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# QUANTITATIVE JACOBIAN DETERMINANT BOUNDS FOR THE CONDUCTIVITY EQUATION IN HIGH CONTRAST COMPOSITE MEDIA

YVES CAPDEBOSCQ AND SHAUN CHEN YANG ONG

ABSTRACT. We consider the conductivity equation in a bounded domain in  $\mathbb{R}^d$  with  $d \geq 3$ . In this study, the medium corresponds to a very contrasted two phase homogeneous and isotropic material, consisting of a unit matrix phase, and an inclusion with high conductivity. The geometry of the inclusion phase is so that the resulting Jacobian determinant of the gradients of solutions  $DU$  takes both positive and negative values. In this work, we exhibit a class of inclusions  $Q$  and boundary conditions  $\phi$  such that the determinant of the solution of the boundary value problem satisfies this sign-changing constraint. We provide lower bounds for the measure of the sets where the Jacobian determinant is greater than a positive constant (or lower than a negative constant). Different sign changing structures were introduced in [9], where the existence of such media was first established. The quantitative estimates provided here are new.

## 1. THE JACOBIAN DETERMINANT AND THE SIGN CHANGING PROPERTY

Given  $\Omega$  a bounded domain in  $\mathbb{R}^d$  consider the following elliptic boundary value problem

$$(1) \quad \begin{aligned} -\operatorname{div}(\gamma DU) &= 0 && \text{in } \Omega, \\ U &= \phi && \text{on } \partial\Omega, \end{aligned}$$

with  $\phi \in C^{1,\sigma}(\mathbb{R}^d; \mathbb{R}^d)$ , for some  $\sigma \in (0, 1)$ . The conductivity  $\gamma$  is a real-valued function such that  $\gamma(\Omega) = \{1, k\}$  with  $k > 0$ . More precisely,

$$(2) \quad \gamma = \left(1 + (k - 1)\chi_Q\right)I_d$$

where  $\chi_Q$  is the indicator function of a sub domain  $Q \subset \Omega$ , which is assumed to be a  $C^2$  open connected inclusion, with  $d(\partial\Omega, Q) > 0$ . We are interested the Jacobian determinant of  $U$ , that is,

$$J = \det DU \text{ in } \Omega \setminus Q,$$

and we consider a class of inclusions  $Q$  for which  $J$  changes sign, that is,

$$(3) \quad |J^{-1}((-\infty, 0))| \times |J^{-1}((0, \infty))| > 0.$$

It is known [16] that in this setting  $J$  is piecewise continuous, so in particular measurable, and the sets appearing in (3) are indeed measurable. Unlike the two dimensional case where such inclusions do not exist [4], when  $d \geq 3$  examples have been provided in [9]. The purpose of our study is to establish lower bounds for the measures of

$$(4) \quad J_- = J^{-1}((-\infty, 0)) \text{ and } J_+ = J^{-1}((0, \infty)),$$

by exhibiting open subsets included in both sets. The bounds we derive depend on some geometrical characteristics of  $\Omega$  and  $Q$ .

**Assumption 1.1.** *The open domain  $\Omega$  is either convex and polygonal, or convex with a  $C^2$  boundary.*

- For any  $x = (x_1, \dots, x_d) \in \Omega$  (resp.  $Q$ ),  $(\pm x_1, \dots, \pm x_d) \in \Omega$  (resp.  $Q$ ).
- The inclusion  $Q \subset \Omega$  is closed with non-empty interior and  $Q$  connected.
- The (hyper)surface  $\partial Q$  is  $C^2$ .
- The sets  $\Omega \setminus Q$  and  $(\Omega \setminus Q) \cap (0, \infty)^d$  are connected.
- For  $i \in \{1, \dots, d\}$ ,  $Q \cap (\mathbb{R}e_i) \neq \emptyset$ .
- The origin is not in  $Q$ , and  $d(0, Q) > 0$ .

Not all boundary conditions lead to sign changing properties: a constant boundary condition leads to a null Jacobian determinant everywhere, for example. The following definition details sufficient assumptions, as we will see.

**Definition 1.2.** Given  $\alpha > 1$ , we say that  $\phi \in C^{1,\sigma}(\mathbb{R}^d; \mathbb{R}^d)$  for some  $\sigma \in (0, 1)$  is an  $\alpha$ -admissible boundary condition if for all  $x$  in  $\partial\Omega$ , and every  $i$  in  $\{1, \dots, d\}$  there holds

$$\begin{aligned} e_i \cdot \phi(x) &= -e_i \cdot \phi(x - 2(x \cdot e_i)e_i), \\ e_i \cdot \phi(x) &= e_i \cdot \phi(x - 2(x \cdot e_j)e_j) \text{ for any } j \neq i, \\ \alpha^{-1} \|\phi\|_{C^{1,\sigma}(\mathbb{R}^d)} e_i \cdot x &\leq e_i \cdot \phi(x) \leq \alpha \|\phi\|_{C^{1,\sigma}(\mathbb{R}^d)} e_i \cdot x. \end{aligned}$$

These assumptions are sufficient to show that the Jacobian determinant changes sign in  $\Omega$ , see Proposition 2.5.

Our result is the following.

**Theorem 1.3.** *Suppose that Assumption 1.1 holds. Then there exists two positive constants  $\tau$  and  $C$  depending on  $\Omega$  and  $Q$  only such that for any  $\alpha > 0$ , any  $k \geq C \cdot \max\{1, \frac{1}{\alpha^d \tau}\}$  and any  $\alpha$ -admissible  $\phi$  in  $C^{1,\sigma}(\mathbb{R}^d)$  the solution of (1) satisfies*

$$B\left(0, \left(\frac{\tau \alpha^d}{C}\right)^{1/\sigma}\right) \cap (\Omega \setminus Q) \subset (\det DU)^{-1}\left(\tau \alpha^d \|\phi\|_{C^{1,\sigma}(\mathbb{R}^d)}^d, \infty\right)$$

and

$$\cup_{i=1}^{2d} B\left(p_i, \left(\frac{\tau \alpha^d}{C}\right)^{1/\sigma}\right) \cap (\Omega \setminus Q) \subset (\det DU)^{-1}\left(-\infty, -\tau \alpha^d \|\phi\|_{C^{1,\sigma}(\mathbb{R}^d)}^d\right),$$

where the points  $p_i \in \Omega \setminus Q$ ,  $i = 1, \dots, 2d$  are all located on different coordinate half-axes.

For a more detailed version of this result see Proposition 4.8. The motivation for this investigation stems from several applications in the field of the so-called hybrid imaging inverse problems. In this context, a non-vanishing Jacobian determinant is key in the assessment of the parameters of the PDE from the knowledge of its solution. Sign changing structures typically renders such approach unsuccessful. Whether the Jacobian determinant vanishes is also an issue in numerical homogenisation: in [8] a Cordes type conditions is imposed, namely

$$\operatorname{esssup}_\Omega \left( d - \frac{(\operatorname{trace} \gamma)^2}{(\operatorname{trace} \gamma^T \gamma)} \right) < C_\Omega$$

for some  $C_\Omega$  small enough, in order to avoid sign changes in the Jacobian determinant. Establishing bounds for  $(\det DU)^{-1}(-\infty, -\tau_0)$  and  $(\det DU)^{-1}(\tau_0, \infty)$  is a first step towards an optimisation scheme aimed at constructing topologically simple structures in which the sign of the Jacobian determinant can be prescribed in most of the domain.

The ambient space dimension plays an important role in this problem. In two dimensions, the Radó–Choquet–Kneser Theorem, and its extension to  $\gamma$ -harmonic maps [3, 2, 4] states that given a bi-continuous map from the boundary  $\partial\Omega$  of a two dimensional domain  $\Omega$  onto the boundary  $\partial\Sigma$  of a convex two dimensional domain  $\Sigma$  generates a harmonic extension of that map (which is defined as the solution to (1)) within the domain which is a diffeomorphism from  $\Omega$  to  $\Sigma$ . In particular, for any  $\gamma \in L^\infty(\Omega)$  bounded below by a positive constant, the logarithm of the Jacobian determinant belongs to the space of functions of bounded mean oscillation

$$\log |\det DU| \in BMO(\Omega),$$

therefore either  $J_+$  or  $J_-$  is null. The situation is completely different in higher dimensions ( $d \geq 3$ ). Counter-examples to the positivity of the Jacobian determinant for harmonic extensions have been known for several decades see [15, 19, 17]. Allowing for two phase materials, one can establish [10] that no choice of boundary data  $\phi \in H^{\frac{1}{2}}(\partial\Omega)^3$  whose harmonic extension satisfies a strong positive determinant constraint could enforce a local positive determinant constraint for all two phase isotropic conductivity matrices. Considering a domain  $\Omega' \Subset \Omega$ ,  $\rho > 0$ ,  $x_0 \in \Omega'$  so that  $B_\rho(x_0) \subset \Omega'$  and the harmonic system

$$\begin{aligned} -\Delta U_\Phi &= 0 \quad \text{in } \Omega, \\ U_\Phi &= \Phi \quad \text{on } \partial\Omega, \end{aligned}$$

and denoting  $A(x_0, \lambda, \rho)$  the set containing all boundary data whose harmonic extension satisfy the strong positive Jacobian determinant bound in the ball of radius  $\rho$  centred at the point  $x_0$

$$A(x_0, \lambda, \rho) := \left\{ \Phi \in H^{\frac{1}{2}}(\partial\Omega)^3 : \det DU_\Phi > \lambda \|\Phi\|_{H^{\frac{1}{2}}(\partial\Omega)}^3 \quad \forall x \in B_\rho(x_0) \right\},$$

the set  $A(x_0, \lambda, \rho)$  is non-empty for  $\lambda$  sufficiently small as the identity map  $I_d$  is in  $A(x_0, \lambda, \rho)$ . Considering a two phase isotropic  $[0, 1]^d$  periodic conductivity  $\gamma$ , which satisfies (3) and some symmetry properties, the following result is established.

**Theorem 1.4** (See [10] and [1]). *Given  $\rho > 0$ ,  $x_0 \in \Omega'$ , such that  $B_\rho(x_0) \subset \Omega'$ , and  $\lambda > 0$ , then there exist  $n > 0$ , depending on  $\rho, \Omega, \Omega'$  and  $\lambda$  only, a universal constant  $\tau > 0$  and two open subsets  $B_+$  and  $B_-$  of  $B_\rho(x_0)$  such that*

$$|B_+| > \tau |B_\rho(x_0)| \quad \text{and} \quad |B_-| > \tau |B_\rho(x_0)|,$$

and for any  $\phi$  in  $A(x_0, \rho, \lambda)$ ,

$$\det(DU_n)(x) < -\tau\lambda \|\phi\|_{H^{\frac{1}{2}}(\partial\Omega)}^d \quad \text{on } B_-,$$

and

$$\det(DU_n)(x) > \tau\lambda \|\phi\|_{H^{\frac{1}{2}}(\partial\Omega)}^d \quad \text{on } B_+,$$

where  $U_n$  is the  $\gamma(n\cdot)$ -harmonic extension of  $\phi$ .

The proof of Theorem 1.4 relied on the key result of the paper in [9] which showed that for a specific inclusion  $Q$  and  $\gamma = (1 + (k - 1)\chi_Q)I_d$  defined for values in the three dimensional unit cube  $Y = [0, 1]^3$ , the periodic corrector matrix  $P = D\zeta$  associated with the two phase homogenization problem which is the solution  $\zeta$  to

$$\begin{aligned} -\operatorname{div}(\gamma D\zeta) &= 0 \quad \text{in } \mathbb{R}^3, \\ \zeta(y) - y &\in H_{\#}^1(Y), \end{aligned}$$

satisfies

$$(5) \quad \det(P)(y) > 2\tau \quad \text{in } Y_+ \quad \text{and} \quad \det(P)(y) < -2\tau \quad \text{in } Y_-$$

for some open neighbourhoods  $Y_+$  and  $Y_-$ . While  $\tau$  is established to be positive, its size is not quantified. Theorem 1.3 provides a lower bound to the measure of  $Y_+$ ,  $Y_-$  and to  $\tau$  in estimates (5) in the case of Dirichlet boundary conditions.

Our paper is devoted to the proof of Theorem 1.3. In Section 2 we show that Assumption 1.1 is sufficient to guarantee that both  $J_+$  and  $J_-$  are non empty provided  $k$  is large enough. We adapt here a method introduced in [9], and show that it is sufficient to focus on the study of the determinant on particular lines. The next two sections are devoted to quantitative estimates. In Section 3 we turn to the case  $k = \infty$ , and establish a counterpart to Theorem 1.3 in that case. In Section 4 we use layer potential estimates to establish our main result by means of an explicit convergence rate of the Jacobian determinant when the contrast  $k$  tends to infinity.

## 2. SYMMETRICAL INCLUSIONS WITH SIGN-CHANGING JACOBIAN DETERMINANTS

The first step, adapted from [9], is to derive a simplified form for the determinant.

**Lemma 2.1.** *Let  $\gamma = (1 + (k - 1)\chi_Q)I_d$  be as above, and suppose Assumption 1.1 holds and that  $\phi$  is  $\alpha$ -admissible. If  $U$  satisfies the Dirichlet problem*

$$(6) \quad \begin{aligned} -\operatorname{div}(\gamma DU) &= 0 \quad \text{in } \Omega, \\ U &= \phi \quad \text{on } \partial\Omega, \end{aligned}$$

then the Jacobian matrix  $DU$  is diagonal along the line  $L := \Omega \cap (\mathbb{R} \times \{0\}^{d-1})$ , and the Jacobian determinant is given by

$$\det DU(s) = \partial_1 U_1(s) \times \cdots \times \partial_d U_d(s) \quad \text{for all } s \in L.$$

*Proof.* We first show that the solution  $U_i$  for  $i = 1, \dots, d$  corresponding to the scalar equation

$$\begin{aligned} -\operatorname{div}(\gamma \nabla U_i) &= 0, \\ U_i &= \phi_i, \end{aligned}$$

is odd with respect to  $x_i$  and even with respect to  $x_j$  for  $i \neq j$ . Letting  $x = (x_i, z)$  where  $z \in \mathbb{R}^{d-1}$  denotes the other remaining  $d - 1$  variables, we denote  $V_i(x) = -U_i(-x_i, z)$ .

