



HAL
open science

Eternal dominating sets on digraphs and orientations of graphs

Guillaume Bagan, Alice Joffard, Hamamache Kheddouci

► **To cite this version:**

Guillaume Bagan, Alice Joffard, Hamamache Kheddouci. Eternal dominating sets on digraphs and orientations of graphs. 2019. hal-02168424

HAL Id: hal-02168424

<https://hal.science/hal-02168424v1>

Preprint submitted on 28 Jun 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Eternal dominating sets on digraphs and orientations of graphs

Guillaume Bagan^a, Alice Joffard^a, Hamamache Kheddouci^a

^a*Lab. LIRIS, UMR CNRS 5205, University of Lyon, F-69003
University of Claude Bernard Lyon 1
43 Bd du 11 Novembre 1918, F-69622, Villeurbanne, France.*

Abstract

The eternal domination game is a problem that answers the following question: Can we eternally defend a graph? The principle is to defend an attacked vertex, that can change every turn, by moving guards along the edges of the graph. In the classical version of the game [1], only one guard can move at a time, but in the m -eternal version [3], any number of guards can move. This problem led to the introduction of two parameters: the eternal and m -eternal domination numbers of the graph, that are the minimum numbers of guards necessary to defend to any eternal sequence of attacks.

In this paper, we first study the (m -)eternal domination number on digraphs. We show that a lot of results proven for undirected graphs can be generalized to digraphs. Then, we introduce the problem of oriented (m -)eternal domination, consisting in finding an orientation of a graph that minimizes its eternal domination number. We prove that computing the oriented eternal domination number is a coNP-hard problem, and we give a complete characterization of the graphs for which the oriented m -eternal domination number is 2. We also study these two parameters on trees, cycles, complete graphs, complete bipartite graphs, trivially perfect graphs, and different kinds of grids and products of graphs.

1. Introduction

1.1. Definitions and notations

In this paper, all graphs and digraphs are considered to be finite, without multiple edges (arcs) and without loops. However, a digraph can contain both arcs (u, v) and (v, u) for some vertices u and v . When there is no ambiguity on the (di)graph G , V is the vertex set of G , E is the edge (arc) set of G , n is the order of G and m is the number of edges in G .

Let $G = (V, E)$ be a graph. Given a vertex $v \in V$, $N(v)$ represents the open neighborhood of v , i.e., the set $\{y \in V : vy \in E\}$ and the closed neighborhood of v is $N[v] = N(v) \cup \{v\}$. A set $S \subseteq V$ is an independent set of G if there is no edge uv for every u, v in S . The independence number $\alpha(G)$ is the size of a largest independent set of G . A set $S \subseteq V$ is a dominating set of G if $\bigcup_{v \in S} N[v] = V$. The domination number $\gamma(G)$ is the size of a smallest dominating set of G . A set $S \subseteq V$ is a k -dominating set of G if for every vertex $v \in V \setminus S$, $|N(v) \cap S| \geq k$. G is k -vertex-connected (resp. k -edge-connected) if G stays connected if one removes any set of at most $k - 1$ vertices (resp. edges). $\gamma_{k,l}(G)$ is the size of the smallest k -dominating l -edge-connected set of G .

Let $G = (V, E)$ be a digraph. For two vertices u and v of G , we say that u dominates v if there is an arc (u, v) in G . $N^+(u) = \{v : (u, v) \in E\}$ and $N^+[u] = N^+(u) \cup \{u\}$. A set $S \subseteq V$ is a dominating set of G if $\bigcup_{v \in S} N^+[v] = V$. The distance $\text{dist}(u, v)$ from u to v is the number of edges of the shortest path from u to v . Given a set $S \subseteq V$, $\text{dist}(u, S) = \min\{\text{dist}(u, v) : v \in S\}$. If G is strongly connected, the diameter $\text{diam}(G)$ of G is the maximum distance from u to v for every vertex u and v in V .

P_n represents a path graph with n vertices, C_n represents a cycle graph with n vertices, K_n represents the complete graph with n vertices and $K_{n,m}$ represents a complete bipartite graph with partitions of size n and m . Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, $G_1 \square G_2$ is the cartesian product of G_1 and G_2 . $G_1 \boxtimes G_2$ is the strong product of G_1 and G_2 . $[k]$ represents the set $\{1, 2, \dots, k\}$. A well-quasi-ordering (wqo) \leq_P is a preorder such that any infinite sequence of elements x_0, x_1, x_2, \dots contains an increasing pair $x_i \leq_P x_j$ with $i < j$. In particular, it does not admit infinite decreasing sequences.

1.2. Eternal domination

The problem of eternal domination on undirected graphs, while being a rather recent problem, has been widely studied (see [10] for a survey).

The eternal domination on a graph G can be seen as an infinite game between two players: the *defender* and the *attacker*. First, the defender chooses a set D_0 of k vertices called the *guards*. At turn i , the attacker chooses a vertex r_i called *attack* in $V \setminus D_{i-1}$ and the defender must *defend* against the attack by moving to r_i a guard on a vertex v_i adjacent to r_i . The new guards configuration is $D_i = D_{i-1} \cup \{r_i\} \setminus \{v_i\}$. The defender wins the game if it can defend against any infinite sequence of attacks. The *eternal domination number*, denoted by $\gamma^\infty(G)$, is the minimum number of guards

necessary for the defender to win. An *eternal dominating set* is a set that can initially be chosen by the defender in a winning strategy.

We now give a more formal definition of these notions.

Definition 1. Let $G = (V, E)$ be a graph. The set $EDS(G)$ of eternal dominating sets of G is the greatest set of subsets of V such that for every $S \in EDS(G)$ and every $r \in V \setminus S$, there is a vertex $v \in S$ such that $\{v, r\} \in E$ and $S \cup \{r\} \setminus \{v\} \in EDS(G)$.

The eternal domination number of G is defined as $\gamma^\infty(G) = \min\{|S|, S \in EDS(G)\}$.

A variant of the eternal domination problem is the m -eternal domination problem [3], where the defender is authorized to move any number of guards at a time. Notice that the m of m -eternal does not represent a parameter here, but stands for multiple, as in multiple guards can move at a time.

Definition 2. Let $G = (V, E)$ be a graph. Given two sets $S_1, S_2 \subseteq V$, a *multimove* f from S_1 to S_2 is a one-to-one mapping from S_1 to S_2 such that for every $x \in S_1$, we have $f(x) = x$ or $\{x, f(x)\} \in E$.

The set $MEDS(G)$ of m -eternal dominating sets of G is the greatest set of subsets of V such that for every $S \in MEDS(G)$ and every $r \in V(G) \setminus S$, there is a multimove f such that $r \in f(S)$ and $f(S) \in MEDS(G)$.

Let G be a graph. The m -eternal domination number of G is defined as $\gamma_m^\infty(G) = \min\{|S|, S \in MEDS(G)\}$.

The two following results compare the values of both the eternal domination number and the m -eternal domination number to the values of other well known graph parameters.

Theorem 1. [1, 3, 6] Given a graph G , we have

$$\gamma(G) \leq \gamma_m^\infty(G) \leq \alpha(G) \leq \gamma^\infty(G) \leq \binom{\alpha(G) + 1}{2}$$

where $\gamma(G)$ is the domination number of G , and $\alpha(G)$ is the independence number of G .

Theorem 2. [1] Given a graph G , we have

$$\gamma^\infty(G) \leq \theta(G)$$

where $\theta(G)$ is the clique covering number of G .

These two theorems are particularly interesting when G is a perfect graph since $\alpha(G) = \gamma^\infty(G) = \theta(G)$ by definition of a perfect graph (and its complement).

The values of γ^∞ and γ_m^∞ have also been studied for many classes of graphs, such as paths, cycles, multipartite graphs and perfect graphs [1, 3]. In particular, a lot of attention has been given to grid graphs.

Another theorem gives an upper bound for γ_m^∞ .

Theorem 3. [3] *Given a graph G , we have*

$$\gamma_m^\infty(G) \leq \gamma_c(G) + 1$$

where $\gamma_c(G)$ is the size of a smallest connected dominating set of G .

1.3. Contributions

To our knowledge, the eternal domination problem has only been studied on undirected graphs. In this paper, we first consider the eternal domination problem on directed graphs, where the guards must follow the direction of the arcs.

Then, as it has been done for many digraph parameters such as diameter, chromatic number, domination number or maximum outgoing degree, we consider the problem, namely oriented (m-)eternal domination, consisting in finding an orientation of an undirected graph which minimizes γ^∞ or γ_m^∞ . Those minimum values will define new parameters, $\overrightarrow{\gamma}^\infty$ and $\overrightarrow{\gamma}_m^\infty$.

We present our contributions: We first generalize Theorem 1 to directed graphs. We also give a generalization of Theorem 3 by introducing the notion of dominating-dominated set. Other bounds by various parameters are also given for digraphs.

Then, we introduce the notion of oriented eternal domination. We show the following complexity result:

Theorem. *Deciding whether $\overrightarrow{\gamma}^\infty(G) \leq k$ is coNP-hard.*

We then give a complete characterization of the graphs for which the oriented m-eternal domination number is 2:

Theorem. *Let G be a graph of order $n \geq 3$. Then, $\overrightarrow{\gamma}_m^\infty(G) = 2$ iff either:*

- $n = 2k$ and G is a complete graph from which at most k disjoint edges are removed.
- $n = 2k + 1$ and G is a complete graph from which at most $k - 1$ disjoint edges are removed.

We also study the oriented eternal domination and oriented m-eternal domination problems on various graph classes, such as trees, complete graphs, complete bipartite graphs, trivially perfect graphs, and different kinds of grids. In particular, we introduce the notion of neighborhood-equitable coloring and use it to find upper bounds.

2. Eternal domination on digraphs

In this section, we consider the eternal domination and m-eternal domination problems on directed graphs. The definitions of $EDS(G)$, $\gamma^\infty(G)$, $MEDS(G)$ and $\gamma_m^\infty(H)$ are straightforward, the only difference is that whenever an edge is considered, we consider an arc instead.

From this, we can already deduce several general results for digraphs.

It is straightforward that the (m-)eternal domination number of a graph is the sum of the (m-)eternal domination number of each of its connected components. We will prove that this result can be extended to the strongly connected components (s.c.c) of a digraph.

Lemma 1. *Let G be a digraph with strongly connected components S_1, \dots, S_l . Then*

$$\gamma^\infty(G) = \sum_{i=1}^l \gamma^\infty(G[S_i])$$

and

$$\gamma_m^\infty(G) = \sum_{i=1}^l \gamma_m^\infty(G[S_i]).$$

Proof. We only prove the first equality. The proof of the second equality is similar. The statement $\gamma^\infty(G) \leq \sum_{i=1}^l \gamma^\infty(G[S_i])$ is straightforward. Let us prove the other part of the equality.

We define the following relation \leq_r on the vertices of G : $x \leq_r y$ if x is reachable from y in G . Since G is finite, \leq_r is a wqo. Let us represent a guard configuration of k guards by a tuple of k vertices and extend the relation \leq_r on (ordered) guard configurations: $(x_1, \dots, x_k) \leq_r (y_1, \dots, y_k)$ iff $x_i \leq_r y_i$ for every $i \in [k]$. This relation is also a wqo since it is the direct product of k wqos. It is easily seen that if the guard configuration c_2 is reachable from the guard configuration c_1 then $c_1 \geq_r c_2$.

Assume now that the defender plays with less than $\sum_{i=1}^l \gamma_m^\infty(G[S_i])$ guards. Let D_0 be the initial configuration. Thus, there is a s.c.c. S_i such that $|S_i \cap D_0| < \gamma^\infty(G[S_i])$. Consequently, the attacker can apply a strategy in $G[S_i]$ such that they can either win, or the defender moves a

guard outside S_i into S_i . In the second case, we obtain a configuration D_j such that $D_0 >_r D_j$. So, the attacker can apply again its strategy on another s.c.c. $S_{i'}$ with $|S_{i'} \cap D_j| < \gamma^\infty(G[S_{i'}])$. Since \leq_r is a wqo, there is no infinite decreasing sequence of configurations. Thus, the attacker eventually wins. \square

This leads to the value of the two parameters for directed acyclic graphs.

