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To cite this version:

Jang Soo Kim. On q-integrals over order polytopes (extended abstract). 28-th International Conference on Formal Power Series and Algebraic Combinatorics, Simon Fraser University, Jul 2016, Vancouver, Canada. 10.46298/dmtcs.6413 . hal-02168131

HAL Id: hal-02168131

https://hal.science/hal-02168131v1

Submitted on 28 Jun 2019

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FPSAC 2016 Vancouver, Canada DMTCS proc. **BC**, 2016, 707–718

On q-integrals over order polytopes (extended abstract)

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Abstract. A q-integral over an order polytope coming from a poset is interpreted as a generating function of linear extensions of the poset. As an application, the q-beta integral and a q-analog of Dirichlet's integral are computed. A combinatorial interpretation of a q-Selberg integral is also obtained.

Résumé. Une q-intégrale sur un polytope provenant d'un poset est interprété comme une série génératrice d'extensions linéaires de la poset. En application, l'intégrale q-bêta et un q-analogue de l'intégrale de Dirichlet sont calculés. Une interprétation combinatoire de une intégrale q-Selberg est également obtenue.

Keywords. q-integral, order polytope, q-Selberg integral

1 Introduction

In this extended abstract we give a combinatorial interpretation of q-integrals over order polytopes. The motivation of this extended abstract is to generalize Stanley's combinatorial interpretation of the Selberg integral.

The Selberg integral is the following integral first evaluated by Selberg [8] in 1944:

$$\begin{aligned} S_n(\alpha,\beta,\gamma) &= \int_0^1 \cdots \int_0^1 \prod_{i=1}^n x_i^{\alpha-1}(1-x_i)^{\beta-1} \prod_{1\le i<j\le n} |x_i-x_j|^{2\gamma} dx_1\cdots dx_n \\ &= \prod_{j=1}^n \frac{\Gamma(\alpha+(j-1)\gamma)\Gamma(\beta+(j-1)\gamma)\Gamma(1+j\gamma)}{\Gamma(\alpha+\beta+(n+j-2)\gamma)\Gamma(1+\gamma)}, \end{aligned} \tag{1}$$

where n is a positive integer and α,β,γ are complex numbers such that $\mathrm{Re}(\alpha)>0$, $\mathrm{Re}(\beta)>0$, and $\mathrm{Re}(\gamma)>-\min\{1/n, \mathrm{Re}(\alpha)/(n-1), \mathrm{Re}(\beta)/(n-1)\}$. Stanley [9, Exercise 1.10 (b)] found a combinatorial interpretation of the above integral when $\alpha-1, \beta-1$ and 2γ are nonnegative integers in terms of permutations. His idea is to interprete the integral as the probability that a random permutation satisfies

†The first author was partially supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2013R1A1A2061006).

‡The second author was supported by NSF grant DMS-1148634.

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certain properties. This idea uses the fact that a real number $x \in (0,1)$ can be understood as the probability that a random number selected from $(0,1)$ lies on an interval of length x is equal to x. Generalizing this fact to q-integrals is not obvious. We instead consider a different approach by interpreting q-integrals as generating functions in q.

Throughout this extended abstract we assume $0 < q < 1$. We will use the following notation for q-series:

$$[n]_q = \frac{1-q^n}{1-q}, \qquad [n]_q! = [1]_q[2]_q \cdots [n]_q, \qquad (a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}).$$

We also use the notation $[n] := \{1, 2, \ldots, n\}$. We denote by $\mathfrak{S}_n$ the set of permutations on $[n]$.

In order to state our main results, we need several definitions. First, recall that the q-integral of a function f over (a,b) is

$$\int_a^b f(x)d_qx = (1-q)\sum_{i=0}^{\infty}\left(f(bq^i)bq^i - f(aq^i)aq^i\right).$$

Note that the limit as $q \to 1$, the q-integral becomes the usual integral. It is well known that

$$\int_a^b x^n d_qx = \frac{b^{n+1}-a^{n-1}}{[n+1]_q}.$$

For a permutation $\pi = \pi_1 \ldots \pi_n \in \mathfrak{S}_n$ we will denote the region

$$\{(x_1, \ldots, x_n) : a \le x_{\pi_1} \le \cdots \le x_{\pi_n} \le b\}$$

by $a \le x_{\pi_1} \le \cdots \le x_{\pi_n} \le b$. The q-integral over this region is defined as follows.

Definition 1.1. For a permutation $\pi = \pi_1 \ldots \pi_n \in \mathfrak{S}_n$, we define

$$\int_{a \le x_{\pi_1} \le \cdots \le x_{\pi_n} \le b} f(x_1, \ldots, x_n)d_qx_1 \cdots d_qx_n = \int_{s_n}^{t_n} \cdots \int_{s_1}^{t_1} f(x_1, \ldots, x_n)d_qx_1 \cdots d_qx_n,$$

where

$$s_i = s_i(x_{i+1}, \ldots, x_n) = \max\left(\{x_j : j > i, \pi_j < \pi_i\} \cup \{a\}\right),$$
$$t_i = t_i(x_{i+1}, \ldots, x_n) = \min\left(\{x_j : j > i, \pi_j > \pi_i\} \cup \{b\}\right).$$

For example,

$$\int_{a \le x_3 \le x_1 \le x_5 \le x_2 \le x_4 \le x_6 \le b} f(x_1, \ldots, x_6)d_qx_1 \cdots d_qx_6$$
$$= \int_a^b \int_a^{x_6} \int_{x_5}^{x_6} \int_a^{x_5} \int_{x_5}^{x_4} \int_{x_3}^{x_5} f(x_1, \ldots, x_6)d_qx_1 \cdots d_qx_6.$$

Note that since s_i and t_i are constants when $x_{i+1}, \ldots, x_n$ and q are fixed, the above definition makes sense as a q-integral.

