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Quasi-Stationary Distributions and Resilience: What to get from a sample?

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Abstract

We study a class of multi-species birth-and-death processes going almost surely to extinction and admitting a unique quasi-stationary distribution (qsd for short). When rescaled by K and in the limit $K \rightarrow +\infty$, the realizations of such processes get close, in any fixed finite-time window, to the trajectories of a dynamical system whose vector field is defined by the birth and death rates. Assuming that this dynamical has a unique attracting fixed point, we analyzed the behavior of these processes for finite K and finite times, “interpolating” between the two limiting regimes just mentioned. In the present work, we are mainly interested in the following question: Observing a realization of the process, can we determine the so-called engineering resilience? To answer this question, we establish two relations which intermingle the resilience, which is a macroscopic quantity defined for the dynamical system, and the fluctuations of the process, which are microscopic quantities. Analogous relations are well known in nonequilibrium statistical mechanics. To exploit these relations, we need to introduce several estimators which we control for times between $\log K$ (time scale to converge to the qsd) and $\exp(K)$ (time scale of mean time to extinction).

Keywords: birth-and-death process; dynamical system; engineering resilience; quasi-stationary distribution; fluctuation-dissipation relation; empirical estimators.

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1 Introduction and main results

1.1 Context and setting

The ability of an ecosystem to return to its reference state after a perturbation stress is given by its resilience, a concept pioneered by Holling. Resilience has several faces and multiple definitions [5]. In the traditional theoretical setting of dynamical systems, that is, differential equations, one of them is the so-called *engineering resilience*. It is concerned with what happens in the vicinity of a fixed point (equilibrium state) of the system, and is given by minus the real part of the dominant eigenvalue of the Jacobian matrix evaluated at the fixed point. It can also be defined as the reciprocal of the characteristic return time to the fixed point after a (small) perturbation. In this paper, we are interested in how to determine the engineering resilience from the data. But which data? The drawback of the notion of engineering resilience is that we do not observe population densities governed by differential equations. Instead, we count individuals which are subject to stochastic fluctuations. Can we nevertheless infer the resilience? The subject of this paper is to show that this is possible in the framework of birth-and-death processes which are, in a sense made precise below, close to the solutions of a corresponding differential equation, at certain time and population size scales.

Let us now describe our framework. We consider a population made of d species interacting with one another. If the state of the process $\underline{N}^K(t) = (N_1^K(t), \dots, N_d^K(t))$ at some time t is $\underline{n} = (n_1, \dots, n_i, \dots, n_d) \in \mathbb{Z}_+^d$, where n_i is the number of individuals of the i th species, then the rate at which the population increases (respectively decreases) by one individual of the j th species is $KB_j(\underline{n}/K)$ (respectively $KD_j(\underline{n}/K)$), where K is a scaling parameter. Under the assumptions we will make, the process goes extinct, *i.e.*, $\underline{0}$ is an absorbing state, with probability one. There are two limiting regimes for the behavior of this process. The first one is to fix K and let t tend to infinity, which leads inevitably to extinction. The second one consists in fixing a time horizon and letting K tend to $+\infty$, after having rescaled the process by K . In this limit, the behavior of the rescaled process is governed by a certain differential equation. More precisely, given any $0 < t_H < +\infty$ and any $\epsilon > 0$, we have

$$\lim_{K \rightarrow +\infty} \mathbb{P} \left(\sup_{0 \leq t \leq t_H} \text{dist} \left(\frac{\underline{N}^K(t)}{K}, \underline{x}(t) \right) > \epsilon \right) = 0$$

where $\text{dist}(\cdot, \cdot)$ is the Euclidean distance in \mathbb{R}_+^d , and $\underline{x}(t)$ is the solution of the differential equation in \mathbb{R}_+^d

$$\frac{d\underline{x}}{dt} = \underline{B}(\underline{x}) - \underline{D}(\underline{x}) \tag{1.1}$$

with initial condition $\underline{x}_0 = \underline{N}^K(0)/K$. We refer to [4, Chapter 11] for a proof. We use the notations $\underline{x} = (x_1, \dots, x_d)$, $\underline{B}(\underline{x}) = (B_1(\underline{x}), \dots, B_d(\underline{x}))$, and so on and so forth. We will make further assumptions (see Subsection 1.4) on the birth and death rates to be in the following situation. The vector field

$$\underline{X} = \underline{B} - \underline{D}$$

has a unique attracting fixed point \underline{x}^* (lying in the interior of \mathbb{R}_+^d). We denote by M^* its differential evaluated at \underline{x}^* , namely

$$M^* = D\underline{X}(\underline{x}^*). \quad (1.2)$$

We then define the (engineering) resilience as

$$\rho^* = -\sup\{\operatorname{Re}(z) : z \in \operatorname{Sp}(M^*)\} \quad (1.3)$$

where $\operatorname{Sp}(M^*)$ denotes the spectrum (set of eigenvalues) of the matrix M^* . Under our assumptions, $\rho^* > 0$.

Now we can formulate more precisely the goal of this paper. Given a *finite-length* realization of the process $(\underline{N}^K(t), t \leq T)$, with large, but *finite* K , we want to build an estimator for ρ^* . To this end, we need a good understanding of the behavior of the Markov process $(\underline{N}^K(t))$ in an intermediate regime between the two limiting regimes described above. This was done in a previous work of ours [3] which can be roughly summarized as follows. All states $\underline{n} \neq \underline{0}$ are transient and $\underline{0}$ is absorbing, hence the only stationary distribution is the Dirac measure sitting at $\underline{0}$. The mean time to extinction behaves like $\exp(\mathcal{O}(1)K)$. If we start in the vicinity of the state $\underline{n}^* = \lfloor K\underline{x}^* \rfloor$, that is, if the initial state has its coordinates of size of order K , then either the process wanders around \underline{n}^* or it gets absorbed at $\underline{0}$. More precisely, there is a unique quasi-stationary distribution (qsd, for short) ν_K which describes the statistics of the process conditioned not to be extinct before time t . Without this conditioning, the law of the process at time t is well approximated a mixture of the Dirac measure at $\underline{0}$ and the qsd ν_K , for $\log K \ll t \ll \exp(\mathcal{O}(1)K)$, $\log K$ being the typical time needed for the law of the process to be very close to ν_K . We will rely on these results that will be recalled precisely later on. We will also have to prove further properties.

1.2 Main results

To estimate the engineering resilience ρ^* , we will establish a matrix relation involving M^* . Let $\underline{\mu}^K = (\mu_1^K, \dots, \mu_d^K)$ be the vector of species sizes averaged with respect to ν_K , that is,

$$\mu_p^K = \int n_p d\nu_K(\underline{n}), \quad p = 1, \dots, d. \quad (1.4)$$

For each $\tau \geq 0$, define the matrix

$$\Sigma_{p,q}^K(\tau) = \mathbb{E}_{\nu_K} [(N_p^K(\tau) - \mu_p^K)(N_q^K(0) - \mu_q^K)], \quad p, q \in \{1, \dots, d\}. \quad (1.5)$$

In Section 4.1, we will prove

Theorem 1.1. *For all $\tau \geq 0$ we have*

$$\Sigma^K(\tau) = e^{\tau M^*} \Sigma^K(0) + \mathcal{O}(\sqrt{K}). \quad (1.6)$$

Later on we will show that $\underline{\mu}^K$ and $\Sigma^K(\tau)$ are of order K , hence the relevant range for τ is $\mathcal{O}(1) \leq \tau \ll \mathcal{O}(1) \log K$. Relation (1.6) allows to determine M^* . Indeed, we have

$$e^{\tau M^*} = \Sigma^K(\tau) \Sigma^K(0)^{-1} + \mathcal{O}\left(\frac{1}{\sqrt{K}}\right). \quad (1.7)$$

Given a finite-length realization of $(N^K(t), 0 \leq t \leq T)$ up to some time $T > 0$, we define the following estimators for $\Sigma_{p,q}^K(\tau)$ and μ_p^K in order to estimate M^* using (1.7). For $T > 0$ and $0 < \tau < T$, $p, q = 1, \dots, d$, $K \in \mathbb{N}^*$, let

$$S_p^\mu(T, K) = \frac{1}{T} \int_0^T N_p^K(s) ds \quad (1.8)$$

$$S_{p,q}^C(T, \tau, K) = \quad (1.9)$$

$$\frac{1}{T - \tau} \int_0^{T-\tau} (N_p^K(s + \tau) - S_p^\mu(T, K))(N_q^K(s) - S_q^\mu(T, K)) ds. \quad (1.10)$$