Corollary 1. *If G is an acyclic digraph with n vertices, then $\gamma^\infty(G) = \gamma_m^\infty(G) = n$.*

An important result concerns the monotony of γ^∞ , that is not verified for γ_m^∞ . This has been proved on graph [9] but the adaptation on digraphs is straightforward.

Lemma 2. [9] *Let G be a digraph and H be an induced subgraph of G . Then, $\gamma^\infty(H) \leq \gamma^\infty(G)$.*

Notice also that γ^∞ and γ_m^∞ do not decrease when one removes edges.

The main result of this section is the generalization of Theorem 1 to directed graphs.

We define α for digraphs as follows.

Definition 3. *Given a digraph G , $\alpha(G)$ is the order of the greatest induced acyclic subgraph of G .*

Notice that this definition of α is, in some sense, a generalization of the one for undirected graphs. Indeed, if we replace every edge of a graph G by two arcs, thus creating the digraph \overleftrightarrow{G} , we have the equality $\alpha(G) = \alpha(\overleftrightarrow{G})$.

Theorem 4. *Given a digraph G , we have*

$$\gamma(G) \leq \gamma_m^\infty(G) \leq \alpha(G) \leq \gamma^\infty(G) \leq \binom{\alpha(G) + 1}{2}.$$

Proof. The inequality $\gamma(G) \leq \gamma_m^\infty(G)$ is straightforward. The inequality $\alpha(G) \leq \gamma^\infty(G)$ is a consequence of Corollary 1 and Lemma 2. The inequality $\gamma^\infty(G) \leq \binom{\alpha(G)+1}{2}$ can easily be proved by adapting the proof in [6]. The remaining inequality is $\gamma_m^\infty(G) \leq \alpha(G)$. For this last inequality, the principle is similar to the proof of Goddard et al. [3] for undirected graphs, but with notable differences, that make the proof a non-trivial adaptation. We consider two cases, depending on the existence of a vertex $v \in V$ that does not belong to any of the greatest induced acyclic subgraphs of G .

If such a vertex does not exist, then every vertex belongs to at least one of the greatest induced acyclic subgraphs of G . We thus prove that for any two distinct subsets $A, B \subseteq V$ of vertices that each induce a greatest induced acyclic subgraph of G , there exists a multimove from A to B , so that we can always defend an attacked vertex a by going from a subset of vertices inducing a greatest induced acyclic subgraph to another that contains a . Let G' be the bipartite undirected graph $G' = (A, B, E')$ with $E' = \{xy, x \in A, y \in B, (x, y) \in E \text{ or } x = y\}$.¹ See Figure 1 for an example.

We show that for any set $S \subseteq A$, $|N(S)| \geq |S|$, where $N(S)$ is the set of neighbors of vertices of S in G' . We will thus be able to use Hall's marriage theorem [4] and prove that there exists a perfect matching between A and B in G' , and therefore a multimove from A to B in G . Suppose that there exists $S \subseteq A$ such that $|N(S)| < |S|$. Let X, Y , and C be the sets of vertices defined by $X = S \setminus B$, $Y = N(S) \setminus A$, and $C = B \cup X \setminus Y$. We have $|C| > |B|$. Indeed, by construction, $S \cap B \subseteq N(S \cap B)$, so that $|S \cap B| \leq |N(S) \cap A|$. Thus, since $|S| > |N(S)|$, we have $|X| > |Y|$, which gives $|C| > |B|$.

Moreover, $G[C]$ is acyclic. Indeed, $G[A]$ and $G[B]$ are acyclic, and for any two vertices $x, y \in C$, $\{x, y\} \notin E'$, so that for all $x \in C \cap A, y \in C \cap B$, $(x, y) \notin E$. Thus, $G[C]$ is an induced acyclic subgraph of order greater than $|B|$, a contradiction, so that Hall's condition is respected. Thus, there exists a perfect matching between A and B in G' , and a multimove from A to B in G .

Now, if there exists a vertex $v \in V$ such that v does not belong to any of the greatest induced acyclic subgraphs of G , we use induction to prove that $\gamma_m^\infty(G) \leq \alpha(G)$. For $n = 1$, we obviously have $\gamma_m^\infty(G) \leq \alpha(G)$. Assume that $n > 1$. We define G' from G by contracting the vertex v as follows: $G' = (V', E')$ with $V' = V \setminus \{v\}$ and $E' = E(G[V']) \cup \{(u, w) \in V'^2, u \neq w \wedge (u, v), (v, w) \in E\}$. By induction hypothesis, we have $\gamma_m^\infty(G') \leq \alpha(G')$.

We have $\alpha(G') \leq \alpha(G) - 1$. Indeed, consider a set $S \subseteq V'$ of cardinality $\alpha(G)$. Let us prove that $G'[S]$ is cyclic. If $G[S]$ is cyclic, $G'[S]$ is cyclic too. Assume that $G[S]$ is acyclic. Then, by maximality of S , $G[S \cup \{v\}]$ is cyclic. Either there exist two distinct vertices $u, w \in S$ such that $(u, v), (v, w) \in E$ and $(u, v), (v, w)$ are part of a cycle in S , or, otherwise, there exists $u \in S$ such that $(u, v), (v, u) \in E$. In this last case, since v does not belong to any of the greatest induced acyclic subgraphs of G (due to the case we are

¹Actually G' is not necessarily a bipartite graph and can contain loops because A and B can have common elements. However, Hall's theorem is still applicable. Moreover, the graph can become bipartite if we make a copy of each element of $A \cap B$.

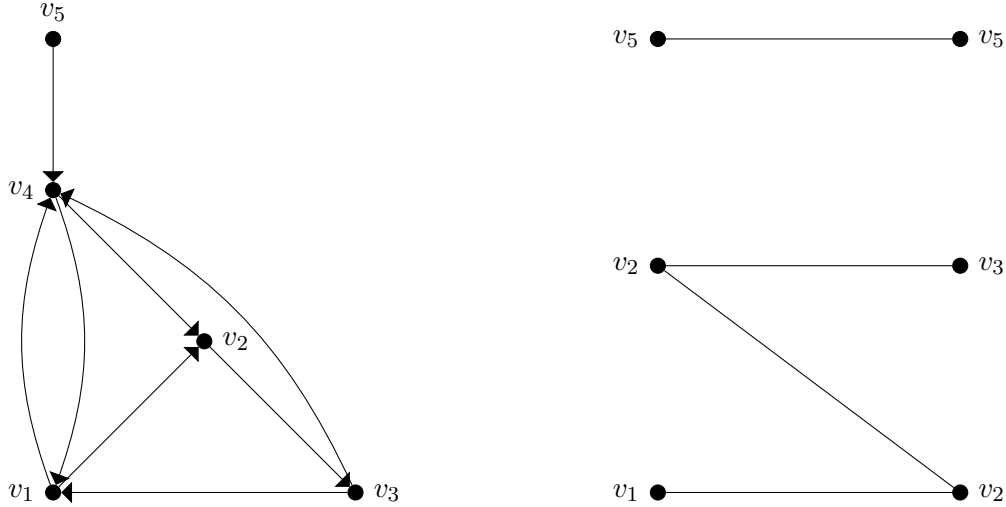


Figure 1: On the left, a graph G for which $G[\{v_1, v_2, v_5\}]$ and $G[\{v_2, v_3, v_5\}]$ are maximum induced acyclic subgraphs. On the right, the bipartite graph constructed in the proof of Theorem 4.

in), $G[S \setminus \{u\} \cup \{v\}]$ is cyclic, so that there exists $w \in S \setminus \{u\}$ such that $(w, v), (v, w) \in E$. In those two cases, by definition of G' , $G'[S]$ is cyclic. Thus, there exists no induced acyclic subgraph of G' of order $\alpha(G)$, and $\alpha(G') \leq \alpha(G) - 1$.

Moreover, $\gamma_m^\infty(G) \leq \gamma_m^\infty(G') + 1$. Indeed, if we can defend G' with k guards, we can defend G with $k + 1$ guards by adding a guard on v and copying the strategy in G . If the defender of G' moves a guard from u to w using an edge (u, w) that has been added in G' (and, thus, not present in G), then the two edges (u, v) and (v, w) belong to G . Thus the defender of G moves a guard from u to v and another one from v to w .

Those three inequalities finally give $\gamma_m^\infty(G) \leq \alpha(G)$. \square

We now generalize Theorem 3 by introducing the notion of a dominating-dominated set of a digraph G . Notice that the notion of domination in digraphs has been defined multiple ways in the literature, also, we focused on the definition where a vertex can only be dominated by itself or its outgoing neighbors, also called out-domination.

Definition 4. Let $G = (V, E)$ be a digraph. A set $S \subseteq V$ is a dominating-dominated set of G if every vertex $v \in V \setminus S$ dominates a vertex of S and is dominated by a vertex of S .

Let $\gamma_{\leftrightarrow}(G)$ be the size of a smallest strongly connected dominating-dominated set of G .

Theorem 5. *Let $G = (V, E)$ be a digraph. Then $\gamma_m^\infty(G) \leq \gamma_{\leftrightarrow}(G) + 1$.*

Proof. Let S be a strongly connected dominating-dominated set of G . We will show that we can defend G with $|S| + 1$ guards. The invariant for a guard configuration is that it contains all vertices of S . Initially, we put the guards on S and an arbitrary vertex u in $V \setminus S$. Consider an attack on a vertex v . By definition of S , there exist two (non-necessarily distinct) vertices $x, y \in S$ such that $(u, x) \in E$ and $(y, v) \in E$. Thus, we move the guard on y to v . Let p be a path from x to y using only vertices of S . We move the guard on u to x and each guard on a vertex w of p to the next vertex. We obtain the guard configuration $S \cup \{v\}$. Thus the invariant is verified. \square

We generalize this notion to dominating- k -dominated sets. The undirected version of Definition 5 has been called distance- k domination, which seems a more natural term.

Definition 5. *Let $G = (V, E)$ be a digraph. A distance k -dominating set of V is a set $S \subseteq V$ such that for any $v \in V \setminus S$, there exists a directed path of length at most k from a vertex of S to v . A set $S \subseteq V$ is a dominating- k -dominated set of G if it is a dominating set of V , and $V \setminus S$ is a distance k -dominating set of V .*

$\gamma'_{\leftrightarrow}(G)$ is the minimum value of $k + |S| - 1$ for every k and S with S a strongly connected dominating- k -dominated set of G .

$\gamma_c(G)$ denotes the size of the smallest strongly connected dominating set of G .

Theorem 6. *Let G be a strongly connected digraph. Then*

$$\gamma_m^\infty(G) \leq \gamma'_{\leftrightarrow}(G) + 1 \leq \gamma_c(G) + \text{diam}(G) \leq (\gamma(G) + 1)\text{diam}(G).$$

Proof. To prove the first inequality, we consider a dominating- k -dominated set S of G . We defend G with $|S| + k$ guards. The invariant is as follows: $|S|$ guards are on S and k guards (v_1, \dots, v_k) are in $V \setminus S$ with $\text{dist}(v_i, S) \leq i$ for every $i \in [k]$. We start with an initial configuration that satisfies the invariant. Assume that a vertex u is attacked. Since S is a dominating set, there exists a vertex $x \in S$ which dominates u . So, we move the guard on x to u . We push the guard on v_1 to x similarly to the proof of Theorem 5

and we move each guard on v_i , $i \geq 2$, to a vertex w_i with $\text{dist}(w_i, S) \leq i - 1$. The invariant is satisfied.

To prove the second inequality, notice that a dominating set is a dominating- $\text{diam}(G)$ -dominated set.