Let P be a poset on a set $\{x_1, x_2, \ldots, x_n\}$. By abuse of notation, we will consider x_i as an element of P and also as a variable. For $i \in [n]$, we denote by P_i the subposet of P consisting of $x_i, x_{i+1}, \ldots, x_n$. Let $\mathcal{O}_a^b(P)$ be the polytope

$$\mathcal{O}_a^b(P) = \{(x_1, \ldots, x_n) \in [a,b]^n : x_i \leq x_j \text{ if } x_i \leq_P x_j\}.$$

We will also use $\mathcal{O}(P)$ in place of $\mathcal{O}_0^1(P)$.

We define the q-integral over this polytope as follows.

Definition 1.2. The q-integral of $f(x_1, \ldots, x_n)$ over the polytope $\mathcal{O}_a^b(P)$ is defined by

$$\int_{\mathcal{O}_a^b(P)} f(x_1, \ldots, x_n) d_q x_1 \cdots d_q x_n = \int_{s_n}^{t_n} \int_{s_{n-1}}^{t_{n-1}} \cdots \int_{s_1}^{t_1} f(x_1, \ldots, x_n) d_q x_1 \cdots d_q x_n,$$

where

$$s_i = s_i(x_{i+1}, \ldots, x_n) = \max\left(\{a\} \cup \{x_k : x_k \lessdot_{P_i} x_i\}\right), \tag{2}$$

$$t_i = t_i(x_{i+1}, \ldots, x_n) = \min\left(\{b\} \cup \{x_k : x_i \lessdot_{P_i} x_k\}\right). \tag{3}$$

By definition we have

$$\lim_{q \to 1} \int_{\mathcal{O}_a^b(P)} f(x_1, \ldots, x_n) d_q x_1 \cdots d_q x_n = \int_{\mathcal{O}_a^b(P)} f(x_1, \ldots, x_n) dx_1 \cdots dx_n.$$

Note that if P is the chain $x_{\pi_1} < x_{\pi_2} < \cdots < x_{\pi_n}$ for a permutation $\pi \in \mathfrak{S}_n$, then $\mathcal{O}_a^b(P)$ is the simplex $\{(x_1, \ldots, x_n) : a \leq x_{\pi_1} \leq \cdots \leq x_{\pi_n} \leq b\}$ and

$$\int_{\mathcal{O}_a^b(P)} f(x_1, \ldots, x_n) d_q x_1 \cdots d_q x_n = \int_{a \leq x_{\pi_1} \leq \cdots \leq x_{\pi_n} \leq b} f(x_1, \ldots, x_n) d_q x_1 \cdots d_q x_n.$$

The remainder of this extended abstract is organised as follows. In Section 2 we will find a formula for the q-integral over a simplex. In Section 3 we show that the q-volume of the order polytope $\mathcal{O}(P)$ can be written as a generating function for linear extensions of the poset P. In Section 4 we find a relation between q-integrals over order polytopes $\mathcal{O}(P)$ and $\mathcal{O}(Q)$ for two posets P and Q such that Q is obtained from P by adding a chain. In Section 5 we show that the q-beta integral and a q-analog of Dirichlet's integral can be computed using our methods. We also discuss a connection with the general q-beta integral of Andrews and Askey [2]. In Section 6 we express a q-Selberg integral in terms of the q-volume of certain order polytope and find a combinatorial interpretation for it.

2 q-integrals over simplices

In this section we compute q-integrals of a multivariate function over the simplex $\{(x_1, \ldots, x_n) : a \leq x_{\pi_1} \leq \cdots \leq x_{\pi_n} \leq b\}$.

Let $\pi = \pi_1 \pi_2 \ldots \pi_n \in \mathfrak{S}_n$. An integer $i \in [n-1]$ is called a *descent* of π if $\pi_i > \pi_{i+1}$. Let $\mathrm{Des}(\pi)$ be the set of descents of π. We define $\mathrm{des}(\pi)$ and $\mathrm{maj}(\pi)$ to be the number of descents of π and the sum of descents of π, respectively.

Lemma 2.1. *For $\pi \in \mathfrak{S}_n$ and $r, s \in \{0, 1, 2, \dots\} \cup \{\infty\}$, we have*

$$\int_{q^r \le x_{\pi_1} \le \cdots \le x_{\pi_n} \le q^s} f(x_{\pi_1}, \dots, x_{\pi_n}) d_q x_1 \cdots d_q x_n = (1-q)^n \sum_{\substack{r > i_1 \ge \cdots \ge i_n \ge s \\ i_j > i_{j+1} \text{ if } j \in \mathrm{Des}(\pi)}} f(q^{i_1}, \dots, q^{i_n}) q^{i_1 + \cdots + i_n}.$$

The following theorem gives a formula for the volume of the simplex.