We write  $H_{\phi_i}^1(\Omega)$  as the affine space  $\phi_i + H_0^1(\Omega)$ . Notice that from the symmetry of  $\Omega$  and  $\phi$ ,  $V_i \in H_{\phi_i}^1(\Omega)$ . Moreover,

$$\begin{aligned} \partial_i V_i(x_i, z) &= \partial_i U_i(-x_i, z), \\ \partial_j V_i(x_i, z) &= -\partial_j U_i(-x_i, z) \quad \text{for } i \neq j, \end{aligned}$$

Using the symmetry of  $\gamma$  and denoting  $T_i : \Omega \mapsto \Omega$  the change of variables  $T_i(x_i, z) = (-x_i, z)$ , a straightforward computation (and a change of variable) yields

$$\begin{aligned}
\int_{\Omega} \gamma DV_i \cdot DV_i \, dx &= \int_{\Omega} \gamma DU_i(-x_i, z) \cdot DU_i(-x_i, z) \, dx_i \, dz \\
&= \int_{T_i(\Omega)} \gamma DU_i(\tau, z) \cdot DU_i(\tau, z) |\det T_i| \, d\tau \, dz \\
(7) \qquad \qquad \qquad &= \int_{\Omega} \gamma DU_i \cdot DU_i \, dx.
\end{aligned}$$

Considering the variational formulation for  $U_i$ , we note that  $U_i$  is the unique minimizer of the Dirichlet energy, namely  $U_i$  satisfies

$$\int_{\Omega} \gamma DU_i \cdot DU_i \, dx = \min_{V \in H_{\phi_i}^1(\Omega)} \int_{\Omega} \gamma DV \cdot DV \, dx.$$

Thus, by the uniqueness of minimisers, we deduce that  $U(x_i, z) = V_i(x_i, z) = -U_i(-x_i, z)$ . The evenness with respect to  $x_j$  is established by similar arguments. In particular, we may apply the same argument to the function  $W_i(x) = U_i(x_i, -z)$  to show that the function  $U_i$  is indeed even with respect to  $x_j \, \forall i \neq j$ . Since  $U_i(x_i, z) = U_i(x_i, -z)$  for any  $(x_i, z) \in \Omega$ , a straightforward computation shows that for  $i \neq j$ ,

$$\partial_i U_j(x_j, z) = -\partial_i U_j(x_j, -z)$$

and for  $i = j$ ,

$$\partial_i U_i(x_i, z) = \partial_i U_i(-x_i, z)$$

altogether, this implies that for  $y = (x_1, -x_2, \dots, -x_d)$ ,

$$DU(y) = \begin{bmatrix} \partial_1 U_1 & -\partial_1 U_2 & \cdots & -\partial_1 U_d \\ -\partial_2 U_1 & \partial_2 U_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -\partial_{d-1} U_d \\ -\partial_d U_1 & \cdots & -\partial_d U_2 & \partial_d U_d \end{bmatrix} (x).$$

In particular, for  $s = (x_1, 0)$ ,

$$DU(s) = \begin{bmatrix} \partial_1 U_1 & 0 & \cdots & 0 \\ 0 & \partial_2 U_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \partial_d U_d \end{bmatrix} (s) \quad \forall s \in L.$$

Subsequently,

$$(8) \qquad \det DU(s) = \partial_1 U_1 \times \cdots \times \partial_d U_d(s).$$

□

**Definition 2.2.** Given  $k > 0$ , we denote by  $U_k \in H^1(\Omega; \mathbb{R}^d)$  the solution to the Dirichlet problem

$$\begin{aligned}
(9) \qquad -\operatorname{div}((1 + (k-1)\chi_Q)DU_k) &= 0 \quad \text{in } \Omega, \\
U_k &= \phi \quad \text{on } \partial\Omega.
\end{aligned}$$

We referred to this solution as  $U$  previously; the  $k$  dependence is underlined in order to compare  $U_k$  to its perfectly conducting limit. We write

$$(10) \quad J_k = \det DU_k$$

and

$$(11) \quad J_\infty = \det DU_\infty$$

where  $U_\infty$  is defined in (12).

Let us first establish the limit problem for (1) as  $k \rightarrow \infty$ . We provide its proof for the reader's convenience. A similar proof can be found in [7]. This can also be deduced from the layer potential approach followed in Section 4.

**Lemma 2.3.** *Suppose Assumption 1.1 holds and that  $\phi$  is  $\alpha$ -admissible. Let  $U_k$  denote the solution to (9) with conductivity  $\gamma_k = (1 + (k-1)\chi_Q)I_d$ . Then  $U_k$  converges strongly in  $H^1(\Omega; \mathbb{R}^d)$  to  $U_\infty$  as  $k \rightarrow \infty$  where  $U_\infty$  satisfies*

$$(12) \quad \begin{aligned} -\Delta U_\infty &= 0 \text{ in } \Omega \setminus Q, \\ U_\infty &= \phi \text{ on } \partial\Omega, \\ U_\infty &= \text{constant in } Q. \end{aligned}$$

Furthermore,  $U_\infty = 0$  on  $Q$ .

*Proof.* Consider the variational formulation for  $U_k$  given by

$$\int_{\Omega} \gamma_k DU_k : DU_k \, dx = \min_{v \in H_{\phi}^1(\Omega; \mathbb{R}^d)} \int_{\Omega} \gamma_k DV : DV \, dx,$$

with  $\gamma_k = (1 + (k-1)\chi_Q)I_d$ . Multiplying by a cut-off function, we may construct a function  $f \in H_{\phi}^1(\Omega; \mathbb{R}^d)$  such that  $f$  is a constant vector on  $Q$  and

$$\begin{aligned} \int_{\Omega} \gamma_k DU_k : DU_k \, dx &= k \int_Q DU_k : DU_k \, dx + \int_{\Omega \setminus Q} DU_k : DU_k \, dx \\ &\leq \int_{\Omega \setminus Q} Df : Df \, dx. \end{aligned}$$

This implies that

$$\|U_k\|_{H^1(\Omega; \mathbb{R}^d)} \leq M$$

for some positive constant  $M$ . Thus by taking a subsequence or otherwise,  $U_k \rightharpoonup U_\infty$  for some function  $U_\infty$  in  $H^1$ . By the uniform  $H^1$  bound of  $U$ , we may infer that

$$\|DU_k\|_{L^2(Q)} \rightarrow 0 \text{ in } Q$$

which implies that the limiting function  $U_\infty$  is a constant vector in  $Q$ . Now, consider the functional

$$I : \mathcal{M} \rightarrow \mathbb{R}$$

defined as

$$I(U) := \int_{\Omega \setminus Q} DU : DU \, dx$$

with arguments taken from the constraint set

$$\mathcal{M} := \{U \in H_{\phi}^1(\Omega; \mathbb{R}^d) : U = \text{constant on } Q\}.$$

Since  $\mathcal{M}$  is a closed subspace of  $H_\phi^1(\Omega; \mathbb{R}^d)$ , the functional  $I$  is convex continuous, and therefore it admits a minimizer in the space  $\mathcal{M}$ . We have

$$\min_{V \in \mathcal{M}} \int_{\Omega} DV \cdot DV \, dx \leq \int_{\Omega \setminus Q} DU_\infty : DU_\infty \, dx,$$

and, on the other hand,

$$(13) \quad \int_{\Omega} \gamma_k DU_k : DU_k \, dx \leq \inf_{V \in \mathcal{M}} \int_{\Omega} DV : DV \, dx$$

as  $\mathcal{M} \subset H_\phi^1(\Omega; \mathbb{R}^d)$ . From the weakly lower semi-continuity of norms, we deduce

$$(14) \quad \begin{aligned} \int_{\Omega} DU_\infty : DU_\infty \, dx &\leq \liminf_{k \rightarrow \infty} \int_{\Omega} DU_k : DU_k \, dx \\ &\leq \liminf_{k \rightarrow \infty} k \int_Q DU_k : DU_k \, dx + \int_{\Omega \setminus Q} DU_k : DU_k \, dx \\ &\leq \inf_{V \in \mathcal{M}} \int_{\Omega} DV : DV \, dx, \end{aligned}$$

hence, the weak limit  $U_\infty$  minimizes the functional  $I$  in  $\mathcal{M}$ . Finally, by (13), we have

$$\limsup_{k \rightarrow \infty} \left( (k-1) \int_Q DU_k : DU_k \, dx + \int_{\Omega \setminus Q} DU_k : DU_k \, dx \right) \leq \int_{\Omega} DU_\infty : DU_\infty \, dx.$$

Combining this fact with (14), the sequence of energies  $I_k(U_k)$  converge to  $I(U_\infty)$ , hence by the uniform convexity of  $L^2$  norms, the sequence  $DU_k$  converges strongly in  $L^2(\Omega)$  to the limit  $DU_\infty$  where  $U_\infty$  satisfies the equation (12) as claimed.

Finally, for any  $i \in \{1, \dots, d\}$ , since  $(U_k)_i$  is odd with respect to the variable  $x_i$ , as shown in Lemma 2.1, and  $U_k \rightarrow U_\infty$  in  $H^1(\Omega)$ , the limit  $(U_\infty)_i$  is also odd in  $x_i$ . Thus  $(U_\infty)_i(\{x_i = 0\} \cap \Omega) = 0$ . Since  $Q$  is a connected component of  $\Omega$ , we infer that  $(U_\infty)_i$  must be identically 0 on the whole of  $Q$ .  $\square$

We now proceed to show that Assumption 1.1 and  $\alpha$ -admissible boundary conditions (1.2) are sufficient to obtain sign changing structures.

**Proposition 2.4.** *Under the assumptions of Lemma 2.3, there holds*

$$(15) \quad |J_\infty^{-1}(0, \infty)| \times |J_\infty^{-1}(-\infty, 0)| > 0.$$

*Proof.* In this proof, we write  $U_\infty = (U_i)_{1 \leq i \leq d}$ . Thanks to Lemma 2.1, and because of the continuity of  $DU$ , it is sufficient to focus on the line  $\mathbb{R} \times \{0\}^{d-1}$ . Given  $i = 1, \dots, d$ , consider the connected region  $D_i := \{x \in \Omega \setminus Q : x_i > 0\} \subset \Omega \setminus Q$  in the upper half space  $x_i \geq 0$ . From (12), we have that  $U_i$  is harmonic in  $D_i$ ,  $U_i|_{\partial D_i} \geq 0$  and  $U_i \geq \alpha^{-1}(\phi)x_i$  on  $\partial\Omega \cap D_i$ . Thus, thanks to weak maximum principle, this implies that  $U_i > 0$  in the interior of  $D_i$ .

Observe that since  $U_1$  is odd with respect to  $x_1$  and even with respect to the other variables, we have  $U_1(0) = 0$ . By assumption,  $B(0, \epsilon) \cap Q = \emptyset$  and  $\mathbb{R}e_1 \cap Q \neq \emptyset$ , therefore there exists  $s_1 > 0$  such that  $x^* = s_1 e_1 \in (\mathbb{R}^+ \times \{0\}^{d-1}) \cap \partial Q$ , and  $U_1(se_1) > 0$  for any  $0 < s < s_1$ . In particular, we have  $F : s \rightarrow U_1(se_1)$  satisfies

$$F(0) = F(s_1) = 0 \text{ and } F(s) > 0 \text{ on } (0, s_1).$$



Since  $Q$  is a  $C^2$  inclusion, elliptic regularity asserts that the solution  $U$  is smooth in the interior of  $\Omega \setminus Q$  and is continuous up to the boundary of  $Q$ , hence, we infer that  $F'(s) = \frac{\partial U_1}{\partial x_1}$  changes sign for some  $s \in (0, s_1)$ .

Next, we show that for  $i \neq 1$ ,  $\frac{\partial U_i}{\partial x_i}$  preserves a constant sign along the line  $\ell = (0, s_1) \times \{0\}^{d-1}$ . To see this, we note that  $\ell \subset (\Omega \setminus Q)$ . For any point  $x \in \ell$ , we may find an open ball centred at  $x$  with radius  $r_x$  so that  $B(x, r_x) \subset \Omega \setminus Q$ . In particular,  $B_{i,x} = B(x + \frac{1}{2}r_x e_i, \frac{1}{2}r_x) \subset D_i$  and

$$\overline{B_{i,x}} \cap \mathbb{R}e_1 = x.$$

Since  $U_i$  is strictly positive in the open ball  $B_{i,x}$ , Hopf's lemma ensures that

$$\frac{\partial U_i}{\partial x_i}(x) > 0.$$

Since  $x$  is chosen arbitrary on the line  $\ell$ , together with the continuity of  $DU$ , we can conclude

$$|\{x \in \Omega \setminus Q : J_\infty > 0\}| \times |\{x \in \Omega \setminus Q : J_\infty < 0\}| > 0.$$

□

**Proposition 2.5.** *Suppose that assumption 1.1 holds and that  $\phi$  is  $\alpha$ -admissible. There exists  $k_0 > 0$  and  $\delta_0 > 0$  such that for  $k \geq k_0$ , the Jacobian determinant  $J_k$  given by (10) satisfies*

$$|\{x \in \Omega \setminus Q : J_k > 0\}| \times |\{x \in \Omega \setminus Q : J_k < 0\}| \geq \delta_0.$$

*Proof.* Observing that the solution  $U_k$  is harmonic and hence smooth away from the inclusion  $Q$ , we may infer from standard regularity estimates that in any compact subset  $K \subset \Omega \setminus Q$ , the sequence of functions  $\{U_k\}_{k \geq k_0}$  is uniformly bounded in  $C^2$  norm. By Ascoli–Arzela's theorem, and the limiting argument of Lemma 2.3,  $U_k$  converges to  $U_\infty$  in  $C^1$  norm as  $k \rightarrow \infty$  in any fixed set  $K$ . Thus, in some compact subset  $K_+ \subset \{x \in \Omega \setminus Q : J_\infty > 0\}$  and  $K_- \subset \{x \in \Omega \setminus Q : J_\infty < 0\}$ , we have

$$\det DU_k \rightarrow \det DU_\infty$$

converging uniformly as  $k \rightarrow \infty$  in  $K_+ \cup K_-$ . Hence, there exist  $k_0(Q) > 1$ , and  $\delta_0 > 0$ , such that for all  $k \geq k_0$

$$|\{x \in \Omega \setminus Q : J_k > 0\}| \times |\{x \in \Omega \setminus Q : J_k < 0\}| > \delta_0,$$

which is our thesis. □

### 3. VOLUME ESTIMATES FOR THE SIGN CHANGING SET OF $\det DU_\infty$

In this section, we present a method which enables us to derive quantitative bounds for the Jacobian determinant of  $DU_\infty$  in the perfectly conducting case. We will make use of the result of Lemma 2.1, namely that the Jacobian determinant is diagonal along the line

$$(16) \quad L := (\Omega \setminus Q) \cap (\mathbb{R}e_1)$$

as

$$\det DU_\infty(s) = \partial_1(U_\infty)_1(s) \times \cdots \times \partial_d(U_\infty)_d(s) \text{ for all } s \in L.$$

Obtaining a pointwise bound for  $\det DU$  along this line corresponds to estimating each of the partial derivatives  $\partial_i U_i$  along the line  $L$ . There are several methods available to do so. One approach is to use Harnack's inequality which states that if

$H$  is a positive harmonic function in some ball  $B(x_0, R)$ , then it follows from the Poisson formula that

$$\frac{1 - R^{-1}|x|}{(1 + R^{-1}|x|)^{d-1}} H(x_0) \leq H(x) \text{ for all } x \in B(x_0, R).$$

In particular, if  $H$  vanishes at some point  $y_0 \in \partial B(x_0, R)$ , this implies the quantitative Hopf's inequality

$$(17) \quad \frac{\partial H}{\partial \nu}(y_0) \leq -\frac{1}{2^{d-1}R} H(x_0).$$

To obtain a quantitative result, we will use that the domain is (for lack of a better word) tube connected, which roughly means that  $\Omega \setminus Q$  is path connected with 'thick' paths.

