SOME RESULTS FOR THE LARGE TIME BEHAVIOR OF HAMILTON-JACOBI EQUATIONS WITH CAPUTO TIME DERIVATIVE

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ABSTRACT. We obtain some Hölder regularity estimates for an Hamilton-Jacobi with fractional time derivative of order $\alpha \in (0,1)$ cast by a Caputo derivative. The Hölder seminorms are independent of time, which allows to investigate the large time behavior of the solutions. We focus on the Namah-Roquejoffre setting whose typical example is the Eikonal equation. Contrary to the classical time derivative case $\alpha = 1$, the convergence of the solution on the so-called projected Aubry set, which is an important step to catch the large time behavior, is not straightforward. Indeed, a function with nonpositive Caputo derivative for all time does not necessarily converge; we provide such a counterexample. However, we establish partial results of convergence under some geometrical assumptions.

1. INTRODUCTION.

In this note we are interested in nonlocal Hamilton-Jacobi equations with the form

$$(1.1) \qquad \partial_t^\alpha u + H(x, Du) = 0 \quad \text{in } Q := \mathbb{T}^N \times (0, +\infty).$$

subject to the initial condition

$$(1.2) \qquad u(\cdot, 0) = g \quad \text{in } \mathbb{T}^N,$$

for some $H \in C(\mathbb{T}^N \times \mathbb{R}^N)$ and $g \in \mathrm{Lip}(\mathbb{T}^N)$ given.

The nonlocal nature of the problem is cast by the operator ∂_t^α, which denotes the Caputo time derivative of order $\alpha \in (0,1)$, starting at time zero. For $\phi \in C^1(0, +\infty)$ it is defined as

$$(1.3) \qquad \partial_t^\alpha \phi(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\phi'(s)}{|t-s|^\alpha} ds,$$

where Γ is the Gamma function that acts as a normalizing constant making ∂_t^α become the usual first order derivative when $\alpha \to 1^-$, see [10, 16] and references therein. Following the ideas of [3], where under appropriate assumption on the function ϕ, its Caputo derivative in (1.3) can be equivalently computed as

$$(1.4) \qquad \partial_t^\alpha \phi(t) = \tilde{c}_\alpha \int_{-\infty}^t \frac{\phi(t) - \phi(s)}{|t-s|^{1+\alpha}} ds,$$

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for some $\tilde{c}_\alpha > 0$, and where we have extended ϕ as $\phi(t) = \phi(0)$ for $t < 0$. This operator has a nonlocal degenerate elliptic nature in the sense of Barles and Imbert [6] that allows to conclude comparison principle among viscosity solutions as it is proved in [23]. This is a powerful reason to consider Caputo derivative instead of other fractional derivatives, as, for example, Riemann-Liouville derivative defined for adequate functions ϕ as

$$\partial_{RL}^\alpha \phi(t) = c_\alpha \frac{d}{dt} \int_0^t \frac{\phi(z)}{(t-z)^\alpha} dz.$$

Moreover, as it can be seen in [10, Chapters 5,6], Caputo derivative is more adequate to deal with the most classical notion of initial condition compared with Riemann-Liouville problems, where the initial condition is understood in a generalized sense.

Coming back to (1.1), and more specifically to the Hamiltonian H, we assume throughout this paper $H(x, p)$ is periodic in x and coercive in p.

We focus our attention into Bellman-type Hamiltonians with the classical structure related to optimal control problems with compact control set, that is, satisfying the regularity/growth condition

$$(1.5) \qquad \begin{cases} |H(x,p) - H(y,p)| \le c_H(1 + |p|)|x - y|, \\ \lim\limits_{|p| \to +\infty} \inf\limits_{x \in \mathbb{T}^N} H(x,p) = +\infty. \end{cases}$$

for some $c_H > 0$, and for all $x, y \in \mathbb{T}^N, p \in \mathbb{R}^N$.

We are also interested in the case H has superlinear growth in the gradient, common in control problems with unrestricted control space. We refer to this case through the following assumption: there exists $m > 1$ (the "gradient growth") and $A > 0$, $c_H > 1$ such that

$$(1.6) \qquad \begin{cases} \mu H(x, \mu^{-1}p) - H(x,p) \ge (1 - \mu)\left(c_H^{-1}|p|^m - A\right), \\ \\ |H(x,p) - H(y, p+q)| \le c_H(1 + |p|^m)|x - y| + c_H|q|(1 + |p|^{m-1}), \end{cases}$$

for all $\mu \in (0,1)$, $x, y \in \mathbb{T}^N$, $p, q \in \mathbb{R}^N$ with $|q| \le 1$. We notice at once that the first condition in (1.6) implies that there exist $C > 0$ such that

$$(1.7) \qquad\qquad H(x,p) \ge C^{-1}|p|^m - C \qquad \text{for all } x \in \mathbb{T}^N, p \in \mathbb{R}^N,$$

and the same inequality holds with $m = 1$ if the second condition holds in (1.6) and H is convex.

Our interest is the analysis of the behavior of the solutions to (1.1)-(1.2) and make a contrast with the classical case $\alpha = 1$, namely, the Hamilton-Jacobi equation

$$(1.8) \qquad\qquad \partial_t u + H(x, Du) = 0 \quad \text{in } Q.$$

There is a vast literature regarding problem (1.8)-(1.2). We refer the surveys of Barles and Ishii in [2] and references therein for the basics about this problem, like existence, uniqueness and regularity.

A natural question that arises is the analysis of the behavior of the solution for long times. There too, there are a lot of references when $\alpha = 1$, see [12, 20, 4, 13, 9, 17, 7, 2] and the references therein. Here, we focus on the nowadays classical framework of Namah and

Roquejoffre paper [20], where the authors address the long time behavior of (1.8) under the assumption that

$$(1.9) \quad \begin{cases} H(x,p) = F(x,p) - f(x) \text{ for all } (x,p), \text{ with } F, f \text{ continuous,} \\ F(x,\cdot) \text{ is convex for all } x \in \mathbb{T}^N, \\ F(x,p) > F(x,0) = 0 \quad \text{for all } x \in \mathbb{T}^N, p \in \mathbb{R}^N \setminus \{0\}, \\ f \geq 0 \text{ in } \mathbb{T}^N. \end{cases}$$

It is possible to generalize slightly the above assumptions but we choose to state them in this form since all the main difficulties are present.

Under these assumptions, (1.1) reads

$$(1.10) \qquad \qquad \partial_t^\alpha u + F(x, Du) = f(x),$$

$f \in \mathrm{Lip}(\mathbb{T}^N)$ and

$$(1.11) \qquad \qquad Z := \{x \in \mathbb{T}^N : f(x) = \min_{\mathbb{T}^N} f\},$$

the so-called projected Aubry set [13], which is a compact subset of $\mathbb{T}^N$. It follows from Lions, Papanicolaou and Varadhan [19] that the so-called *ergodic problem*

$$(1.12) \qquad \qquad F(x, Dv) = f(x) + c \quad x \in \mathbb{T}^N,$$

has a solution $(c, v) \in \mathbb{R} \times W^{1,\infty}(\mathbb{T}^N)$ and c is unique. Actually, under our assumptions, it is easy to see that $c = -\min_{\mathbb{T}^N} f$.

It is then expected that the long time behavior of the solution u of (1.1)-(1.2) is given by the asymptotic expansion

$$(1.13) \qquad u(x,t) + ct^\alpha = v(x) + o(1), \quad o(1) \to 0 \text{ uniformly as } t \to +\infty,$$

where (c, v) is a solution of (1.12).

In the local case $\alpha = 1$, this asymptotic behavior is established in [20, Theorem 1]. The proof relies basically on three steps. The first step is to obtain that the set $\{u(\cdot, t) + ct, t \geq 0\}$ is relatively compact in $W^{1,\infty}(\mathbb{T}^N)$. Let us point out that this property is not sufficient and the main difficulty is to prove the full convergence of $u(\cdot, t) + ct$ as $t \to +\infty$. The second step is to notice that $\partial_t(u(x, t) + ct) \leq 0$ for $x \in Z$, from which one infers that $u(\cdot, t) + ct$ is nonincreasing in time on Z, so it converges uniformly to a Lipschitz continuous function ϕ on Z as $t \to +\infty$. The third step is to take the half-relaxed limits

$$\overline{u}(x) = \limsup_{y \to x, s \to t, \epsilon \to 0} u(y, \frac{s}{\epsilon}) + c\frac{s}{\epsilon} \quad \text{and} \quad \underline{u}(x) = \liminf_{y \to x, s \to t, \epsilon \to 0} u(y, \frac{s}{\epsilon}) + c\frac{s}{\epsilon},$$

which are, respectively, a sub- and a supersolution of (1.12) and then to apply a strong comparison result for (1.12) with the "Dirichlet boundary condition" $\overline{u} = \underline{u} = \phi$ on Z. It follows $\overline{u} = \underline{u}$ in $\mathbb{T}^N$, which gives the desired full convergence.

Our goal is to develop a similar procedure for the Caputo fractional case $\alpha \in (0, 1)$. The first step holds. Indeed, the elliptic properties shown by the expression (1.4) are not only related to comparison principles but also to regularity in the time variable. Using Ishii-Lions method for nonlocal problems presented in [5], we are able to prove that bounded solutions

to (1.1) are α-Hölder continuous in time, uniformly in Q, see Theorem 2.1. We recall that such a property does not come from the contraction principle given by comparison arguments as in the classical case, basically because such a property is not known for fractional problems, where the influence of the "memory" put troubles in the analysis of problems shifted in time. Moreover, in the local case (1.8), Lipschitz estimates in time allows to use Rademacher's Theorem to regard u_t as an L^∞ function, and extract the boundedness of Du through the coercivity of H. Such a program cannot be carried out directly in (1.1) since α-Hölder functions are not sufficient to make $\partial_t^\alpha u$ in L^∞. Nevertheless, we can get equicontinuity in the space variable by a regularization procedure via inf and sup convolutions, see Theorem 2.2.

Similarly, the third step is identical to the one in the classical case since the ergodic problem (1.12) is the same for all $\alpha \in (0, 1]$.

It follows that the limiting step is the second one. We still have $\partial_t^\alpha(u(x, t) + ct^\alpha) \leq 0$ for $x \in Z$ but it is not anymore sufficient to infer neither that $u(\cdot, t) + ct^\alpha$ is nonincreasing in time on Z, nor that it converges. As we show in Section 3, it is possible to have bounded functions with signed α-order Caputo derivative, but not converging as $t \to \infty$, which is a surprising result interesting by itself.

To overcome this difficulty and obtain the convergence on Z, we need to give additional assumptions on the geometry of Z and $\mathrm{argmin}\{g\}$. Precise assumptions are stated in Section 4 but basically, we assume

(1.14) $Z \cap \mathrm{argmin}\{g\} \neq \emptyset,$

(1.15)   for any $z \in Z$, there exists a rectifiable curve in Z joining $\mathrm{argmin}\{g\}$ and z.