Under suitable conditions on \underline{n} , K and T , $S^\mu(T, K)$ well approximates $\underline{\mu}^K$. More precisely, we will prove an estimate of the form (see Theorem 3.4 for a precise statement)

$$\begin{aligned} |\mathbb{E}_{\underline{n}}[S_p^\mu(T, K)] - \mu_p^K| &\leq \\ &C (K + \|\underline{n}\|_1) \left(\frac{1 + \log K}{T} + e^{-c(\|\underline{n}\|_1 \wedge K)} + T e^{-c'K} \right) \end{aligned} \quad (1.11)$$

for every $\underline{n} \in \mathbb{Z}_+^d$, $p = 1, \dots, d$, where C, c, c' are positive constants, and $\|\underline{n}\|_1 = \sum_{i=1}^d n_i$. Let us comment on this bound. For K large, the right-hand side is small provided $\log K \ll \|\underline{n}\|_1 \leq \mathcal{O}(1)K$, $T \gg K \log K$, and $T \ll \exp(\mathcal{O}(1)K)$. Observe that, when $\log T \gg K$, $\mathbb{E}_{\underline{n}}[S_p^\mu(T, K)] \approx 0$, because with high probability the process is absorbed at $\underline{0}$, hence $|\mathbb{E}_{\underline{n}}[S_p^\mu(T, K)] - \mu_p^K|$ is not small. This is the manifestation of the fact that the only stationary distribution is the Dirac measure at $\underline{0}$.

An estimate of the same kind holds for $S^C(T, \tau, K)$ which well approximates $\Sigma^K(\tau)$ in the appropriate regimes.

We can now define an empirical matrix $M_{\text{emp}}^*(T, \tau, K)$ by

$$e^{\tau M_{\text{emp}}^*(T, \tau, K)} = S^C(T, \tau, K) S^C(T, 0, K)^{-1} \quad (1.12)$$

and an empirical resilience by

$$\rho_{\text{emp}}^*(T, \tau, K) = -\sup \{ \text{Re}(z) : z \in \text{Sp}(M_{\text{emp}}^*(T, \tau, K)) \}. \quad (1.13)$$

Our main result (Theorem 5.7) is then the following.

Theorem. *For τ of order one, $T \ll \exp(\mathcal{O}(1)K)$, \underline{n} of order K (initial state), and K large enough, we have*

$$|\rho_{\text{emp}}^*(T, \tau, K) - \rho^*| \leq \mathcal{O}(1) \left(\frac{K^2}{\sqrt{T}} + \frac{1}{\sqrt{K}} \right) \quad (1.14)$$

with a probability larger than $1 - 1/K$. In particular, if $T \gg K^5$, we have

$$|\rho_{\text{emp}}^*(T, \tau, K) - \rho^*| \leq \frac{\mathcal{O}(1)}{\sqrt{K}}.$$

1.3 More results in the case $d = 1$

The above estimator for the engineering resilience, based on (1.7) is valid for any d . In the case $d = 1$ (only one species), we have another, simpler estimator which is based on a ‘‘fluctuation-dissipation relation’’. Let \mathcal{D}^K be a $d \times d$ diagonal matrix given by

$$\mathcal{D}_{p,p}^K = KB_p(\underline{x}^*) = KD_p(\underline{x}^*).$$

We have the following result. We write Σ^K instead of $\Sigma^K(0)$, and the transpose of a matrix M is denoted by M^\top .

Theorem 1.2. *We have*

$$M^* \Sigma^K + \Sigma^K M^{*\top} + 2\mathcal{D}^K = \mathcal{O}(\sqrt{K}). \quad (1.15)$$

This relation will be proved in Section 4.2. For background on fluctuation-dissipation relations in Statistical Physics, we refer to [7, sections 2-3]. Note that the matrix Σ^K is symmetrical, but in general the matrix M^* is not (see [3]). Note also that each term in the left hand side of (1.15) is of order K , as we will see below.

If Σ^K and \mathcal{D}^K are known, the matrix M^* is not uniquely defined, except for $d = 1$ (see for example [8]). Hence, for $d = 1$, (1.15) easily gives the resilience since it becomes a scalar equation:

$$\rho^* = \frac{K(B(\underline{x}^*) + D(\underline{x}^*))}{2\Sigma^K} + \mathcal{O}\left(\frac{1}{\sqrt{K}}\right).$$

Note that $K(B(\underline{x}^*) + D(\underline{x}^*)) = 2KB(\underline{x}^*)$ is the average total jump rate $K\nu_\kappa(B(\underline{n}/K) + D(\underline{n}/K))$ up to $O(1)$. This follows from a Taylor expansion of $B(\underline{n}/K) + D(\underline{n}/K)$ around \underline{x}^* , Theorem 2.6 and Proposition 2.7.

The estimator of \mathcal{D}^κ we use is

$$S_p^{\mathcal{D}}(T, K) = \frac{1}{T} \#\{\text{birth of species } p \text{ for } 0 \leq t \leq T\} \quad (1.16)$$

Indeed, we prove in Section 5 a bound for

$$|\mathbb{E}_{\underline{n}}[S_p^{\mathcal{D}}(T, K)] - KB_p(\underline{x}^*)|$$

which depends T, K and $\|\underline{n}\|_1$, and is small in the relevant regimes. The estimator we use for Σ^κ is

$$S_{p,q}^{\Sigma}(T, K) = \frac{1}{T} \int_0^T (N_p^\kappa(s) - S_p^\mu(T, K))(N_q^\kappa(s) - S_q^\mu(T, K)) ds. \quad (1.17)$$

Of course $S^\Sigma(T, K) = S^C(T, 0, K)$. Again, we can control how well this estimator approximates Σ^κ . Finally one can deduce another estimator for ρ^* and control the error.

1.4 Standing assumptions

The two (regular) vector fields $\underline{B}(\underline{x})$ and $\underline{D}(\underline{x})$ are given in \mathbb{R}_+^d . We assume that their components have second partial derivatives which are polynomially bounded. Obviously, we suppose that $B_j(\underline{x}) \geq 0$ and $D_j(\underline{x}) \geq 0$ for all $j = 1, \dots, d$ and $\underline{x} \in \mathbb{Z}_+^d$. A dynamical system in \mathbb{R}_+^d is defined by the vector field $\underline{X}(\underline{x}) = \underline{B}(\underline{x}) - \underline{D}(\underline{x})$, namely

$$\frac{d\underline{x}}{dt} = \underline{B}(\underline{x}) - \underline{D}(\underline{x}) = \underline{X}(\underline{x}).$$

For $\underline{x} \in \mathbb{R}_+^d$, we use the following standard norms:

$$\|\underline{x}\|_1 = \sum_{j=1}^d x_j, \quad \|\underline{x}\|_2 = \left(\sum_{j=1}^d x_j^2 \right)^{\frac{1}{2}}.$$

We now state our hypotheses.

H.1 The vectors fields \underline{B} and \underline{D} vanish only at $\underline{0}$.

H.2 There exists \underline{x}^* belonging to the interior of \mathbb{R}_+^d (fixed point of \underline{X}) such that

$$\underline{B}(\underline{x}^*) - \underline{D}(\underline{x}^*) = \underline{X}(\underline{x}^*) = \underline{0}.$$

H.3 Attracting fixed point: there exist $\beta > 0$ and $R > 0$ such that $\|\underline{x}^*\|_2 < R$, and for all $\underline{x} \in \mathbb{R}_+^d$ with $\|\underline{x}\|_2 < R$,

$$\langle \underline{X}(\underline{x}), (\underline{x} - \underline{x}^*) \rangle \leq -\beta \|\underline{x}\|_2 \|\underline{x} - \underline{x}^*\|_2^2. \quad (1.18)$$

H.4 The fixed point $\underline{0}$ of the vector field \underline{X} is repelling (locally unstable). Moreover, on the boundary of \mathbb{R}_+^d , the vector field \underline{X} points toward the interior (except at $\underline{0}$).

H.5 Define

$$\widehat{B}(y) = \sup_{\|\underline{x}\|_1=y} \sum_{j=1}^d B_j(\underline{x}), \quad \widehat{D}(y) = \inf_{\|\underline{x}\|_1=y} \sum_{j=1}^d D_j(\underline{x})$$

and for $y > 0$, let

$$F(y) = \frac{\widehat{B}(y)}{\widehat{D}(y)}.$$

We assume that there exists $0 < L < R$ such that $\sup_{y>L} F(y) < 1$ and $\lim_{y \rightarrow \infty} F(y) = 0$.

H.6 There exists $y_0 > 0$ such that $\int_{y_0}^{\infty} \frac{1}{\widehat{D}(y)} dy < +\infty$ and $y \mapsto \widehat{D}(y)$ is increasing on $[y_0, +\infty[$.