**Definition 3.1.** Given a triplet  $(x, S, v)$  with  $x \in \mathbb{R}^d$ ,  $S$  a non-empty bounded open set of a hyperplane of  $\mathbb{R}^d$  containing the origin, and  $v$  a vector normal to  $S$ , we denote  $C(x, S, v)$  the cylinder given by

$$C(x, S, v) := \{x + y + tv \text{ with } y \in S \text{ and } t \in (0, 1)\}.$$

We name its boundaries

$$\begin{aligned} \Sigma_0(x, S) &:= \{x + y \text{ with } y \in S\}, \\ \Sigma_1(x, S, v) &:= \{x + y + v \text{ with } y \in S\}, \\ \Sigma_2(x, S, v) &:= \{x + y + tv \text{ with } y \in \partial S \text{ and } t \in (0, 1)\}. \end{aligned}$$

**Definition 3.2.** The set  $\Omega \setminus Q$  is tube connected to the exterior boundary  $\partial\Omega$  if for all  $a$  in  $(\Omega \setminus Q) \cap [0, \infty)^d$ , and for each direction  $e_i$  with  $i$  in  $\{1, \dots, d\}$ , there exist  $N \in \mathbb{N}$ , and  $N$  cylinder  $C(x_m^i, S_m^i, v_m^i)$  as given in Definition 3.1 such that

- The last cylinder  $C(x_N^i, S_N^i, v_N^i)$  contains  $a$ .
- The first cylinder cuts the exterior boundary:  $\Sigma_0(x_1^i, S_1^i) \subset \mathbb{R}^d \setminus \bar{\Omega}$ , and  $\partial\Omega \cap C(x_1^i, S_1^i, v_1^i) \subset \{x \cdot e_i > \max(Q \cap (\mathbb{R}_+ e_i))\}$ .
- The intermediate cylinders  $\cup_{m=2}^{N-1} C(x_m^i, S_m^i, v_m^i)$  are in  $(\Omega \setminus Q) \cap (0, \infty)^d$ .
- The start of the next cylinder is within the previous one. For each  $2 \leq m \leq N$  there holds  $\Sigma_0(x_m^i, S_m^i) \subset C(x_{m-1}^i, S_{m-1}^i, v_{m-1}^i)$ .
- If  $a \cdot e_i > 0$ ,  $C(x_m^N, S_m^N, v_m^N) \subset (\Omega \setminus Q) \cap (0, \infty)^d$ .

*Remark 3.3.* For explicit constructions, one can consider cylinders whose cross sections are rectangles or discs, see Appendix B.

**Proposition 3.4.** *If  $\Omega$  and  $Q$  satisfy assumption (1.1), then  $\Omega \setminus Q$  is tube connected.*

*Proof.* Consider first the case when  $a \in (\Omega \setminus Q) \cap (0, \infty)^d$ . Pick  $b \in \Omega \cap (0, \infty)^d$  such that  $b \cdot e_i = \frac{1}{2} \max(\Omega \cap (\mathbb{R}_+ e_i)) + \frac{1}{2} \max(Q \cap (\mathbb{R}_+ e_i))$ . Because  $(\Omega \setminus Q) \cap (0, \infty)^d \subset \mathbb{R}^d$  is connected, there exists a finite chain of open balls  $B_2, \dots, B_M$  in  $(\Omega \setminus Q) \cap (0, \infty)^d$  such that  $B_i \cap B_{i+1} \neq \emptyset$  for all  $i \in 1, \dots, M-1$ , with  $a \in B_M$  and  $b \in B_2$ . Choose  $S_1^i = B_2 \cap \{x \cdot e_i = b \cdot e_i\} - b$ . Then  $C(S_1^i, b, 2|b|e_i)$ , satisfies the initial constraint. The construction is straightforward afterwards. To address also the case when  $a \cdot e_i = 0$  for some  $i$ , apply the above construction with  $\tilde{a}$  given by

$$\tilde{a} = a + \sum_{\{j: a \cdot e_j = 0\}} \frac{1}{2} \min\left(d\left(a, \partial Q \cap [0, \infty)^d\right), d\left(a, \partial\Omega \cap [0, \infty)^d\right)\right) e_j,$$

and add a final cylinder to connect to  $a$ .  $\square$

**Lemma 3.5.** *Suppose Assumptions 1.1 holds and that  $\phi$  is  $\alpha$ -admissible are satisfied. Then, there exists  $p_1 \in L$ , where  $L$  is the set given by (16), and two constant  $M_1$  and  $M_0$  such that*

$$J_\infty(p_1) < -\alpha^d \|\phi\|_{C^{1,\sigma}(\mathbb{R}^d)}^d M_1, \text{ and } J_\infty(0) > \alpha^d \|\phi\|_{C^{1,\sigma}(\mathbb{R}^d)}^d M_0.$$

Lower bounds for  $M_0$  and  $M_1$  can be computed depending on geometric invariants of the surface  $\partial Q$ , and depending on the tubes connecting  $p_1$ , another point ( $p_0 \in L$ ) and the origin, to the exterior boundary of  $\Omega$ . Possible locations for  $p_1, p_0 \in L$  can be determined based on exterior ball condition satisfied at  $y_0$ , the point of contact between  $L$  and  $Q$  closest to the origin.

The following quantitative lower bound for the measure of the domains where Jacobian determinant is positive or negative follows.

**Corollary 3.6.** *Under the assumptions of Theorem 1.3, there holds*

$$B\left(p_1, \left(\frac{M_1}{2dC(\Omega, Q)^d} \alpha^d\right)^{1/\sigma}\right) \cap (\Omega \setminus Q) \subset J_\infty^{-1}\left(\left(-\infty, -\frac{1}{2}\alpha^d \|\phi\|_{C^{1,\sigma}(\mathbb{R}^d)}^d M_1\right)\right),$$

and

$$B\left(0, \left(\frac{M_0}{2dC(\Omega, Q)^d} \alpha^d\right)^{1/\sigma}\right) \cap (\Omega \setminus Q) \subset J_\infty^{-1}\left(\left(\frac{1}{2}\alpha^d \|\phi\|_{C^{1,\sigma}(\mathbb{R}^d)}^d M_0, \infty\right)\right).$$

*Proof.* Thanks to a variant of Hadamard's inequality [12] there holds for any  $x, y \in \Omega$ ,

$$\begin{aligned} (18) \quad & |\det DU_\infty(x) - \det DU_\infty(y)| \\ & \leq d |(DU_\infty(x) - DU_\infty(y))| \max(|DU_\infty(x)|, |DU_\infty(y)|)^{d-1} \\ & \leq d |x - y|^\sigma \|U_\infty\|_{C^{1,\sigma}(\Omega \setminus Q)}^d \|U_\infty\|_{C^1(\overline{\Omega \setminus Q})}^{d-1}. \end{aligned}$$

Furthermore, thanks to Lemma 3.8,

$$\|U_\infty\|_{C^1(\overline{\Omega \setminus Q})} \leq \|U_\infty\|_{C^{1,\sigma}(\overline{\Omega \setminus Q})} \leq C(\Omega \setminus Q) \|\phi\|_{C^{1,\sigma}(\mathbb{R}^d)},$$

where  $C(\Omega \setminus Q)$  depends on  $\Omega \setminus Q$  only, and the result follows.  $\square$

*Proof of Lemma 3.5.* In this proof, we write  $U$  for  $U_\infty$ .

First step: construction of the point  $p_1$ . Since  $\partial Q$  is  $C^2$ , it satisfies the exterior ball condition, and we may place a ball  $B(p_0, r_0)$  centred at some (intermediate) point  $p_0 \in L$ , tangential to the boundary of the inclusion  $\partial Q$  at the point of contact  $y_0 \in \partial L$ . Since  $\Omega \setminus Q$  is tube connected, there exists  $N$  tubes  $(C(x_m^1, S_m^1, v_m^1))_{1 \leq i \leq N}$  connecting  $p_0$  to the exterior boundary. Consider the following Dirichlet problems

$$\begin{aligned} -\Delta V_m^1 &= 0 \quad \text{in } C(x_m^1, S_m^1, v_m^1) \\ V_m^1 &= 0 \quad \text{on } \Sigma_1(x_m^1, S_m^1, v_m^1) \cup \Sigma_2(x_m^1, S_m^1, v_m^1), \\ V_m^1 &= 1 \quad \text{on } \Sigma_0(x_m^1, S_m^1). \end{aligned}$$

From Definition 3.2, on  $\partial\Omega \cap C(x_1^1, S_1^1, v_1^1)$  there holds

$$U_1 = \phi \cdot e_1 \geq \alpha(\phi) x \cdot e_1 > c_1,$$

with  $c_1 = \alpha(\phi) \max(Q \cap (\mathbb{R}_+ e_1))$ . By the maximum principle, on  $\partial\Omega \cap C(x_1^1, S_1^1, v_1^1)$

$$c_1 V_1^1 < c_1 < U_1.$$

Since  $U_1 > 0$  ( $\Sigma_1 \cup \Sigma_2$ )  $\cap \Omega$ , we deduce that  $U_1 > c_1 V_1^1$  on  $\Omega \cap C(x_m^1, S_m^1, v_m^1)$ . In particular,  $U_1 > c_1 V_1^1$  on  $\Sigma_0(x_2^1, S_2^1)$ , and in turn, by the maximum principle  $U_1 > c_1 \min_{\Sigma_0(x_2^1, S_2^1)} V_2^1$  on  $C(x_2^1, S_2^1, v_2^1)$ . Iterating this argument, we obtain

$$U_1 > \left( \alpha(\phi) \max(Q \cap (\mathbb{R}_+ e_1)) \prod_{m=1}^{N-1} \min_{\Sigma_0(x_{m+1}^1, S_{m+1}^1)} V_m^1 \right) V_N^1 \text{ on } C_N^1,$$

and in turn

$$U_1 \geq \alpha(\phi) M_2^1 > 0 \text{ on } B(p_0, r_0) \cap L,$$

with

$$(19) \quad M_2^1 = \max(Q \cap (\mathbb{R}_+ e_1)) \left( \prod_{m=1}^{N-1} \min_{\Sigma_0(x_{m+1}^1, S_{m+1}^1)} V_m^1 \right) \min_{L \cap C_N^1(x_1^1, S_{m+1}^1, v_N^1)} V_N^1.$$

Thanks to the quantitative variant of Hopf's Lemma, we may apply Lemma 17 for  $U_1$  in the ball  $B(p_0, r_0)$  and at the point  $y_0$  to obtain

$$\begin{aligned} \frac{\partial U_1}{\partial \nu}(y_0) &= \frac{\partial U_1}{\partial x_1}(y_0) \leq -\frac{1}{2^{d-1} r_0} U_1(p_0) \\ &\leq -\frac{1}{2^{d-1} r_0} \alpha(\phi) M_2^1. \end{aligned}$$

Since  $U_1 \in C^{1,\sigma}(\overline{\Omega \setminus Q})$ , see Lemma 3.8 below, we can choose  $p_1$  between  $p_0$  and  $y_0$  such that

$$\frac{\partial U_1}{\partial x_1}(p_1) \leq -\frac{1}{2^d r_0} \alpha(\phi) M_2^1.$$

Indeed, for every  $\theta \in (0, 1)$

$$\left| \frac{\partial U_1}{\partial \nu}(\theta y_0 + (1-\theta)p_0) - \frac{\partial U_1}{\partial x_1}(y_0) \right| \leq (1-\theta)^\sigma |p_0 - y_0|^\sigma \|U\|_{C^{1,\alpha}(\overline{\Omega \setminus Q})}$$

Second step: lower bound for  $\partial_i U_i(p_1)$  for  $i = 2, \dots, d$ . At the point  $p_1$ , for each  $i = 2, \dots, d$  let  $(C(x_m^i, S_m^i, v_m^i))_{1 \leq m \leq N}$  be the tubes connecting  $p_1$  to the exterior boundary of  $\Omega$  (for a possibly different  $N$  from the one above). We note that due to the evenness of  $\chi_Q$  in the direction  $e_i$ ,  $U_i = 0$  on  $L$ . Consider the following Dirichlet problems for  $m = 1, \dots, N-1$ ,

$$\begin{aligned} -\Delta V_m^i &= 0 \quad \text{in } C(x_m^i, S_m^i, v_m^i) \\ V_m^i &= 0 \quad \text{on } \Sigma_1(x_m^i, S_m^i, v_m^i) \cup \Sigma_2(x_m^i, S_m^i, v_m^i), \\ V_m^i &= 1 \quad \text{on } \Sigma_0(x_m^i, S_m^i), \end{aligned}$$

and, for  $m = N$ ,

$$\begin{aligned} -\Delta V_m^i &= 0 \quad \text{in } C(x_N^i, S_N^i, v_N^i) \cap (0, \infty)^d \\ V_m^i &= 0 \quad \text{on } \partial \left( C(x_N^i, S_N^i, v_N^i) \cap (0, \infty)^d \right) \setminus \Sigma_0(x_N^i, S_N^i), \\ V_m^i &= 1 \quad \text{on } \Sigma_0(x_N^i, S_N^i). \end{aligned}$$