These assumptions are inspired from the classical case, where it is known that the solution u of (1.8) is the value function of an optimal control problem for which assumptions (1.14)-(1.15) means roughly that we can travel with minimal cost on Z. But let us point out that these geometrical assumptions are not required in the classical case and that there is no rigorous link between the fractional case and the expected control problem, which could make these ideas rigorous. See Camilli, De Maio, Iacomini [8] for the precise statement of a related control problem which should be associated with the case $\alpha \in (0, 1)$, and some discussion in this direction.

It follows that we need to translate these ideas in the PDE framework building some suitable supersolutions, which tends to 0 thanks to (1.14)-(1.15). By comparison, we obtain estimates

$$\min_{\mathbb{T}^N} g \leq u(z, t) + ct^\alpha \leq \min_{\mathbb{T}^N} g + \mathrm{Lip}(g)\, \mathrm{length}(\gamma)\, \mathcal{E}(t), \quad \text{for any } z \in Z,$$

where γ is a rectifiable curve on Z joining $\mathrm{argmin}\{g\}$ and z, and $\mathcal{E}$ is the solution of a fractional ODE with limit 0 at $+\infty$; see Theorem 4.2 and Theorem 4.4 for an extension to some possibly infinite length curves. This implies the convergence of u on Z from which we deduce easily (1.13), see Corollary 4.6. This approach is new but we think it does not provide optimal results. In particular, it relies too heavily on the geometry of Z, which is not the case in the classical approach described above for $\alpha = 1$. To go further, one would need some quantitative estimates on how much nonpositive is $\partial_t^\alpha(u(x, t) + ct^\alpha)$, which seems difficult to obtain, even in the case $\alpha = 1$.

The paper is organized as follows. We start by introducing precisely the Caputo fractional operator and recalling some useful properties. Then we establish some regularity estimates for the solution of (1.1) in Section 2. Section 3 is devoted to a counterexample showing that a bounded function with nonnegative Caputo derivative does not necessarily converges. Finally, some positive results for the large time behavior of the solution of (1.1) are proved in Section 4.

Notations and preliminaries. We will write the Caputo derivative using (1.4) (with $\tilde{c}_\alpha = 1$ for simplicity). More precisely, let $\phi : (0, +\infty) \to \mathbb{R}$. We extend ϕ to $(-\infty, 0)$ by setting $\phi(s) = \phi(0)$ for $s < 0$ and define, when it exists, for every $t > 0$ and $0 < \delta < t$,

$$(1.16) \qquad \partial_t^\alpha \phi(t) = \int_{-\infty}^t \frac{\phi(t) - \phi(s)}{|t-s|^{1+\alpha}} ds = \partial_t^\alpha[t-\delta]\phi(t) + \partial_t^\alpha[t-\delta, t]\phi(t),$$

where, for $a < b \le t$, we set

$$\partial_t^\alpha[a]\phi(t) := \int_{-\infty}^a \frac{\phi(t) - \phi(s)}{|t-s|^{1+\alpha}} ds, \qquad \partial_t^\alpha[a, b]\phi(t) = \int_a^b \frac{\phi(t) - \phi(s)}{|t-s|^{1+\alpha}} ds.$$

Notice that $\partial_t^\alpha \phi(t)$ is well-defined as soon as $\phi \in L^1_{\text{loc}}(0, +\infty)$ and ϕ is C^1 in a neighborhood of t. More about the functional formulation of this operator can be found in the references [18, 10, 16]. For the definition of viscosity solutions to (1.1), we refer to [6, 23].

Notice that for all $\beta > 0$, there exists $c_{\alpha,\beta} > 0$ such that

$$(1.17) \qquad \partial_t^\alpha t^\beta = c_{\alpha,\beta} t^{\beta-\alpha}, \quad \text{for all } t \ge 0. \qquad [10, \text{Appendix B}]$$

We introduce the Mittag-Leffler functions $E_\alpha(z)$ of order α as in [10, Definition 4.1]. Recall that E_α is smooth on $\mathbb{R}$, $E_\alpha(0) = 1$ and we have the following useful properties

$$(1.18) \qquad t \in [0, +\infty) \mapsto E_\alpha(-t) \text{ is positive, convex and nonincreasing, [22, Section 6.2]},$$

$$(1.19) \qquad \partial_t^\alpha E_\alpha(\lambda t^\alpha) = \lambda E_\alpha(\lambda t^\alpha), \text{ for all } t > 0 \text{ and } \lambda \in \mathbb{R}, \text{ [10, Theorem 4.3]},$$

$$(1.20) \qquad \frac{1}{\Gamma(1-\alpha)t} \le E_\alpha(-t) \le \frac{1}{\Gamma(1+\alpha)^{-1}t} \text{ for all } t \ge 0, \text{ [22, Theorem 4]}.$$

We will write $\hat{x} = x/|x|$ for every $x \in \mathbb{R}^N \setminus \{0\}$.

2. Regularity Estimates

We start with a regularity result in time for bounded solutions to (1.1). This result is a consequence of the Hölder estimates reported by Barles, Chasseigne and Imbert in [5].

Theorem 2.1. *Assume* (1.5) *or* (1.6), *and* $g \in \text{Lip}(\mathbb{T}^N)$. *Let* u *be bounded, continuous viscosity solution to* (1.1)-(1.2). *Then,* u *is Hölder continuous in time, that is there exists a constant* $L > 1$ *large enough such that*

$$(2.1) \qquad |u(x, s) - u(x, t)| \le L|s - t|^\alpha, \quad s, t \ge 0.$$

The constant L *depends on the data and* $\|u\|_\infty$ *but does not depend on* t.

Proof. Despite the proof we present here is valid for both (1.5) and (1.6), we underline the arguments in the later case. Let $\mu \in (0,1]$ be away from zero, and denote $\bar{u} = \mu u$. If (1.6) is assumed, then we consider $\mu < 1$, and if (1.5) is assumed, then $\mu = 1$ in what follows.

Using the linearity of the Caputo derivative, it is direct to check that $\bar{u}$ solves

$$(2.2) \qquad \partial_t^\alpha \bar{u} + \mu H(x, \mu^{-1} D\bar{u}) = 0 \quad \text{in } Q,$$

with initial condition $\bar{u}(\cdot, 0) = \mu g$.

By (1.17) and $g \in \text{Lip}(\mathbb{T}^N)$, we have that $g(x) \pm Ct^\alpha$ are respectively super- and subsolutions of (1.1)-(1.2) for $C > 0$ large enough depending only on g and H. Therefore, a direct application of the comparison results in [23] leads to the estimates

$$(2.3) \qquad g(x) - Ct^\alpha \le u(x,t) \le g(x) + Ct^\alpha, \quad \text{for all } t \ge 0.$$

These bounds can be readily adapted to $\bar{u}$ by multiplying by μ the last inequality.

By contradiction, assume that for all $L > 1$ we have

$$(2.4) \qquad \sup_{x \in \mathbb{T}^N, s, t \ge 0} \{u(x,s) - u(x,t) - L|s-t|^\alpha\} =: \theta_L > 0.$$

Taking μ very close to 1 in term of θ_L above, we can get

$$\sup_{x \in \mathbb{T}^N, s, t \ge 0} \{\bar{u}(x,s) - u(x,t) - L|s-t|^\alpha\} = \theta_L/2 > 0.$$

For localization purposes, we introduce a function ψ_β with the following properties: we consider $\psi : \mathbb{R} \to \mathbb{R}$ smooth and nondecreasing, such that $\psi(t) = 0$ if $t \le 1$, $\psi(t) \ge 2\|u\|_\infty$ if $t \ge 2$, and for $\beta > 0$ small we denote $\psi_\beta(t) = \psi(\beta t)$. Then, for all β small enough, we have

$$\max_{x \in \mathbb{R}^N, s, t \in \mathbb{R}} \{\bar{u}(x,s) - u(x,t) - L|s-t|^\alpha - \psi_\beta(s)\} = \theta_L/4 > 0,$$

where we recall that we extend $u(x,t)$ as g for negative times t.

Then, for $\beta, \epsilon > 0$ small we define

$$\Phi(x,y,s,t) := \bar{u}(x,s) - u(y,t) - L|s-t|^\alpha - \epsilon^{-2}|x-y|^2 - \psi_\beta(s).$$

We have that Φ attains its maximum at a point $(\bar{x}, \bar{y}, \bar{s}, \bar{t}) \in \bar{Q}^2$ and this maximum is bigger than $\theta_L/4$.

Standard arguments lead to

$$|\bar{x} - \bar{y}| \le \epsilon \omega_\beta(\epsilon), \quad |\bar{s}|, |\bar{t}| \le 2/\beta,$$

where $\omega_\beta(\epsilon) \to 0$ as $\epsilon \to 0$ if β is fixed (ω_β is a modulus of continuity in space of u in the compact set $\mathbb{T}^N \times [0, 2/\beta]$), and

$$(2.5) \qquad |\bar{s} - \bar{t}| \le C_0 L^{-1/\alpha},$$

for some constant $C_0 > 0$ just depending on $\|u\|_\infty$. In particular, for L large enough we may assume that $|\bar{s} - \bar{t}| < 1$.

Below, we use a constant C, which may vary line to line but only depend on the data of the problem and not on ϵ, β nor μ.

Here we claim that $\bar{s}, \bar{t} > 0$ for all L large, and ϵ small in terms of L. In fact, if $\bar{s} = 0$ (the case $\bar{t} = 0$ is analogous), then, using (2.3), we see that

$$\theta_L/4 < \Phi(\bar{x}, \bar{y}, \bar{s}, \bar{t}) \leq \mu g(\bar{x}) - g(\bar{y}) + (C_0 - L)\bar{t}^\alpha \leq C_0 L_0 \epsilon\, \omega_\beta(\epsilon) + C_0(1 - \mu) + (C_0 - L)\bar{t}^\alpha,$$

where L_0 is the Lipschitz constant of g. Taking $L \geq C_0$, we arrive at

$$\theta_L/4 \leq C\epsilon + C(1 - \mu),$$

which is a contradiction if ϵ is taken small, and μ close to 1 in terms of L.

In addition, since u is uniformly continuous in space, then $\bar{s} \neq \bar{t}$ for all L large enough and ϵ small in terms of L. Indeed, if $\bar{s} = \bar{t}$, then we would have

$$0 < \theta_L/4 \leq \bar{u}(\bar{x}, \bar{t}) - u(\bar{y}, \bar{t}) \leq \omega_\beta(\epsilon) + C_0(1 - \mu).$$

Hence, for L and β fixed, taking ϵ small and μ close to 1, we arrive at a contradiction.