Theorem 2.2. *For $\pi \in \mathfrak{S}_n$ and real numbers $a < b$, we have*

$$\int_{a \le x_{\pi_1} \le \cdots \le x_{\pi_n} \le b} d_q x_1 \cdots d_q x_n = \frac{b^n q^{\mathrm{maj}(\pi)}}{[n]_q!} (a q^{-\mathrm{des}(\pi)}/b; q)_n.$$

When $a = 0$ and $b = 1$ in Theorem 2.2 we obtain the following corollary.

Corollary 2.3. *For $\pi \in \mathfrak{S}_n$,*

$$\int_{0 \le x_{\pi_1} \le \cdots \le x_{\pi_n} \le 1} d_q x_1 \cdots d_q x_n = \frac{q^{\mathrm{maj}(\pi)}}{[n]_q!}.$$

3 q-integrals over order polytopes

In this section we consider q-integrals over order polytopes. We need some of the P-partition theory in [9, Chapter 3].

Let $\pi \in \mathfrak{S}_n$. A function $f : [n] \to \mathbb{N}$ is called *π-compatible* if $f(\pi_1) \ge f(\pi_2) \ge \cdots \ge f(\pi_n)$ and $f(\pi_i) > f(\pi_{i+1})$ for all $i \in \mathrm{Des}(\pi)$. Then for any function $f : [n] \to \mathbb{N}$, there is a unique $\pi \in \mathfrak{S}_n$ for which f is π-compatible.

Let P be a poset with n elements. A *labeling* of P is a bijection $\omega : P \to [n]$. A *(P, ω)-partition* is a function $\sigma : P \to \mathbb{N}$ such that

- $\sigma(x) \ge \sigma(y)$ if $x \le_P y$,
- $\sigma(x) > \sigma(y)$ if $x \le_P y$ and $\omega(x) > \omega(y)$.

A *linear extension of* P is an arrangement $(t_1, t_2, \dots, t_n)$ of the elements in P such that if $t_i <_P t_j$ then $i < j$. The *Jordan-Hölder set* $\mathcal{L}(P, \omega)$ is the set of permutations of the form $\omega(t_1)\omega(t_2)\cdots\omega(t_n)$ for some linear extension $(t_1, t_2, \dots, t_n)$ of P.

Let $\mathcal{A}(P, \omega)$ denote the set of (P, ω)-partitions. For a permutation $\pi \in \mathfrak{S}_n$, we denote by $S_\pi(P, \omega)$ the set of functions $\sigma : P \to \mathbb{N}$ such that $\sigma \circ \omega^{-1}$ is π-compatible. Notice that in the definition of $S_\pi(P, \omega)$ we only need the underlying set of P. Thus we can consider $S_\pi(P, \omega)$ when P is a set with n elements and $\omega : P \to [n]$ is a bijection. We will use the following facts [9, Lemma 3.15.3, Theorem 3.15.7]:

$$\mathcal{A}(P, \omega) = \bigcup_{\pi \in \mathcal{L}(P, \omega)} S_\pi(P, \omega), \tag{4}$$

$$\sum_{\sigma\in\mathcal{A}(P,\omega)} q^{|\sigma|} = \frac{\sum_{\pi\in\mathcal{L}(P,\omega)} q^{\mathrm{maj}(\pi)}}{(q;q)_n}. \tag{5}$$

Lemma 2.1 can be restated as follows.

Proposition 3.1. *Let $P = \{x_1, \dots, x_n\}$ with a bijection $\omega : P \to [n]$ defined by $\omega(x_i) = i$. Then for integers $r > s \geq 0$ and a permutation $\pi \in \mathfrak{S}_n$, we have*

$$\int_{q^r \leq x_{\pi_1} \leq \cdots \leq x_{\pi_n} \leq q^s} f(x_1, \dots, x_n) d_q x_1 \cdots d_q x_n = (1-q)^n \sum_{\substack{\sigma\in S_\pi(P,\omega)\\ r>\max(\sigma),\min(\sigma)\geq s}} f(q^{\sigma(x_1)}, \dots, q^{\sigma(x_n)}) q^{|\sigma|}.$$

In particular, when $r \to \infty$ and $s = 0$, we have

$$\int_{0 \leq x_{\pi_1} \leq \cdots \leq x_{\pi_n} \leq 1} f(x_1, \dots, x_n) d_q x_1 \cdots d_q x_n = (1-q)^n \sum_{\sigma\in S_\pi(P,\omega)} f(q^{\sigma(x_1)}, \dots, q^{\sigma(x_n)}) q^{|\sigma|}.$$

Proposition 3.2. *Let P be a poset on $\{x_1, \dots, x_n\}$ with labeling $\omega : P \to [n]$ given by $\omega(x_i) = i$ for all $i \in [n]$. Then for real numbers $a < b$, we have*

$$\int_{\mathcal{O}_a^b(P)} f(x_1, \dots, x_n) d_q x_1 \cdots d_q x_n = \sum_{\pi\in\mathcal{L}(P,\omega)} \int_{a \leq x_{\pi_1} \leq \cdots \leq x_{\pi_n} \leq b} f(x_1, \dots, x_n) d_q x_1 \cdots d_q x_n.$$

Theorem 3.3. *Let P be a poset on $\{x_1, \dots, x_n\}$ with labeling $\omega : P \to [n]$ given by $\omega(x_i) = i$ for all $i \in [n]$. Then, we have*

$$\int_{\mathcal{O}(P)} f(x_1, \dots, x_n) d_q x_1 \cdots d_q x_n = (1-q)^n \sum_{\sigma\in\mathcal{A}(P,\omega)} f(q^{\sigma(x_1)}, \dots, q^{\sigma(x_n)}) q^{|\sigma|}. \tag{6}$$