To prove the last inequality, consider a dominating set $S = \{v_1, \dots, v_l\}$. For each $i \in [l - 1]$, we add to S the vertices of a shortest path from v_i to v_{i+1} . We do the same between v_l and v_1 . We add at most $l(\text{diam}(G) - 1)$ vertices and obtain a set S' of size $l \cdot \text{diam}(G)$ which induces a strongly connected subgraph. \square

3. Eternal domination on orientations of graphs

In this section, we are interested in orientating an undirected graph in order to minimize its eternal domination number, or its m-eternal domination number. An orientation of an undirected graph G is an assignment of exactly one direction to each of the edges of G . This leads to the introduction of three new parameters for undirected graphs:

Definition 6. *Given an undirected graph G ,*
 $\overrightarrow{\gamma}^\infty(G) = \min\{\gamma^\infty(H) : H \text{ is an orientation of } G\}$
 $\overrightarrow{\gamma}_m^\infty(G) = \min\{\gamma_m^\infty(H) : H \text{ is an orientation of } G\}$
 $\overrightarrow{\alpha}(G) = \min\{\alpha(H) : H \text{ is an orientation of } G\}$
 $\overrightarrow{\gamma}_{\leftrightarrow}(G) = \min\{\gamma_{\leftrightarrow}(H) : H \text{ is an orientation of } G\}.$

Notice that, for non-trivial graphs, $\overrightarrow{\gamma}^\infty$ can never be equal to γ .

Proposition 1. *Let G be a graph with at least one edge. Then, $\gamma(G) \leq \alpha(G) < \overrightarrow{\alpha}(G) \leq \overrightarrow{\gamma}^\infty(G)$.*

Proof. We only need to prove that $\alpha(G) < \overrightarrow{\alpha}(G)$. Let H be an orientation of G and S be a maximum independent set of G . Let $S' = S \cup \{v\}$ where v is an arbitrary vertex in $V \setminus S$. $G[S']$ is a union of stars and isolated vertices. Thus, $H[S']$ is acyclic. \square

We conjecture a stronger result.

Conjecture 1. *Let G be a graph with at least one edge. Then, $\theta(G) < \overrightarrow{\alpha}(G)$.*

The conjecture would imply that there is no non-trivial graph with $\gamma^\infty = \overrightarrow{\gamma}^\infty$. It is true for perfect graphs since $\alpha(G) = \gamma^\infty(G) = \theta(G) < \overrightarrow{\alpha}(G)$ (Theorems 1 and 2).

Robbins proved the following theorem.

Theorem 7. [11] *Let G be a graph. Then G admits a strongly connected orientation if and only if G is 2-edge-connected.*

This leads to the following proposition.

Proposition 2. *Let G be a 2-edge-connected graph. Then there exists a strongly connected orientation H of G with $\gamma^\infty(H) = \overrightarrow{\gamma}^\infty(G)$ and $\gamma_m^\infty(H) = \overrightarrow{\gamma}_m^\infty(G)$.*

Proof. Let H be an orientation of G such that $\gamma_m^\infty(H) = \overrightarrow{\gamma}_m^\infty(G)$. It suffices to notice that in H , each guard stays inside its strongly connected component. Thus, we can change in H the orientation of some edges between two components to make it strongly connected. \square

By combining Proposition 2 and Lemma 1, we obtain:

Corollary 2. *Let G be a graph with 2-edge-connected components S_1, \dots, S_l . Then*

$$\overrightarrow{\gamma}^\infty(G) = \sum_{i=1}^l \overrightarrow{\gamma}^\infty(G[S_i])$$

and

$$\overrightarrow{\gamma}_m^\infty(G) = \sum_{i=1}^l \overrightarrow{\gamma}_m^\infty(G[S_i]).$$

We now show that $\overrightarrow{\gamma}_m^\infty$ can be bounded by a natural parameter.

Theorem 8. *Let $G = (V, E)$ be a undirected graph. Then, $\overrightarrow{\gamma}_m^\infty(G) \leq \overrightarrow{\gamma}_{\leftrightarrow}(G) + 1 \leq \gamma_{2,2}(G) + 1$.*

Proof. The first inequality is a consequence of Theorem 5. Let S be a 2-dominating 2-edge-connected set of G . We construct an orientation H of G as follows: we orientate the edges in S such that $H[S]$ is strongly connected. This is possible thanks to Theorem 7 since $G[S]$ is 2-edge-connected. Then, for every vertex $v \in V \setminus S$, we choose two distinct vertices v_1, v_2 in S such that vv_1 and vv_2 are edges of G and orientate vv_1 from v to v_1 and vv_2 from v_2 to v . Thus, S is a strongly connected dominating-dominated set of H . \square

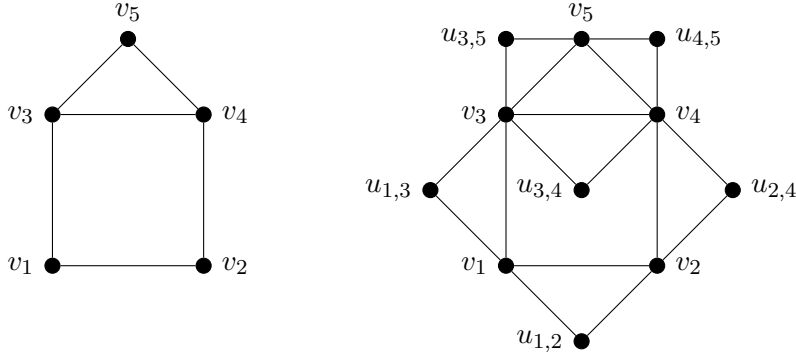


Figure 2: The house graph G on the left and $C(G)$ on the right.

3.1. Hardness results

Let G be an undirected graph. We define $C(G)$ by starting from G , adding one vertex per edge of G and connecting each new vertex to the extremities of the associated edge. See Figure 2 for an example. This definition allows us to present the following result, which leads to complexity results.

Lemma 3. *Let G be an undirected graph with m edges. Then*

- $\overrightarrow{\gamma^\infty}(C(G)) = \gamma^\infty(G) + m$
- $\overrightarrow{\alpha}(C(G)) = \alpha(G) + m$.

Proof. Let P be the set of all added vertices $u_{i,j}$ in $C(G)$.

In $C(G)$, let v_1, \dots, v_n be the vertices of G , and $u_{i,j}$, $i < j$ be the added vertex associated to the edge $\{v_i, v_j\}$. We prove that $\overrightarrow{\gamma^\infty}(C(G)) \leq \gamma^\infty(G) + m$ by showing that there exists an orientation H of $C(G)$ that can always be defended by $\gamma^\infty(G) + m$ guards. To construct H , we orientate $C(G)$ in the following way: we orientate every edge $v_i v_j \in E$ from v_i to v_j iff $i < j$, then $v_j u_{i,j}$ from v_j to $u_{i,j}$, and $v_i u_{i,j}$ from $u_{i,j}$ to v_i . Every triangle induced by some vertices v_i , v_j , and $u_{i,j}$ is therefore an oriented cycle. We will consider a strategy that preserves the following invariant: for each guard configuration D in the strategy, we can always partition D into two sets of guards, a set A of m guards defending each one of the m added vertices $u_{i,j}$, i.e., are either on $u_{i,j}$ or on v_j , and a set $B \subseteq V$ which is an eternal dominating set of G .

We start with the guard configuration $D_0 = A_0 \cup B_0$, with B_0 a minimum eternal dominating set of G , and $A_0 = P$. Clearly, this configuration verifies the invariant. Consider a guard configuration $D = A \cup B$ that verifies the

invariant and let us prove that, for every attacked vertex r , the defender can defend the attack and obtain a configuration D' that verifies the invariant.

If $r = u_{i,j}$, then the defender moves the guard from v_j to $u_{i,j}$. If $r = v_i$, then there is a vertex $v_j \in B$ adjacent to r such that $B \cup \{v_i\} \setminus \{v_j\}$ is an eternal dominating set of G . We have two possible cases that depend on the orientation of $\{v_i, v_j\}$. If $(v_j, v_i) \in E(H)$, then the defender moves the guard from v_j to v_i . If $(v_i, v_j) \in E(H)$, then the defender moves the guard from $u_{i,j}$ to v_i . By choosing $B' = B \cup \{v_i\} \setminus \{v_j\}$, and $A' = A \cup \{v_j\} \setminus \{u_{i,j}\}$, it is easily seen that $D' = A' \cup B'$ satisfies the invariant.

We now prove that $\overrightarrow{\gamma^\infty}(C(G)) \geq \gamma^\infty(G) + m$ by showing that there exists an eternal dominating set X of G such that $|X| = \overrightarrow{\gamma^\infty}(C(G)) - m$. Let H be an orientation of $C(G)$ such that $\gamma^\infty(H) = \overrightarrow{\gamma^\infty}(C(G))$. We call a *clean configuration of H* any eternal dominating set S of H such that $P \subseteq S$. In H , the $\overrightarrow{\gamma^\infty}(C(G))$ guards can always be brought to a clean configuration. Indeed, the attacker can successively attack every vertex of P and be sure that they will all be occupied since P is an independent set. We prove that if a set S is a clean configuration of H , then $S \cap V$ is an eternal dominating set of G .

We first prove that $S \cap V(G)$ is a dominating set of G . Indeed, S is a dominating set of H . Any vertex of G dominated by $v \in S \cap V(G)$ in H is still dominated by it in G , and every vertex $u_{i,j} \in P$ can only dominate v_i or v_j . If $u_{i,j}$ is the only vertex dominating v_i in S , then if v_i is attacked, a guard has to move from $u_{i,j}$ to v_i , which means that $u_{i,j}$ has to be dominated by v_j , so that $v_j \in S$. Since $v_i v_j \in E$, v_j dominates v_i in G , so that every vertex of G is dominated by a vertex of $S \cap V$ in G .

Then, for every attack $r \in V(G) \setminus S$, by attacking r in H and then $u_{i,j}$ if $u_{i,j}$ became unoccupied, we obtain a new clean configuration S' of H so that $r \in S' \cap V(G)$. Indeed, if a guard moves from $v \in V(G)$ to r in H , we obtain a clean configuration $S' = S \setminus \{v\} \cup \{r\}$, and the guard can do the same in G . If a guard moved from $u_{i,j}$ to $r = v_i$ in H , then v_j is the only vertex dominating $u_{i,j}$, so that when $u_{i,j}$ is attacked, a guard must move from v_j to $u_{i,j}$. We therefore obtain a clean configuration $S' = S \setminus \{v_j\} \cup \{v_i\}$, and a guard can directly move from v_j to $r = v_i$ in G to obtain the configuration $S' \cap V(G)$.

Therefore, if we take a clean configuration S of H such that $|S| = \overrightarrow{\gamma^\infty}(C(G))$, we have that $S \cap V(G)$ is an eternal dominating set of G , and since $|S \cap V(G)| = \overrightarrow{\gamma^\infty}(C(G)) - m$, $\overrightarrow{\gamma^\infty}(C(G)) \geq \gamma^\infty(G) + m$.

We now prove that $\alpha(G) \geq \overrightarrow{\alpha}(C(G)) - m$. Let H be an orientation of $C(G)$ such that $\alpha(H) = \overrightarrow{\alpha}(C(G))$. We can assume, without loss of

generality, that all triplets $\{u_{i,j}, v_i, v_j\}$ in H induce an oriented triangle. Indeed, consider an induced acyclic subgraph $H[S]$ in such an orientation. Then, changing the orientation of some edges $\{u_{i,j}, v_i\}$ or $\{u_{i,j}, v_j\}$ does not create new minimal oriented cycles in H . Thus, $H[S]$ stays acyclic, and $\alpha(H)$ cannot decrease. Therefore, at most two of the three vertices of each triplet $\{v_i, v_j, u_{i,j}\}$ belong to S . We can assume, without loss of generality, $P \subseteq S$. Indeed, if $v_i, v_j \in S$, $H[S \setminus \{v_i\} \cup \{u_{i,j}\}]$ is also acyclic, and if $v_i \notin S$, $H[S \cup \{u_{i,j}\}]$ is also acyclic. Thus, $|S \cap V| = \overrightarrow{\alpha}(C(G)) - m$. Moreover, since for all $v_i v_j \in E$, $u_{i,j} \in S$, we cannot have $v_i, v_j \in S$, and $|S \cap V|$ is an independent set of G . Therefore, $\alpha(G) \geq \overrightarrow{\alpha}(C(G)) - m$.