Since $\bar{s}, \bar{t} > 0$ and $\bar{s} \neq \bar{t}$, we can use the penalization defining Φ as a testing for u. For this, in what follows we write $\phi(x, y, s, t) := L|s - t|^\alpha + \epsilon^{-2}|x - y|^2 + \psi_\beta(s)$, from which we see that

$$\Phi(x, y, s, t) = \bar{u}(x, s) - u(y, t) - \phi(x, y, s, t).$$

At this point, denoting $\phi_1(x, s) = \phi(x, \bar{y}, s, \bar{t})$ we notice that the function

$$(x, s) \mapsto \bar{u}(x, s) - (u(\bar{y}, \bar{t}) + \phi_1(x, s))$$

has a local maximum point at $(\bar{x}, \bar{s})$, from which we can use the subsolution's viscosity inequality to write, for all $\delta > 0$, that

$$\partial_t^\alpha[\bar{s} - \delta]\bar{u}(\bar{x}, \cdot)(\bar{s}) + \partial_t^\alpha[\bar{s} - \delta, \bar{s}]\phi_1(\bar{x}, \cdot)(\bar{s}) + \mu H(\bar{x}, \mu^{-1}D\phi_1(\bar{x}, \bar{s})) \leq 0.$$

Similarly, denoting $\phi_2(y, t) = \phi(\bar{x}, y, \bar{s}, t)$ we notice the function

$$(y, t) \mapsto u(y, t) - (\bar{u}(\bar{x}, \bar{s}) - \phi_2(y, t))$$

has a local minimum point at $(\bar{y}, \bar{t})$, from which we can use the supersolution's viscosity inequality to write, for all $\delta > 0$, that

$$\partial_t^\alpha[\bar{t} - \delta]u(\bar{y}, \cdot)(\bar{t}) + \partial_t^\alpha[\bar{t} - \delta, \bar{t}](-\phi_2)(\bar{y}, \cdot)(\bar{t}) + H(\bar{y}, D(-\phi_2)(\bar{y}, \bar{t})) \geq 0.$$

Then, we subtract both inequalities to arrive at

$$(2.6) \qquad\qquad\qquad\qquad\qquad \mathcal{D} \leq \mathcal{H},$$

where, noticing that $D(-\phi_2)(\bar{y}, \bar{t}) - = D\phi_1(\bar{x}, \bar{s}) = 2(\bar{x} - \bar{y})/\epsilon^2$, we write

$$\mathcal{D} = \partial_t^\alpha[\bar{s} - \delta]\bar{u}(\bar{x}, \cdot)(\bar{s}) + \partial_t^\alpha[\bar{s} - \delta, \bar{s}]\phi_1(\bar{x}, \cdot)(\bar{s}) - \partial_t^\alpha[\bar{t} - \delta]u(\bar{y}, \cdot)(\bar{t}) + \partial_t^\alpha[\bar{t} - \delta, \bar{t}]\phi_2(\bar{y}, \cdot)(\bar{t}).$$

$$\mathcal{H} = H(\bar{y}, p) - \mu H(\bar{x}, \mu^{-1}p), \quad p := 2(\bar{x} - \bar{y})/\epsilon^2.$$

From (1.6), we get

$$\mathcal{H} \leq -(1 - \mu)c_H^{-1}|p|^m + c_H(1 + |p|^m)\epsilon + A(1 - \mu)$$

and from here, taking $\epsilon \leq (1 - \mu)c_H^{-2}$, we conclude that

$$(2.7) \qquad\qquad\qquad\qquad\qquad \mathcal{H} \leq C\epsilon + A(1 - \mu),$$

where $C > 0$ just depends on c_H.

Consider $\delta = |\bar{s} - \bar{t}|/2 > 0$ and we split the term $\mathcal{D}$ as

$$\mathcal{D} = \mathcal{D}_1 + \mathcal{D}_2$$

with

$$\mathcal{D}_1 = \partial_t^\alpha[\bar{s} - \delta]\bar{u}(\bar{x}, \cdot)(\bar{s}) - \partial_t^\alpha[\bar{t} - \delta]u(\bar{y}, \cdot)(\bar{t})$$
$$\mathcal{D}_2 = \partial_t^\alpha[\bar{s} - \delta, \bar{s}]\phi_1(\bar{x}, \cdot)(\bar{s}) + \partial_t^\alpha[\bar{t} - \delta, \bar{t}]\phi_2(\bar{y}, \cdot)(\bar{t}).$$

We start with $\mathcal{D}_2$. Directly from the definitions, we see that

$$\mathcal{D}_2 = L\int_{\bar{s}-\delta}^{\bar{s}} \frac{|\bar{s}-\bar{t}|^\alpha - |z-\bar{t}|^\alpha}{|\bar{s}-z|^{1+\alpha}}dz + L\int_{\bar{t}-\delta}^{\bar{t}} \frac{|\bar{s}-\bar{t}|^\alpha - |\bar{s}-z|^\alpha}{|\bar{t}-z|^{1+\alpha}}dz + \int_{\bar{s}-\delta}^{\bar{s}} \frac{\psi_\beta(\bar{s}) - \psi_\beta(z)}{|\bar{s}-z|^{1+\alpha}}dz$$
$$=: \mathcal{D}_2' + o_\beta(1),$$

where $o_\beta(1) \to 0$ as $\beta \to 0$ independent of the rest of the variables by the smoothness of ψ. Now, performing the change of variables $z = \bar{s} + y$ in the first integral, and $z = \bar{t} + y$ in the second, we arrive at

$$\mathcal{D}_2' = L\int_{-\delta}^0 \frac{|\bar{s}-\bar{t}|^\alpha - |\bar{s}-\bar{t}+y|^\alpha}{|y|^{1+\alpha}}dy + L\int_{-\delta}^0 \frac{|\bar{s}-\bar{t}|^\alpha - |\bar{s}-\bar{t}-y|^\alpha}{|y|^{1+\alpha}}dy$$
$$= L\int_{-\delta}^0 \frac{|\bar{s}-\bar{t}|^\alpha - |\bar{s}-\bar{t}+y|^\alpha}{|y|^{1+\alpha}}dy + L\int_0^\delta \frac{|\bar{s}-\bar{t}|^\alpha - |\bar{s}-\bar{t}+y|^\alpha}{|y|^{1+\alpha}}dy$$
$$= -L\int_{-\delta}^\delta \frac{|\bar{s}-\bar{t}+y|^\alpha - |\bar{s}-\bar{t}|^\alpha}{|y|^{1+\alpha}}dy$$
$$= -L\int_{-\delta}^\delta \frac{|\bar{s}-\bar{t}+y|^\alpha - |\bar{s}-\bar{t}|^\alpha - \alpha|\bar{s}-\bar{t}|^{\alpha-1}\widehat{\bar{s}-\bar{t}}\,y}{|y|^{1+\alpha}}dy,$$

where the last equality comes from the symmetry of the kernel. Performing a second order expansion and recalling that $\delta = |\bar{s}-\bar{t}|/2$, we obtain that there exists $\rho(y) \in (-\delta, \delta)$ such that

$$-\left(|\bar{s}-\bar{t}+y|^\alpha - |\bar{s}-\bar{t}|^\alpha - \alpha|\bar{s}-\bar{t}|^{\alpha-1}\widehat{\bar{s}-\bar{t}}\,y\right) = -\frac{\alpha(\alpha-1)}{2}|\bar{s}-\bar{t}+\rho(y)|^{\alpha-2}y^2$$
$$\geq -\frac{\alpha(\alpha-1)}{2^{\alpha-1}}|\bar{s}-\bar{t}|^{\alpha-2}y^2.$$

Therefore,

$$\mathcal{D}_2' \geq -\frac{L\alpha(\alpha-1)}{2^{\alpha-1}}|\bar{s}-\bar{t}|^{\alpha-2}\int_{-\delta}^\delta |y|^{1-\alpha}dy = \frac{\alpha(1-\alpha)}{2-\alpha}L =: cL,$$

and from here

(2.8) $$\mathcal{D}_2 \geq cL - o_\beta(1).$$

Now we deal with $\mathcal{D}_1$, writing

$$
\begin{aligned}
\mathcal{D}_1 =& \partial_t^\alpha[\bar{s}-1]\bar{u}(\bar{x},\cdot)(\bar{s}) - \partial_t^\alpha[\bar{t}-1]u(\bar{y},\cdot)(\bar{t})\\
&+ \partial_t^\alpha[\bar{s}-1,\bar{s}-\delta]\bar{u}(\bar{x},\cdot)(\bar{s}) - \partial_t^\alpha[\bar{t}-1,\bar{t}-\delta]u(\bar{y},\cdot)(\bar{t})\\
\geq& -C||u||_\infty + \partial_t^\alpha[\bar{s}-1,\bar{s}-\delta]\bar{u}(\bar{x},\cdot)(\bar{s}) - \partial_t^\alpha[\bar{t}-1,\bar{t}-\delta]u(\bar{y},\cdot)(\bar{t})\\
=:& -C||u||_\infty + \mathcal{D}_1',
\end{aligned}
$$

where $C > 0$ depends only on α.

To deal with the remaining term $\mathcal{D}_1'$, we assume that $\bar{s} < \bar{t}$ (the case $\bar{t} < \bar{s}$ follows the same lines). Performing similar change of variables as above and using the maximal inequality $\Phi(\bar{x},\bar{y},\bar{s},\bar{t}) \geq \Phi(\bar{x},\bar{y},\bar{s}+y,\bar{t}+y)$ we arrive at

$$
\mathcal{D}_1' = \int_{-1}^{-\delta} \frac{\bar{u}(\bar{x},\bar{s}) - u(\bar{y},\bar{t}) - (\bar{u}(\bar{x},\bar{s}+y) - u(\bar{y},\bar{t}+y))}{|y|^{1+\alpha}}dy \geq \int_{-1}^{-\delta} \frac{\psi_\beta(\bar{s}+y) - \psi_\beta(\bar{s})}{|y|^{1+\alpha}}dy.
$$

Noticing that the smooth function ψ_β satisfies $|\psi_\beta'| \leq C\beta$, we conclude that $\mathcal{D}_1' \geq -o_\beta(1)$. From this

$$
\mathcal{D}_1 \geq -C||u||_\infty - o_\beta(1).
$$

Joining this with (2.7) and (2.8) in (2.6) we get

$$
cL \leq C||u||_\infty + o_\beta(1) + C\epsilon + A(1-\mu).
$$

Then, we let $\epsilon \to 0$ first, then $\mu \to 1$ and finally $\beta \to 0$ and, having taken L large enough just in terms of the data and $||u||_\infty$, we reach a contradiction. It ends the proof. $\qquad\square$

Now we would like to obtain estimates in space.