Moreover, if $Q = \mathcal{O}(P) \cap ([q^{r_1}, q^{s_1}] \times \cdots \times [q^{r_n}, q^{s_n}])$ for $r_i, s_i \in \{0, 1, 2, \dots\} \cup \{\infty\}$, then

$$\int_Q f(x_1, \dots, x_n) d_q x_1 \cdots d_q x_n = (1-q)^n \sum_\sigma f(q^{\sigma(x_1)}, \dots, q^{\sigma(x_n)}) q^{|\sigma|}, \tag{7}$$

where the sum is over all $\sigma \in \mathcal{A}(P, \omega)$ with $s_i \leq \sigma(x_i) < r_i$ for every $i \in \{1, 2, \dots, n\}$.

Thus, when $f(x_1, x_2, \dots, x_n) = 1$ in Theorem 3.3, we obtain the following corollary.

Corollary 3.4. *Let P be a poset on $\{x_1, \dots, x_n\}$ with labeling $\omega : P \to [n]$ given by $\omega(x_i) = i$ for all $i \in [n]$. Then*

$$\int_{\mathcal{O}(P)} d_q x_1 \cdots d_q x_n = \frac{1}{[n]_q!} \sum_{\pi\in\mathcal{L}(P,\omega)} q^{\mathrm{maj}(\pi)}.$$

4 Changing posets

In this section we consider two posets P and Q where Q is obtained from P by adding a chain. The results in this section will be used for the next two sections.

Lemma 4.1. *Let $\rho \in S_m$. Let P be a poset on $\{x_1, \dots, x_n\}$ with $x_s \leq_P x_t$. Define Q to be the poset on $\{x_1, \dots, x_n, y_1, y_2, \dots, y_m\}$ with relations $x_i \leq_Q x_j$ if and only if $x_i \leq_P x_j$, and $x_s \leq_Q y_{\rho_1} \leq_Q \cdots \leq_Q y_{\rho_m} \leq_Q x_t$. Then, we have*

$$\int_{\mathcal{O}(P)} q^{\mathrm{maj}(\rho)} x_t^m (q^{-\mathrm{des}(\rho)} x_s/x_t; q)_m f(x_1, \dots, x_n) d_q x_1 \cdots d_q x_n$$
$$= [m]_q! \int_{\mathcal{O}(Q)} f(x_1, \dots, x_n) d_q y_1 \cdots d_q y_m d_q x_1 \cdots d_q x_n, \quad (8)$$

$$\int_{\mathcal{O}(P)} q^{\mathrm{maj}(\rho)} x_t^m (q^{1-\mathrm{des}(\rho)} x_s/x_t; q)_m f(x_1, \dots, x_n) d_q x_1 \cdots d_q x_n$$
$$= [m]_q! \int_{\mathcal{O}(Q)} f(x_1, \dots, x_n) d_q x_1 \cdots d_q x_s d_q y_1 \cdots d_q y_m d_q x_{s+1} \cdots d_q x_n. \quad (9)$$

Moreover, (9) *holds if the order of integration is obtained from $d_q x_1 \cdots d_q x_n$ by inserting $d_q y_1 \cdots d_q y_m$ anywhere between x_s and x_t.*

Similarly we can prove the following lemma.

Lemma 4.2. *Let $\rho \in S_m$ and $s, t \in [m]$. Let P be a poset on $\{x_1, \dots, x_n\}$. Define Q_1 to be the poset on $\{x_1, \dots, x_n, y_1, y_2, \dots, y_m\}$ with relations $x_i \leq_{Q_1} x_j$ if and only if $x_i \leq_P x_j$, and $x_s \leq_{Q_1} y_{\rho_1} \leq_{Q_1} \cdots \leq_{Q_1} y_{\rho_m}$. Define Q_2 to be the poset on $\{x_1, \dots, x_n, y_1, y_2, \dots, y_m\}$ with relations $x_i \leq_{Q_2} x_j$ if and only if $x_i \leq_P x_j$, and $y_{\rho_1} \leq_{Q_2} \cdots \leq_{Q_2} y_{\rho_m} \leq_{Q_2} x_t$. Then, we have*

$$\int_{\mathcal{O}(P)} q^{\mathrm{maj}(\rho)} (q^{1-\mathrm{des}(\rho)} x_s; q)_m f(x_1, \dots, x_n) d_q x_1 \cdots d_q x_n$$
$$= [m]_q! \int_{\mathcal{O}(Q_1)} f(x_1, \dots, x_n) d_q x_1 \cdots d_q x_n d_q y_1 \cdots d_q y_m, \quad (10)$$

$$\int_{\mathcal{O}(P)} q^{\mathrm{maj}(\rho)} x_t^m f(x_1, \dots, x_n) d_q x_1 \cdots d_q x_n$$
$$= [m]_q! \int_{\mathcal{O}(Q_2)} f(x_1, \dots, x_n) d_q y_1 \cdots d_q y_m d_q x_1 \cdots d_q x_n, \quad (11)$$

5 Examples

In this section we will compute the q-beta integral and a q-analog of Dirichlet's integral using our results. We will then find a connection with the general q-beta integral due to Andrews and Askey [2].

5.1 The q-beta integral

The following is the well known integral called the q-beta integral. We can prove this using our methods.