To see that $\overrightarrow{\alpha}(C(G)) \geq \alpha(G) + m$, consider a maximum independent set S of G . Take $S' = S \cup P$. We obviously have $|S'| = \alpha(G) + m$, and $C(G)[S']$ is acyclic. Indeed, there are no cycles in $C(G)[S] = G[S]$, and an added vertex $u_{i,j}$ of P cannot create any cycle since it is only linked to v_i and v_j , and v_i or v_j is not in S . Thus, for any orientation H of $C(G)$, $H[S']$ is acyclic. \square

Notice that Lemma 3 is not true if we replace $\overrightarrow{\gamma}^\infty$ with $\overrightarrow{\gamma}_m^\infty$. Indeed, the inequality $\overrightarrow{\gamma}_m^\infty(C(G)) \leq \gamma_m^\infty(G) + m$ remains true but the inequality $\overrightarrow{\gamma}_m^\infty(C(G)) \geq \gamma_m^\infty(G) + m$ is not necessarily true. For example, if we consider G as the path graph P_3 , we let the reader verify that $\gamma_m^\infty(G) = 2$ and $\overrightarrow{\gamma}_m^\infty(C(G)) = 3 < \gamma_m^\infty(G) + m$.

The consequences of Lemma 3 are particularly interesting, leading to complexity results. The first consequence is about the coNP-hardness of computing $\overrightarrow{\gamma}^\infty(G)$. To our knowledge, there is no known hardness result about the complexity of computing $\gamma^\infty(G)$. However, given a graph G and a set S , deciding whether S is an eternal dominating set of G is a Π_2^P -hard problem [8].

We will first prove that deciding, given a graph G and an integer k , whether $\gamma^\infty(G) \leq k$ is coNP-hard. We use a reformulation of a theorem which says that $\alpha(G)$ is hard to approximate with a polynomial ratio.

Theorem 9. [13] *Let $\epsilon > 0$ and Γ be a problem with a graph G and an integer $k > 0$ as input and such that:*

- *every instance (G, k) with $\alpha(G) < k$ is negative;*
- *every instance (G, k) with $\alpha(G) \geq k^{1/\epsilon}$ is positive.*

Then, Γ is NP-hard.

Theorem 10. *Given an undirected graph G and an integer k , deciding whether $\gamma^\infty(G) \leq k$ is coNP-hard.*

Proof. We use Theorem 9 and choose $\epsilon = \frac{1}{2}$. We consider the problem Γ : given G, k , do we have $\gamma^\infty(G) > \binom{k+1}{2}$?

Clearly, there is a polynomial reduction from the complement of Γ to the stated problem. Thus, it suffices to prove that Γ satisfies the conditions of Theorem 9. If $\alpha(G) < k$, then $\gamma^\infty(G) \leq \binom{\alpha(G)+1}{2} < \binom{k+1}{2}$ (by Theorem 1), and thus (G, k) is a negative instance of Γ . If $\alpha(G) \geq k^2$, then $\gamma^\infty(G) \geq k^2 > \binom{k+1}{2}$ (by Theorem 1). Thus (G, k) is a positive instance of Γ . \square

From Lemma 3 and Theorem 10, we obtain:

Corollary 3. *Deciding whether $\overrightarrow{\gamma^\infty}(G) \leq k$ is coNP-hard.*

Since deciding whether $\alpha(G) \geq k$ is NP-hard, we also obtain:

Corollary 4. *Deciding whether $\overrightarrow{\alpha}(G) \geq k$ is NP-hard.*

We believe that these lower bounds are loose and these two problems are Π_2^P -hard.

Klostermeyer and MacGillivray [7] proved that there can be an arbitrary gap between α and γ^∞ . From this and as a consequence of Lemma 3, we show the same result between $\overrightarrow{\alpha}$ and $\overrightarrow{\gamma^\infty}$.

Corollary 5. *For every integer $k > 0$, there exists a graph G such that $\overrightarrow{\gamma^\infty}(G) \geq \overrightarrow{\alpha}(G) + k$.*

3.2. Results on some classes of graphs

We are now interested in the values of $\overrightarrow{\gamma^\infty}$ and $\overrightarrow{\gamma_m^\infty}$ for particular classes of graphs.

3.2.1. Cycles and forests

The case of cycles is quite straightforward for both parameters.

Theorem 11. $\overrightarrow{\gamma^\infty}(C_n) = n - 1$ and $\overrightarrow{\gamma_m^\infty}(C_n) = \lceil \frac{n}{2} \rceil$ for every $n \geq 3$.

Proof. By Corollary 1, for any acyclic orientation H of C_n , we have $\gamma^\infty(H) = \overrightarrow{\gamma_m^\infty}(H) = n$. Now consider the cyclic orientation H of C_n .

Since $\alpha(H) = n - 1$, we have $\overrightarrow{\gamma^\infty}(C_n) \geq n - 1$.

To see that $\gamma^\infty(H) \leq n - 1$, consider the following strategy with $n - 1$ guards: every time a vertex is attacked, the guard on its unique incoming neighbor moves to it. Since only the attacked vertex is unoccupied, we

know that the neighbor is occupied so that this defense is always possible, and leads to the exact same configuration that we were in.

Suppose that $\gamma_m^\infty(H) < \lceil \frac{n}{2} \rceil$. Let $D \subseteq V(H)$ be a dominating set of size $\gamma_m^\infty(H)$. Since $\gamma_m^\infty(H) \leq \lceil \frac{n}{2} \rceil$, there exist two vertices $u, v \in V(H) \setminus D$ such that $(u, v) \in E(H)$. Since u is the only vertex which dominates v in H and $u \notin D$, no vertex in D dominates v . Thus, D is not a dominating set of H . Consequently, $\overrightarrow{\gamma}_m^\infty(C_n) \geq \lceil \frac{n}{2} \rceil$.

To see that $\gamma_m^\infty(H) \leq \lceil \frac{n}{2} \rceil$, consider the following strategy with $\lceil \frac{n}{2} \rceil$ guards: place one guard every two vertices on C_n , with eventually two successive vertices if n is odd. Then, every time a vertex is attacked, move every guard to the unique outgoing vertex of their current vertex. Since the attacked vertex is unoccupied, we know that its incoming neighbor is occupied so that this defense is always possible, and leads to the exact same configuration that we were in. \square

We now consider forests. Since they are acyclic for any orientation, we obtain the following result:

Theorem 12. *Let G be a graph of order n . Then, $\overrightarrow{\gamma}^\infty(G) = n$ iff $\overrightarrow{\gamma}_m^\infty(G) = n$ iff G is a forest.*

Proof. If G is a forest, then every orientation of G is acyclic and thus $\overrightarrow{\gamma}^\infty(G) = \overrightarrow{\gamma}_m^\infty(G) = n$ by Corollary 1. If G is not a forest, then G admits a cycle C of k vertices. Consider an orientation H of G where the edges C form an oriented cycle. One can protect C with at most $k-1$ guards and $G-C$ can be protected by at most $n-k$ guards. Thus, $\gamma_m^\infty(H) \leq \gamma^\infty(H) \leq n-1$. \square

3.2.2. Complete graphs and graphs of oriented m -eternal number 2

We will now characterize the graphs G with $\overrightarrow{\gamma}_m^\infty(G) = 2$. Notice that only graphs with one vertex satisfy $\overrightarrow{\gamma}_m^\infty(G) = 1$.

Lemma 4. *Let $G = (V, E)$ be a graph of order $n \geq 2$. Then, $\overrightarrow{\gamma}_m^\infty(G) \geq 2$.*

Proof. Let $H = (V, E')$ be an orientation of G , and suppose that $\gamma_m^\infty(H) = 1$. Let $u, v \in V$. Consider that the guard is on the vertex u and the attack on the vertex v . Thus, necessarily, $(u, v) \in E'$. If the next attack is on u , the defender cannot answer since there is no edge (v, u) in E' . We obtain a contradiction. \square

We now prove a simple but essential lemma.

Lemma 5. *Let $G = (V, E)$ be a digraph of order $n \geq 3$ such that $\gamma_m^\infty(G) = 2$. Let $u, v \in V$. If neither (u, v) nor (v, u) are edges of G , then $\{u, v\}$ is a dominating set of G . Furthermore, the attacker can force the defender to put its two guards on $\{u, v\}$.*

Proof. Let D be an eternal dominating set of G with $|D| = 2$. Consider an attack on u and let D' be the answer of the defender. We have $u \in D'$. Now, consider an attack on v . Since $(u, v) \notin E$, the guard on u cannot move to v , so the second guard must do it. If the defender moves the guard on u to a neighbor u^+ , then $\{v, u^+\}$ is not a dominating set since neither of u^+ nor v dominates u . Therefore the answer of the defender to the attack on v is necessarily $\{u, v\}$. Consequently $\{u, v\}$ is a (eternal) dominating set. \square

We can now characterize the graphs with $\overrightarrow{\gamma}_m^\infty(G) = 2$.

Theorem 13. *Let G be a graph of order $n \geq 3$. Then, $\overrightarrow{\gamma}_m^\infty(G) = 2$ iff either:*

- $n = 2k$ and G is a complete graph from which at most k disjoint edges are removed.
- $n = 2k + 1$ and G is a complete graph from which at most $k - 1$ disjoint edges are removed.

Proof. We first prove that if there exist three distinct vertices $u, v, w \in V$ such that $uv, uw \notin E$, then $\overrightarrow{\gamma}_m^\infty(G) > 2$. Suppose that $\overrightarrow{\gamma}_m^\infty(G) = 2$, and let H be an orientation of G such that $\gamma_m^\infty(H) = 2$. Then, by Lemma 5, $\{u, v\}$ is a dominating set of H . Since $uw \notin E$, $(v, w) \in E(H)$. But, by Lemma 5, $\{u, w\}$ also is a dominating set of H , so $(w, v) \in E(H)$. We thus obtain a contradiction. Therefore, $\overrightarrow{\gamma}_m^\infty(G) > 2$. Consequently, for $n = 2k$ or $n = 2k + 1$, if $\overrightarrow{\gamma}_m^\infty(G) = 2$ then G is a complete graph from which at most k disjoint edges are removed.

Now take $n = 2k + 1$, and let $G = (V, E)$ be the complete graph from which exactly k disjoint edges are removed. Let $x_1, \dots, x_k, y_1, \dots, y_k, z$ be the vertices in V and assume that the non-edges of G are the pairs $x_i y_i$ for $i \in [k]$. We show that $\overrightarrow{\gamma}_m^\infty(G) > 2$. Suppose $\overrightarrow{\gamma}_m^\infty(G) = 2$, and let H be an orientation of G such that $\gamma_m^\infty(H) = 2$.

Claim 1. *For all $i \in [k]$, $\{x_i, y_i\}$ dominates H .*

Proof. This is a direct application of Lemma 5. \diamond

Claim 2. *For every vertex $v \in V \setminus \{z\}$ and for every index $i \in [k]$, v dominates x_i or y_i , but not both.*

Proof. We can assume, without loss of generality, that $v = x_j$. Now, if x_j dominates x_i and y_i , $\{x_i, y_i\}$ does not dominate x_j , which contradicts Claim 1. If v dominates none, by Claim 1, y_j dominates both, so $\{x_i, y_i\}$ does not dominate y_j , which contradicts Claim 1. \diamond

Claim 3. *For all $i \in [k]$, x_i or y_i dominates z , but not both.*

Proof. By Claim 1, x_i or y_i dominates z . Since the attacker can always attack z , there exists $v \in V \setminus \{z\}$ such that $\{z, v\}$ dominates H . Now, if both x_i and y_i dominate z , v dominates x_i and y_i , which contradicts Claim 2. \diamond

Claim 4. *If $\{z, v\}$ dominates H , v dominates z .*

Proof. We can assume, without loss of generality, that $v = x_i$. If z dominates x_i , then, by Claim 3, z does not dominate y_i , but x_i cannot dominate y_i either, we obtain a contradiction. \diamond

Claim 5. *There exists a unique vertex v such that $\{z, v\}$ dominates H .*

Proof. Suppose there exist $v_1, v_2 \in V \setminus \{z\}$ such that $\{z, v_1\}$ and $\{z, v_2\}$ dominate H . By Claim 4, v_1 and v_2 dominate z . But, since $\{z, v_1\}$ dominates v_2 , v_1 dominates v_2 . Similarly, v_2 dominates v_1 . We thus obtain a contradiction. \diamond

We can now assume without loss of generality that $\{z, x_i\}$ dominates H . By Claim 4, x_i dominates z , and by Claim 3, z dominates y_i . By Lemma 5, the attacker can force the guards to be on $\{x_i, y_i\}$. Then, if it attacks z , only the guard on x_i can go to z . The guard on y_i cannot move to x_i , so there exists a vertex v in $V(G) \setminus \{z, x_i\}$ such that $\{z, v\}$ dominates H . This contradicts Claim 5. Thus, $\overrightarrow{\gamma}_m^\infty(G) > 2$. Therefore, if $\overrightarrow{\gamma}_m^\infty(G) = 2$ then $n = 2k$ and G is the complete graph from which are removed at most k disjoint edges, or $n = 2k + 1$ and G is the complete graph from which at most $k - 1$ disjoint edges are removed.