Introducing

$$(20) \quad M_2^i = \max(Q \cap (\mathbb{R}_+ e_i)) \prod_{m=1}^{N-1} \min_{\Sigma_0(x_{m+1}^i, S_{m+1}^i)} V_m^i,$$

and arguing as above we have  $U_i \geq \alpha(\phi) M_2^i V_N^i$  in  $C(x_N^i, S_N^i, v_N^i) \cap (0, \infty)^d$ . In particular, since  $V_N^i = U_i = 0$  on  $L \cap C_N^i$ , we have

$$\partial_i U_i(p_1) \geq \alpha(\phi) M_2^i \partial_i V_N^i(p_1) > 0.$$

Third step: upper bound for  $J_\infty(p_1)$  and lower bound for  $J_\infty(0)$ . In the first two steps, we have obtained that

$$\partial_1 U_1(p_1) \dots \partial_d U_d(p_1) \leq -\frac{\alpha(\phi)^d}{2^d r_0} \Pi_{i=1}^d M_2^i \Pi_{j=2}^d \partial_j V_N^j(p_1),$$

which is the announced upper bound. By the same argument as in the second step, applied at the origin, we have

$$(21) \quad \Pi_{i=2}^d \partial_i U_i(0) \geq \alpha(\phi)^{d-1} \Pi_{i=2}^d \tilde{M}_2^i \partial_i V_N^i(0) > 0.$$

Turning to  $\partial_1 U_1(0)$ , we notice that because of the symmetry of  $\chi_Q$ ,  $U_1(0) = 0$ . We thus repeat once again the second step method, to obtain

$$(22) \quad \partial_1 U_1(0) \geq \alpha(\phi) \tilde{M}_2^1 \partial_1 V_N^1(0) > 0,$$

and the conclusion follows from (21) and (22).  $\square$

*Remark 3.7.* Using explicitly cylinders to connect the points is certainly not necessary. An alternative construction would be to make use auxiliary problems defined in  $\Omega \setminus Q \cap (0, \infty)^d$ . The motivation for this detailed approach is that it allows explicit bounds (as closed form solutions can be written down) in practical cases, see Proposition 3.10.

**Lemma 3.8.** *Let  $Q \Subset \Omega$  be a bounded, connected  $C^2$  subset so that  $\Omega \setminus Q$  remains connected. Given  $\phi$  in  $C^{1,\sigma}(\mathbb{R}^d)$ , the weak solution  $U \in H^1(\Omega)$  to the problem*

$$(23) \quad \begin{aligned} -\Delta U &= 0 & \text{in } \Omega \setminus Q, \\ U &= 0 & \text{on } \partial Q, \\ U &= \phi & \text{on } \partial \Omega, \end{aligned}$$

*satisfies*

$$(24) \quad \|U\|_{C^{1,\sigma}(\overline{\Omega \setminus Q})} \leq C(\Omega \setminus Q, \sigma) \|\phi\|_{C^{1,\sigma}(\mathbb{R}^d)},$$

where the constant  $C(\Omega \setminus Q, \sigma)$  depends on  $\Omega \setminus Q$  and  $\sigma$  only.

*Proof.* See [18], and [13] for the case when  $\Omega$  is a convex polygon.  $\square$

**Example 3.9.** Consider the case  $d = 3$ , Suppose that  $Q$  is a torus centred at the origin with minor radii  $a$ , and major radii  $\ell + a$ , with  $0 < \ell < 1 - 2a$ . We exhibit below  $p_1$ , and provide lower bounds for  $M_0$  and  $M_1$  as defined Lemma 3.5.

**Proposition 3.10.** *Suppose that  $Q$  is a proper torus*

$$x_3^2 = a^2 - \left( \ell + a - \sqrt{x_1^2 + x_2^2} \right)^2,$$

*centred at the origin with minor radii  $a$ , and major radii  $\ell + a$ , with  $0 < \ell < 1 - 2a$ .*

Then we may chose  $y_0 = (\ell, 0, 0)$ , and  $p_1 = (\ell(1 - \kappa), 0, 0)$ , where  $\kappa$  is given by

$$\kappa = \min \left( C_1 / \left( \ell \|DU\|_{C^{0,1}(\overline{\Omega \setminus Q})} \right), \frac{1}{2} \right)$$

with

$$C_1 = \frac{2 - \sqrt{2}}{3\pi^2} \left( \frac{1}{\cosh\left(\frac{2\pi}{\ell}\right)} + \frac{1}{3} \frac{1}{\cosh\left(\frac{6\pi}{\ell}\right)} \right).$$

Furthermore, the constants  $M_0$  and  $M_1$  in Lemma 3.5 satisfy

$$M_1 \geq \frac{\sqrt{3}}{200} \frac{1}{\kappa \ell} \exp \left( -\frac{2\pi}{\ell} \left( 1 + \frac{3}{\kappa} \right) \right) \text{ and } M_0 \geq \frac{1}{400} \frac{1}{\ell} \exp \left( -\frac{5\pi}{\ell} \right).$$

*Proof.* The straightforward but tedious computation is provided in Appendix B.  $\square$

#### 4. CONTINUOUS DEPENDENCE OF THE GRADIENT FIELD IN THE CONDUCTIVITY CONTRAST $k$

In this section, we investigate the continuous dependence of  $DU_k$ , where  $U_k$  is the solution of (9), when  $k$  varies. Several authors have investigated this question including cite [16, 5, 14, 7]. We follow here an integral equation approach. We show  $U_k$  admits a representation formula similar to that in [6], and we use the classical theory of weakly singular boundary integral operators to show the continuous dependence of  $DU_k$  with respect to  $k$  as  $k \rightarrow \infty$ .

In this section we prove the following result.

**Proposition 4.1.** *Under the assumptions of Theorem 1.3, assume that  $U_k \in H^1(\Omega; \mathbb{R}^d)$  solves the conductivity equation (9). Then  $U_k$  converges to  $U_\infty$  in  $C^{1,\sigma}(\overline{\Omega \setminus Q})$ . Furthermore there holds*

$$\|U_k - U_\infty\|_{C^{1,\sigma}(\overline{\Omega \setminus Q})} \leq K_{SL} \frac{1}{k-1} \|\phi\|_{C^{1,\sigma}(\mathbb{R}^d)},$$

where  $K_{SL}$  is a constant independent of the conductivity parameter  $k$  and the boundary value  $\phi$ .

We remind the readers of classical definitions of layer potential operators. Given  $\psi \in C^{0,\sigma}(\partial Q)$ , the single layer potential  $S_Q(\psi)$  and double layer potential  $D_Q(\psi)$  are defined as

$$\begin{aligned} S_Q(\psi) &:= \frac{1}{(2-d)\omega_d} \int_{\partial Q} \frac{1}{|x-y|^{d-2}} \psi(y) \, ds(y) \quad x \in \mathbb{R}^d \setminus \partial Q \\ D_Q(\psi) &:= \frac{1}{\omega_d} \int_{\partial Q} \frac{\langle x-y, \nu(y) \rangle}{|x-y|^d} \psi(y) \, ds(y) \quad x \in \mathbb{R}^d \setminus \partial Q \end{aligned}$$

where  $\omega_d$  denotes the  $d-1$  dimensional surface area of the unit ball.

For a function  $u$  defined on  $\mathbb{R}^d \setminus \partial Q$ , we denote:

$$\begin{aligned} u|_{\pm}(x) &:= \lim_{t \rightarrow 0^+} u(x \pm t\nu_x), \quad x \in \partial Q \\ \frac{\partial}{\partial \nu_x} u|_{\pm}(x) &:= \lim_{t \rightarrow 0^+} \langle \nabla u(x \pm t\nu_x), \nu_x \rangle, \quad x \in \partial Q \end{aligned}$$

where  $\nu_x$  is the outward unit normal to  $\partial Q$  at the point  $x$ . Note that by assumption  $Q$  is orientable and the outer and inner limits  $\frac{\partial}{\partial \nu_x} u|_{\pm}(x)$  are well defined.

The single layer and double layer potential also satisfy the well known jump formulas given by

$$(25) \quad \begin{aligned} [S_Q(\psi)|_+](x) &= [S_Q\psi|_-](x) \quad a.e. x \in \partial Q \\ \left[ (I_d - n \otimes n) \nabla S_Q(\psi)|_+ \right](x) &= \left[ (I_d - n \otimes n) \nabla S_Q(\psi)|_- \right](x) \quad a.e. x \in \partial Q \\ \left[ \nabla S_Q(\psi) \cdot n|_{\pm} \right](x) &= \left[ \left( \pm \frac{1}{2}I + K_Q^* \right) \psi \right](x) \quad a.e. x \in \partial Q \\ \left[ D_Q(\psi)|_{\pm} \right](x) &= \left[ \left( \mp \frac{1}{2}I + K_Q \right) \psi \right](x) \quad a.e. x \in \partial Q \end{aligned}$$

where

$$(26) \quad K_Q^*(\psi) := \frac{1}{\omega_d} \int_{\partial Q} \frac{\langle x-y, \nu_x \rangle}{|x-y|^d} \psi(y) \, ds(y) \quad x \in \partial Q$$

denotes the Poincaré-Neumann operator and  $K_Q(\psi)$  is the  $L^2$ -adjoint of  $K_Q^*$  given by

$$K_Q(\psi) := \frac{1}{\omega_d} \int_{\partial Q} \frac{\langle x-y, \nu_y \rangle}{|x-y|^d} \psi(y) \, ds(y) \quad x \in \partial Q.$$

We refer the reader to [11] for proofs of the jump formulas. Denoting  $u_k$  to be the solution of the following scalar Dirichlet boundary value problem

$$(27) \quad \begin{aligned} -\operatorname{div}((1+(k-1)\chi(Q))\nabla u_k) &= 0 \quad \text{in } \Omega, \\ u_k|_{\partial\Omega} &= f \quad \text{on } \partial\Omega, \end{aligned}$$

with the Dirichlet to Neumann map,  $\Lambda_k : H^{\frac{1}{2}}(\partial\Omega) \rightarrow H^{-\frac{1}{2}}(\partial\Omega)$

$$\Lambda_k(f) = \frac{\partial u_k}{\partial \nu}|_{\partial\Omega},$$

it is proven for example in [6] that the solution  $u_k$  admits a unique layer potential representation given by

$$u_k(x) = H_k(x) + [S_Q(\psi_k)](x), \quad x \in \Omega,$$

for  $k \in (0, \infty]$ , where the function  $H$  is harmonic of the form

$$(28) \quad H_k(x) = (-S_\Omega \circ \Lambda_k + D_\Omega)(f), \quad x \in \Omega,$$

and  $\psi_k$  satisfies the integral equation:

$$(29) \quad \left( \frac{k+1}{2(k-1)}I - K_Q^* \right) \psi_k = \frac{\partial H_k}{\partial \nu}|_{\partial Q} \quad \text{on } \partial Q.$$

Observe that when  $Q$  is perfectly conducting, the jump formulas (25) assert that

$$\begin{aligned} \frac{\partial U}{\partial \nu}|_- &= \frac{\partial}{\partial \nu}|_- (H_k(x) + [S_Q(\psi)]) \\ &= \frac{\partial}{\partial \nu} H_k(x) + \left( -\frac{1}{2}I + K_Q^* \right) \phi \\ &= 0. \end{aligned}$$

**Lemma 4.2.** *The Dirichlet to Neumann map*

$$\begin{aligned} \Lambda_k : H^{\frac{1}{2}}(\partial\Omega) &\rightarrow H^{-\frac{1}{2}}(\partial\Omega), \\ f &\rightarrow \nabla u_k \cdot \nu_\Omega \quad \text{with } u_k \text{ solution of (27)} \end{aligned}$$

is uniformly bounded in  $H^{-\frac{1}{2}}(\partial\Omega)$  with respect to  $k$ . In particular, the following estimates hold

$$(30) \quad \|\nabla u_k \cdot \nu_\Omega\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq \left( \frac{1 + \text{dist}(Q, \partial\Omega)}{\text{dist}(Q, \partial\Omega)} \right)^2 \|f\|_{H^{\frac{1}{2}}(\partial\Omega)},$$

and

$$\left| \int_{\partial\Omega} \nabla u_k \cdot \nu_\Omega \, ds \right| \leq \left( \frac{1 + \text{dist}(Q, \partial\Omega)}{\text{dist}(Q, \partial\Omega)} \right)^2 |\Omega \setminus Q|^{\frac{1}{2}} \|f\|_{H^{\frac{1}{2}}(\partial\Omega)}.$$

*Proof.* Since  $f \in H^{\frac{1}{2}}(\partial\Omega)$ , its harmonic extension  $\tilde{f}$  on  $\Omega$ , which belongs in  $H^1(\Omega)$ , is such that

$$\|\tilde{f}\|_{H^1(\Omega)} = \|f\|_{H^{\frac{1}{2}}(\partial\Omega)},$$

and  $\text{trace}(\tilde{f}) = f$ .

We define  $\xi \in W^{1,\infty}(\Omega; \mathbb{R})$  as the cut-off given by

$$\xi(x) = \max\left(0, 1 - \frac{\text{dist}(x, \partial\Omega)}{\text{dist}(Q, \partial\Omega)}\right)$$

and write  $Q_0 = \overline{\Omega \cap \chi^{-1}(\{0\})}$ .