H.7 There exists $\xi > 0$ such that

$$\inf_{\underline{x} \in \mathbb{R}_+^d} \inf_{1 \leq j \leq d} \frac{D_j(\underline{x})}{\sup_{1 \leq \ell \leq d} x_\ell} > \xi > 0. \quad (H7)$$

H.8 Finally, we assume that

$$\inf_{1 \leq j \leq d} \partial_{x_j} B_j(\underline{0}) > 0. \quad (H8)$$

(By ∂_{x_j} we mean $\frac{\partial}{\partial x_j}$.)

Assumptions **H.5** and **H.6** ensure that the time for “coming down from infinity” for the deterministic dynamical system is finite. Together with **H.3**, this also implies that \underline{x}^* is a globally attracting stable fixed point. More comments on these assumptions can be found in [3].

1.5 A numerical example

We consider the two-dimensional vector fields

$$\underline{B}(x_1, x_2) = \begin{pmatrix} a x_1 + b x_2 \\ e x_1 + f x_2 \end{pmatrix}$$

and

$$\underline{D}(x_1, x_2) = \begin{pmatrix} x_1(c x_1 + d x_2) \\ x_2(g x_1 + h x_2) \end{pmatrix}$$

where all the coefficients are positive. This is a model of competition between two species of Lotka-Volterra type. We have taken

$$\begin{aligned} a &= 0.4569, \quad b = 0.2959, \quad e = 0.5920, \quad f = 0.6449 \\ c &= 0.9263, \quad d = 0.9157, \quad g = 0.9971, \quad h = 0.2905. \end{aligned}$$

Assumptions **H.1** and **H.4** are easily verified numerically. Assumptions **H.5** and **H.6** are true because $\widehat{B}(y) \leq a + b + e + f$ and $\widehat{D}(y) \geq (c \wedge h)y^2/4$. Concerning **H.2**, we checked numerically that there is a unique fixed point inside the positive quadrant, namely $\underline{x}^* = (0.3567, 1.4855)$. It remains to check **H.3**, namely that

$$-\beta = \sup_{\underline{x} \in \mathbb{R}_+^2} R(\underline{x}) < 0$$

where

$$R(\underline{x}) = \frac{\langle \underline{X}(\underline{x}), (\underline{x} - \underline{x}^*) \rangle}{\|\underline{x}\|_2 \|\underline{x} - \underline{x}^*\|_2^2}.$$

We first checked that the numerator $N(\underline{x}) = \langle \underline{X}(\underline{x}), (\underline{x} - \underline{x}^*) \rangle$ is negative and vanishes only in $\underline{0}$ and \underline{x}^* . It is easy to check that $N(\underline{x}) < 0$ for $\|\underline{x}\|_2$ large enough. We have verified numerically that the only solutions of the equations $\partial_{x_1} N = \partial_{x_2} N = 0$ in the closed positive quadrant are \underline{x}^* and $\underline{z} = (0.1739, 0.4361)$, with $N(\underline{z}) = -0.2852$, thus this is negative local minimum. This implies that $N(\underline{x}) < 0$ in the closed positive quadrant, except at $\underline{0}$ and \underline{x}^* where it vanishes. This implies that $R \leq 0$ in the closed positive quadrant. It is easy to check that

$$\limsup_{\|\underline{x}\|_2 \rightarrow +\infty} R(\underline{x}) \leq -(c \wedge h)/\sqrt{2}.$$

This implies that $R < 0$ except perhaps in $\underline{0}$ and \underline{x}^* . Near $\underline{0}$ we have by Taylor expansion

$$R(\underline{x}) = -\frac{\langle D\underline{X}(\underline{0}) \underline{x}, \underline{x}^* \rangle}{\|\underline{x}\|_2 \|\underline{x}^*\|_2^2} (1 + \mathcal{O}(\|\underline{x}\|_2)) = -\frac{\langle \underline{x}, D^\top \underline{B}(\underline{0}) \underline{x}^* \rangle}{\|\underline{x}\|_2 \|\underline{x}^*\|_2^2} (1 + \mathcal{O}(\|\underline{x}\|_2))$$

and, since the vector $D^\top \underline{B}(\underline{0}) \underline{x}^*$ has positive components, there exists $\varrho > 0$ such that for all $\underline{x} \in \mathbb{R}_+^2$

$$\langle \underline{x}, D^\top \underline{B}(\underline{0}) \underline{x}^* \rangle \geq \varrho \|\underline{x}\|_2.$$

If $\underline{y} = \underline{x} - \underline{x}^*$ is small, we have by Taylor expansion (since $\underline{X}(\underline{x}^*) = \underline{0}$)

$$R(\underline{x}) = \frac{\langle M^* \underline{y}, \underline{y} \rangle}{\|\underline{x}^*\|_2 \|\underline{y}\|_2^2} (1 + \mathcal{O}(\|\underline{y}\|_2)) = \frac{\langle \underline{y}, \frac{1}{2} (M^{*\top} + M^*) \underline{y} \rangle}{\|\underline{x}^*\|_2 \|\underline{y}\|_2^2} (1 + \mathcal{O}(\|\underline{y}\|_2)).$$

One can check numerically that the two real eigenvalues of the symmetric matrix

$$M^{*\top} + M^*$$

are strictly negative, the largest being numerically equal to -0.786 . This completes the verification of hypothesis **H.3**.

We have chosen $K = 10^5$ and simulated a realization of the process with $T = 100$ which contains about 5.10^7 jumps. The resilience computed from the vector field is numerically equal to 0.547 . We have computed $\rho_{\text{emp}}^*(100, 1, 10^5)$. The relative error, that is $|\rho_{\text{emp}}^*(100, 1, 10^5) - \rho^*|/\rho^*$, is equal to 0.022 .

We have also checked the size of the error in (1.15). We have computed

$$\varepsilon = \frac{1}{\sqrt{K}} \|M^* S^\Sigma(T, K) + S^\Sigma(T, K) M^{*\top} + 2\mathcal{D}^K\|$$

where $\|\cdot\|$ stands for Hilbert-Schmidt norm. For the above values of T and K , $\varepsilon = 68$, which corresponds to a rather large error term. However, we have also checked that this quantity depends strongly on T . Namely, for $K = 2000$ and $T = 2000$ (which has roughly the same computational cost), we have found $\varepsilon = 3.19$.

1.6 Organization of the paper

In Section 2, we will study the time evolution of the moments of the process and we will prove moment estimates for the qsd. In Section 3, we will obtain control on the large time behavior of averages for the process. In Section 4, we will prove the relations (1.6) and (1.15). In Section 5, we will apply these relations to obtain approximate expressions of the engineering resilience in terms of the covariance matrices for the qsd. From the results of Section 3, we will deduce variance bounds for the estimators (1.9), (1.10) and (1.16), starting either in the qsd or from an initial condition of order K .

2 Time evolution of moments of the process and moments of the QSD

2.1 Time evolution of moments starting from anywhere

The generator \mathcal{L}_K of the birth and death process $\underline{N}^K = (\underline{N}^K(t), t \geq 0)$ is defined by

$$\begin{aligned} \mathcal{L}_K f(\underline{n}) = & \hspace{15em} (2.1) \\ & K \sum_{\ell=1}^d B_\ell \left(\frac{\underline{n}}{K}\right) (f(\underline{n} + \underline{e}^{(\ell)}) - f(\underline{n})) + K \sum_{\ell=1}^d D_\ell \left(\frac{\underline{n}}{K}\right) (f(\underline{n} - \underline{e}^{(\ell)}) - f(\underline{n})) \end{aligned}$$

where $\underline{e}^{(\ell)} = (0, \dots, 0, 1, 0, \dots, 0)$, the 1 being at the ℓ -th position, and $f : \mathbb{Z}_+^d \rightarrow \mathbb{R}$ is a function with bounded support. We denote by $(S_t^K, t \geq 0)$ the semigroup of the process \underline{N}^K acting on bounded functions, that is, for $f : \mathbb{Z}_+^d \rightarrow \mathbb{R}$, we have

$$S_t^K f(\underline{n}) = \mathbb{E} [f(\underline{N}^K(t)) | \underline{N}^K(0) = \underline{n}] = \mathbb{E}_{\underline{n}} [f(\underline{N}^K(t))].$$

For $A > 1$, let

$$\mathcal{T}_A = \inf\{t > 0 : \|\underline{N}^K(t)\|_1 > A\}. \quad (2.2)$$

Notice that we will use either $\|\cdot\|_1$ or $\|\cdot\|_2$ (which are of course equivalent). We have the following result.