We now prove that if $n = 2k$ and G is the complete graph from which at most k disjoint edges are removed, then $\overrightarrow{\gamma}_m^\infty(G) = 2$. Without loss of generality, we assume that exactly k disjoint edges are removed. Let $V(G) = \{v_0, v_1, \dots, v_{2k-1}\}$ and assume that the non-edges of G are the pairs $x_i x_{i+k}$ for $i \in [0, k - 1]$. We construct an orientation H of G in the following way: for all i, j , $i > j$, $(v_i, v_j) \in E(H)$ iff $i - j < k$, and $(v_j, v_i) \in E(H)$ iff $i - j > k$. We will consider a strategy for the guards that preserves the following invariant: for each guard configuration D in the strategy, if

$D = \{v_i, v_j\}$ then $|i - j| = k$. Notice that, by construction of H , every guard configuration that satisfies the invariant is a dominating set of H . We start with $D_0 = \{v_0, v_k\}$. Clearly, this configuration verifies the invariant. Consider a guard configuration $D = \{v_i, v_j\}$ that verifies the invariant and let us prove that, for every attacked vertex v_r , the defender can defend the attack and obtain a configuration D' that verifies the invariant. Without loss of generality, we can assume that $j = i + k$. If $i < r < j$, then we move a guard from v_j to v_r and the other guard from v_i to $v_{r'}$ with $r' = r + k \pmod{2k}$. Otherwise, we move a guard from v_i to v_r and the other guard from v_j to $v_{r'}$. In both cases, we obtain the configuration $\{v_r, v_{r'}\}$ that verifies the invariant.

Assume now that $n = 2k + 1$ and consider G to be the complete graph of order n from which at most $k - 1$ disjoint edges are removed. Without loss of generality, we assume that exactly $k - 1$ disjoint edges are removed. Let us prove that $\overrightarrow{\gamma}_m^\infty(G) = 2$. G has exactly 3 universal vertices. Let z be one of them and let v_0, \dots, v_{2k-1} be the vertices in $V \setminus \{z\}$. Without loss of generality, we assume that v_0 and v_k are the two other universal vertices, and $v_i v_{i+k}$ is not an edge of E for every $i \in [1, k - 1]$.

We create an orientation H of G as follows. We orientate the edges of $G[V \setminus \{z\}]$ except $v_0 v_k$ identically to the orientation of G in the even case. Then, we orientate the edges incident to z except for $z v_0$ and $z v_k$ such that z has the same neighborhood as v_0 . Finally, we orientate the three remaining edges such that $(v_0, v_k), (v_k, z), (z, v_0) \in E(H)$.

We will give a strategy that preserves the following invariant: $D = \{v_i, v_j\}$ with $j - i = k$ or $D = \{z, v_k\}$. There are several cases:

Case 1: if $D = \{v_i, v_{i+k}\}$ and the attacker chooses a vertex v_j , then the defender plays as in the even case.

Case 2: if $D = \{v_i, v_{i+k}\}$ with $i \in [1, k - 1]$ and the attacker plays in z , then, the defender plays as in the even case but by replacing v_0 with z .

Case 3: if $D = \{v_0, v_k\}$ and the attacker plays in z . Then, the defender moves the guard on v_0 to v_k and the guard on v_k to z .

Case 4: if $D = \{z, v_k\}$ and the attacker plays in v_j with $j > 0$, then the defender plays as in the even case but by replacing v_0 with z .

Case 5: if $D = \{z, v_k\}$ and the attacker plays in v_0 , the defender moves the guard on z to v_0 . \square

The class of graphs G with $\overrightarrow{\gamma}_m^\infty(G) = 3$ seems hard to characterize. We now consider complete graphs. Surprisingly, the exact value of $\overrightarrow{\gamma}^\infty$ for complete graphs seems hard to find. However, we can obtain lower and upper bounds using a result from Erdős and Moser concerning $\overrightarrow{\alpha}$.

Theorem 14. [2] For every $n > 0$, $\lfloor \log_2 n \rfloor + 1 \leq \overrightarrow{\alpha}(K_n) \leq 2 \lfloor \log_2 n \rfloor + 1$

By combining this theorem with Theorem 4, we obtain:

Corollary 6. For every integer $n > 0$, $\lfloor \log_2 n \rfloor + 1 \leq \overrightarrow{\gamma}^\infty(K_n) \leq (2^{\lfloor \log_2 n \rfloor + 2})$.

3.2.3. Complete bipartite graphs

The case of complete bipartite graphs, on the other hand, has been fully covered for both parameters.

Theorem 15. $\overrightarrow{\gamma}^\infty(K_{n,m}) = \max\{n, m\} + 1$ for every $n, m \geq 1$.

Proof. Denote by A and B the two parts of $K_{n,m}$. First, we prove that $\overrightarrow{\gamma}^\infty(K_{n,m}) \geq \max\{n, m\} + 1$. Without loss of generality, we assume that $|A| \geq |B|$. Let T be the induced subgraph $G[A \cup \{v\}]$ where v is a vertex of B . It is easily seen that T is a tree. Thus, any orientation of T is acyclic. Consequently, $\overrightarrow{\alpha}(K_{n,m}) \geq |A| + 1$. By Proposition 1, we obtain the desired inequality.

We will now prove that $\overrightarrow{\gamma}^\infty(K_{n,m}) \leq \max\{n, m\} + 1$. By Lemma 2, we assume without loss of generality that $n = m$. Let M be a perfect matching of $K_{n,n}$. We construct an orientation H of $K_{n,n}$ as follows: let $u \in A$, and $v \in B$. If $\{u, v\} \in M$ then $(u, v) \in E(H)$. Otherwise $(v, u) \in E(H)$. We start by putting a guard on every vertex of A and one guard on an arbitrary vertex of B . In the strategy, we preserve the following invariant: there is at least one guard in every edge of the matching M , and exactly one edge e^* of M has a guard on its two extremities. We denote by v^* the extremity of e^* in B . Suppose that a vertex v of B is attacked. Let u be the vertex such that $\{u, v\} \in M$. Then, we move the guard on u to v . Suppose now that a vertex v of A is attacked. Then we move the guard on v^* to v . It is easily seen that the invariant is preserved. \square

Theorem 16. $\overrightarrow{\gamma}_m^\infty(K_{2,2}) = 2$
 $\overrightarrow{\gamma}_m^\infty(K_{2,3}) = \overrightarrow{\gamma}_m^\infty(K_{3,3}) = 3$
 $\overrightarrow{\gamma}_m^\infty(K_{n,m}) = 4$ for every $n \geq 2$, and $m \geq 4$.

Proof. $K_{2,2}$ is isomorphic to C_4 so, by Theorem 11, $\overrightarrow{\gamma}_m^\infty(K_{2,2}) = 2$.

It is easily seen that $K_{2,3}$ and $K_{3,3}$ don't satisfy the conditions of Theorem 13. Thus $\overrightarrow{\gamma}_m^\infty(K_{2,3}) \geq 3$ and $\overrightarrow{\gamma}_m^\infty(K_{3,3}) \geq 3$.

To show that $\overrightarrow{\gamma}_m^\infty(K_{2,3}) \leq 3$, we consider the following defense : we make one guard stay on a_3 , that will therefore always be defended. The subgraph induced by the rest of the vertices $\{a_1, a_2, b_1, b_2\}$ is isomorphic to $K_{2,2}$, and can therefore be defended by the two guards left.

As C_6 is a spanning subgraph of $\overrightarrow{\gamma}_m^\infty(K_{3,3})$, and since, by Theorem 11, $\overrightarrow{\gamma}_m^\infty(C_6) = 3$, we also have $\overrightarrow{\gamma}_m^\infty(K_{3,3}) \leq 3$.

We now consider the case where $n \geq 2$ and $m \geq 4$. We first show that $\overrightarrow{\gamma}_m^\infty(K_{n,m}) \geq 4$. Suppose that $\overrightarrow{\gamma}_m^\infty(K_{n,m}) = 3$, so that there exists an orientation H of $K_{n,m}$ for which any attack can be defended by three guards. Denote by A and B the two parts of $K_{n,m}$, $|A| = m$, $|B| = n$.

Consider a guard configuration D_i where $|D_i \cap A| = 2$ and $|D_i \cap B| = 1$. Let $u, v, s, t \in A$ and $w \in B$. Without loss of generality, we suppose that $D_i = \{u, v, w\}$, and $r_{i+1} = s$ is the attacked vertex. We denote by D_{i+1} a guard configuration reachable from D_i that defends s . Only the guard on w can defend s , so that in H , w dominates s , and the guard on w goes to s . Now the other two guards can either stay on u and v , or go to one of their outgoing neighbors, respectively u^+ , and v^+ . We can therefore have four different configurations for D_{i+1} : $\{u, v, s\}$, $\{u^+, v, s\}$, $\{u, v^+, s\}$ or $\{u^+, v^+, s\}$.

In the first configuration, no vertex dominates t , in the second, none dominates u , and in the third one, none dominates v . Therefore, we necessarily have $D_{i+1} = \{u^+, v^+, s\}$. But, since $u^+, v^+, w \in B$ and w dominates s , no vertex dominates w . Thus, we have either $u^+ = w$ or $v^+ = w$. We can consider, without loss of generality, that $u^+ = w$. Then, u dominates w , and, since $\{w, v^+, s\}$ is a dominating set, with $\{w, v^+\} \in B$ and $s \in A$, w or v^+ must dominate u and v , so v^+ dominates u , and w dominates v .

Suppose now that v^+ dominates s , and the next attacked vertex r_{i+2} is u . Only v^+ can go to u , so that we can once again obtain four different configurations for D_{i+2} : $\{w, u, s\}$, $\{w^+, u, s\}$, $\{w, u, s^+\}$ or $\{w^+, u, s^+\}$. In the first and third configurations, no vertex dominates v^+ , in the second, none dominates either v or t . The only possibility for the last one to dominate s is if $w^+ = s$, but then $\{s, u, s^+\}$ would not dominate v^+ . Therefore, we obtain a contradiction, and, thus, s dominates v^+ .