Corollary 5.1. *We have*

$$\int_0^1 x^n (xq;q)_m d_q x = \frac{[n]_q![m]_q!}{[n+m+1]_q!}.$$

Proof. By (10) and (11) we have

$$\int_0^1 x^n (xq;q)_m d_q x = [n]_q![m]_q! \int_{0\le y_1\le\cdots\le y_n\le x\le z_1\le\cdots\le z_m\le 1} d_q y_1 \cdots d_q y_n d_q x d_q z_1 \cdots d_q z_m.$$

By Corollary 2.3, we get the q-beta integral formula. □

Corollary 5.2. *Let π be a permutation on $[n]$. Let T be the poset obtained from the chain $\{\pi_1 < \pi_2 < \cdots < \pi_n\}$ by adding k_i elements covered by π_i. Then T becomes a tree and for each element v we define the* hook length *h_v of v to be the number of elements u with $u \le_T v$. Let $\mathrm{maj}(T) = \sum_{i\in\mathrm{Des}(\pi)} h_{\pi_i}$. Then*

$$\int_{0\le x_{\pi_1}\le\cdots\le x_{\pi_n}\le 1} x_1^{k_1}\cdots x_n^{k_n} d_q x_1 \cdots d_q x_n = \frac{q^{\mathrm{maj}(T)}}{\prod_{v\in T}[h_v]_q}.$$

Proof. This can be shown by applying (11) to each factor $x_i^{k_i}$. □

5.2 A q-analog of Dirichlet integral

We now consider the simplex

$$\Omega_n = \{(x_1,\ldots,x_n)\in[0,1]^n : x_1+\cdots+x_n \le 1\}.$$

Dirichlet's integral is the following, see [1, Theorem 1.8.1]:

$$\int_{\Omega_n} x_1^{\alpha_1-1}\cdots x_n^{\alpha_n-1} dx_1\cdots dx_n = \frac{\Gamma(\alpha_1)\ldots\Gamma(\alpha_n)}{\Gamma(1+\alpha_1+\cdots+\alpha_n)}. \tag{12}$$

By introducing new variables $y_i = x_1+\cdots+x_i$ and integers $k_i = \alpha_i+1$, we get an equivalent version of (12)

$$\int_{0\le y_1\le y_2\le\cdots\le y_n\le 1} y_1^{k_1}(y_2-y_1)^{k_2}\cdots(y_n-y_{n-1})^{k_n} dy_1\cdots dy_n = \frac{k_1!\cdots k_n!}{(n+k_1+\cdots+k_n)!}. \tag{13}$$

We obtain a q-analog of (13) as follows.

Corollary 5.3. *For nonnegative integers $k_1,\ldots,k_n$, we have*

$$\int_{0\le y_1\le y_2\le\cdots\le y_n\le 1} y_1^{k_1}(qy_1/y_2;q)_{k_2} y_2^{k_2}\cdots(qy_{n-1}/y_n;q)_{k_n} y_n^{k_n} d_q y_1\cdots d_q y_n = \frac{[k_1]_q!\cdots[k_n]_q!}{[n+k_1+\cdots+k_n]_q!}.$$

Proof. By applying (11) to the factor $y_1^{k_1}$ and (9) to the factors $(qy_{i-1}/y_i;q)_{k_i}y_i^{k_i}$ for $2 \le i \le n$, the left hand side is equal to

$$[k_1]_q![k_2]_q!\cdots[k_n]_q!\int_Q d_qx_{1,1}\cdots d_qx_{1,k_1}d_qy_1d_qx_{2,1}\cdots d_qx_{2,k_2}d_qy_2\cdots d_qx_{n,1}\cdots d_qx_{n,k_n}d_qy_n,$$

where

$$Q=\{0\le x_{1,1}\le\cdots\le x_{1,k_1}\le y_1\le x_{2,1}\le\cdots\le x_{2,k_2}\le y_2\le\cdots\le x_{n,1}\le\cdots\le x_{n,k_n}\le y_n\le 1\}.$$

By Corollary 2.3 we obtain the right hand side. □

5.3 The general q-beta integral of Andrews and Askey

In this subsection, we will show that Theorem 2.2 is related to the following result of Andrews and Askey [2] on a generalization of the q-beta integral: for $|q|<1$,

$$\int_a^b \frac{(qx/a;q)_\infty(qx/b;q)_\infty}{(Ax/a;q)_\infty(Bx/b;q)_\infty}d_qx=\frac{(1-q)(q;q)_\infty(AB;q)_\infty ab(a/b;q)_\infty(b/a;q)_\infty}{(A;q)_\infty(B;q)_\infty(a-b)(Ba/b;q)_\infty(Ab/a;q)_\infty}. \tag{14}$$