Theorem 2.1. *There exists a constant $C_{(2.1)} > 0$ such that for K large enough, the operator group S_1^K extends to exponentially bounded functions and*

$$\sup_{\underline{n} \in \mathbb{Z}_+^d} S_1^K \left(e^{\|\cdot\|_1} \right) (\underline{n}) \leq e^{C_{(2.1)}K}.$$

Proof. Let us first introduce the function G_K defined on $[y_0, +\infty)$ by

$$G_K(y) = \int_y^\infty \frac{dz}{\widehat{D}(z)} + \frac{1}{K \widehat{D}(y)}.$$

Assumption **H.6** implies that G_K is well defined and decreasing. We can define its inverse function on $(0, s_0]$ for $s_0 > 0$ small enough (independent of K). Take $0 < \eta \leq s_0 \wedge \frac{1-e^{-1}}{4}$. Then there is a unique positive function y_K defined by

$$y_K(s) = G_K^{-1}(\eta s), \quad s \in (0, 1]. \quad (2.3)$$

Note that $y_K(s) \geq y_0$ and $\lim_{s \downarrow 0} y_K(s) = +\infty$. Let

$$\varphi_K(s) = \frac{e^{-Ky_K(s)}}{K \widehat{D}(y_K(s))}.$$

Note that

$$\lim_{s \downarrow 0} \varphi_K(s) = 0.$$

Using the Lipschitz continuity of \widehat{D} (and then its differentiability almost everywhere) and (2.3), we obtain

$$\dot{\varphi}_K(s) = \frac{d\varphi_K}{ds}(s) = - \left(\frac{e^{-Ky_K(s)}}{\widehat{D}(y_K(s))} + \frac{e^{-Ky_K(s)} \widehat{D}'(y_K(s))}{K \widehat{D}(y_K(s))^2} \right) \frac{dy_K}{ds}(s) = \eta e^{-Ky_K(s)}.$$

We now consider the function

$$f_K(t, \underline{n}) = \varphi_K(t) e^{\|\underline{n}\|_1}$$

to which we apply Itô's formula to f_K at time $t \wedge \mathcal{T}_A$. We get

$$\mathbb{E}_{\underline{n}} \left[\varphi_K(t \wedge \mathcal{T}_A) e^{\|\underline{N}^K(t \wedge \mathcal{T}_A)\|_1} \right] = \mathbb{E}_{\underline{n}} \left[\int_0^{t \wedge \mathcal{T}_A} (\partial_t f_K + \mathcal{L}_K f_K)(s, \underline{N}^K(s)) ds \right].$$

We have

$$\begin{aligned} \partial_t f_K(t, \underline{n}) + \mathcal{L}_K f_K(t, \underline{n}) &= \dot{\varphi}_K(t) e^{\|\underline{n}\|_1} \\ &+ K \varphi_K(t) e^{\|\underline{n}\|_1} \left((e-1) \sum_{\ell=1}^d B_\ell \left(\frac{\underline{n}}{K} \right) + (e^{-1}-1) \sum_{\ell=1}^d D_\ell \left(\frac{\underline{n}}{K} \right) \right). \end{aligned}$$

Note that

$$\begin{aligned} &\partial_t f_K(t, \underline{n}) + \mathcal{L}_K f_K(t, \underline{n}) \\ &\leq e^{\|\underline{n}\|_1} \left(\dot{\varphi}_K(t) + K \varphi_K(t) \left((e-1) \widehat{B} \left(\frac{\|\underline{n}\|_1}{K} \right) - (1-e^{-1}) \widehat{D} \left(\frac{\|\underline{n}\|_1}{K} \right) \right) \right) \\ &\leq e^{\|\underline{n}\|_1} \left(\dot{\varphi}_K(t) - K \varphi_K(t) (1-e^{-1}) \widehat{D} \left(\frac{\|\underline{n}\|_1}{K} \right) \left(1 - e F \left(\frac{\|\underline{n}\|_1}{K} \right) \right) \right). \end{aligned}$$

It follows from **H.5** that there exists a number $\zeta > y_0$ such that if $y > \zeta$, then $F(y) < (2e)^{-1}$.

If $\|\underline{n}\|_1 < \zeta K$ we get

$$\left| \partial_t f_K(t, \underline{n}) + \mathcal{L}_K f_K(t, \underline{n}) \right| \leq \mathcal{O}(1) e^{\zeta K} (\dot{\varphi}_K(t) + K \varphi_K(t)).$$

For $\|\underline{n}\|_1 \geq K(\zeta \vee y_K(t))$ we have

$$\partial_t f_K(t, \underline{n}) + \mathcal{L}_K f_K(t, \underline{n}) \leq 0$$

since $\dot{\varphi}_K(t) = \eta K \widehat{D}(y_K(t)) \varphi_K(t)$ and $\widehat{D}(\|\underline{n}\|_1/K) \geq \widehat{D}(y_K(t))$.

Finally, for $\zeta K \leq \|\underline{n}\|_1 < K y_K(t)$ we get

$$\left| \partial_t f_K(t, \underline{n}) + \mathcal{L}_K f_K(t, \underline{n}) \right| \leq e^{K y_K(t)} \dot{\varphi}_K(t) = \eta.$$

We deduce that

$$\mathbb{E}_{\underline{n}} \left[\varphi_K(1 \wedge \mathcal{T}_A) e^{\|\underline{N}^K(1 \wedge \mathcal{T}_A)\|_1} \right] \leq \mathcal{O}(1) e^{\zeta K}.$$

The result follows by letting A tend to infinity by monotonicity. \square

We deduce moment estimates for the process which are uniform in the starting state, and in time, for times larger than 1.

Corollary 2.2. *For all $t \geq 1$, the semi-group S_t maps functions of polynomially bounded modulus in bounded functions. In particular, for all $q \in \mathbb{N}$, we have*

$$\sup_{t \geq 1} \sup_{\underline{n} \in \mathbb{Z}_+^d} \mathbb{E}_{\underline{n}} \left[\|\underline{N}^K(t)\|_1^q \right] \leq q^q e^{-q} K^q e^{C(2.1)}. \quad (2.4)$$

Proof. We have

$$\begin{aligned}\mathbb{E}_{\underline{n}} [\|\underline{N}^K(1)\|_1^q] &= K^q \mathbb{E}_{\underline{n}} \left[\frac{\|\underline{N}^K(1)\|_1^q}{K^q} e^{-\frac{\|\underline{N}^K(1)\|_1}{K}} e^{\frac{\|\underline{N}^K(1)\|_1}{K}} \right] \\ &\leq K^q q^q e^{-q} \mathbb{E}_{\underline{n}} \left[e^{\frac{\|\underline{N}^K(1)\|_1}{K}} \right]\end{aligned}$$

since for all $x \geq 0$, $x^q e^{-x} \leq q^q e^{-q}$. Inequality (2.4) follows from Hölder's inequality and Theorem 2.1.

Let us now consider $t > 1$. From the Markov property and by using the previous inequality, we deduce that

$$\mathbb{E}_{\underline{n}} [\|\underline{N}^K(t)\|_1^q] = \mathbb{E}_{\underline{n}} \left[\mathbb{E}_{\underline{N}^K(t-1)} [\|\underline{N}^K(1)\|_1^q] \right] \leq q^q e^{-q} K^q e^{C(2.1)}.$$

The proof is finished. \square

For time t less than 1, the moment estimates will depend on the initial state.

