - \eta}{2}}} \right\} W_2^\alpha(\mu, \mu') g(c(t - r), z - y). \end{aligned}$$

For  $B_3$ , from (6.50), (6.55) and the space-time inequality (1.5), we get

$$\begin{aligned} |B_3| &\leq K \left\{ \frac{1}{(t-r)(r-s)^{1-\frac{\eta}{2}}} \wedge \frac{|z-y|^\eta}{(t-r)(r-s)} \right\} \frac{W_2^\alpha(\mu, \mu')}{(r-s)^{\frac{\alpha-\eta}{2}}} g(c(t-r), z-y) \\ &\leq K \left\{ \frac{1}{(t-r)(r-s)^{1-\eta}} \wedge \frac{1}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{1+\frac{\alpha-\eta}{2}}} \right\} W_2^\alpha(\mu, \mu') g(c(t-r), z-y). \end{aligned}$$

Gathering the previous estimates on  $B_1$ ,  $B_2$ ,  $B_3$  and using the induction hypothesis, we finally deduce

$$\begin{aligned} |B| &\leq K \left\{ \left\{ \frac{1}{(t-r)(r-s)^{1+\frac{\alpha-\eta}{2}}} \wedge \frac{1}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{1+\frac{\alpha}{2}}} \right\} + \left\{ \frac{u_m(s, r)}{t-r} \wedge \frac{v_m(s, r)}{(t-r)^{1-\frac{\eta}{2}}} \right\} \right\} W_2^\alpha(\mu, \mu') \\ &\quad \times g(c(t-r), z-y) \\ &\leq K \left\{ \frac{1}{(t-r)(r-s)^{1+\frac{\alpha-\eta}{2}}} \wedge \frac{1}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{1+\frac{\alpha}{2}}} \right\} (1 + C_m(s, r)(r-s)^{\frac{\eta}{2}}) W_2^\alpha(\mu, \mu') \\ &\quad \times g(c(t-r), z-y). \end{aligned}$$

• **Estimates on C:**

For  $C_1$ , from (6.46) applied to the map  $\mu \mapsto b_i(r, y, \mu)$

$$(6.57) \quad \forall \alpha \in [0, 1], \quad |\Delta_{\mu, \mu'} b_i(r, y, [X_r^{s, \xi, (m)}])| \leq K \frac{W_2^\alpha(\mu, \mu')}{(r-s)^{\frac{\alpha-\eta}{2}}}$$

and, similarly to (6.36)

$$\begin{aligned} &\left| \partial_\mu^2 H_1^i \left( \int_r^t a(v, z, [X_v^{s, \xi, (m)}]) dv, z-y \right) (v) \right| \\ &\leq K \left\{ \frac{|z-y|}{(t-r)^2} \int_r^t \max_{i,j} \left| \partial_\mu^2 [a_{i,j}(v, z, [X_v^{s, \xi, (m)}])](v) \right| dv \right. \\ &\quad \left. + \frac{|z-y|}{(t-r)^3} \left( \int_r^t \max_{i,j} \left| \partial_\mu [a_{i,j}(v, z, [X_v^{s, \xi, (m)}])](v_1) \right| dv \right) \left( \int_r^t \max_{i,j} \left| \partial_\mu [a_{i,j}(v, z, [X_v^{s, \xi, (m)}])](v_2) \right| dv \right) \right\} \\ (6.58) \quad &\leq K \frac{|z-y|}{(t-r)(r-s)^{1-\frac{\eta}{2}}}. \end{aligned}$$

Combining both estimates and using the space-time inequality (1.5), we obtain

$$|C_1| \leq K \frac{W_2^\alpha(\mu, \mu')}{(t-r)^{\frac{1}{2}}(r-s)^{1+\frac{\alpha}{2}-\eta}} g(c(t-r), z-y).$$

We now make use of (6.47) and (6.48) as well as the mean-value theorem and the induction hypothesis

$$\begin{aligned} |C_2| &\leq K \left\{ \frac{1}{(t-r)^{\frac{3}{2}}} \int_r^t \max_{i,j} |\Delta_{\mu, \mu'} \partial_\mu^2 a_{i,j}(v', z, [X_{v'}^{s, \xi, (m-1)}])](v)| dv' + \frac{W_2^\alpha(\mu, \mu')}{(t-r)^{\frac{1}{2}}(r-s)^{1+\frac{\alpha-\eta}{2}}} \right\} g(c(t-r), z-y) \\ &\leq \frac{K}{(t-r)^{\frac{1}{2}}(r-s)^{1+\frac{\alpha-\eta}{2}}} \left( 1 + (t-r)^{-1} \int_r^t C_m(s, v)(v-s)^{\frac{\eta}{2}} dv' \right) W_2^\alpha(\mu, \mu') g(c(t-r), z-y). \end{aligned}$$

For  $C_3$ , from (6.50), (6.58) and the space-time inequality, we obtain

$$\begin{aligned} |C_3| &\leq K \frac{W_2^\alpha(\mu, \mu')}{(t-r)^{\frac{3}{2}}(r-s)^{\frac{\alpha-\eta}{2}}} \int_r^t \max_{i,j} |\partial_\mu^2 a_{i,j}(v', z, [X_{v'}^{s, \xi, (m)}])](v)| dv' g(c(t-r), z-y) \\ &\leq K \frac{W_2^\alpha(\mu, \mu')}{(t-r)^{\frac{1}{2}}(r-s)^{1+\frac{\alpha}{2}-\eta}} g(c(t-r), z-y). \end{aligned}$$

Gathering the previous estimates on  $C_1$ ,  $C_2$  and  $C_3$ , we thus conclude

$$|C| \leq \frac{K}{(t-r)^{\frac{1}{2}}(r-s)^{1+\frac{\alpha-\eta}{2}}} \left( 1 + (t-r)^{-1} \int_r^t C_m(s, v)(v-s)^{\frac{\eta}{2}} dv' \right) W_2^\alpha(\mu, \mu') g(c(t-r), z-y).$$

• **Estimates on D:**

In order to deal with  $D_1$ , we first remark that from (6.51) and the computations shortly after, distinguishing the two cases  $W_2(\mu, \mu') \geq (r-s)^{\frac{1}{2}}$  and  $W_2(\mu, \mu') \leq (r-s)^{\frac{1}{2}}$ , we get

$$\forall \alpha \in [0, 1], \quad |\Delta_{\mu, \mu'}(a_{i,j}(r, y, [X_r^{s, \xi, (m)}]) - a_{i,j}(r, z, [X_r^{s, \xi, (m)}]))| \leq K \frac{|z-y|^\eta}{(r-s)^{\frac{\alpha}{2}}} W_2^\alpha(\mu, \mu').$$

From (6.46), we also get

$$\forall \alpha \in [0, 1], \quad |\Delta_{\mu, \mu'} a_{i,j}(r, y, [X_r^{s, \xi, (m)}])| + |\Delta_{\mu, \mu'} a_{i,j}(r, z, [X_r^{s, \xi, (m)}])| \leq K \frac{W_2^\alpha(\mu, \mu')}{(r-s)^{\frac{\alpha-\eta}{2}}}.$$