Let $\pi\in\mathfrak{S}_n$. We will compute the integral in Theorem 2.2 in a different way. First we decompose π into $\pi=\sigma n\tau$ using the largest integer n. Suppose that σ and τ have r and s letters respectively and $\mathrm{des}(\sigma)=k_1$, $\mathrm{des}(\tau)=k_2$. Then $n=r+s+1$ and $k=k_1+k_2+1$ if $k_2\ge 1$ and $k=k_1$ if $k_2=0$. The integral in Theorem 2.2 can be written as

$$\int_a^b\left(\int_{a\le x_{\sigma_1}\le\cdots\le x_{\sigma_r}\le x_n} d_qy_1\cdots d_qy_r\int_{x_n\le x_{\tau_1}\le\cdots\le x_{\tau_s}\le b} d_qz_1\cdots d_qz_s\right)d_qx_n,$$

where $y_1,\dots,y_r$ and $z_1,\dots,z_s$ are obtained by rearranging $x_{\sigma_1},\dots,x_{\sigma_r}$ and $x_{\tau_1},\dots,x_{\tau_s}$ respectively so that subscripts are increasing. By applying Theorem 2.2 to the two inside integrals, the above is equal to

$$\int_a^b \frac{x^rq^{\mathrm{maj}(\sigma)}}{[r]_q!}(aq^{-k_1}/x;q)_r\frac{b^sq^{\mathrm{maj}(\tau)}}{[s]_q!}(xq^{-k_2}/b;q)_sd_qx. \tag{15}$$

Note that $\mathrm{maj}(\pi)=\mathrm{maj}(\sigma)+\mathrm{maj}(\tau)+(r+1)(k_2+1)$ if $s\ge 1$ and $\mathrm{maj}(\pi)=\mathrm{maj}(\sigma)$ if $s=0$. In either case we can write $\mathrm{maj}(\pi)=\mathrm{maj}(\sigma)+\mathrm{maj}(\tau)+(r+1)(k-k_1)$. Since (15) is equal to the integral in Theorem 2.2 we obtain the following.

Proposition 5.4. *Let n,r,s,k_1,k_2 be nonnegative integers such that $n=r+s+1$, $k_1\le r$, $k_2\le s$, and $k=k_1+k_2+1$ if $s\ge 1$ and $k=k_1$ if $s=0$. Then*

$$\int_a^b x^r(aq^{-k_1}/x;q)_r(xq^{-k_2}/b;q)_sd_qx=\frac{[r]_q![s]_q!}{[n]_q!}b^{r+1}q^{(k-k_1)(r+1)}(aq^{-k}/b;q)_n.$$

We now consider the case $s\ge 1$ in Proposition 5.4 so that $k=k_1+k_2+1$. One can rewrite the integral in Proposition 5.4 as

$$\begin{aligned}(-1)^ra^rq^{\binom{r}{2}-k_1r}&\int_a^b\frac{(xq^{1-r+k_1}/a;q)_\infty(xq^{-k_2}/b;q)_\infty}{(xq^{k_1+1}/a;q)_\infty(xq^{s-k_2}/b;q)_\infty}d_qx\\&=(-1)^ra^rq^{\binom{r}{2}-k_1r}\int_{aq^{r-k_1}}^{bq^{k_2+1}}\frac{(xq^{1-r+k_1}/a;q)_\infty(xq^{-k_2}/b;q)_\infty}{(xq^{k_1+1}/a;q)_\infty(xq^{s-k_2}/b;q)_\infty}d_qx,\end{aligned}$$

where the equality follows from the fact that the integrand is 0 if $x = bq^j$ for $0 \le j \le k_2$ and $x = aq^j$ for $0 \le j \le r - k_1 - 1$. Thus Proposition 5.4 is equivalent to

$$\int_{aq^{r-k_1}}^{bq^{k_2+1}} \frac{(xq^{1-r+k_1}/a;q)_\infty (xq^{-k_2}/b;q)_\infty}{(xq^{k_1+1}/a;q)_\infty (xq^{s-k_2}/b;q)_\infty} d_q x = \frac{(-1)^r b^{r+1} [r]_q! [s]_q! q^{kr+k-k_1-\binom{r}{2}} (aq^{-k}/b;q)_n}{a^r [n]_q!}.$$

One can check that the above equation is the special case of (14) with substitution

$$(a, b, A, B) \mapsto (aq^{r-k_1}, bq^{k_2+1}, q^{r+1}, q^{s+1}).$$

6 q-Selberg integrals

In this section we will find a combinatorial interpretation for a q-Selberg integral.

For a set of variables $x = (x_1, \ldots, x_n)$ we denote

$$\Delta(x) = \prod_{1 \le i < j \le n} (x_i - x_j).$$

There are many generalizations of the Selberg integral, see [4]. In this section we consider the following two q-Selberg integrals:

$$\mathrm{AS}_n(\alpha, \beta, m) = \int_0^1 \cdots \int_0^1 \prod_{i=1}^n x_i^{\alpha-1} (qx_i;q)_{\beta-1} \prod_{1 \le i < j \le n} x_j^{2m} \left(q^{1-m} x_i/x_j; q\right)_{2m} d_q x_1 \cdots d_q x_n,$$

$$\mathrm{KS}_n(\alpha, \beta, m) = \int_0^1 \cdots \int_0^1 \prod_{i=1}^n x_i^{\alpha-1} (qx_i;q)_{\beta-1} \Delta_{n,m}(x) d_q x_1 \cdots d_q x_n,$$

where

$$\Delta_{n,m}(x) = \prod_{1 \le i < j \le n} x_j^m \left(q^{1-m} x_i/x_j; q\right)_m x_j^m \left(x_i/x_j; q\right)_m = \Delta(x) \prod_{1 \le i < j \le n} x_j^{2m-1} \left(q^{1-m} x_i/x_j; q\right)_{2m-1}.$$

It is easy to check that $\Delta_{n,m}(x)$ is symmetric in the variables $x_1, x_2, \ldots, x_n$.