Proposition 2.3. *For each integer q , there exists a constant $c_q > 0$ such that for all $K > 1$, $t \geq 0$ and $\underline{n} \in \mathbb{Z}_+^d$*

$$\mathbb{E}_{\underline{n}} [\|\underline{N}^K(t)\|_2^q] \leq c_q K^q + \|\underline{n}\|_2^q \mathbf{1}_{\{t < 1\}}.$$

Proof. We have only to study the case $t < 1$, the other case being given in (2.4). We prove the result for q even, namely $q = 2q'$. The general for q odd follows from Cauchy-Schwarz inequality. Letting

$$f_{q'}(\underline{n}) = \|\underline{n}\|_2^{2q'}$$

we have

$$\begin{aligned}\mathcal{L}_K f_{q'}(\underline{n}) &= K \sum_{\ell=1}^d B_{\ell} \left(\frac{\underline{n}}{K} \right) \left((\|\underline{n}\|_2^2 + 2\underline{n}_{\ell} + 1)^{q'} - \|\underline{n}\|_2^{2q'} \right) \\ &\quad + K \sum_{\ell=1}^d D_{\ell} \left(\frac{\underline{n}}{K} \right) \left((\|\underline{n}\|_2^2 - 2\underline{n}_{\ell} + 1)^{q'} - \|\underline{n}\|_2^{2q'} \right).\end{aligned}$$

Using **H.5** and the equivalence of norms, we see that there exists a constant $c_{q'} > 0$ such that if $\|\underline{n}\|_2 > c_{q'} K$

$$\mathcal{L}_K f_{q'}(\underline{n}) < 0.$$

We can moreover choose $c_{q'}$ larger such that for all \underline{n}

$$\mathcal{L}_K f_{q'}(\underline{n}) \leq c_{q'} K^{2q'}.$$

Applying Itô's formula to $f_{q'}$ we get as in the proof of Theorem 2.1

$$\begin{aligned} \mathbb{E}_{\underline{n}} [\|\underline{N}^K(t \wedge \mathcal{T}_A)\|_2^{2q'}] &\leq \|\underline{n}\|_2^{2q'} + \mathbb{E}_{\underline{n}} \left[\int_0^{t \wedge \mathcal{T}_A} c_{q'} K^{2q'} ds \right] \\ &\leq \|\underline{n}\|_2^{2q'} + t c_{q'} K^{2q'}. \end{aligned}$$

(Recall that \mathcal{T}_A is defined in (2.2).) The result follows by letting A tend to infinity. \square

2.2 Moments estimates for the qsd

Let us first recall (cf. [3]) that, under the assumptions of Section 1.4, there exists a unique qsd ν_K with support $\mathbb{Z}_+^d \setminus \{0\}$. Further, starting from the qsd, the extinction time is distributed according to an exponential law with parameter $\lambda_0(K)$ satisfying (Theorem 3.2 in [3])

$$e^{-d_1 K} \leq \lambda_0(K) \leq e^{-d_2 K} \quad (2.5)$$

where $d_1 > d_2 > 0$ are constants independent of K . Recall also that for all $t > 0$,

$$\mathbb{P}_{\nu_K}(\underline{N}^K(t) \in \cdot, T_0 > t) = e^{-\lambda_0(K)t} \nu_K(\cdot) \quad (2.6)$$

where

$$T_0 = \inf\{t > 0 : \underline{N}^K(t) = \underline{0}\}.$$

Finally, for all f in the domain of the generator

$$\mathcal{L}_K^\dagger \nu_K(f) = \nu_K(\mathcal{L}_K f) = -\lambda_0(K) \nu_K(f), \quad (2.7)$$

with the notation

$$\nu_K(f) = \int f(\underline{n}) d\nu_K(\underline{n}).$$

We use several notations from [3]. Let

$$\underline{n}^* = \lfloor K \underline{x}^* \rfloor.$$

For $\underline{x} \in \mathbb{R}_+^d$ and $r > 0$, $\mathcal{B}(\underline{x}, r)$ is the ball of center \underline{x} and radius r . We consider the sets

$$\Delta = \mathcal{B}(\underline{n}^*, \rho\sqrt{K}), \quad \mathcal{D} = \mathcal{B}\left(\underline{n}^*, \frac{\min_j n_j}{2}\right) \cap \mathbb{Z}_+^d \quad (2.8)$$

where $\rho > 0$ is a constant defined in [3, Corollary 4.2]. Note that since \underline{n}^* is of order K , we have $\Delta \subset \mathcal{D}$ for K large enough. The first entrance time in Δ (resp. \mathcal{D}) will be denoted by T_Δ (resp. $T_{\mathcal{D}}$).

We first prove that the support of the qsd is for large K essentially included in \mathcal{D} . That will be a key result to obtain moment properties.

Proposition 2.4. *There exists a constant $c_{(2.4)} > 0$ such that for all K large enough*

$$\nu_K(\mathcal{D}^c) \leq e^{-c_{(2.4)}K}.$$

Proof. We first recall two results from [3].

From Lemma 5.1 in [3], there exist $\gamma > 0$ and $\delta \in (0, 1)$ such that for all K large enough

$$\sup_{\underline{n} \in \Delta^c \setminus \underline{0}} \mathbb{P}_{\underline{n}}(T_{\Delta} > \gamma \log K, T_{\underline{0}} > T_{\Delta}) \leq \delta. \quad (2.9)$$

By Sublemma 5.8 in [3], there exist two constants $C > 0$ and $c > 0$ such that for all K large enough, and for all $t > 0$

$$\sup_{\underline{n} \in \Delta} \mathbb{P}_{\underline{n}}(T_{\mathcal{D}^c} < t) \leq C(1+t)e^{-cK}. \quad (2.10)$$

For $q \in \mathbb{N} \setminus \{0\}$ define

$$t_q = q\gamma \log K.$$

We will first estimate $\sup_{\underline{n}} \mathbb{P}_{\underline{n}}(\underline{N}^K(t_q) \in \mathcal{D}^c, T_{\underline{0}} > t_q)$.

Note that $\underline{N}^K(t_q) \in \mathcal{D}^c$ implies $T_{\mathcal{D}^c} \leq t_q$. We distinguish two cases.

1. Let $\underline{n} \in \Delta$. It follows from (2.10) that

$$\mathbb{P}_{\underline{n}}(\underline{N}^K(t_q) \in \mathcal{D}^c) \leq C(1+t_q)e^{-cK}.$$

2. Let $\underline{n} \in \Delta^c \setminus \{\underline{0}\}$. We have

$$\begin{aligned} \mathbb{P}_{\underline{n}}(\underline{N}^K(t_q) \in \mathcal{D}^c \setminus \{\underline{0}\}) &= \\ &\mathbb{P}_{\underline{n}}(\underline{N}^K(t_q) \in \mathcal{D}^c \setminus \{\underline{0}\}, T_{\Delta} \leq t_q) + \mathbb{P}_{\underline{n}}(\underline{N}^K(t_q) \in \mathcal{D}^c \setminus \{\underline{0}\}, T_{\Delta} > t_q). \end{aligned}$$

Using the strong Markov property at time T_{Δ} and (2.10) we obtain

$$\begin{aligned} &\mathbb{P}_{\underline{n}}(\underline{N}^K(t_q) \in \mathcal{D}^c \setminus \{\underline{0}\}, T_{\Delta} \leq t_q) \\ &= \mathbb{E}_{\underline{n}} \left[\mathbf{1}_{\{T_{\Delta} \leq t_q\}} \mathbb{P}_{\underline{N}^K(T_{\Delta})}(\underline{N}^K(t_q - T_{\Delta}) \in \mathcal{D}^c \setminus \{\underline{0}\}) \right] \\ &\leq C(1+t_q)e^{-cK}. \end{aligned}$$

We bound the second term recursively in q .

$$\begin{aligned} &\mathbb{P}_{\underline{n}}(T_{\Delta} > t_q, T_{\underline{0}} > T_{\Delta}) \\ &= \mathbb{E}_{\underline{n}} \left[\mathbf{1}_{\{T_{\Delta} > t_{q-1}\}} \mathbf{1}_{\{T_{\underline{0}} > T_{\Delta}\}} \mathbb{P}_{\underline{N}^K(t_{q-1})}(T_{\Delta} > t_1, T_{\underline{0}} > T_{\Delta}) \right] \\ &\leq \delta \sup_{\underline{n} \in \Delta^c \setminus \{\underline{0}\}} \mathbb{P}_{\underline{n}}(T_{\Delta} > t_{q-1}, T_{\underline{0}} > T_{\Delta}) \end{aligned}$$

where we used the strong Markov property at time t_{q-1} and (2.9). This implies

$$\begin{aligned} & \sup_{\underline{n} \in \Delta^c \setminus \{0\}} \mathbb{P}_{\underline{n}}(\underline{N}^K(t_q) \in \mathcal{D}^c \setminus \{0\}, T_{\Delta} > t_q) \\ & \leq \sup_{\underline{n} \in \Delta^c \setminus \{0\}} \mathbb{P}_{\underline{n}}(T_{\Delta} > t_q, T_{\underline{0}} > T_{\Delta}) \leq \delta^q. \end{aligned}$$

Therefore

$$\sup_{\underline{n} \neq \underline{0}} \mathbb{P}_{\underline{n}}(\underline{N}^K(t_q) \in \mathcal{D}^c \setminus \{0\}) \leq C(1 + t_q) e^{-cK} + \delta^q.$$

By taking $q = [K]$ we conclude that there exists a constant $c' > 0$ such that for K large enough

$$\sup_{\underline{n} \neq \underline{0}} \mathbb{P}_{\underline{n}}(\underline{N}^K(t_{[K]}) \in \mathcal{D}^c \setminus \{0\}) \leq e^{-c'K}.$$