Gathering the two previous bounds, one obtains

(6.59)

$$\forall \alpha \in [0, 1], \quad |\Delta_{\mu, \mu'}(a_{i,j}(r, y, [X_r^{s, \xi, (m)}]) - a_{i,j}(r, z, [X_r^{s, \xi, (m)}]))| \leq K \left\{ \frac{|z-y|^\eta}{(r-s)^{\frac{\alpha}{2}}} \wedge \frac{1}{(r-s)^{\frac{\alpha-\eta}{2}}} \right\} W_2^\alpha(\mu, \mu').$$

Moreover, from (6.36) (bounding  $C_m$  by a constant  $K$  independent of  $m$ ), one has

$$\left| \partial_\mu^2 H_2^{i,j} \left( \int_r^t a(v', z, [X_{v'}^{s, \xi, (m)}]) dv', z-y \right) (v) \widehat{p}_{m+1}(\mu, s, r, t, y, z) \right| \leq \frac{K}{(t-r)(r-s)^{1-\frac{\eta}{2}}} g(c(t-r), z-y).$$

The two previous estimates and the space-time inequality (1.5) thus yield

$$|D_1| \leq K \left\{ \frac{1}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{1+\frac{\alpha-\eta}{2}}} \wedge \frac{1}{(t-r)(r-s)^{1+\frac{\alpha}{2}-\eta}} \right\} W_2^\alpha(\mu, \mu') g(c(t-r), z-y).$$

For  $D_2$ , we handle  $\Delta_{\mu, \mu'} \partial_\mu^2 H_2^{i,j}(\int_r^t a(v', z, [X_{v'}^{s, \xi, (m)}]) dv', z-y)(v)$  in a similar way as we did for  $\Delta_{\mu, \mu'} \partial_\mu^2 H_1^i(\int_r^t a(v', z, [X_{v'}^{s, \xi, (m)}]) dv', z-y)(v)$ , that is, from the mean-value theorem, (6.47) and (6.48), skipping some technical details, one gets

$$\begin{aligned} |D_2| &\leq K \left\{ \frac{1}{(t-r)^{2-\frac{\eta}{2}}} \int_r^t \max_{i,j} |\Delta_{\mu, \mu'} \partial_\mu^2 a_{i,j}(v', z, [X_{v'}^{s, \xi, (m-1)}])|(v) dv' + \frac{W_2^\alpha(\mu, \mu')}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{1+\frac{\alpha-\eta}{2}}} \right\} g(c(t-r), z-y) \\ &\leq \frac{K}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{1+\frac{\alpha-\eta}{2}}} \left( 1 + (t-r)^{-1} \int_r^t C_m(s, v')(v'-s)^{\frac{\eta}{2}} dv' \right) W_2^\alpha(\mu, \mu') g(c(t-r), z-y). \end{aligned}$$

To deal with  $D_3$ , we employ (6.36) (bounding  $C_m$  by a constant  $K$  independent of  $m$ ), the  $\eta$ -Hölder regularity of  $x \mapsto a(t, x, \mu)$  (combined with the space time inequality (1.5)) and (6.50). We obtain

$$|D_3| \leq K \frac{W_2^\alpha(\mu, \mu')}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{1+\frac{\alpha}{2}-\eta}} g(c(t-r), z-y).$$

Gathering the previous estimates on  $D_1$ ,  $D_2$  and  $D_3$ , we get

$$\begin{aligned} |D| &\leq C \left\{ \frac{1}{(t-r)(r-s)^{1+\frac{\alpha-\eta}{2}}} \wedge \frac{1}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{1+\frac{\alpha-\eta}{2}}} \right\} \left( 1 + (t-r)^{-1} \int_r^t C_m(s, v')(v'-s)^{\frac{\eta}{2}} dv \right) \\ &\quad \times W_2^\alpha(\mu, \mu') g(c(t-r), z-y). \end{aligned}$$

#### • Estimates on E:

For  $E_1$ , we proceed as for the previous terms. To be more specific, from (6.57), the mean-value theorem and (6.46) (with  $h = a_{i,j}$ ) as well as (6.37) (note that one may bound the constant  $C_m$  by a constant  $K$  independent of  $m$ ), we have

$$|E_1| \leq K \frac{W_2^\alpha(\mu, \mu')}{(t-r)^{\frac{1}{2}}(r-s)^{1+\frac{\alpha}{2}-\eta}} g(c(t-r), z-y).$$

For  $E_2$ , from (6.59) on the one hand and (6.37) as well as (6.56) on the other hand, we get

$$|E_2| \leq K \left\{ \frac{1}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{1+\frac{\alpha-\eta}{2}}} \wedge \frac{1}{(t-r)(r-s)^{1+\frac{\alpha}{2}-\eta}} \right\} W_2^\alpha(\mu, \mu') g(c(t-r), z-y).$$

For the last term  $E_3$ , from (6.45) and the induction hypothesis, one obtains

$$|E_3| \leq \frac{K}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{1+\frac{\alpha-\eta}{2}}} \left( 1 + (t-r)^{-1} \int_r^t C_m(s, v')(v'-s)^{\frac{\eta}{2}} dv' \right) W_2^\alpha(\mu, \mu') g(c(t-r), z-y).$$

Gathering the previous estimates, we finally deduce

$$|\mathbf{E}| \leq \frac{K}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{1+\frac{\alpha-\eta}{2}}} \left( 1 + (t-r)^{-1} \int_r^t C_m(s, v')(v'-s)^{\frac{\eta}{2}} dv' \right) W_2^\alpha(\mu, \mu') g(c(t-r), z-y).$$

We now collect all the previous estimates on A, B, C, D and E. Using the fact that  $v' \mapsto C_m(s, v')$  is non-decreasing, we finally obtain the following bound

$$\begin{aligned} & |\Delta_{\mu, \mu'} \partial_\mu^2 \mathcal{H}_{m+1}(\mu, s, r, t, y, z)(v)| \\ & \leq K \left\{ \frac{1}{(t-r)(r-s)^{1+\frac{\alpha-\eta}{2}}} \wedge \frac{1}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{1+\frac{\alpha}{2}}} \right\} \left( 1 + \frac{1}{t-r} \int_r^t C_{m,n}(s, v')(v'-s)^{\frac{\eta}{2}} dv' \right) \\ & \quad \times W_2^\alpha(\mu, \mu') g(c(t-r), z-y) \end{aligned}$$

which in turn, after a space-time convolution with  $p_{m+1}$  (separating the time integral into the two disjoint parts  $[s, \frac{t+s}{2}]$  and  $[\frac{t+s}{2}, t]$  as we did before), implies

$$\begin{aligned} & |p_{m+1} \otimes \Delta_{\mu, \mu'} \partial_\mu^2 \mathcal{H}_{m+1}(\mu, s, t, x, z)(v)| \\ & \leq \frac{K}{(t-s)^{1+\frac{\alpha-\eta}{2}}} \left( B\left(\frac{\eta}{2}, \frac{\eta-\alpha}{2}\right) + \sum_{k=1}^m C^k \prod_{i=1}^{k+1} B\left(\frac{\eta}{2}, \frac{\eta-\alpha}{2} + (i-1)\frac{\eta}{2}\right) (t-s)^{k\frac{\eta}{2}} \right) \\ & \quad \times W_2^\alpha(\mu, \mu') g(c(t-s), z-x). \end{aligned}$$