Theorem 2.2. *Assume hypotheses of Theorem 2.1 hold. For each bounded viscosity solution to (1.1), there exists a modulus of continuity $m \in C([0,+\infty))$ independent of t such that*

$$
|u(x,t) - u(y,t)| \leq m(|x-y|) \quad \text{for all } x,y \in \mathbb{T}^N,\ t \geq 0.
$$

If, in addition, H satisfies (1.7) for some $m \geq 1$, then, for each $\beta \in (0,1)$, there exists a constant $C > 0$ such that

$$
|u(x,t) - u(y,t)| \leq C|x-y|^\beta \quad \text{for all } x,y \in \mathbb{T}^N,\ t \geq 0.
$$

The modulus m and the constant C depend on the datas and $||u||_\infty$ but do not depend on t.

Proof. For $\epsilon \in (0,1)$, we introduce the sup-convolution

$$
u^\epsilon(x,t) = \sup_{s \geq 0}\{u(x,s) - \epsilon^{-1}|s-t|^2\}.
$$

We collect some properties of this regularization of u:

    (i) u^ϵ is still Hölder continuous in time satisfying (2.1) like u,

    (ii) $||u^\epsilon - u||_\infty \leq C\epsilon^{\alpha/2}$,

    (iii) u^ϵ is Lipschitz continuous in time with Lipschitz constant $C\epsilon^{-1}$ with $C > 0$ just depending on $||u||_\infty$,

    (iv) u^ϵ is a viscosity subsolution to (1.1) in $\mathbb{T}^N \times (a_\epsilon,+\infty)$ for some $a_\epsilon \to 0$ as $\epsilon \to 0$.

The proofs of (i) and (ii) are easy consequences of Theorem 2.1, (iii) is a classical property of the sup-convolution regularization and (iv) is proved in [23].

Now we prove the desired regularity of u by adapting the standard viscosity procedure to get regularity estimates from the coercivity of H. For any $x_0 \in \mathbb{T}^N$, $s_0 > 0$, $\beta > 0$ and $L > 0$ to be chosen, we consider

$$(2.9) \qquad \sup_{x \in \mathbb{T}^N, s \geq 0} \{ u^\epsilon(x,s) - u^\epsilon(x_0, s_0) - L|x - x_0| - \frac{|s - s_0|^2}{\beta^2} \},$$

where ϵ is chosen small enough in order that $s_0 > a_\epsilon$ in (iv). Classical results imply that this maximum is achieved at $(\bar{x}, \bar{s})$ with $\bar{s} \to s_0$ as $\beta \to 0$. We take β small enough in order that $\bar{s} > a_\epsilon$.

If $\bar{x} \neq x_0$, then we use $(x, s) \mapsto u^\epsilon(x_0, s_0) + L|x - x_0| + \frac{|s-s_0|^2}{\beta^2}$ as a test function for the subsolution u^ϵ of (1.1) at $(\bar{x}, \bar{s})$ to get that, for every $\bar{\delta} \in (0, 1)$,

$$(2.10) \qquad \partial_t^\alpha[\bar{s} - \bar{\delta}]u^\epsilon(\bar{x}, \cdot)(\bar{s}) + \partial_t^\alpha[\bar{s} - \bar{\delta}, \bar{s}]\frac{|\cdot - s_0|^2}{\beta^2}(\bar{s}) + H(\bar{x}, L\widehat{\bar{x} - x_0}) \leq 0.$$

Actually, since $\frac{|\cdot - s_0|^2}{\beta^2}$ is smooth and u^ϵ is Lipschitz continuous, we can send $\bar{\delta} \to 0$ in the previous inequality. In other words, due to the Lipschitz continuity of u^ϵ, we can use u^ϵ itself as a test-function in the fractional derivative in the viscosity inequality, see [23, Proposition 2.4] for details.

It follows that it is enough to estimate the fractional term $\partial_t^\alpha u^\epsilon(\bar{x}, \cdot)(\bar{s})$ that we expand, for $\delta > 0$, as

$$\partial_t^\alpha u^\epsilon(\bar{x}, \cdot)(\bar{s}) = \partial_t^\alpha[\bar{s} - 1]u^\epsilon(\bar{x}, \cdot)(\bar{s}) + \partial_t^\alpha[\bar{s} - 1, \bar{s} - \delta]u^\epsilon(\bar{x}, \cdot)(\bar{s}) + \partial_t^\alpha[\bar{s} - \delta, \bar{s}]u^\epsilon(\bar{x}, \cdot)(\bar{s}).$$

At first, from (ii),

$$\partial_t^\alpha[\bar{s} - 1]u^\epsilon(\bar{x}, \cdot)(\bar{s}) \geq -C\|u^\epsilon\|_\infty \geq -C(\|u\|_\infty + \epsilon^{\alpha/2}).$$

Then, using (i),

$$\partial_t^\alpha[\bar{s} - 1, \bar{s} - \delta]u^\epsilon(\bar{x}, \cdot)(\bar{s}) = \int_{\bar{s}-1}^{\bar{s}-\delta} \frac{u^\epsilon(\bar{x}, \bar{s}) - u^\epsilon(\bar{x}, s)}{|\bar{s} - s|^{1+\alpha}} ds \geq -C \int_{\bar{s}-1}^{\bar{s}-\delta} \frac{1}{|\bar{s} - s|} ds \geq -C|\log(\delta)|.$$

For the third term, we use (iii) to obtain

$$\partial_t^\alpha[\bar{s} - \delta, \bar{s}]u^\epsilon(\bar{x}, \cdot)(\bar{s}) \geq -C \int_{\bar{s}-\delta}^{\bar{s}} \frac{1}{|\bar{s} - s|^\alpha} ds \geq -\frac{C}{\epsilon}\delta^{1-\alpha}.$$

Plugging these estimates in (2.10), we obtain

$$H(\bar{x}, L\widehat{\bar{x} - x_0}) \leq C(1 + |\log(\delta)| + \epsilon^{-1}\delta^{1-\alpha},$$

where $C > 0$ just depends on the data and $\|u\|_\infty$. Taking the minimum for $\delta > 0$ we arrive at

$$(2.11) \qquad H(\bar{x}, L\widehat{\bar{x} - x_0}) \leq C(1 + |\log(\epsilon)|).$$

From the coercivity of H, we reach a contradiction if $L = L(\epsilon)$ is large enough.

It follows that the maximum in (2.9) is achieved for $\bar{x} = x_0$, which implies

$$u^\epsilon(x, \bar{s}) - u^\epsilon(x_0, \bar{s}) \leq L(\epsilon)|x - x_0|, \qquad \text{for all } x \in \mathbb{T}^N.$$

Sending $\beta \to 0$, recalling $\bar{s} \to s_0$ as $\beta \to 0$ and that x_0, s_0 are arbitrary, we finally obtain that, for all $x, y \in \mathbb{T}^N$, $t > 0$,

$$(2.12) \quad u(x, t) - u(y, t) \leq u^\epsilon(x, t) - u^\epsilon(y, t) + C\epsilon^{\alpha/2} \leq L(\epsilon)|x - y| + C\epsilon^{\alpha/2}, \quad 0 < \epsilon < 1.$$

This latter inequality means that u is uniformly continuous with respect to x independently of t.

In addition, if H satisfies (1.7) for some $m \geq 1$, then (2.11) and (2.12) lead to

$$u(x, t) - u(y, t) \leq C(1 + |\log(\epsilon)|)|x - y| + C\epsilon^{\alpha/2}, \quad 0 < \epsilon < 1.$$

Thus, taking the infimum with respect to $0 < \epsilon < 1$, we conclude that

$$|u(x, t) - u(y, t)| \leq C(1 + |\log|x - y||)|x - y|,$$

from which the result follows. $\qquad \square$

Remark 2.3. When H has a sublinear growth, it is possible to obtain some better regularity estimates in space, namely, Lipschitz estimates. More precisely, if (1.5) holds and $g \in \text{Lip}(\mathbb{T}^N)$, then every bounded viscosity solution to (1.1) satisfies

$$|u(x, t) - u(y, t)| \leq (1 + \text{Lip}(g)) E_\alpha (2c_H t^\alpha) |x - y| \quad \text{for all } x, y \in \mathbb{T}^N, t \geq 0.$$

Such a result was already showed in Giga and Namba [15]. We do not focus on such results because the Lipschitz constant depends heavily on time, a dependence we want to avoid in order to obtain the large time behavior of the solution.

3. Oscillating function with positive Caputo derivative.

In this section we construct a bounded function $u : [0, +\infty) \to \mathbb{R}$ such that $\partial_t^\alpha u \geq 0$ but such that

$$\liminf_{t \to +\infty} u(t) < \limsup_{t \to +\infty} u(t),$$

which prevents u to have any limit as $t \to +\infty$. This result shows a great contrast with the standard case $\alpha = 1$ in which $\partial_t u \geq 0$ implies that u is a nondecreasing function and therefore it is convergent.

In what follows, for any $\alpha \in (0, 1)$, we define the incomplete regularized beta function (see [1, Chapter 6]) by

$$B_\alpha[z_0, z_1] := \frac{1}{\pi \csc(\alpha\pi)} \int_{z_0}^{z_1} t^{-\alpha}(1 - t)^{\alpha-1}dt, \quad \text{for all } 0 \leq z_0 \leq z_1 \leq 1,$$

and we simply write $B_\alpha[z] = B_\alpha[0, z]$. We remark that $B_\alpha[0, 1] = 1$ ([1, 6.1.17 and 6.2.2]).

We also define

$$b_\alpha := B_\alpha^{-1}[1/2] \in (0, 1),$$

where $B_\alpha^{-1}[\cdot]$ is the inverse function of $B_\alpha[\cdot]$. As an example, if $\alpha = 1/2$, then $b_\alpha = 1/2$.