Since $\{u, v, w\}$ dominates H , with $\{u, v\} \in A$ and $w \in B$, w dominates $A \setminus \{u, v\}$, and since w dominates v , only u dominates w . Thus, for any k , $u \in D_k$, or $w \in D_k$, or both. Since $\{w, v^+, s\}$ is a dominating set, with $\{w, v^+\} \in B$ and $s \in A$, s dominates $B \setminus \{w, v^+\}$, and since s dominates v^+ , only w dominates s . Thus, for any k , $w \in D_k$, or $s \in D_k$, or both. Similarly, since we could have had $r_{i+1} = t$, for any k , $w \in D_k$, or $t \in D_k$, or both. Therefore, for any k , $D_k = \{u, s, t\}$, or $w \in D_k$. But, in $\{u, s, t\}$, no vertex dominates v . Thus, for any k , $w \in D_k$. But, from $D_{i+1} = \{w, v^+, s\}$, if $r_{i+2} = v$, only w dominates v so the guard on w has to move to v , and since neither s nor v^+ dominates w , we necessarily have $w \notin D_{i+2}$ and obtain a

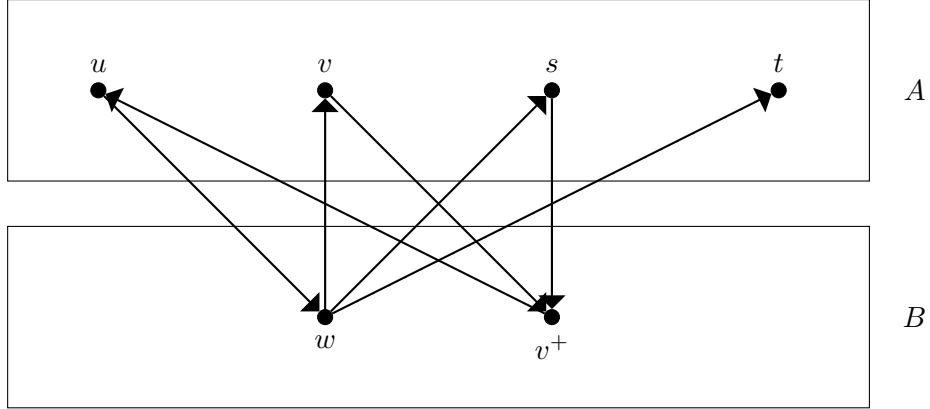


Figure 3: The vertices u, v, s, t, w, v^+ , and some of their corresponding arcs in the orientation H of $K_{m,n}$, for $D_i = \{u, v, w\}$, $r_{i+1} = s$, $D_{i+1} = \{w, v^+, s\}$, and $r_{i+2} = u$. We can see here that from $D_{i+1} = \{w, v^+, s\}$, if we take $r_{i+2} = v$, we have $w \notin D_{i+2}$.

contradiction.

All considered vertices and some of the arcs in H are represented in Figure 3.

We now have to prove that the attacker can always bring the guards to a set D_i such that $|D_i \cap A| = 2$ and $|D_i \cap B| = 1$. Suppose there exists i such that $|D_i \cap A| = 1$ and $|D_i \cap B| = 2$. Let $u, v \in B$ and $w, s, t \in A$. We take $D_i = \{u, v, w\}$, and $r_{i+1} = s$. Only the guards on u or v can go to s . We can suppose, without loss of generality, that the guard on u goes to s , so that u dominates s . We can obtain four different configurations for D_{i+1} : $\{s, v, w\}$, $\{s, v^+, w\}$, $\{s, v, w^+\}$ or $\{s, v^+, w^+\}$.

In the first and last configurations, we have $|D_{i+1} \cap A| = 2$, and $|D_{i+1} \cap B| = 1$, so we are done. Let us suppose it is not the case. In the second configuration, since $s, v^+, w \in A$, and $|A| = m \geq 4$, there exists a vertex in A that is not dominated by any of the three vertices. Therefore, we necessarily have $D_{i+1} = \{s, v, w^+\}$. But, since $u, w^+, v \in B$ and u dominates s , to dominate u , we must have $w^+ = u$, w dominates u , and $D_{i+1} = \{s, v, u\}$.

Now, similarly, if we take $r_{i+2} = t$, we must have $D_{i+2} = \{v, t, u\}$, and either u or v dominates t and is dominated by s . Since u dominates s , it must be v . Similarly, if we take $r_{i+3} = w$, we must have $D_{i+3} = \{v, w, u\}$, and either u or v dominates w and is dominated by t . Since v dominates t , and w dominates u , we obtain a contradiction. Therefore, if there exists a configuration D_i such that $|D_i \cap A| = 1$ and $|D_i \cap B| = 2$, then there exists a sequence of attacks such that either the attacker wins or we obtain a guard

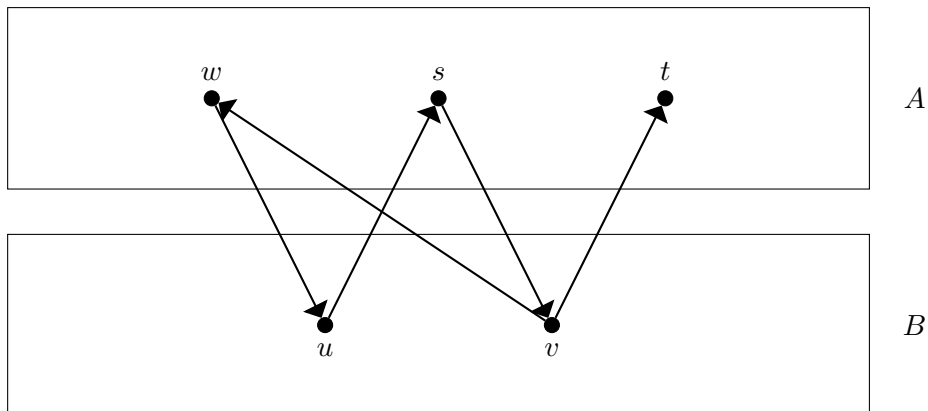


Figure 4: The vertices u, v, w, s, t , and some of their corresponding arcs in the orientation H of $K_{m,n}$, if we suppose there is no j s.t. $|D_j \cap A| = 2$ and $|D_j \cap B| = 1$, and have $D_i = \{u, v, w\}$, $r_{i+1} = s$, $D_{i+1} = \{s, v, u\}$, $r_{i+2} = t$, and $D_{i+2} = \{v, t, u\}$. We can see here that if we take $r_{i+3} = w$, only the guard on v can move to w , but then the guard on t cannot move to v .

configuration D_j , $j > i$ with $|D_j \cap A| = 2$ and $|D_j \cap B| = 1$. All considered vertices and some of their arcs are represented in Figure 4.

Now, consider a guard configuration D_i with $|D_i \cap A| = 3$. Since $|A| = m \geq 4$, there exists a vertex in A that is not dominated by any of the three vertices, so we obtain a contradiction. If D_i is a guard configuration with $|D_i \cap B| = 3$ and the next attack is on a vertex in A , for any answer of the defender, we obtain one of the previously seen possibilities. Therefore, for any defense, there exists i such that $|D_i \cap A| = 1$ and $|D_i \cap B| = 2$, so that we have $\overrightarrow{\gamma}_m^\infty(K_{n,m}) \geq 4$.

We now prove that $\overrightarrow{\gamma}_m^\infty(K_{n,m}) \leq 4$. To see it, consider the following strategy with 4 guards: partition A into two non-empty sets A_1 and A_2 , and B into two non-empty sets B_1 and B_2 . Orientate every edge of $K_{n,m}$ either from A_1 to B_1 , from B_1 to A_2 , from A_2 to B_2 or from B_2 to A_1 . Start with a guard in each of the four sets of vertices. Exactly one guard dominates one set. Every time a vertex is attacked, move the guard who dominates its corresponding set to the vertex, and move all the three other guards to the set they dominate. This leads to the exact same configuration that we were into, so that we can always defend any attack like this. \square

3.2.4. Trivially perfect graphs

Trivially perfect graphs can be characterized in various ways. We use the following definition due to Wolk [12]. A graph G is trivially perfect if every connected induced subgraph of G admits a universal vertex. For this class of graphs, we obtain the exact value of $\overrightarrow{\gamma}_m^\infty$.

Theorem 17. *Let G be a connected trivially perfect graph with at least 2 vertices and l 2-vertex-connected components. Then, we have:*

- if G admits exactly one 2-vertex-connected component G_i of size at least 3 then $\overrightarrow{\gamma}_m^\infty(G) = \overrightarrow{\gamma}_m^\infty(G_i) + l - 1$ with $\overrightarrow{\gamma}_m^\infty(G_i) \leq 3$.
- if every 2-vertex-connected component is of size at most 3, then $\overrightarrow{\gamma}_m^\infty(G) = l + 1$.
- if G admits at least two 2-vertex-connected components of size at least 4 and 3, respectively, then $\overrightarrow{\gamma}_m^\infty(G) = l + 2$.

We split the proof of this theorem into several lemmas.

Lemma 6. $\overrightarrow{\gamma}_m^\infty(G) \leq l + 2$.

Proof. Let x be a universal vertex of G . If $l \geq 2$, then x is unique, since if another vertex y was universal, then removing any vertex would keep G connected, and we would have $l = 1$. Moreover, every 2-vertex-connected component G_i contains x and another vertex x_i universal for G_i . Indeed, G_i being maximal, it necessarily contains the universal vertex x , and the connected induced subgraph of G obtained by removing x to G_i has a universal vertex x_i , since G is trivially perfect. We orientate G as follows, for every vertex v different from x and every x_i , we orientate the edge xv from x to v . For every vertex x_i we orientate xx_i from x_i to x . For every vertex v in a component G_i and different from x and x_i , we orientate the edge x_iv from v to x_i . The other edges are oriented arbitrarily, as they are not useful to the defense of G .

The strategy of the defender consists in permanently satisfying the following invariant:

- 1) there is a guard on x ;
- 2) every component G_i contains at least two guards (including the one on x);
- 3) the component G_i with three guards has a guard on x_i .

We denote by $G^* = G_i$ the component with three guards, on the vertices x , $x^* = x_i$ and another vertex v . Let us prove that the defender can maintain

the invariant. If the attack is on a vertex x_i different from x^* , then the defender moves the guard on the vertex in the component G_i that is not x to x_i . If the attack is on a vertex v in $V(G_i) \setminus \{x, x_i\}$, then the defender moves the guard on x to v , the guard on x^* to x and eventually the guard on a vertex in $V(G_i) \setminus \{x\}$ to x_i if it is not already on it. Thus, the invariant is maintained. \square

Lemma 7. *If every 2-vertex-connected component is of size at most 3, then $\overrightarrow{\gamma}_m^\infty(G) = l + 1$.*

Proof. We first prove that $\overrightarrow{\gamma}_m^\infty(G) \leq l + 1$. We put a guard on x . We orientate each component isomorphic to K_3 cyclically and we put a guard on the vertex which is an out-neighbor of x . We orientate each component isomorphic to P_2 arbitrarily and we put a guard on the vertex that is not x . It is easily seen that the defender can permanently defend and maintain a guard on x .

Now, we prove that $\overrightarrow{\gamma}_m^\infty(G) \geq l + 1$. By Corollary 2, we can assume that G has no component isomorphic to P_2 since we need to permanently put a guard on the pendent vertex. Thus, G is 2-arc-connected, and we can consider a strongly connected orientation of G (Theorem 7 and Proposition 2). If there are less than $l + 1$ guards, then there is a component G_i with at most one guard. Since the orientation is strongly connected, one vertex of G_i is not dominated. \square

Lemma 8. *If G admits exactly one 2-vertex-connected G_i component of size at least 3 then $\overrightarrow{\gamma}_m^\infty(G) = \overrightarrow{\gamma}_m^\infty(G_i) + l - 1$.*

Proof. This is a straightforward consequence of Corollary 2. \square

Lemma 9. *If G admits two 2-vertex-connected components of size at least 4 and 3, respectively, then $\overrightarrow{\gamma}_m^\infty(G) \geq l + 2$.*

Proof. Consider an orientation H of G and a defense of H with $l + 1$ guards. Let x be the universal vertex of G . By Proposition 2, we assume that H is strongly connected. Let G_1, G_2 be the 2-vertex-connected components of G of size at least 4 and 3 respectively and H_1, H_2 be the corresponding subgraphs in H .

First, we will prove that there exists a vertex $v_1 \in H_1 \setminus \{x\}$ such that $\{x, v_1\}$ is not a dominating set of H_1 . Assume this is not the case. We consider two cases.

Case 1: $H_1 \setminus \{x\}$ admits no vertex with outdegree 0. Let v be a vertex of $H_1 \setminus \{x\}$ and u be an out-neighbor of v . Then v is not dominated by

u . Since $\{x, u\}$ dominates it, there is an edge (x, v) . Thus x dominates all vertices in H_1 . That contradicts the fact that H is strongly connected.

Case 2: There is a vertex v with outdegree 0 in $H_1 \setminus \{x\}$. Thus, v dominates x and x dominates all vertices in H_1 except v . Let a and b be two other vertices of H_1 with $(a, b) \notin E(H)$. The attacker first chooses the vertex a . So the defender must move a guard to a . Then the attacker chooses the vertex b . If the defender can answer, then it must move the guard that is not on a to b . If it moves the second guard on v , then a is not dominated and if it moves the second guard elsewhere, x is not dominated. Notice that the guards on other components cannot participate in the defense of H_1 . Indeed, if such a guard moves in x , then its component is not dominated.