From standard computations, we deduce that the series  $\sum_{k \geq 0} (p_{m+1} \otimes \Delta_{\mu, \mu'} \partial_\mu^2 \mathcal{H}_{m+1}) \otimes \mathcal{H}_{m+1}^{(k)}(\mu, s, t, x, z)(v)$  converges absolutely and uniformly with respect to  $s, x, v$  on any compact set  $\mathcal{K} \subset [0, t) \times \mathbb{R}^d \times (\mathbb{R}^d)^2$ . Moreover, there exist constants  $K := K((\mathbf{HE}), (\mathbf{HR}_+), T)$ ,  $c := c(\lambda) > 0$  such that for any  $\alpha \in [0, \eta)$

$$(6.60) \quad \begin{aligned} & \sum_{k \geq 0} \left| (p_{m+1} \otimes \Delta_{\mu, \mu'} \partial_\mu^2 \mathcal{H}_{m+1}) \otimes \mathcal{H}_{m+1}^{(k)}(\mu, s, t, x, z)(v) \right| \\ & \leq \frac{K}{(t-s)^{1+\frac{\alpha-\eta}{2}}} \left\{ B\left(\frac{\eta}{2}, \frac{\eta-\alpha}{2}\right) + \sum_{k=1}^m C^k \prod_{i=1}^{k+1} B\left(\frac{\eta}{2}, \frac{\eta-\alpha}{2} + (i-1)\frac{\eta}{2}\right) (t-s)^{k\frac{\eta}{2}} \right\} \\ & \quad \times W_2^\alpha(\mu, \mu') g(c(t-s), z-x). \end{aligned}$$

From (6.43) and (4.24) (at step  $m+1$ ), separating the computations into the two disjoint intervals  $[s, \frac{t+s}{2}]$  and  $[\frac{t+s}{2}, t]$  to balance the time singularity, we get

$$|(\partial_\mu^2 p_{m+1} \otimes \Delta_{\mu, \mu'} \mathcal{H}_{m+1})(\mu, s, t, x, z)(v)| \leq \frac{K}{(t-s)^{1+\frac{\alpha}{2}-\eta}} W_2^\alpha(\mu, \mu') g(c(t-s), z-x)$$

so that

$$(6.61) \quad \sum_{k \geq 0} |(\partial_\mu^2 p_{m+1} \otimes \Delta_{\mu, \mu'} \mathcal{H}_{m+1}) \otimes \mathcal{H}_{m+1}^{(k)}(\mu, s, t, x, z)(v)| \leq \frac{K}{(t-s)^{1+\frac{\alpha}{2}-\eta}} W_2^\alpha(\mu, \mu') g(c(t-s), z-x)$$

for any  $\alpha \in [0, \eta)$ . Similarly, from the estimates (6.42), (6.38) (bounding  $C_{m,n}$  by a constant  $K$  independent of  $m$ ), separating the time integral into two disjoint intervals as previously done, after some standard computations, we get

$$|\Delta_{\mu, \mu'} p_{m+1} \otimes \partial_\mu^2 \mathcal{H}_{m+1}(\mu, s, t, x, z)(v)| \leq \frac{K}{(t-s)^{1+\frac{\alpha}{2}-\eta}} W_2^\alpha(\mu, \mu') g(c(t-s), z-x)$$

which in turn implies

$$(6.62) \quad \sum_{k \geq 0} |(\Delta_{\mu, \mu'} p_{m+1} \otimes \partial_\mu^2 \mathcal{H}_{m+1}) \otimes \mathcal{H}_{m+1}^{(k)}(\mu, s, t, x, z)(v)| \leq \frac{K}{(t-s)^{1+\frac{\alpha}{2}-\eta}} W_2^\alpha(\mu, \mu') g(c(t-s), z-x).$$

If we differentiate twice with respect to the measure argument the relation  $p_{m+1} = \widehat{p}_{m+1} + p_{m+1} \otimes \mathcal{H}_{m+1}$ , we obtain  $\partial_\mu^2 p_{m+1} = \partial_\mu^2 \widehat{p}_{m+1} + p_{m+1} \otimes \partial_\mu^2 \mathcal{H}_{m+1} + \partial_\mu^2 p_{m+1} \otimes \mathcal{H}_{m+1}$  so that

$$\begin{aligned} \Delta_{\mu, \mu'} \partial_\mu^2 p_{m+1}(\mu, s, t, x, z)(v) &= \Delta_{\mu, \mu'} \partial_\mu^2 \widehat{p}_{m+1}(\mu, s, t, x, z)(v) + p_{m+1} \otimes \Delta_{\mu, \mu'} \partial_\mu^2 \mathcal{H}_{m+1}(\mu, s, t, x, z)(v) \\ & \quad + \Delta_{\mu, \mu'} p_{m+1} \otimes \partial_\mu^2 \mathcal{H}_{m+1}(\mu, s, t, x, z)(v) + \partial_\mu^2 p_{m+1} \otimes \Delta_{\mu, \mu'} \mathcal{H}_{m+1}(\mu, s, t, x, z)(v) \\ & \quad + \Delta_{\mu, \mu'} \partial_\mu^2 p_{m+1} \otimes \mathcal{H}_{m+1}(\mu, s, t, x, z)(v). \end{aligned}$$

One may then iterate the previous relation and thus obtain the following representation

$$\begin{aligned} \Delta_{\mu,\mu'} \partial_{\mu}^2 p_{m+1}(\mu, s, t, x, z)(v) &= \sum_{k \geq 0} \left[ \Delta_{\mu,\mu'} \partial_{\mu}^2 \widehat{p}_{m+1} + p_{m+1} \otimes \Delta_{\mu,\mu'} \partial_{\mu}^2 \mathcal{H}_{m+1} \right. \\ &\quad \left. + \Delta_{\mu,\mu'} p_{m+1} \otimes \partial_{\mu}^2 \mathcal{H}_{m+1} + \partial_{\mu}^2 p_{m+1} \otimes \Delta_{\mu,\mu'} \mathcal{H}_{m+1} \right] \otimes \mathcal{H}_{m+1}^{(k)}(\mu, s, t, x, z)(v). \end{aligned}$$

Gathering the estimates (6.49), (6.60), (6.62), (6.61), we deduce that the above series converges absolutely and satisfies

$$\begin{aligned} &|\Delta_{\mu,\mu'} \partial_{\mu}^2 p_{m+1}(\mu, s, t, x, z)(v)| \\ &\leq \frac{K}{(t-s)^{1+\frac{\alpha}{2}-\eta}} W_2^{\alpha}(\mu, \mu') g(c(t-s), z-x) \\ &\quad + \frac{K}{(t-s)^{1+\frac{\alpha-\eta}{2}}} \left\{ B\left(\frac{\eta}{2}, \frac{\eta-\alpha}{2}\right) + \sum_{k=1}^m C^k \prod_{i=1}^{k+1} B\left(\frac{\eta}{2}, \frac{\eta-\alpha}{2} + (i-1)\frac{\eta}{2}\right) (t-s)^{k\frac{\eta}{2}} \right\} \\ &\quad \times W_2^{\alpha}(\mu, \mu') g(c(t-s), z-x) \\ &\leq \frac{K}{(t-s)^{1+\frac{\alpha-\eta}{2}}} \left\{ B\left(\frac{\eta}{2}, \frac{\eta-\alpha}{2}\right) + \sum_{k=1}^m C^k \prod_{i=1}^{k+1} B\left(\frac{\eta}{2}, \frac{\eta-\alpha}{2} + (i-1)\frac{\eta}{2}\right) (t-s)^{k\frac{\eta}{2}} \right\} \\ &\quad \times W_2^{\alpha}(\mu, \mu') g(c(t-s), z-x) \end{aligned}$$