Recast  $\tilde{f} := \xi \tilde{f}$  so that  $\tilde{f}|_{Q_0} = 0$ , and

$$\|\tilde{f}\|_{H^1(\Omega)} \leq \frac{1 + \text{dist}(Q, \partial\Omega)}{\text{dist}(Q, \partial\Omega)} \|f\|_{H^{\frac{1}{2}}(\partial\Omega)}.$$

Let  $W$  be the solution to the auxiliary problem

$$\begin{aligned} -\Delta W &= 0 & \text{in } \Omega \setminus Q_0, \\ W &= g & \text{on } \partial\Omega, \\ W &= 0 & \text{on } \partial Q_0. \end{aligned}$$

If we call  $\tilde{g}$  the harmonic extension of  $g$ ,  $W$  being the minimiser of a Dirichlet problem,

$$\|W\|_{H^1(\Omega \setminus \Omega_0)} \leq \|\xi \tilde{g}\|_{H^1(\Omega \setminus \Omega_0)} \leq \frac{1 + \text{dist}(Q, \partial\Omega)}{\text{dist}(Q, \partial\Omega)} \|g\|_{H^{\frac{1}{2}}(\partial\Omega)}.$$

Testing the equation (27) with  $W$ , we observe that

$$\int_{\partial\Omega} g \nabla u_k \cdot \nu_\Omega \, ds = \int_{\Omega \setminus \Omega_0} \nabla u_k \cdot \nabla W \, dx$$

for any arbitrary function  $g \in H^{\frac{1}{2}}(\partial\Omega)$ . In particular, denoting  $\langle \cdot, \cdot \rangle_{H^{\frac{1}{2}}, H^{-\frac{1}{2}}}$  to be the duality bracket between  $H^{\frac{1}{2}}(\partial\Omega)$  and  $H^{-\frac{1}{2}}(\partial\Omega)$ , we have

$$\left| \langle g, \nabla u_k \cdot \nu_\Omega \rangle_{H^{\frac{1}{2}}, H^{-\frac{1}{2}}} \right| \leq \|\nabla u_k\|_{L^2(\Omega \setminus \Omega_0)} \|\nabla W\|_{L^2(\Omega \setminus \Omega_0)}$$

Since  $u_k$  is a minimiser of a Dirichlet problem, we have

$$\|\nabla u_k\|_{L^2(\Omega \setminus \Omega_0)} \leq \|\nabla \tilde{f}\|_{L^2(\Omega \setminus \Omega_0)}.$$



Altogether, we have obtained

$$\begin{aligned} \left| \langle g, \nabla u_k \cdot \nu_\Omega \rangle_{H^{\frac{1}{2}}, H^{-\frac{1}{2}}} \right| &\leq \left\| \nabla \tilde{f} \right\|_{L^2(\Omega \setminus Q)} \left\| \nabla W \right\|_{L^2(\Omega \setminus \Omega_0)} \\ &\leq \left( \frac{1 + \text{dist}(Q, \partial\Omega)}{\text{dist}(Q, \partial\Omega)} \right)^2 \|f\|_{H^{\frac{1}{2}}(\partial\Omega)} \|g\|_{H^{\frac{1}{2}}(\partial\Omega)} \end{aligned}$$

which implies (30). The second inequality follows from the same argument, with  $g = 1$ . In that case  $\|W\|_{H^1(\Omega \setminus \Omega_0)} \leq \|\xi\|_{H^1(\Omega \setminus \Omega_0)} \leq \left( \frac{1 + \text{dist}(Q, \partial\Omega)}{\text{dist}(Q, \partial\Omega)} \right) |\Omega \setminus Q|^{\frac{1}{2}}$ .  $\square$

**Lemma 4.3.** *Let  $u_k$  denote the solution to (27). Given  $f \in H^{\frac{1}{2}}(\partial\Omega)$ , the function  $H_k$  given by (28) is bounded in  $C^2(\bar{Q})$  independently of  $k$  in  $[1, \infty)$ . Furthermore, there holds*

$$\|D^2 H_k\|_{C^0(\bar{Q})} \leq C_0 \|f\|_{H^{\frac{1}{2}}(\partial\Omega)},$$

where  $C_0$  depends only on  $\text{dist}(Q, \partial\Omega)$ ,  $|\Omega \setminus Q|$ ,  $|\partial\Omega|$  and  $d$ , and is given by (31).

*Proof.* Let  $x$  be an arbitrary point in  $\bar{Q}$ . A straightforward computation of  $\partial_{\alpha\beta} H_k(x)$  gives

$$\begin{aligned} \partial_{\alpha\beta} H_k(x) &= \partial_{\alpha\beta} \left( \frac{1}{(d-2)\omega_d} \int_{\partial\Omega} \frac{1}{|x-y|^{d-2}} \nabla u_k \cdot \nu_\Omega(y) \, ds(y) \right. \\ &\quad \left. + \frac{1}{\omega_d} \int_{\partial\Omega} \frac{\langle x-y, \nu_y \rangle}{|x-y|^d} f(y) \, ds(y) \right) \\ &= \frac{1}{(d-2)\omega_d} \int_{\partial\Omega} \partial_{\alpha\beta} \frac{1}{|x-y|^{d-2}} \nabla u_k \cdot \nu_\Omega(y) \, ds(y) \\ &\quad + \frac{1}{\omega_d} \int_{\partial\Omega} \partial_{\alpha\beta} \frac{\langle x-y, \nu_y \rangle}{|x-y|^d} f(y) \, ds(y) \\ &= \frac{1}{\omega_d} \int_{\partial\Omega} \left( \frac{(-d)(x_\alpha - y_\alpha)(x_\beta - y_\beta)}{|x-y|^{d+2}} + \frac{\delta_{\alpha\beta}}{|x-y|^d} \right) \nabla u_k \cdot \nu_\Omega(y) \, ds(y) \\ &\quad + \frac{1}{\omega_d} \int_{\partial\Omega} (2-d) \langle x-y, \nu_y \rangle \left( \frac{(-d)(x_\alpha - y_\alpha)(x_\beta - y_\beta)}{|x-y|^{d+2}} \right. \\ &\quad \left. + \frac{\delta_{\alpha\beta}}{|x-y|^d} \right) f(y) \, ds(y) \\ &\quad + \frac{1}{\omega_d} \int_{\partial\Omega} \left( (2-d) \frac{(x_\alpha - y_\alpha)(x_\beta - y_\beta)}{|x-y|^d} (\nu_y^\beta + \nu_y^\alpha) \right) f(y) \, ds(y). \end{aligned}$$

Write  $\delta_0 = \text{dist}(Q, \partial\Omega)$ . Using that  $|x - y| \leq \delta_0$  for all  $y \in \partial\Omega$ , and the bounds derived in Lemma 4.2 we obtain

$$\begin{aligned} \sup_{x \in \overline{Q}} |\partial_{\alpha\beta} H_k(x)| &\leq \frac{d+1}{\omega_d} \left| \int_{\partial\Omega} \frac{1}{|x-y|^d} \nabla u_k \cdot \nu_\Omega(y) \, d\sigma(y) \right| \\ &\quad + \frac{(d+1)(d-2)}{\omega_d} \left| \int_{\partial\Omega} \frac{1}{|x-y|^{d-1}} f(y) \, ds(y) \right| \\ &\quad + \frac{2(d-2)}{\omega_d} \left| \int_{\partial\Omega} \frac{1}{|x-y|^{d-2}} f(y) \, ds(y) \right| \\ &\leq C_0 \|f\|_{H^{\frac{1}{2}}(\partial\Omega)}, \end{aligned}$$

with

$$(31) \quad C_0 = \frac{1}{\omega_d} \left( \frac{(d+1)(1+\delta_0)^2 |\Omega \setminus Q|^{\frac{1}{2}}}{\delta_0^{d+2}} + \frac{(d-2) |\partial\Omega|^{\frac{1}{2}}}{\delta_0^{d-1}} (d+1+2\delta_0) \right).$$

□

*Remark.* Since  $\text{dist}(Q, \partial\Omega) > 0$ , the function  $H_k = (-S_\Omega \circ \Lambda_k + D_\Omega) f$  which is defined as a boundary layer potential on  $\partial\Omega$  is trivially smooth on  $\overline{Q}$ . However, in the spirit of tracking down constants, the purpose of the lemma above is to provide a quantitative version for the second order derivatives of  $H_k$  on the set  $\overline{Q}$ .

**Proposition 4.4.** *Suppose that  $\psi, \psi_k \in C^{0,\sigma}(\partial Q)$  solve the integral equations*

$$\left( \frac{k+1}{2(k-1)} I - K_Q^* \right) \psi_k = \frac{\partial H_k}{\partial \nu} \Big|_{\partial Q}$$

and

$$\left( \frac{1}{2} I - K_Q^* \right) \psi = \frac{\partial H_k}{\partial \nu} \Big|_{\partial Q}$$

respectively, with  $H_k$  be the harmonic function defined in (28). Then for any  $k > 1$ , there holds

$$\|\psi_k - \psi\|_{C^{0,\sigma}(\partial Q)} \leq C_Q \frac{1}{k-1} \|f\|_{H^{\frac{1}{2}}(\partial\Omega)}$$

where the constant  $C_Q$  is a geometrical constant which depends on the spectral radius of the operator  $K_Q^*$ .

*Proof.* Thanks to Proposition 5.3 from the appendix, a straightforward computation shows that for any  $\phi \in C^{0,\sigma}(\partial Q)$ ,  $\lambda_n, \lambda \in \mathbb{R}$  such that  $\lambda_n > \lambda$  and  $\lambda$  is greater than the spectral radius of  $K_Q^*$  we have

$$\begin{aligned} \left( (\lambda_n I - K_Q^*)^{-1} - (\lambda I - K_Q^*)^{-1} \right) \phi &= \left( \frac{1}{\lambda_n} \sum_{i=0}^{\infty} \left( \frac{K_Q^*}{\lambda_n} \right)^i - \frac{1}{\lambda} \sum_{i=0}^{\infty} \left( \frac{K_Q^*}{\lambda} \right)^i \right) \phi \\ &= \sum_{i=0}^{\infty} \left( \frac{1}{\lambda_n^{i+1}} - \frac{1}{\lambda^{i+1}} \right) (K_Q^*)^i \phi \end{aligned}$$

The triangle inequality then shows that

$$\begin{aligned}
& \left| \left( (\lambda_n I - K_Q^*)^{-1} - (\lambda I - K_Q^*)^{-1} \right) \phi \right| \\
& \leq |\lambda - \lambda_n| \left( \sum_{i=0}^{\infty} \sum_{k=0}^{k=i} \left( \frac{1}{\lambda_n} \right)^{i-k} \cdot \left( \frac{1}{\lambda} \right)^k \left\| (K_Q^*)^i \right\| \right) \|\phi\|_{C^{0,\sigma}(\partial Q)} \\
& \leq |\lambda - \lambda_n| \left( \sum_{i=0}^{\infty} (i+1) \cdot \frac{1}{\lambda^i} \left\| (K_Q^*)^i \right\| \right) \|\phi\|_{C^{0,\sigma}(\partial Q)} \\
& = M(\rho, |\lambda|) \|\phi\|_{C^{0,\sigma}(\partial Q)} |\lambda - \lambda_n|,
\end{aligned}$$

where  $\rho = \limsup_n \left\| (K_Q^*)^n \right\|^{\frac{1}{n}}$  and  $M(\rho, \lambda) \leq \frac{C(Q)}{|\lambda| - \rho}$  with  $C(Q)$  independent of  $\lambda$ . Using Proposition 5.3, we know that the spectral radius of  $K_Q^*$  on the set  $C_0^{0,\sigma}(\partial Q) := \left\{ f \in C^{0,\sigma}(\partial Q) : \int_{\partial Q} f = 0 \right\}$  is less than  $\frac{1}{2}$ . Applying the above argument to  $\lambda = \frac{1}{2}$  and  $\lambda_n = \frac{k+1}{2(k-1)}$ , we obtain

$$\begin{aligned}
& \|\psi_k - \psi\|_{C^{0,\sigma}(\partial Q)} \\
& = \left\| \left( \left( \frac{k+1}{2(k-1)} I - K_Q^* \right)^{-1} - \left( \frac{1}{2} I - K_Q^* \right)^{-1} \right) \left( \frac{\partial H_k}{\partial \nu} \Big|_{\partial Q} \right) \right\|_{C^{0,\sigma}(\partial Q)} \\
& \leq M \left( \rho, \frac{1}{2} \right) \left| \frac{1}{k-1} \right| \left\| \frac{\partial H_k}{\partial \nu} \Big|_{\partial Q} \right\|_{C^{0,\sigma}(\partial Q)} \\
& \leq C_Q \frac{1}{k-1} \|f\|_{H^{\frac{1}{2}}(\partial \Omega)},
\end{aligned}$$

as announced.  $\square$

**Lemma 4.5.** *Let  $\Omega$  be an open connected subset of  $\mathbb{R}^d$ . Suppose further that  $\Omega$  has cubic symmetry, then the single layer potential  $S_\Omega$  and double layer potential  $D_\Omega$  map odd functions to odd functions.*

*Proof.* Let  $\psi$  be an arbitrary odd function belonging to  $H^{\frac{1}{2}}(\partial \Omega)$ . Denoting the map  $z(y) = -y$  for  $y \in \mathbb{R}^d$ , observe that if  $\psi$  is odd and  $\Omega$  has cubic symmetry, a straightforward computation shows that for any  $x \in \mathbb{R}^d$ ,

$$\begin{aligned}
S_\Omega(\psi)(-x) & := \frac{1}{(2-d)\omega_d} \int_{\partial Q} \frac{1}{|x+y|^{d-2}} \psi(y) \, ds(y) \\
& = \frac{1}{(2-d)\omega_d} \int_{z(\partial Q)} \frac{1}{|x-z|^{d-2}} \psi(-z) |\det z| \, ds(z) \\
& = \frac{1}{(2-d)\omega_d} \int_{\partial Q} \frac{1}{|x-z|^{d-2}} \psi(-z) \, ds(z) \\
& = -\frac{1}{(2-d)\omega_d} \int_{\partial Q} \frac{1}{|x-z|^{d-2}} \psi(z) \, ds(z) \\
& = -S_\Omega(\psi)(x).
\end{aligned}$$

and similarly,

$$\begin{aligned}
D_\Omega(\psi)(-x) &:= \frac{1}{\omega_d} \int_{\partial Q} \frac{\langle -x-y, \nu(y) \rangle}{|x+y|^d} \psi(y) \, ds(y) \\
&= \frac{1}{\omega_d} \int_{\partial Q} \frac{\langle -x+z, \nu(-z) \rangle}{|x-z|^d} \psi(-z) \, ds(z) \\
&= -\frac{1}{\omega_d} \int_{\partial Q} \frac{\langle x-z, \nu(z) \rangle}{|x-z|^d} \psi(z) \, ds(z) \\
&= -D_\Omega(\psi)(x)
\end{aligned}$$

as  $\nu(-y) = -\nu(y)$  for all  $y \in \partial\Omega$ , thus establishing our thesis.  $\square$

**Corollary 4.6.** *Let  $\Omega$  be as the above and let  $K_\Omega^*$  be the Poincaré-Neumann operator given in (26). Denoting*

$$C_{\text{odd}}^{0,\sigma}(\partial\Omega) := \{f \in C^{0,\sigma}(\partial\Omega) : f(-x) = -f(x) \text{ for } x \in \partial\Omega\},$$

the integral equation given by

$$\left( \frac{k+1}{2(k-1)} I - K_\Omega^* \right) \psi_k = f \quad \forall x \in \partial\Omega$$

is uniquely solvable for any  $f \in C_{\text{odd}}^{0,\sigma}(\partial\Omega)$ , moreover, the solution  $\psi_k$  belongs to  $C_{\text{odd}}^{0,\sigma}(\partial\Omega)$ .