Askey [3] conjectured that

$$\mathrm{AS}_n(\alpha, \beta, m) = q^{\alpha m \binom{n}{2} + 2m^2 \binom{n}{3}} \prod_{j=1}^n \frac{\Gamma_q(\alpha + (j-1)m) \Gamma_q(\beta + (j-1)m) \Gamma_q(1 + jm)}{\Gamma_q(\alpha + \beta + (n+j-2)m) \Gamma_q(1+m)}, \tag{16}$$

which has been proved by Habsieger [5] and Kadell [6] independently. Kadell [7, Eq. (4.11)] showed that

$$\mathrm{AS}_n(\alpha, \beta, m) = \frac{[n]_{q^m}!}{n!} \mathrm{KS}_n(\alpha, \beta, m). \tag{17}$$

Since the integrand of $\mathrm{KS}_n(\alpha, \beta, m)$ is symmetric under any permutation of $x_1, \ldots, x_n$ and zero whenever $x_i = x_j$, we have

$$\mathrm{KS}_n(\alpha, \beta, m) = n! \int_{0 \le x_1 \le \cdots \le x_n \le 1} \prod_{i=1}^n x_i^{\alpha-1} (qx_i;q)_{\beta-1} \Delta_{n,m}(x) d_q x_1 \cdots d_q x_n.$$

Thus we have

$$\begin{aligned}\mathrm{AS}_n(\alpha,\beta,m) &= \int_0^1 \cdots \int_0^1 \prod_{i=1}^n x_i^{\alpha-1}(qx_i;q)_{\beta-1} \prod_{1\le i<j\le n} x_i^{2m}\left(q^{1-m}x_j/x_i;q\right)_{2m} d_qx_1\cdots d_qx_n \\ &= [n]_{q^m}! \int_{0\le x_1\le\cdots\le x_n\le 1} \prod_{i=1}^n x_i^{\alpha-1}(qx_i;q)_{\beta-1}\Delta_{n,m}(x) d_qx_1\cdots d_qx_n.\end{aligned}$$

By (16) and (17), we have

$$\begin{aligned}&\int_{0\le x_1\le\cdots\le x_n\le 1} \prod_{i=1}^n x_i^{\alpha-1}(qx_i;q)_{\beta-1} \prod_{1\le i<j\le n} x_j^{2m-1}\left(q^{1-m}x_i/x_j;q\right)_{2m-1} \Delta(x) d_qx_1\cdots d_qx_n \\ &\qquad = q^{\alpha m\binom{n}{2}+2m^2\binom{n}{3}} \prod_{j=1}^n \frac{\Gamma_q(\alpha+(j-1)m)\Gamma_q(\beta+(j-1)m)\Gamma_q(jm)}{\Gamma_q(\alpha+\beta+(n+j-2)m)\Gamma_q(m)}. \qquad (18)\end{aligned}$$

We define the *Selberg poset* $P(n,r,s,m)$ to be the poset in which the elements are $x_i, y_i^{(a)}, z_i^{(b)}, w_{i,j}^{(k)}$ for $i,j\in[n], a\in[r], b\in[s], k\in[m]$ with $i\ne j$, and the covering relations are as follows:

- $x_i < w_{i,j}^{(1)} < \cdots < w_{i,j}^{(m)} < x_j$ for $1\le i<j\le n$,
- $x_i < w_{j,i}^{(m)} < \cdots < w_{j,i}^{(1)} < x_j$ for $1\le i<j\le n$,
- $y_i^{(1)} < \cdots < y_i^{(r)} < x_i < z_i^{(1)} < \cdots < z_i^{(s)}$ for $1\le i\le n$.

For an example of $P(n,r,s,m)$, see Figure 1.

The following theorem implies that the q-Selberg integral is the q-volume of an order polytope up to a certain factor.

Theorem 6.1. *We have*

$$\begin{aligned}&\int_{0\le x_1\le\cdots\le x_n\le 1} \prod_{i=1}^n x_i^r(qx_i;q)_s \prod_{1\le i<j\le n} x_j^m\left(q^{1-m}x_i/x_j;q\right)_m x_j^m\left(x_i/x_j;q\right)_m d_qx_1\cdots d_qx_n \\ &\qquad = q^{-\binom{m}{2}\binom{n}{2}}([r]_q!)^n([s]_q!)^n([m]_q!)^{2\binom{n}{2}} \int_{\mathcal{O}(P(n,r,s,m))} d_qW d_qY d_qX d_qZ,\end{aligned}$$

where the order of integration $d_qW d_qY d_qX d_qZ$ is given by

$$\left(\prod_{1\le i\ne j\le n} d_qw_{i,j}^{(1)}\cdots d_qw_{i,j}^{(m)}\right)\left(\prod_{i=1}^n d_qy_i^{(1)}\cdots d_qy_i^{(r)}\right)\left(\prod_{i=1}^n d_qx_i\right)\left(\prod_{i=1}^n d_qz_i^{(1)}\cdots d_qz_i^{(s)}\right). \qquad (19)$$

In the order of integration (19), the order of $d_qw_{i,j}^{(k)}$ and $d_qw_{i',j'}^{(k)}$ for $(i,j)\ne(i',j')$ does not matter.

The following corollary gives a combinatorial interpretation for the q-Selberg integral in terms of linear extensions.