Hence, in the general case, with this choice of b_α, we have

$$(3.1) \qquad\qquad B_\alpha[0, b_\alpha] = B_\alpha[b_\alpha, 1] = 1/2.$$

Then, we define

$$\eta_\alpha := \pi \csc(\alpha\pi) B_\alpha[b_\alpha^3, b_\alpha^2] \in (0, 1),$$

and consider the continuous functions

$$f_1(t) = \begin{cases} 1 & \text{if } t \in [0, 1], \\ t^{-\alpha} & \text{if } t \geq 1, \end{cases}$$

and

$$f_2(t) = \begin{cases} \frac{t - a_{2k}}{\epsilon_k} & \text{if } t \in [a_{2k}, a_{2k} + \epsilon_k), \\ 1 & \text{if } t \in [a_{2k} + \epsilon_k, a_{2k+1} - \epsilon_k), \\ \frac{a_{2k+1} - t}{\epsilon_k} & \text{if } t \in [a_{2k+1} - \epsilon_k, a_{2k+1}), \\ 0 & \text{if not,} \end{cases}$$

where

$$a_k := (1/b_\alpha)^k \quad \text{and} \quad \epsilon_k := \frac{1 - b_\alpha^2}{4} \frac{\eta_\alpha}{a_{2k}}, \quad \text{for all } k \geq 0.$$

Next, we consider the continuous function $f := f_1 f_2$ (see Figure 1) and we define u as

$$(3.2) \qquad u(t) = \int_0^t \frac{f(z)}{(t - z)^{1-\alpha}} dz, \quad \text{for all } t \geq 0.$$
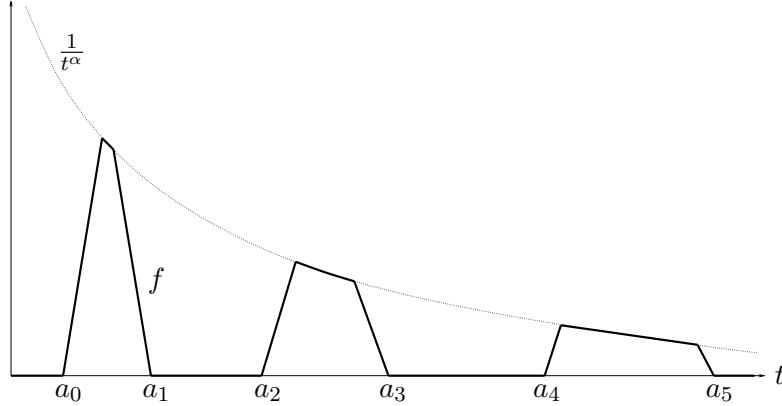

FIGURE 1. Behavior of $f = f_1 f_2$

The function f is regular enough (locally Lipschitz) to use the representation formula in [10, Theorem 3.7], meaning that u solves the fractional ODE

$$(3.3) \qquad \partial_t^\alpha u = f \quad \text{in } (0, +\infty), \qquad u(0) = 0,$$

where ∂_t^α is the Caputo derivative of order $\alpha \in (0, 1)$.

Notice that $u = 0$ in $[0, 1]$ and $0 \leq u$ is bounded in $\mathbb{R}^+$. In fact, for $t > 1$ we see that

$$u(t) \leq \int_1^t \frac{f_1(z)}{(t - z)^{1-\alpha}} dz,$$

from which we get that

$$u(t) \le t^{\alpha-1} \int_1^t \frac{z^{-\alpha}}{(1-z/t)^{1-\alpha}} dz = t^{\alpha-1} \int_{1/t}^1 \frac{t^{-\alpha}y^{-\alpha}}{(1-y)^{1-\alpha}} t dy \le \int_0^1 \frac{dy}{y^\alpha(1-y)^{1-\alpha}} = \pi \csc(\alpha\pi),$$

from which u is bounded.

Now we compare $u(a_{2N+1})$ and $u(a_{2N+2})$, for N large enough such that

$$(3.4) \qquad \int_{1-\epsilon_N/a_{2N+1}}^1 \frac{dy}{y^\alpha(1-y)^{1-\alpha}} < \eta_\alpha/4 \quad \text{and} \quad a_{2N}(a_1 - 1) = (1/b_\alpha)^{2N}(1/b_\alpha - 1) \ge 2.$$

Using the definition of f we see that

$$u(a_{2N+1}) = \int_1^{a_{2N+1}} \frac{z^{-\alpha} f_2(z)}{(a_{2N+1} - z)^{1-\alpha}} dz$$

$$= \sum_{k=0}^N \left(\int_{a_{2k}}^{a_{2k}+\epsilon_k} \frac{z^{-\alpha} f_2(z)}{(a_{2N+1} - z)^{1-\alpha}} dz + \int_{a_{2k}+\epsilon_k}^{a_{2k+1}-\epsilon_k} \frac{z^{-\alpha}}{(a_{2N+1} - z)^{1-\alpha}} dz \right.$$

$$\left. + \int_{a_{2k+1}-\epsilon_k}^{a_{2k+1}} \frac{z^{-\alpha} f_2(z)}{(a_{2N+1} - z)^{1-\alpha}} dz \right)$$

$$= \sum_{k=0}^N \int_{a_{2k}}^{a_{2k+1}} \frac{z^{-\alpha}}{(a_{2N+1} - z)^{1-\alpha}} dz - \sum_{k=0}^N \int_{a_{2k}}^{a_{2k}+\epsilon_k} \frac{z^{-\alpha}(1 - f_2(z))}{(a_{2N+1} - z)^{1-\alpha}} dz$$

$$- \sum_{k=0}^N \int_{a_{2k+1}-\epsilon_k}^{a_{2k+1}} \frac{z^{-\alpha}(1 - f_2(z))}{(a_{2N+1} - z)^{1-\alpha}} dz$$

$$=: v_1(a_{2N+1}) - v_2(a_{2N+1}) - v_3(a_{2N+1}),$$

and similarly

$$u(a_{2N+2}) =: v_1(a_{2N+2}) - v_2(a_{2N+2}) - v_3(a_{2N+2}).$$

From here, by simple integration we get

$$v_1(a_{2N+1}) = \sum_{k=0}^N \int_{a_{2k}/a_{2N+1}}^{a_{2k+1}/a_{2N+1}} \frac{dy}{y^\alpha(1-y)^{1-\alpha}}$$

$$= \pi \csc(\alpha\pi) \sum_{k=0}^N B_\alpha[a_{2k}/a_{2N+1}, a_{2k+1}/a_{2N+1}].$$

Moreover,

$$v_1(a_{2N+2}) = \pi \csc(\alpha\pi) \sum_{k=0}^N B_\alpha[a_{2k}/a_{2N+2}, a_{2k+1}/a_{2N+2}].$$

Now, we estimate the term $v_1(a_{2N+2}) - v_1(a_{2N+1})$. For this, using the definition of b_α, η_α and (3.1), we notice that

$$\frac{v_1(a_{2N+2}) - v_1(a_{2N+1})}{\pi \csc(\alpha\pi)} = \sum_{k=0}^{N} \left(B_\alpha[a_{2k}/a_{2N+2}, a_{2k+1}/a_{2N+2}] - B_\alpha[a_{2k}/a_{2N+1}, a_{2k+1}/a_{2N+1}]\right)$$

$$\leq B_\alpha[0, b_\alpha] - \sum_{k=0}^{N} B_\alpha[a_{2k}/a_{2N+1}, a_{2k+1}/a_{2N+1}]$$

$$= B_\alpha[0, b_\alpha] - B_\alpha[b_\alpha, 1] - \sum_{k=0}^{N-1} B_\alpha[a_{2k}/a_{2N+1}, a_{2k+1}/a_{2N+1}]$$

$$= -\sum_{k=0}^{N-1} B_\alpha[a_{2k}/a_{2N+1}, a_{2k+1}/a_{2N+1}]$$

$$\leq - B_\alpha[b_\alpha^3, b_\alpha^2].$$

Therefore, we have that $v_1(a_{2N+2}) - v_1(a_{2N+1}) \leq -\eta_\alpha$. Hence, using the above result and the fact that $v_2, v_3 \geq 0$, we have that

$$u(a_{2N+2}) - u(a_{2N+1}) \leq - \eta_\alpha + v_2(a_{2N+1}) + v_3(a_{2N+1}).$$

Finally, we estimate the last two terms

$$v_2(a_{2N+1}) = \sum_{k=0}^{N} \int_{a_{2k}}^{a_{2k}+\epsilon_k} \frac{z^{-\alpha}(1 - \frac{z-a_{2k}}{\epsilon_k})}{(a_{2N+1} - z)^{1-\alpha}} dz$$

$$= \sum_{k=0}^{N} \epsilon_k \int_0^1 \frac{1-y}{(a_{2k} + \epsilon_k y)^\alpha (a_{2N+1} - a_{2k} - \epsilon_k y)^{1-\alpha}} dy.$$

We have $a_{2k} + \epsilon_k y \geq a_0 = 1$ and, using $\epsilon_k \leq 1$ and (3.4), $a_{2N+1} - a_{2k} - \epsilon_k y \geq a_{2N+1} - a_{2N} - 1 = a_{2N}(a_1 - 1) - 1 \geq 1$. It follows

$$v_2(a_{2N+1}) \leq \sum_{k=0}^{N} \epsilon_k \int_0^1 dy = \sum_{k=0}^{\infty} \epsilon_k = \frac{\eta_\alpha}{4}.$$

Similarly, we have that

$$v_3(a_{2N+1}) \leq \sum_{k=0}^{N-1} \int_{a_{2k+1}-\epsilon_k}^{a_{2k+1}} \frac{z^{-\alpha}(1 - \frac{a_{2k+1}-z}{\epsilon_k})}{(a_{2N+1} - z)^{1-\alpha}} dz + \int_{a_{2N+1}-\epsilon_N}^{a_{2N+1}} \frac{z^{-\alpha}}{(a_{2N+1} - z)^{1-\alpha}} dz$$

$$= \sum_{k=0}^{N-1} \epsilon_k \int_0^1 \frac{1-y}{(a_{2k+1} - \epsilon_k y)^\alpha (a_{2N+1} - a_{2k+1} + \epsilon_k y)^{1-\alpha}} dy$$

$$+ \int_{a_{2N+1}-\epsilon_N}^{a_{2N+1}} \frac{z^{-\alpha}}{a_{2N+1}^{1-\alpha}(1 - z/a_{2N+1})^{1-\alpha}} dz.$$

To estimate the first term above we notice first that $a_{2k+1} - \epsilon_k y - 1 \geq a_1 - (1 - b_\alpha^2)/4 = (b_\alpha - 1)(b_\alpha^2 - b_\alpha - 4)/(4b_\alpha) \geq 0$ since $0 < b_\alpha < 1$. Moreover, for $k \leq N-1$, $a_{2N+1} - a_{2k+1} + \epsilon_k y \geq a_{2N+1} - a_{2N-1} \geq a_{2N}(a_1 - 1) \geq 2$ by (3.4). To estimate the second term, we notice

$$\int_{a_{2N+1}-\epsilon_N}^{a_{2N+1}} \frac{z^{-\alpha}}{a_{2N+1}^{1-\alpha}(1 - z/a_{2N+1})^{1-\alpha}} dz = \int_{1-\epsilon_N/a_{2N+1}}^{1} \frac{1}{y^\alpha(1-y)^{1-\alpha}} dz \leq \frac{\eta_\alpha}{4},$$

using again (3.4). It follows

$$v_3(a_{2N+1}) \leq \sum_{k=0}^{N-1} \epsilon_k \int_0^1 dy + \frac{\eta_\alpha}{4} = \frac{\eta_\alpha}{2}.$$

Therefore, $u(a_{2N+2}) - u(a_{2N+1}) \leq -\eta_\alpha/4 < 0$, and this means that

$$\liminf_{t\to\infty} u(t) - \limsup_{t\to\infty} u(t) \leq -\eta_\alpha/4,$$

from which u does not have any limit at infinity.