This implies

$$\mathbb{P}_{\nu_K}(\underline{N}^K(t_{[K]}) \in \mathcal{D}^c, T_{\underline{0}} > t_{[K]}) \leq e^{-c'K}$$

but

$$\mathbb{P}_{\nu_K}(\underline{N}^K(t_{[K]}) \in \mathcal{D}^c, T_{\underline{0}} > t_{[K]}) = e^{-\lambda_0(K)t_{[K]}} \nu_K(\mathcal{D}^c)$$

by (2.6) and the result follows from (2.5). \square

Corollary 2.5. *For each $q \in \mathbb{N}$, there exists a constant C_q such that for all K large enough*

$$\int_{\mathcal{D}^c} \|\underline{n}\|_1^q d\nu_K(\underline{n}) \leq C_q K^q e^{-c(2.4)K} \quad \text{and} \quad \int \|\underline{n}\|_1^q d\nu_K(\underline{n}) \leq C_q K^q.$$

Proof. It follows at once from (2.6) (at time 1) and Theorem 2.1 that

$$\int e^{\|\underline{n}\|_1} d\nu_K(\underline{n}) \leq e^{\lambda_0(K)} e^{C(2.1)K} \leq 2 e^{C(2.1)K} \quad (2.11)$$

for K large enough. We have

$$\begin{aligned} \int_{\mathcal{D}^c} \|\underline{n}\|_1^q d\nu_K(\underline{n}) &= K^q \int_{\mathcal{D}^c} \left(\frac{\|\underline{n}\|_1}{K} \right)^q e^{-\frac{\|\underline{n}\|_1}{K}} e^{\frac{\|\underline{n}\|_1}{K}} d\nu_K(\underline{n}) \\ &\leq K^q q^q e^{-q} \int e^{\frac{\|\underline{n}\|_1}{K}} \mathbf{1}_{\mathcal{D}^c}(\underline{n}) d\nu_K(\underline{n}). \end{aligned}$$

We use Hölder inequality to get

$$\int_{\mathcal{D}^c} \|\underline{n}\|_1^q d\nu_K(\underline{n}) \leq K^q q^q e^{-q} \left(\int e^{\|\underline{n}\|_1} d\nu_K(\underline{n}) \right)^{\frac{1}{K}} \left(\int \mathbf{1}_{\mathcal{D}^c}(\underline{n}) d\nu_K(\underline{n}) \right)^{1 - \frac{1}{K}}.$$

The first result follows from (2.11) and Proposition 2.4. The second estimate follows from the first one and the bound $\sup_{\underline{n} \in \mathcal{D}} \|\underline{n}\|_1 \leq \mathcal{O}(1)K$. \square

We now estimate centered moments.

Theorem 2.6. *For each $q \in \mathbb{Z}_+$, there exists a constant $C_q > 0$ such that for all K large enough*

$$\int \|\underline{n} - K\underline{x}^*\|_2^{2q} d\nu_K(\underline{n}) \leq C_q K^q.$$

Proof. The proof is a recursion over q . The bound is trivial for $q = 0$. For $q \in \mathbb{N}$ define the function

$$f_q(\underline{n}) = \|\underline{n} - K\underline{x}^*\|_2^{2q} \mathbf{1}_{\mathcal{D}_1}(\underline{n})$$

where

$$\mathcal{D}_1 = \mathcal{B}\left(K\underline{x}^*, \frac{2K}{3} \min_j x_j^*\right) \cap \mathbb{Z}_+^d.$$

Recall that $\underline{e}^{(j)}$ is the vector with 1 at the j th coordinate and 0 elsewhere. From the trivial identity

$$\|\underline{n} - K\underline{x}^* \pm \underline{e}^{(j)}\|_2^2 = \|\underline{n} - K\underline{x}^*\|_2^2 \pm 2(n_j - Kx_j^*) + 1 \quad (2.12)$$

it follows that

$$\begin{aligned} & \left| \|\underline{n} - K\underline{x}^* \pm \underline{e}^{(j)}\|_2^{2q} - \|\underline{n} - K\underline{x}^*\|_2^{2q} \pm 2q(n_j - Kx_j^*)\|\underline{n} - K\underline{x}^*\|_2^{2q-2} \right| \\ & \leq 3^q 2^q (1 + \|\underline{n} - K\underline{x}^*\|_2^{2q-2}). \end{aligned}$$

Indeed, applying the trinomial expansion to (2.12), we obtain

$$\begin{aligned} & \left| \|\underline{n} - K\underline{x}^* \pm \underline{e}^{(j)}\|_2^{2q} - \|\underline{n} - K\underline{x}^*\|_2^{2q} \pm 2q(n_j - Kx_j^*)\|\underline{n} - K\underline{x}^*\|_2^{2q-2} \right| \\ & \leq q! \sum_{\substack{p_1 \leq q-2 \\ p_1+p_2+p_3=q}} \frac{\|\underline{n} - K\underline{x}^*\|_2^{2p_1} (2\|\underline{n} - K\underline{x}^*\|_2)^{p_2}}{p_1! p_2! p_3!} + q \|\underline{n} - K\underline{x}^*\|_2^{2q-2}. \end{aligned}$$

Observe that if $p_1 \leq q-2$, $p_1 + p_2 + p_3 = q$ and $p_1 \leq q-2$, then $2p_1 + p_2 = p_1 + q - p_3 \leq 2q - 2 - p_3 \leq 2q - 2$ since $p_3 \geq 0$. This implies that

$$\|\underline{n} - K\underline{x}^*\|_2^{2p_1} (2\|\underline{n} - K\underline{x}^*\|_2)^{p_2} \leq 2^q (1 + \|\underline{n} - K\underline{x}^*\|_2^{2q-2}).$$

It follows that

$$\mathcal{L}_K f_q(\underline{n}) = 2qK \sum_{j=1}^d X_j \left(\frac{\underline{n}}{K}\right) (n_j - Kx_j^*) \|\underline{n} - K\underline{x}^*\|_2^{2q-2} \mathbf{1}_{\mathcal{D}_1}(\underline{n}) + R_q(\underline{n}) \quad (2.13)$$

where

$$|R_q(\underline{n})| \leq \mathcal{O}(1) K 6^q (1 + \|\underline{n} - K\underline{x}^*\|_2^{2q-2}) \mathbf{1}_{\mathcal{D}_1}(\underline{n}) + \mathcal{O}(1) q K^{2q+1} \mathbf{1}_{\mathcal{D}^c}(\underline{n}) \quad (2.14)$$

where we used the fact that

$$\sup_{j=1,\dots,d} |\mathbf{1}_{\mathcal{D}_1}(\underline{n} \pm \underline{e}^{(j)}) - \mathbf{1}_{\mathcal{D}_1}(\underline{n})| \leq \mathbf{1}_{\mathcal{D}^c}(\underline{n}).$$

Using (1.18) we get

$$\begin{aligned} & K \sum_{j=1}^d X_j \left(\frac{\underline{n}}{K} \right) (n_j - Kx_j^*) \|\underline{n} - K\underline{x}^*\|_2^{2q-2} \mathbf{1}_{\mathcal{D}_1}(\underline{n}) \\ & \leq -\beta' \|\underline{n} - K\underline{x}^*\|_2^{2q} \mathbf{1}_{\mathcal{D}_1}(\underline{n}) = -\beta' f_q(\underline{n}) \end{aligned} \quad (2.15)$$

where

$$\beta' = \frac{\beta}{3} \min_j x_j^*.$$

Integrating the equation (2.13) with respect to ν_K and using (2.7), (2.14), (2.15) and Proposition 2.4 we obtain

$$\begin{aligned} & (2q\beta' - \lambda_0(K)) \nu_K(f_q) \\ & \leq \mathcal{O}(1) K 6^q (1 + \nu_K(f_{q-1})) + \mathcal{O}(1) 6^q K^{2q+1} e^{-c(2.4)K}. \end{aligned}$$

Observing that $\nu_K(f_0) \leq 1$, it follows by recursion over q that for each integer q there exists $C'_q > 0$ such that for all K large enough $\nu_K(f_q) \leq C'_q K^q$. Finally we have

$$\begin{aligned} \nu_K(\|\underline{n} - K\underline{x}^*\|_2^{2q}) &= \nu_K(f_q) + \nu_K(\|\underline{n} - K\underline{x}^*\|_2^{2q-2} \mathbf{1}_{\mathcal{D}_1^c}(\underline{n})) \\ &\leq \nu_K(f_q) + \nu_K(\|\underline{n} - K\underline{x}^*\|_2^{2q-2} \mathbf{1}_{\mathcal{D}^c}(\underline{n})). \end{aligned}$$

The result follows using the previous estimate and Corollary 2.5. \square

The next result gives a more precise estimate for the average of \underline{n} (instead of an error of order \sqrt{K}). (Recall that $\underline{\mu}^K$ is defined in (1.4).)