so that

$$u_{m+1}(s, t) \leq \frac{K}{(t-s)^{1+\frac{\alpha}{2}-\eta}} \left\{ B\left(\frac{\eta}{2}, \frac{\eta-\alpha}{2}\right) + \sum_{k=1}^m C^k \prod_{i=1}^{k+1} B\left(\frac{\eta}{2}, \frac{\eta-\alpha}{2} + (i-1)\frac{\eta}{2}\right) (t-s)^{k\frac{\eta}{2}} \right\}$$

and similarly,

$$v_{m+1}(s, t) \leq \frac{K}{(t-s)^{1+\frac{\alpha-\eta}{2}}} \left\{ B\left(\frac{\eta}{2}, \frac{\eta-\alpha}{2}\right) + \sum_{k=1}^m C^k \prod_{i=1}^{k+1} B\left(\frac{\eta}{2}, \frac{\eta-\alpha}{2} + (i-1)\frac{\eta}{2}\right) (t-s)^{k\frac{\eta}{2}} \right\}.$$

Since the constant  $K$  does not depend either on the constant  $C$  appearing in the definition of  $C_m(s, t)$  or  $m$ , one may change  $C$  once for all and derive the induction hypothesis at step  $m+1$  for  $u_m$  and  $v_m$ . This completes the proof of (4.25) at step  $m+1$ .

In order to derive (4.26) at step  $m+1$ , we proceed similarly.

We first point that the uniform  $\beta$ -Hölder regularity,  $\beta \in [0, \eta)$ , of the map  $x \mapsto \partial_{\mu}^2 p_m(\mu, s, t, x, z)(v)$  directly stems from the representation formula (6.17). Indeed, from (6.24) and (6.23), distinguishing the two disjoint cases  $|x-x'| \leq (t-s)^{\frac{1}{2}}$  and  $|x-x'| > (t-s)^{\frac{1}{2}}$ , after some standard computations that we omit, the following estimate holds: there exists a positive constant  $K$  such that for any  $m \geq 1$ , for any  $x, x' \in \mathbb{R}^d$ , for any  $\beta \in [0, \eta)$ , one has

$$|\partial_{\mu}^2 \widehat{p}_m(\mu, s, t, x, z)(y) - \partial_{\mu}^2 \widehat{p}_m(\mu, s, t, x', z)(y)| \leq K \frac{|x-x'|^{\beta}}{(t-s)^{1+\frac{\beta-\eta}{2}}} \{g(c(t-s), z-x) + g(c(t-s), z-x')\}$$

which in turn implies the absolute and local uniform convergence of the series  $\sum_{k \geq 0} |\partial_{\mu}^2 \widehat{p}_{m+1} \otimes \mathcal{H}_{m+1}^{(k)}(\mu, s, t, x, z)(y) - \partial_{\mu}^2 \widehat{p}_{m+1} \otimes \mathcal{H}_{m+1}^{(k)}(\mu, s, t, x', z)(y)|$  along with the following estimate

$$\begin{aligned} &\sum_{k \geq 0} |\partial_{\mu}^2 \widehat{p}_{m+1} \otimes \mathcal{H}_{m+1}^{(k)}(\mu, s, t, x, z)(y) - \partial_{\mu}^2 \widehat{p}_{m+1} \otimes \mathcal{H}_{m+1}^{(k)}(\mu, s, t, x', z)(y)| \\ &\leq K \frac{|x-x'|^{\beta}}{(t-s)^{1+\frac{\beta-\eta}{2}}} \{g(c(t-s), z-x) + g(c(t-s), z-x')\}. \end{aligned}$$

Similarly, from (4.11) (with  $n=0$  and  $\beta \in [0, \eta)$ ), (6.28) and similar lines of reasonings, we get

$$\begin{aligned} &\sum_{k \geq 0} |p_{m+1} \otimes \partial_{\mu}^2 \mathcal{H}_{m+1} \otimes \mathcal{H}_{m+1}^{(k)}(\mu, s, t, x, z)(y) - p_{m+1} \otimes \partial_{\mu}^2 \mathcal{H}_{m+1} \otimes \mathcal{H}_{m+1}^{(k)}(\mu, s, t, x', z)(y)| \\ &\leq K \frac{|x-x'|^{\beta}}{(t-s)^{1+\frac{\beta-\eta}{2}}} \{g(c(t-s), z-x) + g(c(t-s), z-x')\}. \end{aligned}$$

Hence, combining the two previous estimates, we conclude

(6.63)

$$|\partial_\mu^2 p_m(\mu, s, t, x, z)(v) - \partial_\mu^2 p_m(\mu, s, t, x', z)(v)| \leq C \frac{|x - x'|^\beta}{(t - s)^{1 + \frac{\beta - \eta}{2}}} \{g(c(t - s), z - x) + g(c(t - s), z - x')\}.$$

We now introduce the quantities

$$u_m(s, t) := \sup_{(y, y') \in (\mathbb{R}^d)^2 \times (\mathbb{R}^d)^2, y \neq y'} \int_{(\mathbb{R}^d)^2} (|y'' - x'|^\eta \wedge 1) \frac{|\Delta_{y, y'} \partial_\mu^2 p_m(\mu, s, t, x', y'')(y)|}{|y - y'|^\beta} dy'' \mu(dx'),$$

$$v_m(s, t) := \sup_{(y, y') \in (\mathbb{R}^d)^2 \times (\mathbb{R}^d)^2, y \neq y'} \int_{(\mathbb{R}^d)^2} \frac{|\Delta_{y, y'} \partial_\mu^2 p_m(\mu, s, t, x', y'')(y)|}{|y - y'|^\beta} dy'' \mu(dx'),$$

for any fixed  $\beta \in [0, \eta)$  and for any  $m \geq 1$ , where we introduced the notation  $\Delta_{y, y'} \partial_\mu^2 h(\mu)(y) := \partial_\mu^2 h(\mu)(y) - \partial_\mu^2 h(\mu)(y')$ , for  $h \in \mathcal{C}_f^2(\mathcal{P}_2(\mathbb{R}^d))$ ,  $y, y' \in (\mathbb{R}^d)^2$ . We will also use the notation  $\Delta_{y, y'} \partial_\mu h(\mu)(y) = \partial_\mu h(\mu)(y) - \partial_\mu h(\mu)(y')$  for  $h \in \mathcal{C}^1(\mathcal{P}_2(\mathbb{R}^d))$  and  $y, y' \in \mathbb{R}^d$ . We prove by induction the following key inequalities:

$$u_m(s, t) \leq C_m(s, t)(t - s)^{-(1 + \frac{\beta}{2} - \eta)}, \quad \text{and} \quad v_m(s, t) \leq C_m(s, t)(t - s)^{-(1 + \frac{\beta - \eta}{2})},$$

with  $C_m(s, t) := \sum_{k=1}^m C^k \prod_{i=1}^k B\left(\frac{\eta}{2}, \frac{\eta - \beta}{2} + (i - 1)\frac{\eta}{2}\right) (t - s)^{(k-1)\frac{\eta}{2}}$ ,  $C := C((\mathbf{HE}), (\mathbf{HR}_+), T)$  being a positive constant independent of  $m$ . The result being straightforward for  $m = 1$ , we assume that it holds at step  $m$ . From (6.14) and (6.15) (both with  $n = 0$  and  $n = 1$ ) as well as (4.16), (4.10), (4.11), we get

$$(6.64) \quad |\partial_\mu a_{i,j}(t, x, [X_t^{s, \xi, (m)}])(y_1) - \partial_\mu a_{i,j}(t, x, [X_t^{s, \xi, (m)}])(y'_1)|$$

$$+ |\partial_\mu b_i(t, x, [X_t^{s, \xi, (m)}])(y_1) - \partial_\mu b_i(t, x, [X_t^{s, \xi, (m)}])(y'_1)| \leq K \frac{|y_1 - y'_1|^\beta}{(t - s)^{\frac{1 + \beta - \eta}{2}}}.$$