4. Ergodic Large Time Behavior.

In this section we present some cases for which ergodic large time behavior (1.13) holds. The main assumption here follows the classical requirements of Namah and Roquejoffre [20], see Assumptions (1.9).

We have in mind the classical Eikonal case

$$(4.1) \qquad H(x,p) = F(x,p) - f(x) = a(x)|p| - f(x),$$

where $a, f : \mathbb{T}^N \to \mathbb{R}$ are Lipschitz continuous, $a(x) \geq \underline{a} > 0$ and $f(x) \geq \min_{\mathbb{T}^N} f = 0$. However, we are able to deal with Hamiltonians with superlinear growth on the gradient.

We notice that the convexity and the coercivity condition in (1.5) leads to a quantitative growth for the Hamiltonian, that is

$$H(x,p) \geq C^{-1}|p| - C, \qquad \text{for all } x \in \mathbb{T}^N, \, p \in \mathbb{R}^N,$$

for some constants $C > 0$. Since a similar condition is found when (1.6) holds, throughout this section we assume the existence of a constants $C_H > 1$ and $m \geq 1$ such that

$$(4.2) \qquad F(x,p) \geq C_H^{-1}|p|^m - C_H, \qquad \text{for all } x \in \mathbb{T}^N, \, p \in \mathbb{R}^N.$$

We also require some assumption on the behavior of F near $p = 0$ which is not reflected by (4.2). In order to be able to deal with more general Hamiltonian, e.g., smooth ones which are nonnegative and nondegenerate near $p = 0$, we introduce an additional assumption:

(4.3) There exists $\nu, r > 0$ and $k \geq 1$ such that $F(x,p) \geq \nu|p|^k$ for all $x \in \mathbb{T}^N, p \in B(0,r)$.

Thus, if (1.5) or (1.6) holds, then the later condition together with (1.9) lead to

(4.4) for every $R > 0$, there exists $\nu_R > 0$ such that $F(x,p) \geq \nu_R|p|^k$ for all $p \in B(0,R)$.

This is a sort of nondegeneracy condition in the sense that F is not too flat around $p = 0$.

By replacing f with $f - \min_{\mathbb{T}^N} f$ in (1.10), we may assume without loss of generality that

$$\min_{\mathbb{T}^N} f = 0.$$

It follows that $c = 0$ in (1.12).

As we will see later in the proof of Lemma 4.3, the solutions of (1.1) for $x \in Z$ are strongly related to the solutions of the ODE $\partial_t^\alpha \mathcal{E}(t) + A|\mathcal{E}(t)|^k = 0$, for which we state a technical lemma.

Lemma 4.1. *For every $A > 0$ and $k \geq 1$, there exists a unique positive solution $\mathcal{E} \in C([0,\infty)) \cap C^1((0,\infty))$ to*

$$(4.5) \qquad \partial_t^\alpha \mathcal{E}(t) + A|\mathcal{E}(t)|^k = 0, \qquad \mathcal{E}(0) = 1,$$

such that $\mathcal{E}(t) \searrow 0$ as $t \to +\infty$. Moreover, there exists $C = C(A, k, \alpha) > 0$ and, for all $\epsilon > 0$, there exists $C_\epsilon = C_\epsilon(A, k, \alpha)$ such that

$$(4.6) \qquad \frac{C}{t^{\alpha/k}} \leq \mathcal{E}(t) \leq \frac{C_\epsilon}{t^{\alpha/k - \epsilon}} \quad \text{for t large enough.}$$

Proof of Lemma 4.1. Existence and uniqueness of the positive decreasing solution $\mathcal{E}$ of (4.5) satisfying the lower bound in (4.6) is given by [14, Theorem 5.10].

Concerning the (upper) estimates in (4.6), we start with the case $k = 1$. In this case the related ODE (4.5) reads $\partial_t^\alpha \mathcal{E}(t) + A\mathcal{E}(t) = 0$, for which we have the explicit solution $t \mapsto E_\alpha(-At^\alpha)$, see (1.19). In this case, the estimate (4.6) is then optimal and given by (1.20).

Now we concentrate on the case $k > 1$. Below, c is a positive constant which may change line to line. Also, by $\partial_t^\alpha (1 + t)^{-p}$ we mean $\partial_t^\alpha (1 + \cdot)^{-p}(t)$.

Claim: For each $p > 0$, there exists $c > 0$ just depending on α and p such that

$$-ct^{-\alpha} \leq \partial_t (1 + t)^{-p} \leq 0 \quad \text{for all $t > 1$.}$$

The upper bound is obvious. The lower bound can be obtained by a combination of [10, p.193] and [1, Chapter 15, 7.3], but we present here an alternative proof for completeness.

Using the definition of Caputo derivative, for $t > 1$ we see that

$$\partial_t^\alpha (1+t)^{-p} \geq \partial_t^\alpha [t/2, t](1+t)^{-p} + \partial_t^\alpha [0, t/2](1+t)^{-p} + \int_{-\infty}^{0} \frac{-1}{|t - z|^{1+\alpha}} dz$$

$$\geq \partial_t^\alpha [t/2, t](1+t)^{-p} + \partial_t^\alpha [0, t/2](1+t)^{-p} - ct^{-\alpha},$$

for some $c > 0$ just depending on α. Using the Mean Value Theorem, there exists a constant c depending on p such that

$$\partial_t^\alpha [t/2, t](1+t)^{-p} \geq -c(1+t)^{-p-1} \int_{t/2}^{t} |t - z|^{-\alpha} dz \geq -ct^{-p-\alpha},$$

meanwhile, neglecting positive terms, we see that

$$\partial_t^\alpha [t/2, t](1+t)^{-p} \geq -\int_0^{t/2} \frac{(1+z)^{-p}}{|t - z|^{1+\alpha}} dz \geq -\int_0^{t/2} \frac{dz}{|t - z|^{1+\alpha}} \geq -ct^{-\alpha},$$

and joining the above inequalities we conclude the Claim.

Let $\epsilon > 0$ be small enough in order to have $p_\epsilon := \alpha/k - \epsilon > 0$. Take $C > 0$ large enough such that $C(1+t)^{-p_\epsilon} \geq C2^{-p_\epsilon} \geq \mathcal{E}$ in $[0,1]$. By the Claim, it is possible to take C larger if it is necessary to get

$$\partial_t^\alpha C(1+t)^{-\alpha/k+\epsilon} + (C(1+t)^{-\alpha/k+\epsilon})^k \geq 0 \quad \text{in } [1,+\infty).$$

Then, by comparison, we arrive at $0 \leq \mathcal{E}(t) \leq C_\epsilon(1+t)^{-\alpha/k+\epsilon}$ for all $t \geq 0$. This concludes the proof. $\qquad\square$

In order to state our key result to obtain the large time behavior, we need some definitions. Given two points $x_0, x_1 \in \mathbb{R}^N$, we denote $[x_0, x_1]$ the line segment joining x_0 and x_1. For a set of points $x_0, x_1, ..., x_n$, with $n \in \mathbb{N}$, we denote

$$[x_0, ..., x_n] = \bigcup_{i=1}^{n}[x_i, x_{i-1}],$$

that is, the polygonal curve joining the points x_i, $i = 0, ..., n$. The length of a finite polygonal line $[x_0, x_1, \cdots, x_n]$, $x_i \in \mathbb{T}^N$, is given by $\ell([x_0, x_1, \cdots, x_n]) = \sum_{i=1}^{n}|x_i - x_{i-1}|$. A continuous curve $\gamma : [0,1] \to \mathbb{T}^N$ is said to be *rectifiable* if

$$\ell(\gamma) := \sup_{\substack{n \in \mathbb{N} \\ 0 := t_0 < t_1 < \cdots < t_n = 1}} \ell([\gamma(t_0), \gamma(t_1), \cdots, \gamma(t_n)]) < +\infty.$$

We call $\ell(\gamma)$ the length of γ.

As explained in the introduction, the proof of Namah-Roquejoffre Theorem [20, Theorem 1] relies on three steps, the limiting one being the second one, i.e., to prove that $u(\cdot, t)$ converges on $Z = \{f = 0\}$ when $t \to +\infty$ (recall that we assume $c = 0$). We prove now this key result under the additional assumption

(4.7) $$\operatorname{argmin}\{g\} \cap Z \neq \emptyset.$$

The large time behavior is an easy consequence, see Corollary 4.6.

Theorem 4.2. *Assume* (1.5) *or* (1.6), (1.9), (4.3) *and* (4.7). *Assume further that for each* $z \in Z$, *there exists* $x_0 \in Z \cap \operatorname{argmin}\{g\}$ *and a rectifiable curve* $\gamma : [0,1] \to \mathbb{T}^N$ *such that* $\gamma(0) = x_0$, $\gamma(1) = z$ *and* $\gamma(t) \in Z$ *for all* $t \in [0,1]$. *Then, the unique solution* u *to* (1.1)-(1.2) *converges on* Z, *i.e.,*

$$\text{for every } x \in Z, \ u(x,t) \to \min\{g\} \text{ as } t \to +\infty.$$

Before giving the proof of the theorem, we state the following key lemma.

Lemma 4.3. *Assume hypotheses of Theorem 4.3 hold. Let* $z \in Z$, $x_0 \in Z \cap \operatorname{argmin}\{g\}$ *and assume that there exists a finite polygonal line* $\gamma := [x_0, x_1, \cdots, x_n := z]$ *lying in* Z *and joining* x_0 *to* z. *Then, the unique solution* u *to* (1.1)-(1.2) *satisfies*

(4.8) $$\min\{g\} \leq u(z,t) \leq \min\{g\} + \operatorname{Lip}(g)\ell(\gamma)\mathcal{E}(t),$$

where $\mathcal{E}(t) \searrow 0$ *as* $t \to +\infty$ *is a function which depends on* H, $||f||_\infty$, $\operatorname{Lip}(g)$, N *and* $\ell(\gamma)$.

Proof of Lemma 4.3. Without loss of generality we can assume $\min\{g\} = 0$. From this and (1.9), we obtain that 0 is a subsolution of (1.1)-(1.2) in Q. Therefore, by comparison, $0 \leq u$ in Q.

For the upper bound, the idea is to construct a function U such that $U(z,t) \searrow 0$ as $t \to +\infty$ and such that $u(z,t) \leq U(z,t)$ for all $t > 0$. This is performed by an inductive procedure, building a sequence of functions $(U_i)_{0 \leq i \leq n-1}$ which are supersolutions for the equation solved by u but in the set $Q_i := Q \setminus \{x_i\} \times [0, +\infty)$, with some control on the line $\{x_i\} \times [0, +\infty)$ in order to use comparison principles for the Cauchy-Dirichlet problem.

We divide the proof in several steps.