*Proof.* Since  $\partial\Omega$  has cubic symmetry, it is clear that the set  $C_{\text{odd}}^{0,\sigma}(\partial\Omega)$  is non-empty. Thanks to Lemma 4.5,

$$\left( \frac{k+1}{2(k-1)} I - K_\Omega^* \right) : C_{\text{odd}}^{0,\sigma}(\partial\Omega) \longrightarrow C_{\text{odd}}^{0,\sigma}(\partial\Omega),$$

moreover, as  $C_{\text{odd}}^{0,\sigma}$  is a closed subspace of

$$C_0^{0,\sigma}(\partial\Omega) := \left\{ f \in C^{0,\sigma}(\partial\Omega) : \int_{\partial Q} f \, d\sigma = 0 \right\},$$

Proposition 5.2 from the appendix ensures that the spectral radius of  $K_\Omega^*$  is less than  $\frac{1}{2}$ , thus, by Theorem 5.1, the operator  $\left( \frac{k+1}{2(k-1)} I - K_\Omega^* \right)$  is invertible on the closed subspace  $C_{\text{odd}}^{0,\sigma}(\partial\Omega)$  in which,  $\left( \frac{k+1}{2(k-1)} I - K_\Omega^* \right)^{-1}$  may be given by its Neumann series.  $\square$

**Proposition 4.7.** *Under the assumptions of Theorem 1.3, the solution  $U_k$  to (9) converges to  $U_\infty$ , the solution of the perfect conductivity equation (12) in  $C^{1,\sigma}(\overline{\Omega \setminus Q})$ . Furthermore, there holds*

$$\|U_k - U_\infty\|_{C^{1,\sigma}(\overline{\Omega \setminus Q})} \leq K_{SL} \frac{1}{k-1} \|\phi\|_{C^{1,\sigma}(\mathbb{R}^d)},$$

where  $K_{SL}$  is independent of the conductivity parameter  $k$ .

*Proof.* Let  $U_{i,k}$  denote the  $i$ -th component of the vector  $U_k$ . By the representation formula (28),

$$U_{i,k} = H_{i,k} + (S_Q(\psi_k))_i$$

where  $\psi_k$  satisfies the integral equation:

$$(32) \quad \left( \frac{k+1}{2(k-1)} I - K_Q^* \right) \psi_k = \frac{\partial H_k}{\partial \nu} \Big|_{\partial Q} \quad \text{on } \partial Q.$$

Observe from the proof of Lemma 2.1 that  $U_{i,k}$  is an odd function in  $\Omega$  and therefore its gradient  $\nabla U_{i,k}$  is even for all  $x \in \Omega$ . The symmetry property of  $\partial\Omega$  then implies that  $\nabla U_{i,k} \cdot \nu_\Omega$  is an odd function so that as shown in Lemma 4.5, the function

$$H_{i,k} = -S_\Omega(\nabla U_{i,k} \cdot \nu_\Omega) + D_\Omega(\phi_i)$$

is also odd. Since  $Q$  is a symmetrical inclusion by Assumption 1.1, we may reiterate the argument to show that the function  $(H_{i,k} \cdot \nu) \Big|_{\partial Q}$  is also odd. Since  $H_k$  is in  $C^2(\overline{Q})$  by Lemma 4.3,  $(H_{i,k} \cdot \nu) \Big|_{\partial Q}$  is in  $C_{\text{odd}}^{0,\sigma}$  for any  $\sigma \in (0, 1)$ . Then thanks to [11, Theorem 2.30] together with Corollary 4.6 we deduce the existence of the unique  $\psi_k$  in  $C_{\text{odd}}^{0,\sigma}$ . Next, write  $V_{i,k} := H_{i,k} + S_Q(\psi_i)$  where  $\psi_i$  solves the integral equation

$$\left( \frac{1}{2} I - K_Q^* \right) \psi_i = \frac{\partial H_{i,k}}{\partial \nu} \Big|_{\partial Q} \quad \text{on } \partial Q.$$

The function  $V_{i,k}$  is odd thanks to Lemma 4.5. Thus  $V_{i,k} = 0$  on  $\overline{Q}$ . Consider the function  $W_{i,k} = U_{i,k} - V_{i,k}$ . By construction, we have

$$W_{i,k}(x) = (S_Q(\psi_k - \psi))_i \quad \forall x \in \mathbb{R}^d \setminus \partial Q,$$

moreover, as  $\partial Q$  is  $C^2$  and  $\psi_k, \psi \in C^{0,\sigma}(\partial Q)$ , the following estimate holds, see [11, Theorem 2.17],

$$\|S_Q(\psi_k - \psi)\|_{C^{1,\sigma}(\overline{\Omega \setminus Q})} \leq C_S \|\psi_k - \psi\|_{C^{0,\sigma}(\partial Q)}.$$

Now, observe that the function  $F_{i,k} := V_{i,k} - U_{\infty,i}$  satisfies

$$\begin{aligned} -\Delta F_{i,k} &= 0 \text{ in } \Omega \setminus Q \\ F_{i,k} &= 0 \text{ on } \partial Q \\ F_{i,k} &= -(S_Q(\psi_k - \psi))_i \text{ on } \partial\Omega. \end{aligned}$$

Lemma 3.8 implies that

$$\begin{aligned} \|F_{i,k}\|_{C^{1,\sigma}(\overline{\Omega \setminus Q})} &\leq C(\Omega \setminus Q, \sigma) \|S_Q(\psi_k - \psi)\|_{C^{1,\sigma}(\mathbb{R}^d)} \\ &\leq C(\Omega \setminus Q, \sigma) C_S \|\psi_k - \psi\|_{C^{0,\sigma}(\partial Q)}. \end{aligned}$$

Thus, we have for any  $i \in \{1, \dots, d\}$ ,

$$\begin{aligned} \|U_{i,k} - U_i\|_{C^{1,\sigma}(\overline{\Omega \setminus Q})} &\leq \|W_{i,k}\|_{C^{1,\sigma}(\overline{\Omega \setminus Q})} + \|F_{i,k}\|_{C^{1,\sigma}(\overline{\Omega \setminus Q})} \\ &\leq (C_S + C(\Omega \setminus Q, \sigma) C_S) \|\psi_k - \psi\|_{C^{0,\sigma}(\partial Q)}. \\ &\leq K_{SL} \frac{1}{k-1} \|\phi\|_{C^{1,\sigma}(\partial\Omega)} \end{aligned}$$

which is our statement.  $\square$

We are now ready to prove our main result, which follows from the following proposition.

**Proposition 4.8.** *Assume that Assumption (1.1) holds and that  $\phi$  is  $\alpha$ -admissible. Let  $M_0$  and  $M_1$  denote the values given by Lemma 3.5, and  $K_{SL}$  be the constant given by Proposition 4.7. Write*

$$k_0 := 1 + 4 \cdot K_{SL} \cdot \max \left\{ 1, \frac{d2^{d-1}}{\alpha^d \min \{M_0, M_1\}} \right\} \text{ and } \tau_0 := \frac{1}{4} \alpha^d \min \{M_0, M_1\}.$$

*There exists non-empty sets  $A_-^1, \dots, A_-^d$  and  $A_+$  such that for all  $k > k_0$ , for all  $\alpha$ -admissible  $\phi$ , the solution  $U$  of (1) satisfies*

$$\cup_{i=1}^d A_-^i \subset (\det DU)^{-1} \left( -\infty, -\tau_0 \|\phi\|_{C^{1,\sigma}(\mathbb{R}^d)}^d \right),$$

and

$$A_+ \subset (\det DU)^{-1} \left( \tau_0 \|\phi\|_{C^{1,\sigma}(\mathbb{R}^d)}^d, \infty \right).$$

Furthermore, we have

$$B(0, r_0) \cap (\Omega \setminus Q) \subseteq A_+,$$

where  $r_0 = \left( \frac{M_0}{2dC(\Omega, Q)} \alpha^d \right)^{1/\sigma}$ . The constant  $C(\Omega, Q)$  depends on the regularity of  $\Omega$  and  $Q$  only. Furthermore

$$(B(p_i, r_1) \cup B(-p_i, r_1)) \cap (\Omega \setminus Q) \subseteq A_-^i,$$

with  $r_1 = \left( \frac{M_1}{2dC(\Omega, Q)} \alpha^d \right)^{1/\sigma}$ , and where the point  $p_i$  is constructed on  $\mathbb{R}^+ e_i \cap (\Omega \setminus Q)$ .

*Proof.* Using Corollary 3.6, we observe that, writing

$$B_1 := B(p_1, r_1) \cap (\Omega \setminus Q)$$

and

$$B_0 := B(0, r_0) \cap (\Omega \setminus Q),$$

there holds

$$(33) \quad B_1 \subset J_\infty^{-1} \left( \left( -\infty, -\frac{1}{2} \alpha^d \|\phi\|_{C^{1,\sigma}(\mathbb{R}^d)}^d M_1 \right) \right)$$

and

$$B_0 \subset J_\infty^{-1} \left( \left( \frac{1}{2} \alpha^d \|\phi\|_{C^{1,\sigma}(\mathbb{R}^d)}^d M_1, \infty \right) \right).$$

Thanks to Proposition 4.7, we have

$$\|U_k - U_\infty\|_{C^{1,\sigma}(\overline{\Omega \setminus Q})} \leq K_{SL} \frac{1}{k-1} \|\phi\|_{C^{1,\sigma}(\mathbb{R}^d)},$$

therefore arguing as in the proof of Corollary 3.6,

$$\begin{aligned} & \sup_{x \in \overline{\Omega \setminus Q}} |\det DU_k(x) - \det DU_\infty(x)| \\ & \leq d \sup_{x \in \overline{\Omega \setminus Q}} |(DU_k(x) - DU_\infty(x))| \max(|DU_k(x)|, |DU_\infty(x)|)^{d-1} \\ & \leq K_{SL} \frac{1}{k-1} \cdot d \left( 1 + \frac{K_{SL}}{k-1} \right)^{d-1} \|\phi\|_{C^{1,\sigma}(\mathbb{R}^d)}^d, \end{aligned}$$

and for any positive  $k$  such that

$$K_{SL} \frac{1}{k-1} \cdot d \left( 1 + \frac{K_{SL}}{k-1} \right)^{d-1} \leq \frac{1}{4} \alpha^d \min(M_0, M_1),$$

the triangle inequality and (33) shows for all  $x \in B_1$ ,

$$\begin{aligned} \det DU_k(x) &\leq \det DU_\infty(x) + \frac{K_{SL}}{k-1} d 2^{d-1} \|\phi\|_{C^{1,\sigma}(\mathbb{R}^d)}^d \\ &\leq -\frac{1}{4} \alpha^d \|\phi\|_{C^{1,\sigma}(\mathbb{R}^d)}^d M_1, \end{aligned}$$

and for all  $x \in B_0$  there holds

$$\begin{aligned} \det DU_k(x) &\geq \det DU_\infty(x) - K_{SL} \frac{1}{k-1} \cdot d \left(1 + \frac{K_{SL}}{k-1}\right)^{d-1} \|\phi\|_{C^{1,\sigma}(\mathbb{R}^d)}^d \\ &\geq \frac{1}{4} \alpha^d \|\phi\|_{C^{1,\sigma}(\mathbb{R}^d)}^d M_0. \end{aligned}$$

Combining these two statements, and writing  $\tau_0 = \frac{1}{4} \alpha^d \min\{M_0, M_1\}$ , we have the set containments

$$B_1 \subset J_k^{-1} \left( \left( -\infty, -\tau_0 \|\phi\|_{C^{1,\sigma}(\mathbb{R}^d)}^d \right) \right) \text{ and } B_0 \subset J_k^{-1} \left( \left( \|\phi\|_{C^{1,\sigma}(\mathbb{R}^d)}^d, \tau_0, \infty \right) \right),$$

which holds uniformly for all  $k > k_0 := 1 + 4 \cdot K_{SL} \cdot \max\left\{1, \frac{d 2^{d-1}}{\min\{M_0, M_1\}} \alpha^d\right\}$ .

Finally, because Assumption 1.1 imposes axial symmetry and because  $\phi$  is  $\alpha$ -admissible, all of the above developments are valid for every canonical direction  $e_1, \dots, e_d$  and both in the positive and negative half lines, thus,

$$A_-^i = \left( B \left( p_i, \left( \frac{M_1}{2dC(\Omega, Q)} \alpha^d \right)^{1/\sigma} \right) \cup B \left( -p_i, \left( \frac{M_1}{2dC(\Omega, Q)} \alpha^d \right)^{1/\sigma} \right) \right) \cap (\Omega \setminus Q)$$

where  $p_i$  is constructed similarly to  $p_1$  on the half-axes  $\mathbb{R}e_i$ .  $\square$

## 5. ADDITIONAL REMARKS.

The method presented here does not provide optimal bounds for  $J_k^{-1}((-\infty, 0))$  and  $J_k^{-1}((0, \infty))$ , as the restriction to tubes we have used is not necessary. Its advantage is that it delivers explicit bounds in practical cases. It does give a rule of thumb to obtain larger bounds: the larger the cross section of the connecting tubes, the better, by comparison.