From the decomposition (2.11) applied to the maps  $h = a_{i,j}$ ,  $b_i$  and  $\Theta(s, \mu) = [X_t^{s, \xi, (m)}]$ , with the estimates (4.11), (4.16), (4.19) and some standard computations that we omit, we get

$$(6.65) \quad |\Delta_{y, y'} \partial_\mu^2 a_{i,j}(t, x, [X_t^{s, \xi, (m)}])(y)| + |\Delta_{y, y'} \partial_\mu^2 b_i(t, x, [X_t^{s, \xi, (m)}])(y)|$$

$$\leq K \left\{ \frac{|y - y'|^\beta}{(t - s)^{1 + \frac{\beta - \eta}{2}}} + \int_{(\mathbb{R}^d)^2} (|y'' - x'|^\eta \wedge 1) |\Delta_{y, y'} \partial_\mu^2 p_m(\mu, s, t, x', y'')(y)| dy'' d\mu(x') \right\}.$$

Hence, combining (6.24) with (6.64) and (6.65)

$$(6.66) \quad |\Delta_{y, y'} \partial_\mu^2 \widehat{p}_{m+1}(\mu, s, r, t, x, z)(y)| \leq K |y - y'|^\beta \left\{ \frac{1}{(t - r)^2} \int_r^t (v - s)^{-\frac{1 + \beta - \eta}{2}} dv \int_r^t (v - s)^{-\frac{1 - \eta}{2}} dv \right.$$

$$\left. + \frac{1}{t - r} \int_r^t \left[ \frac{1}{(v - s)^{1 + \frac{\beta - \eta}{2}}} + (|y'' - x'|^\eta \wedge 1) \frac{|\Delta_{y, y'} \partial_\mu^2 p_m(\mu, s, v, x', y'')(y)|}{|y - y'|^\beta} dy'' d\mu(x') \right] dv \right\}$$

$$\times g(c(t - r), z - x)$$

so that, from the previous inequality with  $r = s$  and the induction hypothesis,

$$(6.67) \quad |\Delta_{y, y'} \partial_\mu^2 \widehat{p}_{m+1}(\mu, s, t, x, z)(y)|$$

$$\leq K |y - y'|^\beta \left\{ \frac{1}{(t - s)^{1 + \frac{\beta - \eta}{2}}} + \frac{1}{t - s} \int_s^t \frac{C_m(s, r)}{(r - s)^{1 + \frac{\beta}{2} - \eta}} dr \right\} g(c(t - s), z - x)$$

$$\leq K |y - y'|^\beta \left\{ \frac{1}{(t - s)^{1 + \frac{\beta - \eta}{2}}} + \frac{1}{(t - s)^{\frac{\eta}{2}}} \int_s^t \frac{C_m(s, r)}{(t - r)^{1 - \frac{\eta}{2}} (r - s)^{1 + \frac{\beta}{2} - \eta}} dr \right\} g(c(t - s), z - x)$$

$$= K \frac{|y - y'|^\beta}{(t - s)^{1 + \frac{\beta - \eta}{2}}} \left\{ 1 + \sum_{k=1}^m C^k \prod_{i=1}^{k+1} B\left(\frac{\eta}{2}, \frac{\eta - \beta}{2} + (i - 1)\frac{\eta}{2}\right) (t - s)^{k\frac{\eta}{2}} \right\} g(c(t - s), z - x).$$

By induction on  $r$ , there exists a positive constant  $K := K((\mathbf{HE}), (\mathbf{HR}_+), T)$  (which may change from lines to lines but is independent of  $m$  and  $C$ ) such that

$$\begin{aligned} & |(\Delta_{y,y'} \partial_\mu^2 \widehat{p}_{m+1} \otimes \mathcal{H}_{m+1}^{(r)})(\mu, s, t, x, z)(y)| \\ & \leq K^r |y - y'|^\beta \left\{ 1 + \sum_{k=1}^m C^k \prod_{i=1}^{k+1} B\left(\frac{\eta}{2}, \frac{\eta - \beta}{2} + (i-1)\frac{\eta}{2}\right) (t-s)^{k\frac{\eta}{2}} \right\} (t-s)^{-1 + \frac{(\eta-\beta)}{2} + r\frac{\eta}{2}} \\ & \quad \times \prod_{i=1}^r B\left(\frac{\eta}{2}, \frac{\eta - \beta}{2} + (i-1)\frac{\eta}{2}\right) g(c(t-s), z-x) \end{aligned}$$

which in turn implies

$$\begin{aligned} & \sum_{r \geq 0} |(\Delta_{y,y'} \partial_\mu^2 \widehat{p}_{m+1} \otimes \mathcal{H}_{m+1}^{(r)})(\mu, s, t, x, z)(y)| \\ & \leq K \frac{|y - y'|^\beta}{(t-s)^{1 + \frac{(\beta-\eta)}{2}}} \left\{ 1 + \sum_{k=1}^m C^k \prod_{i=1}^{k+1} B\left(\frac{\eta}{2}, \frac{\eta - \beta}{2} + (i-1)\frac{\eta}{2}\right) (t-s)^{k\frac{\eta}{2}} \right\} \\ & \quad \times g(c(t-s), z-x) \\ & \leq K \frac{|y - y'|^\beta}{(t-s)^{1 + \frac{(\beta-\eta)}{2}}} \left\{ B\left(\frac{\eta}{2}, \frac{\eta - \beta}{2}\right) + \sum_{k=1}^m C^k \prod_{i=1}^{k+1} B\left(\frac{\eta}{2}, \frac{\eta - \beta}{2} + (i-1)\frac{\eta}{2}\right) (t-s)^{k\frac{\eta}{2}} \right\} \\ (6.68) \quad & \quad \times g(c(t-s), z-x). \end{aligned}$$

Then, from (6.27), we write

$$\begin{aligned} \Delta_{y,y'} \partial_\mu^2 \mathcal{H}_m(\mu, s, r, t, x, z)(y) &= \text{I}(y) - \text{I}(y') + \text{II}(y) - \text{II}(y') + \text{III}(y) - \text{III}(y') \\ &= \text{I}_1(y) - \text{I}_1(y') + \text{I}_2(y) - \text{I}_2(y') + \text{II}_1(y) - \text{II}_1(y') + \text{II}_2(y) - \text{II}_2(y') \\ & \quad + \text{III}(y) - \text{III}(y') \end{aligned}$$

and prove appropriate estimates for each term. With our notations, one has

$$\text{I}_1(y) - \text{I}_1(y') = \sum_{i=1}^d H_1^i \left( \int_r^t a(v, z, [X_v^{s,\xi,(m)}]) dv, z-x \right) \Delta_{y,y'} \partial_\mu^2 b_i(r, x, [X_r^{s,\xi,(m)}])(y) \widehat{p}_{m+1}(\mu, r, t, x, z).$$