Step 1. Definition of $\mathcal{E}(t)$. By Lemma 4.1, for every $A > 0$ and $k \geq 1$, there exists a unique positive solution $\mathcal{E} \in C([0,\infty)) \cap C^1((0,\infty))$ to (4.5) such that $\mathcal{E}(t) \searrow 0$ as $t \to +\infty$. Notice that, since $\mathcal{E}$ is nonincreasing, $\partial_t^\alpha \mathcal{E}(t) \leq 0$.

We now define A and other constants, the definition of which will be clear below. We set $L := \mathrm{Lip}(g)$ and

$$(4.9) \qquad M := L + C_H^2 + C_H \|f\|_\infty + L C_H (\sqrt{N} + \ell(\gamma)),$$

where C_H appears in (4.2) and $\sqrt{N} = \mathrm{diam}(\mathbb{T}^N)$.

From (4.3) and (4.4), we may define $\nu_{L+M} > 0$ such that

$$(4.10) \qquad F(x,p) \geq \nu_{L+M}|p|^k \quad \text{for } |p| \leq L + M.$$

We then fix

$$(4.11) \qquad A = \frac{\nu_{L+M} L^{k-1}}{\sqrt{N} + \ell(\gamma)}$$

in (4.5). Notice that we may assume without loss of generality that that $A \leq 1$ by decreasing $\nu_{L+M} > 0$ if necessary.

Step 2. Definition of the function U_i, $0 \leq i \leq n-1$. We set

$$U_0(x,t) = L|x - x_0|\mathcal{E}(t) + M d_{[x_0,x_1]}(x)$$

and

$$(4.12) \quad U_i(x,t) = L \sum_{j=1}^{i} |x_j - x_{j-1}|\mathcal{E}(t) + L|x - x_i|\mathcal{E}(t) + M d_{[x_i,x_{i+1}]}(x), \qquad 1 \leq i \leq n-1,$$

where M is given by (4.9) and $d_{[x_i,x_{i+1}]}$ denotes the (periodic) distance function to the segment $[x_i, x_{i+1}]$, that is, for each $x \in \mathbb{T}^N$ (cast as a point in $[0,1)^N$), we write

$$d_{[x_i,x_{i+1}]}(x) = \inf_{y \in [x_i,x_{i+1}], k \in \mathbb{Z}^N} |x + k - y|.$$

This is a 1-Lipschitz continuous. At the points where it is differentiable we have the gradient meets $\widehat{x - p_i}$, where p_i is the projection of x to the segment, from which $|Dd_{[x_i,x_{i+1}]}(x)| = 1$. In addition, for each point in the set of non differentiability of $d_{[x_i,x_{i+1}]}$ which do not lie on the segment $[x_i, x_{i+1}]$, there is not C^1 function touching the function from below.

Step 3. The initial supersolution U_0. We prove actually that U_0 is a supersolution. At first, for $t = 0$ and all $x \in \mathbb{T}^N$, since $L = \text{Lip}(g)$ and $g(x_0) = 0$, we have

$$U_0(x, 0) \geq L|x - x_0| \geq g(x) - g(x_0) = g(x).$$

If $t > 0$, $x \notin [x_0, x_1]$ and U_0 is C^1 at (x, t), by the choice of the constant M_0 we use the coercivity properties of the Hamiltonian to write the following computation holds

$$
\begin{aligned}
\partial_t^\alpha U_0 &+ F(x, DU_0) - f(x) \\
&= L|x - x_0|\partial_t^\alpha \mathcal{E}(t) + F\left(x, L\mathcal{E}(t)\widehat{x - x_0} + M_0 Dd_{[x_0,x_1]}(x)\right) - f(x) \\
&\geq L\sqrt{N}\partial_t^\alpha \mathcal{E}(t) + \frac{1}{C_H}\left|L\mathcal{E}(t)\widehat{x - x_0} + M_0 Dd_{[x_0,x_1]}(x)\right| - C_H - ||f||_\infty \\
&\geq \frac{M}{C_H} - \frac{L}{C_H} - C_H - ||f||_\infty + L\sqrt{N}\left(\partial_t^\alpha \mathcal{E}(t) + A|\mathcal{E}(t)|^k\right) - L\sqrt{N}A|\mathcal{E}(t)|^k \\
&\geq \frac{M}{C_H} - \frac{L}{C_H} - C_H - ||f||_\infty - L\sqrt{N}A,
\end{aligned}
$$

where we used (4.2), $0 \leq \mathcal{E}(t) \leq 1$, $\partial_t^\alpha \mathcal{E}(t) \leq 0$, (4.5) and $\sqrt{N} = \text{diam}\,\mathbb{T}^N$. By the choice of M in (4.9) and since $A \leq 1$, the right hand side of the previous inequality is nonnegative. So U_0 is a supersolution outside $[x_0, x_1]$.

Now, let $t > 0$, $x \in [x_0, x_1]$ and consider any C^1 test-function φ touching U_0 from below at (x, t), i.e., $U_0(y, s) \geq \varphi(y, s)$ and $U_0(x, t) = \varphi(x, t)$. Since U_0 is sufficiently smooth in time, the following computation holds in a classical way,

$$
\begin{aligned}
\partial_t^\alpha \varphi(x, t) = \int_{-\infty}^t \frac{\varphi(x, t) - \varphi(x, s)}{|t - s|^{1+\alpha}}ds &\geq \int_{-\infty}^t \frac{U_0(x, t) - U_0(x, s)}{|t - s|^{1+\alpha}}ds \\
&= \partial_t^\alpha U_0(x, t) = L|x - x_0|\partial_t^\alpha \mathcal{E}(t).
\end{aligned}
$$

Since the right hand side of the above inequality is zero at $x = x_0$ and $F(x, p) \geq 0$, the supersolution property holds at $x = x_0$.

It remains to prove that U_0 is a supersolution for $x \in [x_0, x_1] \setminus \{x_0\}$. In this case, $x - h\widehat{x - x_0} \in [x_0, x_1]$ for $h > 0$ enough small and $d_{[x_0,x_1]}(x) = d_{[x_0,x_1]}(x - h\widehat{x - x_0}) = 0$. It follows

$$
\begin{aligned}
\varphi(x - h\widehat{x - x_0}, t) - \varphi(x, t) &\leq U_0(x - h\widehat{x - x_0}, t) - U_0(x, t) \\
&= L|x - h\widehat{x - x_0} - x_0|\mathcal{E}(t) - L|x - x_0|\mathcal{E}(t) = -h\mathcal{E}(t).
\end{aligned}
$$

Dividing by h and letting $h \searrow 0$ we obtain a lower-bound for $|D\varphi(x, t)|$,

$$(4.13) \qquad\qquad |D\varphi(x, t)| \geq \langle D\varphi(x, t), \widehat{x - x_0}\rangle \geq L\mathcal{E}(t).$$

Noticing that U_0 is Lipschitz continuous with constant $L + M$ in space, we have also an upper-bound $|D\varphi(x, t)| \leq L + M$. For $x \in [x_0, x_1] \setminus \{x_0\}$, recalling $f(x) = 0$, we use the behavior of

F near the origin in (4.4) together with (4.13) to write

$$\partial_t^\alpha \varphi(x,t) + F(x, D\varphi(x,t)) - f(x)$$
$$\geq \partial_t^\alpha U_0(x,t) + \nu_{L+M}|D\varphi(x,t)|^k$$
$$= L|x - x_0|\partial_t^\alpha \mathcal{E}(t) + \nu_{L+M}L^k|\mathcal{E}(t)|^k$$
$$\geq L\sqrt{N}\left(\partial_t^\alpha \mathcal{E}(t) + A|\mathcal{E}(t)|^k\right) - L\sqrt{N}A|\mathcal{E}(t)|^k + \nu_{L+M}L^k|\mathcal{E}(t)|^k$$
$$\geq L\left(\nu_{L+M}L^{k-1} - A\sqrt{N}\right)|\mathcal{E}(t)|^k$$

by (4.5). Recalling the choices of A in (4.11), the right hand side of the above inequality is nonnegative.

Since the viscosity inequality follows at once in the points where the distance function cannot be touched from below, the previous discussion leads to conclude that U_0 is a super-solution of (1.1)-(1.2) in Q. By comparison, we obtain $u(x,t) \leq U_0(x,t)$ for all $(x,t) \in \bar{Q}$, from which, in particular we get that

$$u(x_1, t) \leq L|x_1 - x_0|\mathcal{E}(t) \quad \text{for all } t.$$

Step 4. Proof by induction for U_i, $i \geq 1$. By induction, we will prove that U_i in (4.12) satisfies

(4.14) $$\partial_i^\alpha U_i + H(x, DU_i) \geq 0 \quad \text{in } Q_i, \qquad u \leq U_i \quad \text{in } \partial_p Q_i,$$

where $\partial_p Q_i$ is the parabolic boundary $\{x_i\} \times (0,t) \cup \mathbb{T}^N \times \{0\}$.

We first deal with the Cauchy-Dirichlet condition. By assumption, we have $0 \leq u(x_i, t) \leq U_{i-1}(x_i, t)$ for every $t \geq 0$. When $i = 1$, it follows

$$u(x_1, t) \leq U_0(x_1, t) = L|x_1 - x_0|\mathcal{E}(t) \leq \sqrt{N}L\mathcal{E}(t) \leq U_1(x_1, t).$$

When $i \geq 2$, we have

$$u(x_i, t) \leq U_{i-1}(x_i, t) = L\sum_{j=1}^{i-1}|x_j - x_{j-1}|\mathcal{E}(t) + L|x_i - x_{i-1}|\mathcal{E}(t) = U_i(x_i, t).$$

For $t = 0$ and all $x \in \mathbb{T}^N$, we have, using the triangle inequality,

$$U_i(x, 0) = L\sum_{j=1}^{i}|x_j - x_{j-1}|\mathcal{E}(0) + L|x - x_i|\mathcal{E}(0) \geq L|x - x_0| \geq g(x) - g(x_0) = g(x),$$

so the initial condition is satisfied.

Now we deal with the PDE in (4.14). The proof follows the same lines as the one in Step 4, so we only sketch it.

If $t > 0$ and $x \notin [x_i, x_{i+1}]$, then U_i is C^1 at (x, t) and we can do the same computation as in Step 4 to obtain

$$\partial_t{}^\alpha U_i + F(x, DU_i) - f(x)$$
$$= L\ell(\gamma)\partial_t^\alpha \mathcal{E}(t) + L|x - x_i|\partial_t^\alpha \mathcal{E}(t) + F\left(x, L\widehat{x - x_i}\mathcal{E}(t) + MDd_{[x_i,x_{i+1}](x)}\right) - f(x)$$
$$\geq L(\ell(\gamma) + \sqrt{N})\partial_t^\alpha \mathcal{E}(t) + \frac{M - L}{C_H} - C_H - ||f||_\infty$$
$$\geq L(\ell(\gamma) + \sqrt{N})\left(\partial_t^\alpha \mathcal{E}(t) + A|\mathcal{E}(t)|^k\right) + \frac{M}{C_H} - \frac{L}{C_H} - C_H - ||f||_\infty - L(\ell(\gamma) + \sqrt{N})A|\mathcal{E}(t)|^k.$$

By the choice of M in (4.9), recalling that $A \leq 1$, we obtain that the right hand side of the above inequality is nonnegative.