Proposition 2.7. *We have*

$$\underline{\mu}^K - K\underline{x}^* = \mathcal{O}(1)$$

where $\underline{\mu}^K$ is defined in (1.4). Moreover, since $\|\underline{n}^* - K\underline{x}^*\|_2 = \mathcal{O}(1)$, we have

$$\underline{\mu}^K - \underline{n}^* = \mathcal{O}(1). \quad (2.16)$$

Proof. We define the functions

$$g_j(\underline{n}) = \langle \underline{n} - K\underline{x}^*, \underline{e}^{(j)} \rangle, \quad 1 \leq j \leq d.$$

By Taylor expansion and the polynomial bounds on \underline{B} and \underline{D} we get

$$\begin{aligned} \mathcal{L}_K g_j(\underline{n}) &= K(B_j(\underline{n}/K) - D_j(\underline{n}/K)) \\ &= \sum_{m=1}^d (\partial_m B_j(\underline{x}^*) - \partial_m D_j(\underline{x}^*)) g_m(\underline{n}) \mathbf{1}_{\mathcal{D}}(\underline{n}) + \mathcal{O}(1) \frac{\|\underline{n} - K\underline{x}^*\|_2^2}{K} \mathbf{1}_{\mathcal{D}}(\underline{n}) \\ &\quad + \mathcal{O}(1) (K^p + \|\underline{n}\|_2^p) \mathbf{1}_{\mathcal{D}^c}(\underline{n}) \end{aligned}$$

for some positive integer p independent of K . Using Cauchy-Schwarz inequality, identity (2.7), Corollary 2.5 and Proposition 2.4 we get

$$\int (1 + \|\underline{n}\|_2^p) \mathbb{1}_{\mathcal{D}^c}(\underline{n}) \, d\nu_K(\underline{n}) = o(1).$$

From Proposition 2.4, Theorem 2.6 and (2.5) we get

$$\sum_{m=1}^d (\partial_m B_j(\underline{x}^*) - \partial_m D_j(\underline{x}^*)) \nu_K(g_m) = \mathcal{O}(1).$$

The result follows from the invertibility of the $d \times d$ matrix $(\partial_m B_j(\underline{x}^*) - \partial_m D_j(\underline{x}^*))$ which follows from **H.3**. The other inequalities follow immediately. \square

Corollary 2.8. *We have*

$$\|\Sigma^K\| \leq \int \|\underline{n} - \underline{\mu}^K\|_2^2 \, d\nu_K(\underline{n}) = \int \|\underline{n} - K\underline{x}^*\|_2^2 \, d\nu_K(\underline{n}) + \mathcal{O}(1) \leq \mathcal{O}(1)K.$$

Proof. Combine Proposition 2.7 and Theorem 2.6. \square

We now show that Σ^K is indeed of order K .

Proposition 2.9. *For K large enough, the matrix Σ^K satisfies*

$$\Sigma^K \geq \mathcal{O}(1) K \text{Id}$$

for the order among positive definite matrices, Id being the identity matrix. In particular,

$$\int \|\underline{n} - \underline{\mu}^K\|_2^2 \, d\nu_K(\underline{n}) \geq \mathcal{O}(1)K.$$

Proof. We denote by $\tilde{\Sigma}^K$ the positive definite matrix

$$\tilde{\Sigma}_{p,q}^K = \int (n_p - n_p^*)(n_q - n_q^*) \, d\nu_K(\underline{n}).$$

By (2.16) we have

$$\|\tilde{\Sigma}^K - \Sigma^K\| = \mathcal{O}(1). \quad (2.17)$$

Let \underline{v} be a unit vector in \mathbb{R}^d . We have

$$\langle \underline{v}, \tilde{\Sigma}^K \underline{v} \rangle = \int \langle \underline{v}, (\underline{n} - \underline{n}^*) \rangle^2 \, d\nu_K(\underline{n}) \geq \int_{\Delta} \langle \underline{v}, (\underline{n} - \underline{n}^*) \rangle^2 \, d\nu_K(\underline{n}).$$

From Lemma 5.3 in [3] there exists a constant $c > 0$ such that for all K large enough and all $\underline{n} \in \Delta$,

$$\nu_K(\{\underline{n}\}) \geq c U_{\Delta}(\{\underline{n}\})$$

where U_Δ is the uniform distribution on Δ . Therefore

$$\langle \underline{v}, \tilde{\Sigma}^K \underline{v} \rangle \geq c \int_{\Delta} \langle \underline{v}, (\underline{n} - \underline{n}^*) \rangle^2 dU_\Delta(\underline{n})$$

and we get

$$\langle \underline{v}, \tilde{\Sigma}^K \underline{v} \rangle \geq \mathcal{O}(1)K \|\underline{v}\|_2^2.$$

The result follows. \square

3 Controlling time averages of the estimators

For $T > 0$, we define the time average of a function $f : \mathbb{Z}_+^d \rightarrow \mathbb{R}$ by

$$S_f(T, K) = \frac{1}{T} \int_0^T f(\underline{N}^K(s)) ds. \quad (3.1)$$

The goal of this section is to obtain a control of $|S_f(T, K) - \nu_K(f)|$ for a suitable class of functions.

We recall the following result from [3, Theorem 3.1].

Theorem 3.1 ([3]). *There exist $a > 0$, $K_0 > 1$ such that, for all $t \geq 0$ and for all $K \geq K_0$, we have*

$$\sup_{\underline{n} \in \mathbb{Z}_+^d \setminus \{0\}} \left\| \mathbb{P}_{\underline{n}}(\underline{N}^K(t) \in \cdot, t < T_0) - \mathbb{P}_{\underline{n}}(t < T_0) \nu_K(\cdot) \right\|_{\text{TV}} \leq 2e^{-\frac{at}{\log K}}. \quad (3.2)$$

It is also proved in [3] that, for a time much larger than $\log K$ and much smaller than the extinction time (which is of order $\exp(\mathcal{O}(1)K)$), the law of the process is close to the qsd with a positive probability. The accuracy of the approximation depends on the initial condition. This suggests to study the distance between the law of the process at time s and the qsd as a function of the initial condition, K and s . This will result from (3.2) if we can estimate $\mathbb{P}_{\underline{n}}(T_0 \leq t)$. In fact we prove a more general result.

Lemma 3.2. *For $\gamma \geq 0$, define $\tau_\gamma = \inf \{t \geq 0 : \|\underline{N}^K(t)\|_1 \leq \gamma K\}$. There exist $\delta > 0$, $\alpha > 0$ and $C > 0$ such that for all $\underline{n} \in \mathbb{Z}_+^d$, $K \geq 1$, $0 \leq \gamma \leq 1 \wedge \frac{\alpha}{\|\underline{x}^*\|_1}$ and $t \geq 0$, we have*

$$\mathbb{P}_{\underline{n}}(\tau_\gamma \leq t) \leq C \left(e^{-\delta \left(\zeta \left(\frac{\|\underline{n}\|_1}{K} \wedge \alpha \right) - \gamma \|\underline{x}^*\|_1 \right) K} + t e^{-\delta (\alpha - \gamma \|\underline{x}^*\|_1) K} \right) \quad (3.3)$$

where

$$\zeta = \min_{1 \leq j \leq d} x_j^* > 0. \quad (3.4)$$

Taking $\gamma = 0$ in (3.3), we get

$$\mathbb{P}_{\underline{n}}(T_0 \leq t) \leq C \left(e^{-\delta \left(\zeta \frac{\|\underline{n}\|_1}{K} \wedge \alpha \right) K} + t e^{-\alpha \delta K} \right). \quad (3.5)$$

Proof. It follows from **H.1** and **H.3** (using Taylor's expansion of $\underline{X}(\underline{x})$ near $\underline{0}$) that there exists $\alpha_0 \in (0, R)$ (where R was introduced in Assumption **H.3**) such that for all $\underline{x} \in \mathbb{R}_+^d$ satisfying $\|\underline{x}\|_2 \leq \alpha_0$ we have

$$\begin{aligned} \langle \underline{X}(\underline{x}), \underline{x}^* \rangle &\geq \beta \|\underline{x}^*\|_2^2 \|\underline{x}\|_2 - 2\beta \|\underline{x}\|_2 \langle \underline{x}, \underline{x}^* \rangle + \beta \|\underline{x}\|_2^3 + \langle \underline{X}(\underline{x}), \underline{x} \rangle \\ &\geq \beta \|\underline{x}^*\|_2^2 \|\underline{x}\|_2 + \mathcal{O}(1) \|\underline{x}\|_2^2 \geq \frac{\beta \|\underline{x}^*\|_2^2}{2} \|\underline{x}\|_2. \end{aligned} \quad (3.6)$$