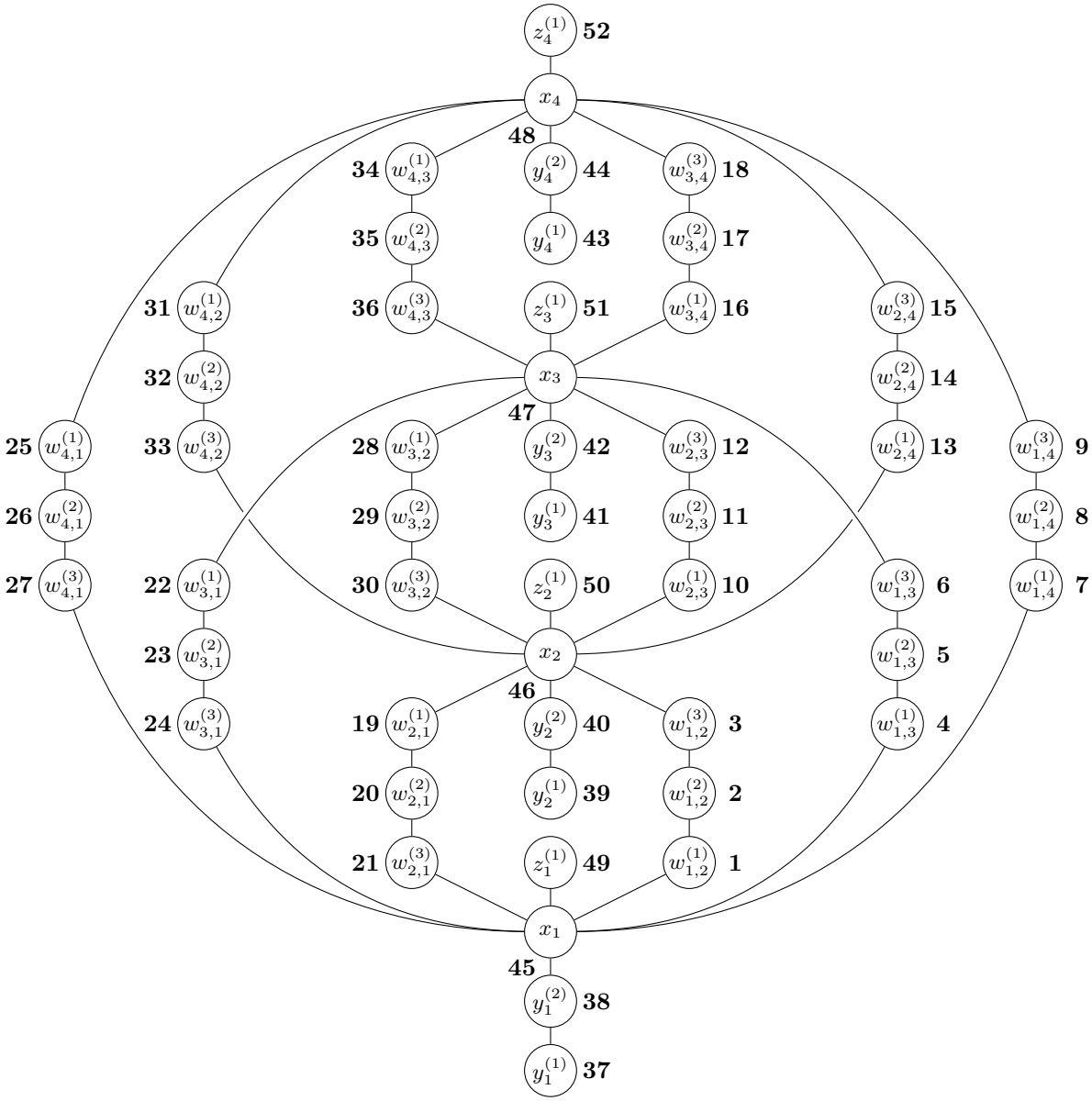

Fig. 1: The Selberg poset $P(n, r, s, m)$ for $n = 4, r = 2, s = 1, m = 3$ with labeling.

Corollary 6.2. *We have*

$$\int_{0\le x_1\le\cdots\le x_n\le 1}\prod_{i=1}^{n} x_i^r (qx_i;q)_s \prod_{1\le i<j\le n} x_j^m \left(q^{1-m}x_i/x_j;q\right)_m x_j^m \left(x_i/x_j;q\right)_m d_qx_1\cdots d_qx_n$$
$$= \frac{q^{-\binom{m}{2}\binom{n}{2}}([r]_q!)^n([s]_q!)^n([m]_q!)^{2\binom{n}{2}}}{[N]_q!}\sum_{\pi\in\mathcal{L}(P(n,r,s,m),\omega)} q^{\mathrm{maj}(\pi)},$$

where

$$N = n(r+s+1)+2m\binom{n}{2},$$

and, ω is the labeling of the Selberg poset $P(n,r,s,m)$ as shown in Figure 1.

Corollary 6.3. *We have*

$$\sum_{\pi\in\mathcal{L}(P(n,r,s,m),\omega)} q^{\mathrm{maj}(\pi)}$$
$$= \frac{q^{\binom{m}{2}\binom{n}{2}+(r+1)m\binom{n}{2}+2m^2\binom{n}{3}}[N]_q!}{([r]_q!)^n([s]_q!)^n([m]_q!)^{2\binom{n}{2}}}\prod_{j=1}^{n}\frac{[r+(j-1)m]_q![s+(j-1)m]_q![jm-1]_q!}{[r+s+1+(n+j-2)m]_q![m-1]_q!},$$

where N and ω are the same as in Corollary 6.2.

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