The amplitude of the boundary condition does not play a role in the definition of the sets where the sign of the Jacobian determinant is controlled. It appears as a multiplicative factor for the (positive or negative) value of the determinant on these sets.

The fact that  $\gamma$  is piecewise constant is not required for Proposition 2.5 to hold. We can argue by a perturbation argument. Consider a regularised  $\gamma$  using the standard mollifier  $\eta \in C^\infty(\mathbb{R})$ , where  $\eta_\epsilon(x) = c_\eta \exp\left(\frac{-1}{1-|x/\epsilon|^2}\right)$  for  $|x| < \epsilon$  and 0 otherwise, in which, the constant  $c_\eta$  is a normalizing constant so that  $\|\eta_\epsilon\|_{L^1(\mathbb{R}^d)} = 1$ . Observing that the regularised conductivity matrix  $\tilde{\gamma} = \gamma \star \eta_\epsilon$  is smooth and yet satisfies Assumption 1.1, we observe the Jacobian determinant of the solution  $D\tilde{U}$  with coefficient matrix  $\tilde{\gamma}$  would still satisfy the sign changing phenomenon. The main result, however, makes use of layer potential techniques which are ill-suited for variable coefficients.

It is worth noting that for the perfectly conducting case, the measure of the sets where the Jacobian determinant is positive or negative is not controlled by

the Lebesgue measure of the inclusion  $Q$ ; in particular it may happen when  $Q$  has zero measure. Consider the case  $d = 3$  where  $\Sigma$  is a crack inclusion, or a surface of co-dimension 1 which has a  $C^2$  boundary edge and satisfies Assumption 1.1, for example lying in the plane  $x_3 = 0$ . The perfect conductivity equation in this case reads

$$(34) \quad \begin{aligned} -\Delta U &= 0 && \text{in } \Omega \setminus \Sigma \\ U &= I_d && \text{on } \partial\Omega \\ U &= 0 && \text{on } \overline{\Sigma}. \end{aligned}$$

The solution to (34) is non-trivial and a classical solution of  $C^2(\Omega \setminus \Sigma) \cap C(\overline{\Omega \setminus \Sigma})$  may be obtained by Perron's method of sub-solutions, since surfaces of co-dimension 1 with a  $C^2$  boundary edge consists only of regular points to the Laplace operator. By the continuity of the solution and the fact that the weak maximum principle still holds here, the proof of Proposition 2.4 still applies even though the gradient  $DU$  is likely to be unbounded as it approaches the boundary edge of  $\Sigma$ .

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## APPENDIX A

**Theorem 5.1.** (*Colton & Kress*) *Let  $A : X \rightarrow X$  be a bounded linear operator mapping the Banach space  $X$  onto itself. Then the Neumann series*

$$(\lambda I - K_Q^*)^{-1} = \sum_{n=0}^{\infty} \lambda^{-n-1} A^n$$

*converges in the uniform operator norm for all  $\lambda > r(A)$  where  $r(A)$  is the spectral radius of  $A$  in  $X$ .*

**Proposition 5.2.** *Let  $\partial\Omega$  be a  $C^{1,\alpha}$  domain, consider the space*

$$\mathcal{H}_0 := \left\{ f \in L^2(\partial\Omega) : \int_{\partial\Omega} f = 0 \right\}$$

*endowed with the inner product*

$$\langle \phi, \psi \rangle_{\mathcal{H}} := - \int_{\partial\Omega} \phi \cdot S_{\Omega}(\psi) ds(y),$$

*then  $\mathcal{H}_0$  is a Hilbert space in which its norm is equivalent to the usual space  $H^{\frac{1}{2}}(\partial\Omega)$ . Moreover, the operator*

$$K_{\Omega}^* : \mathcal{H}_0 \rightarrow \mathcal{H}_0$$

*is compact, self-adjoint, and only has real discrete spectrum accumulating to 0.*

**Proposition 5.3.** *Let  $Q$  be a  $C^2$  domain and let*

$$C_0^{0,\alpha}(\partial Q) := \left\{ f \in C^{0,\alpha}(\partial Q) : \int_{\partial Q} f d\sigma(y) = 0 \right\}.$$

*Then the spectrum of the operator*

$$K_Q^* : C_0^{0,\alpha}(\partial Q) \rightarrow C_0^{0,\alpha}(\partial Q)$$

*consists only of discrete eigenvalues  $\sigma_n$  where  $\sigma_n \in (-\frac{1}{2}, \frac{1}{2})$ , moreover, the Neumann series*

$$(\lambda I - K_Q^*)^{-1} = \frac{1}{\lambda} \sum_{i=0}^{\infty} \left( \frac{K_Q^*}{\lambda} \right)^i$$

*converges in operator norm for all  $\lambda$  such that  $|\lambda| > \sup_n \sigma_n$ .*

*Proof.* Suppose that  $\phi \in C_0^{0,\alpha}(\partial Q)$  and  $\phi$  satisfies  $(\sigma_n I - K_Q^*)(\phi) = 0$ , where  $\phi \neq 0$ . Consider the following quantities

$$(35) \quad A = \int_Q |\nabla S_Q(\phi)|^2 dx, \quad B = \int_{\mathbb{R}^3 \setminus Q} |\nabla S_Q(\phi)|^2 dx,$$

since

$$\Delta S_Q(\phi) = 0 \quad \text{in } \mathbb{R}^d \setminus \partial Q$$

we may apply the divergence theorem and the jump formula of  $S_Q(\phi)$  to find

$$(36) \quad A = \int_{\partial Q} \left( -\frac{1}{2}I + K_Q^* \right) (\phi) \cdot S_Q(\phi) \, dx, \quad B = - \int_{\partial Q} \left( \frac{1}{2}I + K_Q^* \right) (\phi) \cdot S_Q(\phi) \, dx.$$

Since  $K_Q^*(\phi) = \sigma_n \phi$ , we have

$$\sigma_n = \frac{1}{2} \frac{B - A}{B + A}.$$

Thus, if  $A$  and  $B$  are not identically 0, this shows that  $|\sigma_n| < \frac{1}{2}$  and we are done.

Now consider the case that  $\sigma_n = -\frac{1}{2}$  which corresponds to the case  $B = 0$ . Since  $B = 0$ , we deduce from (35) that  $S_Q(\phi)$  is constant on  $\mathbb{R}^d \setminus Q$ . Moreover, by the usual decay estimates

$$|S_Q(\phi)|(x) \leq O\left(\frac{1}{|x|^{d-1}}\right) \quad \text{for } |x| \text{ large,}$$

thus  $S_Q(\phi) \equiv 0$  on  $\mathbb{R}^d \setminus Q$ , which further implies that  $\phi$  is identically 0 by the injectivity of  $S_Q$ . Since we have assumed that  $\phi \neq 0$ , this implies a contradiction and thanks to (36), this also implies that  $\sigma_n = -\frac{1}{2}$  is not an eigenvalue.

When  $\sigma_n = \frac{1}{2}$ , we have that  $A = 0$ , so we deduce again from (35) that  $S_Q(\phi) = m$ , a constant in  $\overline{Q}$ . This, in turn, yields

$$\begin{aligned} B &= - \int_{\partial Q} \left( \frac{1}{2}I + K_Q^* \right) (\phi) \cdot S_Q(\phi) \, dx \\ &= -m \int_{\partial Q} \phi \, dx. \end{aligned}$$

Since by our assumption  $\phi \in C_0^0(\partial Q)$ , it must be that  $B = 0$ , which further implies that  $S_Q(\phi)$  is a constant in  $\mathbb{R}^3 \setminus Q$ . By the same argument as before  $S_Q(\phi)$  must be 0 in  $\mathbb{R}^3 \setminus Q$ , which implies  $m = 0$  by the continuity of  $S_Q(\phi)$ , therefore implying  $\phi = 0$  and hence a contradiction. Combining these two facts, we deduce that the spectrum of  $K_Q^*$  in  $C_0(\partial Q)$  lies in the open interval  $\sigma_n \in (-\frac{1}{2}, \frac{1}{2})$ .

Lastly, consider the space

$$\mathcal{H}_0 := \left\{ f \in L^2(\partial\Omega) : \int_{\partial\Omega} f = 0 \right\}$$

endowed with the inner product

$$\langle \phi, \psi \rangle_{\mathcal{H}} := - \int_{\partial\Omega} \phi \cdot S_{\Omega}(\psi) \, ds(y).$$

By Proposition 5.2 in the appendix, the spectrum of  $K_Q^*$  in  $\mathcal{H}_0$  consists only of real discrete eigenvalues converging to 0 with spectral radius  $\sup_n \sigma_n < \frac{1}{2}$ . Since

$$\sigma(K_{\partial Q}^*|_{C_0^0}) \subseteq \sigma(K_{\partial Q}^*|_{\mathcal{H}_0}),$$

we deduce that the spectrum of  $K_{\partial Q}^*$  in the space  $C_0^0(\partial Q)$  also consists only of discrete eigenvalues with spectral radius  $\sup_n \sigma_n < \frac{1}{2}$ . Thus, we may apply Theorem 5.1  $\square$

## APPENDIX B

**Proposition 5.4.** *Let  $d = 3$ ,  $\Omega = [-1, 1]^3$ , and  $U_\infty$  be given by (12) with  $\phi = I_3$ . Suppose that  $Q$  is a proper torus*

$$x_3^2 = a^2 - \left( \ell + a - \sqrt{x_1^2 + x_2^2} \right)^2,$$

*centred at the origin with minor radii  $a$ , and major radii  $\ell + a$ , with  $0 < \ell < 1 - 2a$ . Then  $y_0 = \inf(\Omega \setminus Q \cap (\mathbb{R}^+ e_1)) = (\ell, 0, 0)$  and there holds*

$$\frac{\partial U_1}{\partial x_1}(0) \geq C_1, \text{ and } \frac{\partial U_1}{\partial x_1}(y_0) \leq -C_1,$$

*with*

$$(37) \quad C_1 = \frac{4}{3\pi^2} \frac{\sqrt{2}-1}{\sqrt{2}} \left( \frac{1}{\cosh\left(\frac{2\pi}{\ell}\right)} + \frac{1}{3} \frac{1}{\cosh\left(\frac{6\pi}{\ell}\right)} \right).$$

*Proof.* We parametrize  $Q$  as follows

$$Q(a, c, \theta, \phi) := \begin{bmatrix} (c + a \cos \theta) \cos \phi \\ (c + a \cos \theta) \sin \phi \\ a \sin \theta \end{bmatrix}; \quad 0 < a < c < 1, \text{ and } \phi \in [0, 2\pi].$$

Set  $p_0 = (\frac{1}{2}\ell, 0, 0)$ . The ball  $B(p_0, \frac{1}{2}\ell)$  is tangential to the boundary of the torus  $Q$  at  $y_0$ . Consider the cuboid  $D = p_0 + \left(-\frac{1}{2\sqrt{2}}\ell, \frac{1}{2\sqrt{2}}\ell\right)^2 \times (-1, 1)$  so that  $D \cap Q = \emptyset$  and  $D \subset (\Omega \cap x_1 > 0)$ . Take  $V_1$  to be the solution to

$$\begin{aligned} \Delta V_1 &= 0 \text{ in } D, \\ V_1 &= \ell \frac{\sqrt{2}-1}{2\sqrt{2}} \text{ on } \partial D \cap \{x_3 = \pm 1\}, \\ V_1 &= 0 \text{ on } \partial D \setminus \{x_3 = \pm 1\}. \end{aligned}$$

By the weak maximum principle, we have  $U_1|_{\partial D} \geq V_1|_{\partial D}$  so that  $V_1$  is a sub-solution with respect to  $U_1$ . We construct the solution to  $V_1$  by the standard method of separation of variables in  $D$  and obtain

$$\begin{aligned} &V_1(x_1, x_2, x_3) \\ &= \sum_{m=\text{odd}}^{\infty} \sum_{n=\text{odd}}^{\infty} C_{m,n} \sin\left(\frac{m\pi}{2r} \left(x_1 - \ell \frac{\sqrt{2}-1}{2\sqrt{2}}\right)\right) \sin\left(\frac{n\pi}{2r} \left(x_2 + \frac{1}{2\sqrt{2}}\ell\right)\right) \cosh(k_{m,n} x_3) \end{aligned}$$

where

$$\begin{aligned} k_{m,n} &= \frac{\pi\sqrt{2}}{\ell} \sqrt{m^2 + n^2} \\ C_{m,n} &= 8 \frac{\sqrt{2}-1}{\sqrt{2}} \frac{\ell}{mn\pi^2} \frac{1}{\cosh(k_{m,n})} \end{aligned}$$

$$\begin{aligned}
V_1(p_0) &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} C_{2m+1, 2n+1} (-1)^{m+n} \\
&\geq \frac{1}{3} C_{1,1} + C_{3,3} \\
&= 8 \frac{\sqrt{2}-1}{\sqrt{2}} \frac{\ell}{3\pi^2} \left( \frac{1}{\cosh\left(\frac{2\pi}{\ell}\right)} + \frac{1}{3} \frac{1}{\cosh\left(\frac{6\pi}{\ell}\right)} \right) \\
&= 2\ell C_1,
\end{aligned}$$

where we used Lemma 5.7 to derive a lower bound. Subsequently, we infer by the comparison principle that

$$U_1(p_0) \geq V_1(p_0) \geq 2\ell C_1,$$

so that by (17) (and by symmetry),

$$(38) \quad \frac{\partial U_1}{\partial x_1}(0) \geq C_1 \text{ and } \frac{\partial U_1}{\partial x_1}(y_0) \leq -C_1,$$

which is our thesis.  $\square$

**Proposition 5.5.** *Let  $p_1 = (\ell - \kappa\ell, 0, 0)$  with  $\kappa = \min\left(C_1 / \left(2\ell \|DU\|_{C^{0,1}(\overline{\Omega \setminus Q})}\right), \frac{1}{2}\right)$ .*

*There holds*

$$\det DU(p_1) \leq -M_1,$$

where  $M_1$  is given in Proposition 3.10.