As a consequence, from (6.65) and the induction hypothesis

$$\begin{aligned} |\text{I}_1(y) - \text{I}_1(y')| &\leq K \frac{|y - y'|^\beta}{(t-r)^{\frac{1}{2}} (r-s)^{1 + \frac{(\beta-\eta)}{2}}} \left[ 1 + (r-s)^{1 + \frac{(\beta-\eta)}{2}} u_m(s, r) \right] g(c(t-r), z-x) \\ &\leq K \frac{|y - y'|^\beta}{(t-r)^{\frac{1}{2}} (r-s)^{1 + \frac{(\beta-\eta)}{2}}} \left[ 1 + C_m(s, r) (r-s)^{\frac{\eta}{2}} \right] g(c(t-r), z-x). \end{aligned}$$

Next, we proceed similarly. From (6.65) and the mean-value theorem

$$\begin{aligned} & |\text{I}_2(y) - \text{I}_2(y')| \\ & := \left| \sum_{i=1}^d b_i(r, x, [X_r^{s,\xi,(m)}]) \Delta_{y,y'} \partial_\mu^2 H_1^i \left( \int_r^t a(v, z, [X_v^{s,\xi,(m)}]) dv, z-x \right) (y) \widehat{p}_m(\mu, r, t, x, z) \right. \\ & \leq K \left\{ \frac{|y - y'|^\beta}{(t-r)^{\frac{1}{2}} (r-s)^{1 + \frac{\beta-\eta}{2}}} + \frac{1}{(t-r)^{\frac{3}{2}}} \int_r^t \max_{i,j} \left| \partial_\mu^2 a_{i,j}(v, z, [X_v^{s,\xi,(m)}])(y) - \partial_\mu^2 a_{i,j}(v, z, [X_v^{s,\xi,(m)}])(y') \right| dv \right\} \\ & \quad \times g(c(t-r), z-x) \\ & \leq K \frac{|y - y'|^\beta}{(t-r)^{\frac{1}{2}} (r-s)^{1 + \frac{\beta-\eta}{2}}} \left\{ 1 + \frac{1}{t-r} \int_r^t C_m(s, v) (v-s)^{\frac{\eta}{2}} dv \right\} g(c(t-r), z-x). \end{aligned}$$

so that, gathering the two previous estimates and using the fact that  $v \mapsto C_m(s, v)$  is non-decreasing, we conclude

$$|\text{I}(y) - \text{I}(y')| \leq K \frac{|y - y'|^\beta}{(t-r)^{\frac{1}{2}} (r-s)^{1 + \frac{(\beta-\eta)}{2}}} \left( 1 + (t-r)^{-1} \int_r^t C_m(s, v) (v-s)^{\frac{\eta}{2}} \right) g(c(t-r), z-x).$$



Still using our notations, we have

$$\Pi_1(y) - \Pi_1(y') = \sum_{i,j=1}^d \left\{ H_2^{i,j} \left( \int_r^t a(v, z, [X_v^{s,\xi,(m)}]) dv, z - x \right) \times (J_{i,j}(y) - J_{i,j}(y')) \right\} \widehat{p}_{m+1}(\mu, r, t, x, z).$$

From the decomposition (6.32), on the one hand, one may use the  $\eta$ -Hölder of the maps  $x \mapsto \frac{\delta}{\delta m} a_{i,j}(r, x, \mu)$ ,  $\frac{\delta^2}{\delta m^2} a_{i,j}(r, x, \mu)$  and the estimates (4.10), (4.11), (4.18), (4.19) as well as the induction hypothesis to get

$$|J_{i,j}(y) - J_{i,j}(y')| \leq K \frac{|z - x|^\eta |y - y'|^\beta}{(r - s)^{1 + \frac{\beta}{2}}} (1 + C_m(s, r)(r - s)^{\frac{\eta}{2}}).$$

On the other hand, still from (6.32), separating the computations into the two disjoint cases:  $|y - y'| \geq (r - s)^{\frac{1}{2}}$  and  $|y - y'| < (r - s)^{\frac{1}{2}}$ , using (6.34) (bounding  $u_m(s, r)$  by  $K(r - s)^{-1 + \frac{\eta}{2}}$ ) in the first case and the uniform  $\eta$ -Hölder regularity of the maps  $\mathbb{R}^d \ni z \mapsto \frac{\delta}{\delta m} a_{i,j}(t, x, \mu)(z)$ ,  $(\mathbb{R}^d)^2 \ni z \mapsto \frac{\delta^2}{\delta m^2} a_{i,j}(t, x, \mu)(z)$  together with (4.11), (4.18), (4.19) in the second case, we obtain

$$\sum_{\ell=1}^6 |J_{i,j}^\ell(y) - J_{i,j}^\ell(y')| \leq K \frac{|y - y'|^\beta}{(r - s)^{1 + \frac{\beta - \eta}{2}}}$$

and, from the  $\eta$ -Hölder regularity of the map  $\mathbb{R}^d \ni z \mapsto \frac{\delta}{\delta m} a_{i,j}(t, x, \mu)(z)$  and the induction hypothesis

$$\begin{aligned} |J_{i,j}^7(y) - J_{i,j}^7(y')| &\leq K \int_{(\mathbb{R}^d)^2} (|z' - x'|^\eta \wedge 1) |\partial_\mu^2 p_m(\mu, s, r, x', z')(y) - \partial_\mu^2 p_m(\mu, s, r, x', z')(y')| dz' d\mu(x') \\ &\leq K \frac{|y - y'|^\beta}{(r - s)^{1 + \frac{\beta}{2} - \eta}} C_m(s, r). \end{aligned}$$

Combining the three previous estimates yields

$$|\Pi_1(y) - \Pi_1(y')| \leq K \left\{ \frac{|y - y'|^\beta}{(t - r)^{1 - \frac{\eta}{2}} (r - s)^{1 + \frac{\beta}{2}}} \wedge \frac{|y - y'|^\beta}{(t - r)(r - s)^{1 + \frac{\beta - \eta}{2}}} \right\} \left[ 1 + C_m(s, r)(r - s)^{\frac{\eta}{2}} \right] g(c(t - r), z - x)$$

for all  $y, y' \in (\mathbb{R}^d)^2$ .