Now, let $t > 0$, $x \in [x_i, x_{i+1}] \setminus \{x_i\}$ and consider any C^1 test-function φ touching U_i from below at (x, t), i.e., $U_i(y, s) \geq \varphi(y, s)$ and $U_i(x, t) = \varphi(x, t)$. As in Step 4, we check easily that $\partial_t^\alpha \varphi(x, t) \geq \partial_t^\alpha U_i(x, t)$ and, since $d_{[x_i,x_{i+1}]}(x - h\widehat{x - x_i}) = 0$ for $h > 0$ small enough, that $|D\varphi(x, t)| \geq L\mathcal{E}(t)$. Moreover, $|D\varphi(x, t)| \leq L + M$. It follows

$$\partial_t^\alpha \varphi(x, t) + F(x, D\varphi(x, t)) - f(x)$$
$$\geq L(\ell(\gamma) + \sqrt{N})\left(\partial_t^\alpha \mathcal{E}(t) + A|\mathcal{E}(t)|^k\right) - L(\ell(\gamma) + \sqrt{N})A|\mathcal{E}(t)|^k + \nu_{L+M}L^k|\mathcal{E}(t)|^k$$
$$\geq L\left(\nu_{L+M}L^{k-1} - (\ell(\gamma) + \sqrt{N})A\right)|\mathcal{E}(t)|^k$$

recalling that $f = 0$ on $[x_i, x_{i+1}]$. By the choice of A in (4.11), the right hand side of the above inequality is nonnegative.

It ends the proof of the supersolution property for U_i, and therefore the inductive process.

Then, using Cauchy-Dirichlet comparison principles in [21] (or standard adaptation of the comparison principles presented in [23] when the Dirichlet condition is satisfied pointwisely), we conclude that $u \leq U_i$ in Q_i, from which we get $u(z, t) \leq U_i(z, t)$ for all $t > 0$. This concludes the proof. □

Proof of Theorem 4.2. We may assume without loss of generality that $\min g = 0$. It follows that $0 \leq u$ in Q and we need to prove that $u(x, t) \to 0$ on Z as $t \to +\infty$.

Let $\epsilon > 0$ and $f_\epsilon := (f - C_f \epsilon)_+$, where $C_f := \mathrm{Lip}(f)$. Then f_ϵ is a periodic function that satisfies the following properties,

$$0 \leq f_\epsilon \leq f, \qquad ||f_\epsilon - f||_\infty \leq C_f \epsilon,$$
$$Z = \{f = 0\} \subset \{d_Z(x) \leq \epsilon\} \subset Z_\epsilon = \{f_\epsilon = 0\}.$$

To prove the last inclusion, let $x \in \mathbb{T}^N$ such that $d_Z(x) \leq \epsilon$. Then there exists $x_Z \in Z$ such that $|x - x_Z| \leq \epsilon$ and $0 \leq f(x) \leq f(x_Z) + C_f|x - x_Z| \leq C_f \epsilon$. Thus $f_\epsilon(x) = 0$.

Now we consider (1.10)-(1.2) by replacing f with f_ϵ. Notice that the assumptions of Theorem 4.2 and Lemma 4.3 still holds. In particular, there exists a unique solution u_ϵ. Moreover, $u_\epsilon \pm ||f - f_\epsilon||c_{\alpha,\alpha}^{-1}t^\alpha$, where $c_{\alpha,\alpha}$ appears in (1.17), are respectively a super- and a subsolution

of (1.10)-(1.2) with f. By comparison, we get

$$(4.15) \qquad\qquad ||u - u_\epsilon|| \le \frac{C_f}{c_{\alpha,\alpha}} \epsilon t^\alpha.$$

Let $z \in Z$ and $\gamma : [0,1] \to Z$ be a rectifiable polygonal curve such that $\gamma(0) = x_0 \in Z \cap \mathrm{argmin}\{g\}$ and $\gamma(1) = z$. By assumption, there exists a sequence of subdivision $t_0^k := 0 < t_1^k < \cdots < t_{n_k}^k := 1$, $k \in \mathbb{N}$ and a finite polygonal line $\gamma_k := [\gamma(t_0^k), \cdots, \gamma(t_{n_k}^k)]$ which satisfies $\ell(\gamma_k) \nearrow \ell(\gamma)$ as $k \to \infty$.

In particular, we can prove that $\mathrm{dist}(\gamma_k, \gamma) \to 0$ as $k \to \infty$. It follows that there exists $k_\epsilon \in \mathbb{N}$ such that $\gamma_{k_\epsilon} \subset Z_\epsilon$ and γ_{k_ϵ} is a finite polygonal line joining x_0 and z. We can apply Lemma 4.3 to obtain

$$(4.16) \qquad\qquad 0 \le u_\epsilon(z,t) \le \mathrm{Lip}(g)\ell(\gamma_{k_\epsilon})\mathcal{E}(t), \qquad \text{for all } t \ge 0.$$

A priori, $\mathcal{E}$ depends on γ_{k_ϵ} through the dependence of A with respect to $\ell(\gamma_{k_\epsilon})$ in (4.5) but, since $\ell(\gamma_k) \le \ell(\gamma) < +\infty$, we can fix A and $\mathcal{E}$ independently of ϵ. From (4.15), we finally obtain

$$0 \le u(z,t) \le \frac{C_f}{c_{\alpha,\alpha}} \epsilon t^\alpha + \mathrm{Lip}(g)\ell(\gamma)\mathcal{E}(t), \qquad \text{for all } t \ge 0.$$

Sending first $\epsilon \to 0$ and then $t \to +\infty$, we conclude. $\qquad\qquad\qquad\square$

In the Eikonal case (4.1), we have an explicit formula for $\mathcal{E}$ in Lemma 4.3, see the proof of Lemma 4.1, which allows to deal with more involved Z. More precisely, we consider subsets Z satisfying the following assumption

$$(4.17) \qquad \begin{array}{c} \text{There exists } D \ge 1 \text{ and } C > 0 \text{ such that, for all } \epsilon > 0 \text{ and } x \in Z(\epsilon) := \{d_Z < \epsilon\}, \\ \text{there exists } x_0 \in Z \cap \mathrm{argmin}\{g\} \text{ and a finite polygonal line } \gamma_\epsilon \subset Z(\epsilon) \\ \text{such that } \gamma_\epsilon \text{ is formed by at most } C\epsilon^{-D} \text{ lines of length } \epsilon. \end{array}$$

We remark that a set Z satisfying Assumption (4.17) is a curve of box-counting dimension D, see Falconer [11]. In several interesting cases, box-counting dimension agrees with Hausdorff dimension and when $D > 1$ then the curve have infinite length.

Theorem 4.4. *(Eikonal case) Assume* (1.5)*,* (1.9)*,* (4.4) *with* $k = 1$*, and* (4.17) *for* $1 \le D < \frac{3}{2}$*. Then, the unique solution* u *to* (1.1)-(1.2) *converges on* Z*, i.e.,*

$$\text{for every } x \in Z, \ u(x,t) \to \min\{g\} \ \text{as } t \to +\infty.$$

Proof of Theorem 4.4. We proceed exactly as in the proof of Theorem 4.2, where γ_{k_ϵ} is replacing with γ_ϵ given by Assumption (4.17). From (4.15) and (4.16), we arrive at

$$(4.18) \qquad\qquad 0 \le u(z,t) \le \frac{C_f}{c_{\alpha,\alpha}} \epsilon t^\alpha + \mathrm{Lip}(g)\ell(\gamma_\epsilon)\mathcal{E}(t), \qquad \text{for all } t \ge 0.$$

The difference with the proof of Theorem 4.2 is that γ is not necessarily rectifiable anymore. But we can estimate the length of γ_ϵ thanks to (4.17) and take profit of the explicit formula for the solution of (4.5) in the Eikonal case $k = 1$.

The constant $C > 0$ below may change line to line but it does not depend neither on ϵ nor on t. We have

$$\ell(\gamma_\epsilon) \leq C\epsilon^{-D}\epsilon, \qquad \text{from Assumption (4.17)},$$

$$\mathcal{E}(t) = E_\alpha(-At^\alpha) \leq \frac{C}{At^\alpha}, \qquad \text{from (1.19)-(1.20)},$$

$$A \geq \frac{1}{C\ell(\gamma_\epsilon)}, \qquad \text{from (4.11) and Assumption (4.4) with } k = 1.$$

Plugging all the estimates in (4.18), we end up with

(4.19) $$0 \leq u(z,t) \leq C\left(\epsilon t^\alpha + \frac{1}{\epsilon^{2(D-1)}t^\alpha}\right), \quad \text{for all } t \geq 0, \ \epsilon > 0.$$

Minimizing over $\epsilon > 0$, we obtain

$$0 \leq u(z,t) \leq Ct^{\alpha\frac{2D-3}{2D-1}},$$

and the right hand side tends to 0 as $t \to +\infty$ when $D < \frac{3}{2}$. $\qquad\square$

Remark 4.5. Notice that the condition on D does not depend on $\alpha \in (0,1)$. If we use the same approach in the classical case ($\alpha = 1$), then we can repeat the above proof with $\mathcal{E}(t) = e^{-At}$ and the right hand side of (4.19) now reads $C\epsilon t + \epsilon^{1-D}e^{-t\epsilon^{D-1}/C}$. We can prove that the minimum over $\epsilon > 0$ tends to 0 as $t \to +\infty$ if and only if $D < 2$. Even, if we obtain a more general result than in the fractional case $\alpha \in (0,1)$, this result is not optimal since the convergence on Z holds for any Z and $\mathrm{argmin}\{g\}$ even without any connectedness requirement (see Introduction).

Corollary 4.6. *Under the assumptions of Theorem 4.2 or 4.4, the unique solution u of* (1.1)-(1.2) *satisfies*

$$u(x,t) + ct^\alpha - v(x) \to 0 \quad \text{uniformly as } t \to +\infty,$$

where $c = -\min_{\mathbb{T}^N} f$ and v is the unique solution of (1.12) *satisfying $v = \min_{\mathbb{T}^N} g$ on Z.*

The proof of the corollary follows the procedure described in the introduction. We only sktech it. By comparison, we obtain that $u + ct^\alpha$ is uniformly bounded in $\mathbb{T}^N \times [0, +\infty)$. We then apply Theorems 2.1 and 2.2 to $u + ct^\alpha$ to prove Step 1. Step 2 follows from Theorem 4.2 or 4.4 and Step 3 is classical. For details, we refer the reader to the survey of Barles in [2] or Namah-Roquejoffre [20].

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