At any time, by choosing v_1 , since $\{x, v_1\}$ is not a dominating set of H_1 , the attacker can force the defender to move two guards to $H_1 \setminus \{x\}$. If there are two guards in $H_1 \setminus \{x\}$, necessarily, we are in the following configuration. Each component H_i except H_1 contains exactly one guard on a vertex x_i . Since H is strongly connected, the guard on x_i cannot dominate its whole component H_i but it can eventually dominate $H_i \setminus \{x\}$. Notice that means $x_i \neq x$ and there is no guard on x . Thus, the strategy of the attacker is as follows: first, it attacks v_1 so that we are in the previously described situation. Then, it chooses a vertex y in H_2 different from x_2 and then it attacks v_1 again, which forces the defender to move two guards to $H_1 \setminus \{x\}$. The guard on y cannot come back to x_2 and there is no guard on x (unless the guard on y moved to x , in which case y is not dominated). Thus H_2 is not dominated and the attacker wins. \square

This concludes the proof of Theorem 17.

By combining Theorems 17 and 13, we obtain a full characterization of the value of $\overrightarrow{\gamma}_m^\infty$ for trivially perfect graphs. It is easily seen that these characterizations can be checked in linear time [5]. So we obtain the following result.

Corollary 7. $\overrightarrow{\gamma}_m^\infty$ can be computed in linear time on trivially perfect graphs.

3.2.5. Grids and products of graphs

We now consider $\overrightarrow{\gamma}^\infty$ on grids. Contrary to γ^∞ (which is $\lceil \frac{nm}{2} \rceil$), we think that the exact value of $\overrightarrow{\gamma}^\infty$ cannot be expressed by a simple formula. We give here lower and upper bounds.

The following proposition has been verified using a computer.

Proposition 3. $\overrightarrow{\gamma}^\infty(P_3 \square P_3) = 7$.

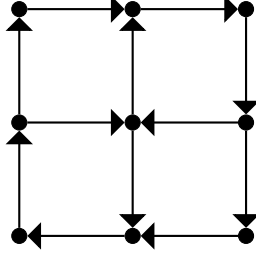


Figure 5: An orientation \bar{H} of $P_3 \square P_3$ with $\gamma^\infty(H) = 7$.

The unique orientation H with $\gamma^\infty(H) = 7$ (up to isomorphism) is shown in Figure 5.

We will show that $\frac{2nm}{3} \leq \overrightarrow{\gamma^\infty}(P_n \square P_m) \leq \frac{7nm}{9} + O(n + m)$. The next two theorems give more precise bounds ².

Theorem 18.

$$\left\lceil \frac{n}{2} \right\rceil m + \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{m}{3} \right\rceil \leq \overrightarrow{\gamma^\infty}(P_n \square P_m).$$

Proof. Consider the graph $G = P_n \square P_m$ with n lines and m columns. We denote by $v_{i,j}$ the vertex at line i and column j . To prove the lower bound, we will show that $\alpha(H) \geq \left\lceil \frac{n}{2} \right\rceil m + \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{m}{3} \right\rceil$ for any orientation H of G . We construct a set S such that $H[S]$ is acyclic as follows. First, we put odd lines vertices into S . Obviously, $H[S]$ is acyclic and $|S| = \left\lceil \frac{n}{2} \right\rceil m$. Now, we will show that one can add $\left\lceil \frac{m}{3} \right\rceil$ vertices for each even line and $H[S]$ remains acyclic. Let $i \in [n]$ be even. If $i = n$, clearly, we can add to S the vertices $v_{i,j}$ with j odd without creating cycles. Assume that $i < n$. We split the vertices in the line i into 3 types. A is the set of vertices $v_{i,j}$ where $(v_{i-1,j}, v_{i,j})$ and $(v_{i,j}, v_{i+1,j})$ are arcs of H . B is the set of vertices $v_{i,j}$ where $(v_{i+1,j}, v_{i,j})$ and $(v_{i,j}, v_{i-1,j})$ are arcs of H . C is the set of vertices $v_{i,j}$ that does not belong to A or B .

There are three cases.

Case 1: $|A| \geq \left\lceil \frac{m}{3} \right\rceil$. We add all vertices of type A to S and we don't create cycles.

Case 2: $|B| \geq \left\lceil \frac{m}{3} \right\rceil$. Similar to case 1.

Case 3: $|A| < \left\lceil \frac{m}{3} \right\rceil$ and $|B| < \left\lceil \frac{m}{3} \right\rceil$. Without loss of generality, we assume

²Notice that Theorem 18 is not necessarily optimized according to the respective values of n and m , but, since $\overrightarrow{\gamma^\infty}(P_n \square P_m) = \overrightarrow{\gamma^\infty}(P_m \square P_n)$, we can always take the maximum of the two obtained lower bounds.

that $|A| \geq |B|$. We construct a set D from $A \cup C$ by picking one vertex out of two in the ordered sequence of vertices of $A \cup C$. That is, the odd (positioned) vertices in this ordered sequence of vertices are chosen. Since $|A \cup C| \geq \lceil \frac{2m}{3} \rceil$, we have $|D| \geq \lceil \frac{m}{3} \rceil$. We add every vertex of D in S . The only manner to create cycles with elements of $A \cup C$ is to choose two consecutive elements but it is not possible by the construction of D . \square

Theorem 19. *For $m = 3p + 2x$ and $n = 3q + 2y$ with $p, q \in \mathbb{N}$ and $x, y \in \{0, 1, 2\}$, we have:*

$$\overrightarrow{\gamma}_2^\infty(P_n \square P_m) \leq 7pq + \left\lceil \frac{9p}{2} \right\rceil y + \left\lceil \frac{9q}{2} \right\rceil x + 3xy$$

Proof. We divide the grid into 4 parts of size $3p \times 3q$, $3p \times 2y$, $2x \times 3q$ and $2x \times 2y$ respectively. We already know from Proposition 3 that $\overrightarrow{\gamma}^\infty(P_3 \square P_3) = 7$ and from Theorem 11 that $\overrightarrow{\gamma}^\infty(P_2 \square P_2) = 3$. Thus, the grid of size $3p \times 3q$ can be protected with $7pq$ guards by dividing it into squares of size 3×3 . Similarly, the grid of size $2x \times 2y$ can be protected with $3xy$ guards. The two remaining parts can be covered by squares of size 2×2 and 1×1 and can therefore be protected with $\left\lceil \frac{9p}{2} \right\rceil y$ and $\left\lceil \frac{9q}{2} \right\rceil x$ guards respectively. \square

For grids of size $2 \times n$, $3 \times n$ and $4 \times n$, the lower bound of Theorem 18 and the upper bound of Theorem 19 coincide and we have the exact value of $\overrightarrow{\gamma}^\infty$.

Corollary 8. *Let $n \geq 2$. Then,*

$$\begin{aligned} \overrightarrow{\gamma}^\infty(P_2 \square P_n) &= \left\lceil \frac{3n}{2} \right\rceil, \\ \overrightarrow{\gamma}^\infty(P_3 \square P_n) &= \left\lceil \frac{7n}{3} \right\rceil, \\ \overrightarrow{\gamma}^\infty(P_4 \square P_n) &= 2 \left\lceil \frac{3n}{2} \right\rceil. \end{aligned}$$

We don't know the value of $\overrightarrow{\gamma}^\infty(P_5 \square P_5)$. Using Theorems 18 and 19, we obtain $19 \leq \overrightarrow{\gamma}^\infty(P_5 \square P_5) \leq 20$.

We now study $\overrightarrow{\gamma}_m^\infty$ on various kinds of grids.

Theorem 20. *For every $n \geq 2$ and $m \geq 2$, we have*

$$\overrightarrow{\gamma}_m^\infty(P_n \square P_m) \leq \left\lceil \frac{nm}{2} \right\rceil.$$

Proof. If n or m are even, $P_n \square P_m$ admits a hamiltonian cycle and thus $\overrightarrow{\gamma}_m^\infty(P_n \square P_m) \leq \frac{nm}{2}$ by Theorem 11. Otherwise, $P_n \square P_m$ admits a hamiltonian cycle if we remove a corner vertex. If we keep a guard on the corner vertex and defend the remaining vertices with $\frac{nm-1}{2}$ vertices, we obtain the desired bound. \square

We don't have lower bounds except the straightforward bound $\lceil \frac{mn}{4} \rceil$. On the other hand, the upper bound seems loose but we have verified, using a computer, that $\overrightarrow{\gamma}_m^\infty(P_n \square P_m) = \lceil \frac{nm}{2} \rceil$ for every n and m between 2 and 5. No counterexample has been found for other values. We lack tools to find tight lower bounds.

We now consider upper bounds on $\overrightarrow{\gamma}_m^\infty$ for toroidal grids, rook's graphs, toroidal king's grid and toroidal hypergrids. We present a general method based on the neighborhood-equitable coloring, a notion we introduce.

Definition 7. Let k and l be two integers and G be a $(k-1)l$ -regular graph. A (k, l) -NE coloring of G is a proper coloring (V_1, \dots, V_k) of G with k colors such that for every vertex v and color $i \in [k]$ such that $v \notin V_i$, we have $|N(v) \cap V_i| = l$.

Theorem 21. Let $G = (V, E)$ be a graph that admits a $(k, 2l)$ -NE coloring. Then $\overrightarrow{\gamma}_m^\infty(G) \leq \frac{n}{k}$.

Proof. Consider a $(k, 2l)$ -NE coloring (V_1, \dots, V_k) of G . Let G_{ij} be the subgraph of G induced by $V_i \cup V_j$. By construction, G_{ij} is a $2l$ -regular bipartite graph. We orientate G_{ij} such that the indegree and outdegree of every vertex is l . Indeed, each component G_{ij} is eulerian. So we can orientate each component to obtain eulerian orientations. We do this for every distinct i and j in $[k]$ and we obtain an orientation H of G . Let us prove that $\overrightarrow{\gamma}_m^\infty(H) \leq \frac{n}{k}$. We initially put all guards in an arbitrary color class V_i . If a vertex $v \in V_j$ is attacked, we move all guards from V_i to V_j . Indeed, consider the graph B_{ij} with vertices $V_i \cup V_j$ and where we put an edge between two vertices $u \in V_i$ and $v \in V_j$ iff $(u, v) \in E(H)$. B_{ij} is l -regular by construction. Thus, by application of Hall's marriage theorem [4], B_{ij} admits a perfect matching between V_i and V_j . Consequently, there is a multimove from V_i to V_j in H . \square

Products of graphs admit this nice property.

Theorem 22. Let G_1 be a graph that admits a (k, l_1) -NE coloring and G_2 be a graph that admits a (k, l_2) -NE coloring. Then, $G_1 \square G_2$ admits a $(k, l_1 + l_2)$ -NE coloring.

Proof. We assume that the vertices of G_1 and G_2 are colored with integers chosen in the set $[0, k-1]$. Let v_1, \dots, v_n be the vertices of G_1 , and u_1, \dots, u_m be the vertices of G_2 . For a vertex v_i of G_1 and a vertex u_j of G_2 , we denote by $w_{i,j}$ the vertex associated to (v_i, u_j) in $G_1 \square G_2$. Let p be the color of v_i and q be the color of u_j . Then, we assign to $w_{i,j}$ the color $r = p + q$

mod k . Let r' be a color different from r . Then $w_{i,j}$ has exactly l_1 (resp. l_2) neighbors $w_{i',j'}$ of color r' with $i = i'$ (resp. $j = j'$) and thus $l_1 + l_2$ neighbors of color r' . \square

This notion of coloring has direct consequences on toroidal grids, i.e., cartesian products of two cycles, and rook's graphs, i.e., cartesian products of two complete graphs).

Theorem 23. *When m and n are both multiples of 3, we have:*

$$\overrightarrow{\gamma}_m^\infty(C_n \square C_m) \leq \frac{nm}{3}.$$

In general, we have:

$$\overrightarrow{\gamma}_m^\infty(C_n \square C_m) \leq \left\lceil \frac{nm}{3} \right\rceil + O(n + m).$$

Proof. The first inequality is a direct consequence of Theorems 21 and 22 with the fact that a cycle graph of order that is a multiple of 3 admits a $(3, 1)$ -NE coloring.