From (6.64) and (6.65), for any  $\beta \in [0, \eta)$  and for any  $y, y' \in (\mathbb{R}^d)^2$ , one gets

$$\begin{aligned} |\Pi_2(y) - \Pi_2(y')| &\leq K \left\{ \frac{1}{(t - r)^{2 - \frac{\eta}{2}}} \int_r^t \max_{i,j} |\Delta_{y,y'} \partial_\mu^2 a_{i,j}(v', z, [X_{v'}^{s,\xi,(m-1)}])|(y) dv' + \frac{|y - y'|^\beta}{(t - r)^{1 - \frac{\eta}{2}} (r - s)^{1 + \frac{\beta - \eta}{2}}} \right\} \\ &\quad \times g(c(t - r), z - x) \\ &\leq K \frac{|y - y'|^\beta}{(t - r)^{1 - \frac{\eta}{2}} (r - s)^{1 + \frac{\beta - \eta}{2}}} \left( 1 + (t - r)^{-1} \int_r^t C_m(s, v)(v - s)^{\frac{\eta}{2}} dv \right) g(c(t - r), z - x). \end{aligned}$$

Finally, for the last term, from (6.66) and the induction hypothesis, we obtain

$$\begin{aligned} \forall \beta \in [0, \eta), \forall (y, y') \in (\mathbb{R}^d)^2, \quad &|\text{III}(y) - \text{III}(y')| \\ &\leq K \frac{1}{(t - r)^{1 - \frac{\eta}{2}}} |\Delta_{y,y'} \partial_\mu^2 \widehat{p}_{m+1}(\mu, s, r, t, x, z)(y)| \\ &\leq K \frac{|y - y'|^\beta}{(t - r)^{1 - \frac{\eta}{2}} (r - s)^{1 + \frac{\beta - \eta}{2}}} \left[ 1 + \int_r^t C_m(s, v)(v - s)^{\frac{\eta}{2}} dv \right] g(c(t - r), z - x). \end{aligned}$$

Gathering all the previous computations, we finally conclude

$$\begin{aligned} \left| \Delta_{y,y'} \partial_\mu^2 \mathcal{H}_{m+1}(\mu, s, r, t, x, z)(y) \right| &\leq K \left\{ \frac{|y - y'|^\beta}{(t - r)^{1 - \frac{\eta}{2}} (r - s)^{1 + \frac{\beta}{2}}} \wedge \frac{|y - y'|^\beta}{(t - r)(r - s)^{1 + \frac{\beta - \eta}{2}}} \right\} \\ &\quad \times \left[ 1 + (t - r)^{-1} \int_r^t C_m(s, v)(v - s)^{\frac{\eta}{2}} dv \right] \times g(c(t - r), z - x) \end{aligned}$$

for any  $y, y' \in (\mathbb{R}^d)^2$ . We again separate the space-time convolution into the two disjoint cases:  $r \in [s, \frac{t+s}{2}]$  and  $r \in [\frac{t+s}{2}, t]$ . Skipping some technical details, we obtain

$$\begin{aligned} & |p_{m+1} \otimes \Delta_{y,y'} \partial_\mu^2 \mathcal{H}_{m+1}(\mu, s, t, x, z)(y)| \\ & \leq K |y - y'|^\beta \left( \frac{1}{(t-s)^{1+\frac{(\beta-\eta)}{2}}} B\left(\frac{\eta}{2}, \frac{\eta-\beta}{2}\right) + \frac{1}{(t-s)^{\frac{\eta}{2}}} \int_s^t \frac{C_m(s, r)(r-s)^{\frac{\eta}{2}}}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{1+\frac{(\beta-\eta)}{2}}} dr \right) \\ & \quad \times g(c(t-s), z-x) \\ & \leq K \frac{|y-y'|^\beta}{(t-s)^{1+\frac{(\beta-\eta)}{2}}} \left( B\left(\frac{\eta}{2}, \frac{\eta-\beta}{2}\right) + \sum_{k=1}^m C^k \prod_{i=1}^{k+1} B\left(\frac{\eta}{2}, \frac{\eta-\beta}{2} + (i-1)\frac{\eta}{2}\right) (t-s)^{k\frac{\eta}{2}} \right) \\ & \quad \times g(c(t-s), z-x) \end{aligned}$$

so that

$$\begin{aligned} & \sum_{r \geq 0} |(p_{m+1} \otimes \Delta_{y,y'} \partial_\mu^2 \mathcal{H}_{m+1}) \otimes \mathcal{H}_{m+1}^{(r)}(\mu, s, t, x, z)(y)| \\ (6.69) \quad & \leq K \frac{|y-y'|^\beta}{(t-s)^{1+\frac{(\beta-\eta)}{2}}} \left( B\left(\frac{\eta}{2}, \frac{\eta-\beta}{2}\right) + \sum_{k=1}^m C^k \prod_{i=1}^{k+1} B\left(\frac{\eta}{2}, \frac{\eta-\beta}{2} + (i-1)\frac{\eta}{2}\right) (t-s)^{k\frac{\eta}{2}} \right) g(c(t-s), z-x). \end{aligned}$$

Now, combining (6.68) and (6.69) with the following representation

$$\Delta_{y,y'} \partial_\mu^2 p_{m+1}(\mu, s, t, x, z)(y) = \sum_{r \geq 0} [\Delta_{y,y'} \partial_\mu^2 \hat{p}_{m+1} + p_{m+1} \otimes \Delta_{y,y'} \partial_\mu^2 \mathcal{H}_{m+1}] \otimes \mathcal{H}_{m+1}^{(r)}(\mu, s, t, x, z)(y)$$

we deduce that there exist two constants  $K, c$  (independent of  $C$  and  $m$ ) such that

$$\begin{aligned} & |\Delta_{y,y'} \partial_\mu^2 p_{m+1}(\mu, s, t, x, z)(y)| \\ & \leq K \frac{|y-y'|^\beta}{(t-s)^{1+\frac{(\beta-\eta)}{2}}} \left\{ B\left(\frac{\eta}{2}, \frac{\eta-\beta}{2}\right) + \sum_{k=1}^m C^k \prod_{i=1}^{k+1} B\left(\frac{\eta}{2}, \frac{\eta-\beta}{2} + (i-1)\frac{\eta}{2}\right) (t-s)^{k\frac{\eta}{2}} \right\} g(c(t-s), z-x) \end{aligned}$$

so that

$$u_{m+1}(s, t) \leq \frac{K}{(t-s)^{1+\frac{\beta}{2}-\eta}} \left\{ B\left(\frac{\eta}{2}, \frac{\eta-\beta}{2}\right) + \sum_{k=1}^m C^k \prod_{i=1}^{k+1} B\left(\frac{\eta}{2}, \frac{\eta-\beta}{2} + (i-1)\frac{\eta}{2}\right) (t-s)^{k\frac{\eta}{2}} \right\}$$

and similarly,

$$v_{m+1}(s, t) \leq \frac{K}{(t-s)^{1+\frac{\beta-\eta}{2}}} \left\{ B\left(\frac{\eta}{2}, \frac{\eta-\beta}{2}\right) + \sum_{k=1}^m C^k \prod_{i=1}^{k+1} B\left(\frac{\eta}{2}, \frac{\eta-\beta}{2} + (i-1)\frac{\eta}{2}\right) (t-s)^{k\frac{\eta}{2}} \right\}.$$

Again, since the constant  $K$  does not depend either on the constant  $C$  appearing in the definition of  $C_m(s, t)$  or  $m$ , one may change  $C$  once for all and derive the induction hypothesis at step  $m$  for  $u_{m+1}(s, t)$  and  $v_{m+1}(s, t)$ . This completes the proof of (4.26) at step  $m+1$ . As previously mentioned, we omit the proof of (4.27). We thus conclude that for all  $m \geq 1$ ,  $(s, x, \mu) \mapsto p_m(\mu, s, t, x, z) \in \mathcal{C}_f^{1,2,2}([0, t] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$  and that the estimates (4.24) to (4.27) are valid.