*Proof.* Thanks to Lemma 3.8 there holds

$$\begin{aligned}
\frac{\partial U_1}{\partial x_1}(p_1) - \frac{\partial U_1}{\partial x_1}(y_0) &\leq \|DU\|_{C^{0,1}(\overline{\Omega \setminus Q})} |p_1 - y_0|, \\
&\leq \kappa \|DU\| \ell, \\
&\leq \frac{1}{2} C_1,
\end{aligned}$$

which together with 38 yields

$$(39) \quad \frac{\partial U_1}{\partial x_1}(p_1) \leq -\frac{1}{2} C_1.$$

The point  $p_1$  lies on the line  $(s, 0, 0)$  and at a distance  $r_2 = \kappa\ell$  away from the boundary of  $\partial Q$ . In the  $x_1 x_2$  plane, consider the triangle with vertices  $E$

$$\begin{aligned}
E &:= p_1 + \{(-r_2, 0, 0), (r_2, 0, 0), (0, h, 0)\} \\
&= \{(\ell(1-2\kappa), 0, 0), (\ell, 0, 0), (\ell(1-\kappa), h, 0)\}
\end{aligned}$$

with  $h = \sqrt{\ell^2 - (\ell - r_2)^2}$ , so that the maximal circle contained in  $E$  has radius

$$(40) \quad \tau = \kappa\ell \sqrt{\frac{\sqrt{2} - \sqrt{\kappa}}{\sqrt{2} + \sqrt{\kappa}}}.$$

Introducing the cube

$$D_2 := p_1 + (-\tau, +\tau) \times \left(0, \frac{1}{2}\tau\right) \times (-1, 1)$$

and taking  $V_2$  to be the solution to the Dirichlet problem

$$\begin{aligned}\Delta V_2 &= 0 \text{ in } D_2 \\ V_2 &= x_2 \text{ on } \partial D_2 \cap \{x_3 = \pm 1\} \\ V_2 &= 0 \text{ on } \partial D_2 \setminus \{x_3 = \pm 1\},\end{aligned}$$

we have  $0 \leq V_2|_{\partial D_2} \leq U_2|_{\partial D_2}$  and  $V_2$  is a sub-solution with respect to  $U_2$ . We compute the solution  $V_2$  as

$$V_2(x_1, x_2, x_3) = \sum_{m=\text{odd}}^{\infty} \sum_{n=\text{odd}}^{\infty} D_{m,n} \sin\left(\frac{m\pi}{2\tau}(x_1 - (|p_1| - \tau))\right) \sin\left(\frac{2n\pi}{\tau}(x_2)\right) \cosh\left(k_{m,n}x_3\right),$$

where

$$\begin{aligned}k_{m,n} &= \frac{\pi}{\tau} \sqrt{\frac{m^2}{4} + 4n^2}, \\ D_{m,n} &= \frac{4\tau}{mn\pi^2} \frac{1}{\cosh\left(k_{m,n}\right)}.\end{aligned}$$

At the point  $(|p_1|, \frac{1}{4}\tau, 0)$ , this expansion reduces to

$$\begin{aligned}V_2(|p_1|, \frac{1}{4}\tau, 0) &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} D_{2m+1, 2n+1} (-1)^{n+m} \\ &\geq \frac{1}{3} D_{1,1} + D_{3,3} \\ &= \frac{4\tau}{3\pi^2} \left( \frac{1}{\cosh\left(\frac{\pi}{\tau} \frac{\sqrt{17}}{2}\right)} + \frac{1}{3} \frac{1}{\cosh\left(3\frac{\pi}{\tau} \frac{\sqrt{17}}{2}\right)} \right)\end{aligned}$$

thanks to Lemma 5.7. The quantitative Hopf's bound (17) yields in turn

$$\begin{aligned}\frac{\partial U_2}{\partial x_2}(p_1) &\geq \frac{1}{\tau} \cdot \frac{4\tau}{3\pi^2} \left( \frac{1}{\cosh\left(\frac{\pi}{\tau} \frac{\sqrt{17}}{2}\right)} + \frac{1}{3} \frac{1}{\cosh\left(3\frac{\pi}{\tau} \frac{\sqrt{17}}{2}\right)} \right) \\ (41) \quad &= \frac{4}{3\pi^2} \left( \frac{1}{\cosh\left(\frac{\pi}{\tau} \frac{\sqrt{17}}{2}\right)} + \frac{1}{3} \frac{1}{\cosh\left(3\frac{\pi}{\tau} \frac{\sqrt{17}}{2}\right)} \right) \\ &= C_2.\end{aligned}$$

by Harnack's inequality. Indeed, by our choice for  $D_2$ , there is a ball centred at  $(|p_1|, \frac{1}{4}\tau, 0)$ , tangent to the point at  $p_1$ , and yet lies in half space  $x_2 > 0$ .

Finally, choose

$$D_3 := p_1 + (-\tau, +\tau) \times \left(-\frac{\tau}{2}, \frac{\tau}{2}\right) \times (0, 1),$$

and  $V_3$  to be the solution to

$$\begin{aligned}\Delta V_3 &= 0 \text{ in } D_3 \\ V_3 &= 1 \text{ on } \partial D_3 \cap \{x_3 = 1\} \\ V_3 &= 0 \text{ on } \partial D_3 \setminus \{x_3 = 1\}.\end{aligned}$$

We have

$$V_3(x_1, x_2, x_3) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} E_{m,n} \sin\left(\frac{m\pi}{2\tau}(x_1 - (|p_1| + \tau))\right) \sin\left(\frac{n\pi}{\tau}(x_2 + \frac{\tau}{2})\right) \sinh(k_{m,n}x_3)$$

with

$$k_{m,n} = \frac{\pi}{\tau} \sqrt{\frac{m^2}{4} + n^2}$$

$$E_{m,n} = \frac{16}{mn\pi^2} \frac{1}{\sinh(k_{m,n})}$$

and

$$\frac{\partial V_3}{\partial x_3}(x) = \sum_{m=\text{odd}}^{\infty} \sum_{n=\text{odd}}^{\infty} k_{m,n} E_{m,n} \sin\left(\frac{m\pi}{2\tau}(x_1 - (|p_1| + \tau))\right) \sin\left(\frac{n\pi}{\tau}(x_2 + \frac{\tau}{2})\right) \cosh(k_{m,n}x_3),$$

as this series expansion converges uniformly for all  $x \in D_3$ . Thus,

$$\begin{aligned} \frac{\partial V_3}{\partial x_3}(p_1) &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} k_{2m+1, 2n+1} E_{2m+1, 2n+1} (-1)^{n+m} \\ &> \frac{1}{3} k_{1,1} E_{1,1} + k_{3,3} E_{3,3} \\ (42) \quad &> \frac{8\sqrt{5}}{3\pi\tau} \left( \frac{1}{\sinh\left(\frac{\pi}{\tau} \frac{\sqrt{5}}{2}\right)} + \frac{1}{\sinh\left(3\frac{\pi}{\tau} \frac{\sqrt{5}}{2}\right)} \right) \\ &= C_3 \end{aligned}$$

using Lemma 5.7. Altogether, we combine the estimates (39),(41) and (42) to obtain

$$\begin{aligned} \det DU(p_1) &\leq -\frac{1}{2} C_1 C_2 C_3 \\ &= -\frac{32\sqrt{10}(\sqrt{2}-1)}{27\pi^5\tau} \left( \frac{1}{\cosh\left(\frac{2\pi}{\ell}\right)} + \frac{1}{3} \frac{1}{\cosh\left(\frac{6\pi}{\ell}\right)} \right) \times \\ &\quad \left( \frac{1}{\cosh\left(\frac{\pi}{\tau} \frac{\sqrt{17}}{2}\right)} + \frac{1}{3} \frac{1}{\cosh\left(3\frac{\pi}{\tau} \frac{\sqrt{17}}{2}\right)} \right) \left( \frac{1}{\sinh\left(\frac{\pi}{\tau} \frac{\sqrt{5}}{2}\right)} + \frac{1}{\sinh\left(3\frac{\pi}{\tau} \frac{\sqrt{5}}{2}\right)} \right) \\ &\leq -\frac{1}{200} \frac{1}{\tau} \exp\left(-\frac{2\pi}{\ell} - \frac{\pi}{\tau} \frac{\sqrt{17} + \sqrt{5}}{2}\right). \end{aligned}$$

Now, since  $\kappa \leq \frac{1}{2}$ , formula 40 shows that  $\frac{1}{\sqrt{3}}\kappa\ell \leq \tau \leq \kappa\ell$ , thus

$$\det DU(p_1) \leq -\frac{\sqrt{3}}{200} \frac{1}{\kappa\ell} \exp\left(-\frac{2\pi}{\ell} \left(1 + \frac{3}{\kappa}\right)\right).$$

□

**Proposition 5.6.** *There holds*

$$\det DU(0) \geq M_0$$

where  $M_1$  is given in Proposition 3.10.

*Proof.* By the symmetry of  $U_1$  and  $U_2$  we have  $U_1(s, 0, 0) = U_2(0, s, 0)$  for all  $s$  in  $[-1, 1]$ , thus

$$\frac{\partial U_1}{\partial x_1}(s, 0, 0) = \frac{\partial U_2}{\partial x_2}(0, s, 0)$$

so we may use Proposition 5.4 to provide the same quantitative lower bounds for  $\frac{\partial U_2}{\partial x_2}(0)$  as for  $\frac{\partial U_1}{\partial x_1}(0)$ . To estimate  $\frac{\partial U_3}{\partial x_3}(0)$ , consider the domain

$$D_4 := \left(-\frac{\ell}{\sqrt{2}}, \frac{\ell}{\sqrt{2}}\right) \times \left(-\frac{\ell}{\sqrt{2}}, \frac{\ell}{\sqrt{2}}\right) \times (0, 1)$$

where  $l = c - a$ , and take  $W$  to satisfy

$$\begin{aligned} -\Delta W &= 0 \quad \text{in } D_4, \\ W\left(-\frac{\ell}{\sqrt{2}}, \cdot, \cdot\right) &= W\left(\frac{\ell}{\sqrt{2}}, \cdot, \cdot\right) = 0, \\ W\left(\cdot, -\frac{\ell}{\sqrt{2}}, \cdot\right) &= W\left(\cdot, \frac{\ell}{\sqrt{2}}, \cdot\right) = 0, \\ W(\cdot, \cdot, 1) &= 1, \\ W(\cdot, \cdot, 0) &= 0. \end{aligned}$$

we have that  $W|_{\partial D_4} \leq U_3|_{\partial D_4}$  so that  $W$  is a sub-solution with respect to  $U_3$ . We compute  $W$  as

$$W(x_1, x_2, x_3) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} F_{m,n} \sin\left(\frac{m\pi}{\sqrt{2}l}\left(x_1 + \frac{\ell}{\sqrt{2}}\right)\right) \sin\left(\frac{n\pi}{\sqrt{2}l}\left(x_2 + \frac{\ell}{\sqrt{2}}\right)\right) \sinh(k_{m,n}x_3)$$

&

$$\begin{aligned} k_{m,n} &= \frac{\pi}{\sqrt{2}l} \sqrt{m^2 + n^2} \\ F_{m,n} &= \frac{16}{mn\pi^2} \frac{1}{\sinh(k_{m,n})} \end{aligned}$$

$$\frac{\partial W}{\partial x_3}(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} k_{m,n} F_{m,n} \sin\left(\frac{m\pi}{\sqrt{2}l}\left(x_1 + \frac{\ell}{\sqrt{2}}\right)\right) \sin\left(\frac{n\pi}{\sqrt{2}l}\left(x_2 + \frac{\ell}{\sqrt{2}}\right)\right) \cosh(k_{m,n}x_3),$$

hence, using Lemma 5.7

$$\begin{aligned} \frac{\partial W}{\partial x_3}(0) &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} k_{2m+1, 2n+1} F_{2m+1, 2n+1} \\ &> \frac{1}{3} k_{11} F_{11} + k_{33} F_{33} \\ &\geq \frac{16}{3\pi\ell} \left( \frac{1}{\sinh\left(\frac{\pi}{\ell}\right)} + \frac{1}{\sinh\left(\frac{3\pi}{\ell}\right)} \right) \\ &= C_4. \end{aligned}$$

Altogether, we obtain

$$\begin{aligned}
\det DU(0) &= \partial_1 U_1 \partial_2 U_2 \partial_3 U_3(0) \\
&\geq C_1^2 C_4 \\
&= \frac{128}{27} \cdot \frac{3 - 2\sqrt{2}}{\pi^5 \ell} \left( \frac{1}{\sinh(\frac{\pi}{\ell})} + \frac{1}{\sinh(\frac{3\pi}{\ell})} \right) \left( \frac{1}{\cosh(\frac{2\pi}{\ell})} + \frac{1}{3 \cosh(\frac{6\pi}{\ell})} \right)^2 \\
&\geq \frac{1}{400} \frac{1}{\ell} \exp\left(-\frac{5\pi}{\ell}\right).
\end{aligned}$$

□

**Lemma 5.7.** *Assume that  $(u_{n,m})_{n,m}$  is a positive sequence such that for each  $n$   $mu_{n,m}$  is decreasing in  $n$  for each  $m$  and in  $m$  for each  $n$  towards zero, then*

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} u_{2n+1,2m+1} (-1)^{n+m} > u_{1,1} + u_{3,3} - u_{1,3} - u_{3,1} \geq \frac{1}{3} u_{1,1} + u_{3,3}.$$

*Proof.* We apply the alternating series estimation theorem three times:

$$\begin{aligned}
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} u_{2n+1,2m+1} (-1)^{n+m} &> \sum_{m=0}^{\infty} u_{1,2m+1} (-1)^m - \sum_{m=0}^{\infty} u_{3,2m+1} (-1)^m \\
&> u_{1,1} - u_{1,3} - (u_{3,1} - u_{3,3}) \\
&= \frac{1}{3} u_{1,1} + u_{3,3} + \frac{1}{3} (u_{1,1} - 3u_{3,1}) + \frac{1}{3} (u_{1,1} - 3u_{1,3}) \\
&\geq \frac{1}{3} u_{1,1} + u_{3,3}.
\end{aligned}$$

□

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