If n or m is not a multiple of 3, consider the grid $C_{3n+x} \square C_{3m+y}$ with $x, y \in \{0, 1, 2\}$ and $x > 0$ or $y > 0$. Let H' be an orientation of $C_{3n} \square C_{3m}$ as described in the previous case. We construct an orientation H of $C_{3n+x} \square C_{3m+y}$ as follows. We orientate each edge between (i, j) and $(i, j + 1)$ for $i \in [3n]$, $j \in [3m - 1]$, and between (i, j) and $(i + 1, j)$ for $i \in [3n - 1]$, $j \in [3m]$ in the same direction as in H' . For every $i \in [3n]$, if $(v_{i,3m}, v_{i,1}) \in E(H')$, then we orientate H such that $(v_{i,3m}, v_{i,3m+1}, \dots, v_{i,3m+y}, v_{i,1})$ is an oriented path. Otherwise, we orientate H such that $(v_{i,1}, v_{i,3m+y}, v_{i,3m+y-1}, \dots, v_{i,3m})$ is an oriented path. We do the same for every edge $v_{3n,j}v_{1,j}$ with $j \in [3m]$. We arbitrarily orientate the remaining edges. An example of an orientation is described in Figure 6.

Consider the set of vertices S of H including an m -eternal dominating set of H' , as described in the previous case, and containing every vertex (i, j) with $i > 3n$ or $j > 3m$. Then, S is an m -eternal dominating set of H . Indeed, we mimic the strategy of the defender for H' . The only difference is when a guard in H' goes from a "border" of the grid to the opposite border. For example, a guard goes from a vertex $(i, 3m)$ to the vertex $(i, 1)$. Then, we push every guard, except for the last one, in the path $(v_{i,3m}, v_{i,3m+1}, \dots, v_{i,3m+y}, v_{i,1})$ to the next vertex. One can easily generalize to the other borders. \square

Rook's graphs are cartesian products of two complete graphs. They received their names from the legal moves of the rook chess piece on a chessboard. For square rook's graphs, we obtain the exact value of $\overrightarrow{\gamma}_m^\infty$.

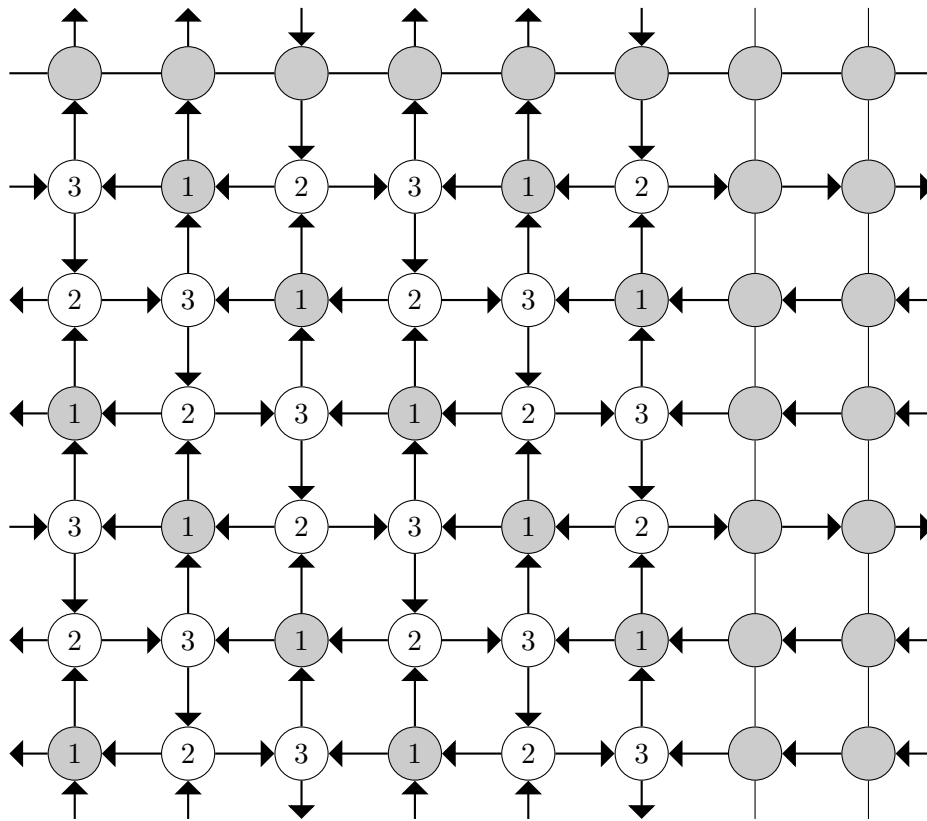


Figure 6: orientation of the toroidal grid $C_7 \square C_8$.

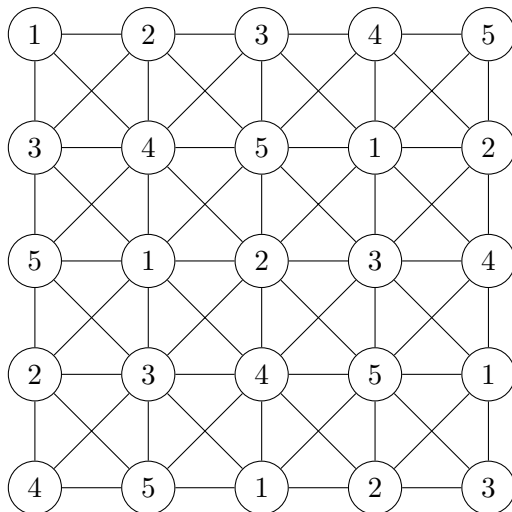


Figure 7: $(5, 2)$ -NE coloring of a square of a king's grid.

Theorem 24. For every $n \geq 1$, we have $\overrightarrow{\gamma}_m^\infty(K_n \square K_n) = \gamma(K_n \square K_n) = n$.

Proof. It easily seen that $\gamma(K_n \square K_n) \geq n$. Indeed, a set of size lower than n does not dominate at least a line and a column. Thus, it does not dominate the vertex which is at the intersection of this line and this column. The upper bound is a direct consequence of Theorems 21 and 22 with the fact that a complete graph of order n admits a $(n, 1)$ -NE coloring. \square

Toroidal king's grids are the strong product of two cycles. They received their names from the legal moves of the king chess piece on a (toroidal) chessboard. For this class of graphs, we obtain the following result.

Theorem 25. Let m and n be two multiples of 5. Then, we have:

$$\overrightarrow{\gamma}_m^\infty(C_n \boxtimes C_m) \leq \frac{nm}{5}.$$

Proof. If we split the grid in squares of size 5×5 , each square can be colored as in Figure 7. An easy case study permits to conclude that we obtain a $(5, 2)$ -NE coloring. \square

Notice that we can obtain an upper bound $\frac{nm}{5} + O(n + m)$ when there is no condition on n and m . The idea is similar to the proof of Theorem 23.

However, the proof is quite complicated and the result not essential so we omit it in this paper.

We also generalize Theorem 23 to toroidal hypergrids.

Theorem 26. $\overrightarrow{\gamma}_m^\infty(C_{n_1} \square \dots \square C_{n_k}) \leq \frac{n}{k+1}$ where n is the order of the graph and all n_i are multiples of $k+1$.

Proof. Let v be a vertex at position (i_1, \dots, i_k) in the hypergrid. We affect to v the color

$$\sum_{j=1}^k j i_j \pmod{(k+1)}.$$

It is easily seen that this coloring is proper. Additionally, for every distinct colors i, j , and vertex v of color i , v has exactly two neighbors of color j . Indeed, if v is at position (i_1, \dots, i_k) , then the two neighbors are at positions $(i_1, \dots, i_{p-1}, i_p + 1, i_{p+1}, \dots, i_k)$ where $p = j - i \pmod{(k+1)}$ and $(i_1, \dots, i_{q-1}, i_q - 1, i_{q+1}, \dots, i_k)$ where $q = i - j \pmod{(k+1)}$. Thus, we obtain a $(k+1, 2)$ -NE coloring. \square

We conjecture that the upper bounds in Theorems 25, 26, and 23 correspond to the exact value of $\overrightarrow{\gamma}_m^\infty$. A way to prove this would be to show that any orientation that minimizes γ_m^∞ is eulerian. More generally, we think that the following proposition is true.

Conjecture 2. *Let G be a graph that admits a $(k, 2)$ -NE coloring. Then $\overrightarrow{\gamma}_m^\infty(G) = \frac{n}{k}$.*

This conjecture is verified for even cycles (Theorem 11) and for square rook's grids (Theorem 24).

4. Future works and open questions

Besides the two conjectures given in this paper, we enumerate some future works and open questions.

- Give a tight upper bound of $\overrightarrow{\gamma}_m^\infty(n)$ for 2-edge-connected graphs.
- Is there a natural parameter for digraphs that is an upper bound of γ^∞ as the clique covering number is for graphs?
- Give better bounds for $\overrightarrow{\gamma}^\infty$ on complete graphs and grids.

- We have proved that $\overrightarrow{\gamma^\infty} = \overrightarrow{\alpha}$ for trees, cycles, complete bipartite graphs and grids $2 \times n$, $3 \times n$, $4 \times n$. Is it true for complete graphs or (rectangular) grids in general?
- Can we characterize the graphs for which $\overrightarrow{\gamma_m^\infty} = \gamma$? Such examples are rook's graphs and non-complete graphs with $\overrightarrow{\gamma_m^\infty} = 2$.
- Study the complexity of deciding whether $\overrightarrow{\gamma_m^\infty}(G) \leq k$ in the general case and when k is fixed. Notice that, for $\overrightarrow{\gamma^\infty}$, the problem is coNP-hard in the general case (Corollary 3) and trivial when k is fixed. Indeed, thanks to Corollary 6, we know that $\overrightarrow{\gamma^\infty}(G) \leq k$ implies $n \leq 2^k$, so when k is fixed, there is only a finite number of positive instances to test.

- [1] A. Burger, E. Cockayne, W. Grundlingh, C. Mynhardt, J. Van Vuuren, and W. Winterbach. Infinite order domination in graphs. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 50:179–194, 2004.
- [2] P. Erdős and L. Moser. On the representation of directed graphs as unions of orderings. *Math. Inst. Hung. Acad. Sci.*, 9:125–132, 1964.
- [3] W. Goddard, S. M. Hedetniemi, and S. T. Hedetniemi. Eternal security in graphs. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 52:169–180, 2005.
- [4] P. Hall. On representatives of subsets. *Journal of the London Mathematical Society*, 1(1):26–30, 1935.
- [5] J. E. Hopcroft and R. E. Tarjan. Dividing a graph into triconnected components. *SIAM Journal on Computing*, page 135, 1973.
- [6] W. Klostermeyer and G. MacGillivray. Eternal security in graphs of fixed independence number. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 63, 2007.
- [7] W. Klostermeyer and G. MacGillivray. Eternal dominating sets in graphs. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 68:97–111, 2009.
- [8] W. F. Klostermeyer. Complexity of eternal security. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 61:135, 2007.

- [9] W. F. Klostermeyer and G. MacGillivray. Eternally secure sets, independence sets and cliques. *AKCE Int. J. Graphs Comb*, 2:119–122, 2005.
- [10] W. F. Klostermeyer and C. M. Mynhardt. Protecting a graph with mobile guards. *Applicable Analysis and Discrete Mathematics*, 10(1):1–29, 2016.
- [11] H. E. Robbins. A theorem on graphs, with an application to a problem of traffic control. *The American Mathematical Monthly*, 46(5):281–283, 1939.
- [12] E. S. Wolk. The comparability graph of a tree. *Proceedings of the American Mathematical Society*, 13(5):789–795, 1962.
- [13] D. Zuckerman. Linear degree extractors and the inapproximability of max clique and chromatic number. In *Proceedings of the thirty-eighth annual ACM symposium on Theory of computing*, pages 681–690. ACM, 2006.