## REFERENCES

- [BCCH19] P. Briand, P. Cardaliaguet, P.-É. Chaudru de Raynal, and Y. Hu. Forward and Backward Stochastic Differential Equations with normal constraint in law. *arXiv e-prints*, Mar 2019.
- [BLPR17] R. Buckdahn, J. Li, S. Peng, and C. Rainer. Mean-field stochastic differential equations and associated pdes. *Ann. Probab.*, 45(2):824–878, 2017.
- [Car13] P. Cardaliaguet. Notes on mean field games. <https://www.ceremade.dauphine.fr/cardaliaguet/MFG20130420.pdf>, 2013.
- [CD18] R. Carmona and F. Delarue. *Probabilistic Theory of Mean Field Games with Applications I: Mean Field FBSDEs, Control, and Games*. Probability Theory and Stochastic Modelling. Springer International Publishing, 2018.
- [CDLL15] P. Cardaliaguet, F. Delarue, J.-M. Lasry, and P.-L. Lions. The master equation and the convergence problem in mean field games. Technical report, To appear in *Annals of Mathematics Studies*, 2015.
- [CdR19] P. E. Chaudru de Raynal. Strong well posedness of McKean–Vlasov stochastic differential equations with Hölder drift. *Stochastic Processes and their Applications*, 2019.
- [CdRF18] P.-E. Chaudru de Raynal and N. Frikha. Well-posedness for some non-linear diffusion processes and related PDE on the Wasserstein space. Technical report, arXiv:1811.06904. Under review, 2018.

- [CM17] D. Crisan and E. McMurray. Smoothing properties of McKean–Vlasov SDEs. *Probability Theory and Related Fields*, Apr 2017.
- [CST19] J.-F. Chassagneux, L. Szpruch, and A. Tse. Weak quantitative propagation of chaos via differential calculus on the space of measures. Technical report, arXiv:1901.02556, 2019.
- [FG15] N. Fournier and A. Guillin. On the rate of convergence in wasserstein distance of the empirical measure. *Probability Theory and Related Fields*, 162(3):707–738, Aug 2015.
- [Fri64] A. Friedman. *Partial differential equations of parabolic type*. Prentice-Hall, 1964.
- [Fun84] T. Funaki. A certain class of diffusion processes associated with nonlinear parabolic equations. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 67(3):331–348, Oct 1984.
- [Gär88] J. Gärtner. On the McKean–Vlasov Limit for Interacting Diffusions. *Mathematische Nachrichten*, 137(1):197–248, 1988.
- [Hol16] T. Holding. Propagation of chaos for Hölder continuous interaction kernels via Glivenko–Cantelli. Technical report, arXiv:1608.02877, 2016.
- [HvS18] W. Hammersley, D. Šiška, and L. Szpruch. McKean–Vlasov SDEs under Measure Dependent Lyapunov Conditions. Preprint Arxiv arXiv:1802.03974, 2018.
- [JM98] B. Jourdain and S. Méléard. Propagation of chaos and fluctuations for a moderate model with smooth initial data. *Annales de l’I.H.P. Probabilités et statistiques*, 34(6):727–766, 1998.
- [Jou97] B. Jourdain. Diffusions with a nonlinear irregular drift coefficient and probabilistic interpretation of generalized burgers’ equations. *ESAIM: PS*, 1:339–355, 1997.
- [JW18] P.-E. Jabin and Z. Wang. Quantitative estimates of propagation of chaos for stochastic systems with  $W^{-1,\infty}$  kernels. *Inventiones mathematicae*, 214(1):523–591, 2018.
- [Kac56] M. Kac. Foundations of kinetic theory. In *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, Volume 3: Contributions to Astronomy and Physics*, pages 171–197, Berkeley, Calif., 1956. University of California Press.
- [Kol10] V. N. Kolokoltsov. *Nonlinear Markov processes and kinetic equations*, volume 182 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2010.
- [KTY14] V. N. Kolokoltsov, M. Troeva, and W. Yang. On the rate of convergence for the mean-field approximation of controlled diffusions with large number of players. *Dyn. Games Appl.*, 4(2):208–230, 2014.
- [L86] C. Léonard. Une loi des grands nombres pour des systèmes de diffusions avec interaction et à coefficients non bornés. *Annales de l’I.H.P. Probabilités et statistiques*, 22(2):237–262, 1986.
- [Lac18] D. Lacker. On a strong form of propagation of chaos for mckean-vlasov equations. *Electron. Commun. Probab.*, 23:11 pp., 2018.
- [Lio14] P.-L. Lions. Cours au collège de France. <http://www.college-de-france.fr/site/pierre-louis-lions/seminar-2014-11-14-11h1> 2014.
- [LM16] J. Li and H. Min. Weak solutions of mean-field stochastic differential equations and application to zero-sum stochastic differential games. *SIAM Journal on Control and Optimization*, 54(3):1826–1858, 2016.
- [M96] S. Méléard. Asymptotic behaviour of some interacting particle systems; McKean–Vlasov and Boltzmann models. In *Probabilistic models for nonlinear partial differential equations (Montecatini Terme, 1995)*, volume 1627 of *Lecture Notes in Math.*, pages 42–95. Springer, Berlin, 1996.
- [Mal03] F. Malrieu. Convergence to equilibrium for granular media equations and their euler schemes. *Ann. Appl. Probab.*, 13(2):540–560, 2003.
- [McK66] H. P. McKean. A class of markov processes associated with nonlinear parabolic equations. *Proceedings of the National Academy of Sciences of the United States of America*, 56:1907–1911, 1966.
- [McK67] H. P. McKean. Propagation of chaos for a class of non-linear parabolic equations. *Stochastic Differential Equations (Lecture Series in Differential Equations, Session 7, Catholic Univ., 1967)*, pages 41–57, 1967.
- [MM13] S. Mischler and C. Mouhot. Kac’s program in kinetic theory. *Invent. Math.*, 193(1):1–147, 2013.
- [MMW15] S. Mischler, C. Mouhot, and B. Wennberg. A new approach to quantitative propagation of chaos for drift, diffusion and jump processes. *Probab. Theory Related Fields*, 161(1-2):1–59, 2015.
- [MV18] Y. S. Mishura and A. Y. Veretennikov. Existence and uniqueness theorems for solutions of McKean–Vlasov stochastic equations. Preprint Arxiv arXiv:1603.02212, 2018.
- [Oel84] K. Oelschläger. A martingale approach to the law of large numbers for weakly interacting stochastic processes. *Ann. Probab.*, 12(2):458–479, 1984.
- [SV79] D. W. Stroock and S. R. S. Varadhan. *Multidimensional diffusion processes*, volume 233 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin-New York, 1979.
- [Szn91] A.-S. Sznitman. Topics in propagation of chaos. In Paul-Louis Hennequin, editor, *Ecole d’Eté de Probabilités de Saint-Flour XIX — 1989*, pages 165–251, Berlin, Heidelberg, 1991. Springer Berlin Heidelberg.
- [Tan78] H. Tanaka. Probabilistic treatment of the Boltzmann equation of Maxwellian molecules. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 46(1):67–105, 1978.
- [Ver80] A. Y. Veretennikov. Strong solutions and explicit formulas for solutions of stochastic integral equations. *Matematicheski Sbornik. Novaya Seriya*, 111(153)(3):434–452, 1980.
- [Zvo74] A. K. Zvonkin. A transformation of the phase space of a diffusion process that will remove the drift. *Mat. Sb. (N.S.)*, 93(135):129–149, 152, 1974.

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