For $\alpha \in (0, \alpha_0]$ and $\delta > 0$ to be chosen later on, we define

$$\psi(\underline{n}) = e^{-\delta \langle \underline{n}, \underline{x}^* \rangle \wedge \alpha K}.$$

It is easy to verify that if $\langle \underline{n}, \underline{x}^* \rangle > \alpha K + \|\underline{x}^*\|_2$ we have

$$\mathcal{L}_K \psi = 0.$$

If $\alpha K - \|\underline{x}^*\|_2 \leq \langle \underline{n}, \underline{x}^* \rangle \leq \alpha K + \|\underline{x}^*\|_2$ we have

$$|\mathcal{L}_K \psi| \leq \mathcal{O}(1) K e^{-\alpha \delta K}.$$

For $\langle \underline{n}, \underline{x}^* \rangle \leq \alpha K - \|\underline{x}^*\|_2$, we have $\|\underline{n}\|_1 \leq \langle \underline{n}, \underline{x}^* \rangle / \zeta \leq \alpha K / \zeta$, where ζ is defined in (3.4), and

$$\mathcal{L}_K \psi(\underline{n}) = K \mathfrak{g}\left(\delta, \frac{\underline{n}}{K}\right) e^{-\delta \langle \underline{n}, \underline{x}^* \rangle}$$

where the function \mathfrak{g} is defined by

$$\mathfrak{g}(s, \underline{x}) = \sum_{j=1}^d B_j(\underline{x}) (e^{-sx_j^*} - 1) + \sum_{j=1}^d D_j(\underline{x}) (e^{sx_j^*} - 1).$$

We have

$$\begin{aligned} \mathfrak{g}(s, \underline{x}) &= -s \sum_{j=1}^d (B_j(\underline{x}) - D_j(\underline{x})) x_j^* \\ &\quad + \sum_{j=1}^d B_j(\underline{x}) (e^{-sx_j^*} - 1 + sx_j^*) + \sum_{j=1}^d D_j(\underline{x}) (e^{sx_j^*} - 1 - sx_j^*). \end{aligned}$$

From the differentiability of the vector fields \underline{B} and \underline{D} and using (3.6), it follows that there exists a constant $\Gamma > 0$ such that, for all $0 \leq s \leq 1$ and $\|\underline{x}\|_2 < \alpha_0$ we have

$$\begin{aligned} \mathfrak{g}(s, \underline{x}) &= -s \langle \underline{X}(\underline{x}), \underline{x}^* \rangle + \mathcal{O}(1) s^2 \|\underline{x}\|_2 \\ &\leq -s \frac{\beta \|\underline{x}^*\|_2^2}{2} \|\underline{x}\|_2 + \Gamma s^2 \|\underline{x}\|_2. \end{aligned}$$

Therefore we can choose $\delta > 0$ and $0 < \alpha < \alpha_0$ such that

$$\sup_{\|\underline{x}\|_2 \leq \alpha} \mathbf{g}(\delta, \underline{x}) < 0.$$

Therefore, for all \underline{n}

$$\mathcal{L}_K \psi(\underline{n}) \leq \mathcal{O}(1)K e^{-\alpha\delta K}.$$

For $\tilde{\gamma} > 0$ (independent of K), we define

$$\tilde{\tau}_{\tilde{\gamma}} = \inf \{t \geq 0 : \langle \underline{N}^K(t), \underline{x}^* \rangle \leq \tilde{\gamma}K\}.$$

We apply Ito's formula to ψ to get

$$\mathbb{E}_{\underline{n}}[\psi(\underline{N}^K(t \wedge \tilde{\tau}_{\tilde{\gamma}}))] = \psi(\underline{n}) + \mathbb{E}_{\underline{n}} \left[\int_0^{t \wedge \tilde{\tau}_{\tilde{\gamma}}} \mathcal{L}_K \psi(\underline{N}^K(s)) ds \right].$$

We have

$$\tilde{\gamma}K - \zeta \leq \langle \underline{N}^K(\tilde{\tau}_{\tilde{\gamma}}), \underline{x}^* \rangle \leq \tilde{\gamma}K$$

hence

$$\psi(\underline{N}^K(\tilde{\tau}_{\tilde{\gamma}})) \geq e^{-\delta(\tilde{\gamma} \wedge \alpha)K} e^{-\delta\zeta}.$$

We have

$$\mathbb{E}_{\underline{n}}[\psi(\underline{N}^K(t \wedge \tilde{\tau}_{\tilde{\gamma}}))] \geq \mathbb{P}_{\underline{n}}(\tilde{\tau}_{\tilde{\gamma}} \leq t) e^{-\delta(\tilde{\gamma} \wedge \alpha)K} e^{-\delta\zeta}.$$

Therefore

$$\mathbb{P}_{\underline{n}}(\tilde{\tau}_{\tilde{\gamma}} \leq t) e^{-\delta(\tilde{\gamma} \wedge \alpha)K} e^{-\delta\zeta} \leq e^{-\delta(\langle \underline{n}, \underline{x}^* \rangle \wedge \alpha K)} + t \mathcal{O}(1)K e^{-\alpha\delta K}.$$

To conclude, observe that

$$\mathbb{P}_{\underline{n}}(\tau_{\gamma} \leq t) \leq \mathbb{P}_{\underline{n}}(\tilde{\tau}_{\tilde{\gamma}} \leq t)$$

for $\tilde{\gamma} = \gamma \|\underline{x}^*\|_1$ because for all $\underline{n} \in \mathbb{Z}_+^d$,

$$0 < \zeta \|\underline{n}\|_1 \leq \langle \underline{n}, \underline{x}^* \rangle \leq \|\underline{n}\|_1 \sup_{j=1, \dots, d} x_j^* \leq \|\underline{n}\|_1 \|\underline{x}^*\|_1$$

and $\|\underline{N}^K(\tau_{\gamma})\|_1 \leq \gamma K$. □

We have the following result.

Proposition 3.3. *For all bounded functions $h : \mathbb{Z}_+^d \rightarrow \mathbb{R}$, $t \geq 0$, $\underline{n} \in \mathbb{Z}_+^d$, and $K > K_0$, we have*

$$|\mathbb{E}_{\underline{n}}[h(\underline{N}^K(t))] - \nu_K(h)| \leq \mathcal{O}(1)\|h\|_{\infty} \left(e^{-\delta(\zeta \frac{\|\underline{n}\|_1}{K} \wedge \alpha)K} + t e^{-\alpha\delta K} + e^{-\frac{at}{\log K}} \right)$$

where α, δ and ζ are defined in Lemma 3.2, and a and K_0 are defined in Theorem 3.1.

Proof. From the bound (3.2) we get

$$\left| \mathbb{E}_{\underline{n}} \left[h(\underline{N}^K(t)) \mathbf{1}_{\{T_0 > t\}} \right] - \mathbb{P}_{\underline{n}}(t < T_0) \nu_K(h) \right| \leq \mathcal{O}(1) \|h\|_{\infty} e^{-\frac{at}{\log K}}.$$

This implies

$$\begin{aligned} & \left| \mathbb{E}_{\underline{n}} \left[h(\underline{N}^K(t)) \right] - \nu_K(h) \right| \\ & \leq \left| \mathbb{E}_{\underline{n}} \left[h(\underline{N}^K(t)) \mathbf{1}_{\{T_0 \leq t\}} \right] \right| + \mathbb{P}_{\underline{n}}(t \geq T_0) \nu_K(h) + \mathcal{O}(1) \|h\|_{\infty} e^{-\frac{at}{\log K}} \\ & \leq \mathcal{O}(1) \|h\|_{\infty} \left(\mathbb{P}_{\underline{n}}(t \geq T_0) + e^{-\frac{at}{\log K}} \right) \\ & \leq \mathcal{O}(1) \|h\|_{\infty} \left(e^{-\delta \left(\zeta \frac{\|\underline{n}\|_1}{K} \wedge \alpha \right) K} + e^{-\alpha \delta K} + e^{-\frac{at}{\log K}} \right) \end{aligned}$$

using (3.5). \square

We now extend Proposition 3.3 to more general functions. For $q \in \mathbb{Z}_+$, we define the Banach space $\mathcal{F}_{K,q}$ by

$$\mathcal{F}_{K,q} = \left\{ f : \mathbb{Z}_+^d \rightarrow \mathbb{R} : \|f\|_{\kappa,q} := \sup_{\underline{n} \neq \underline{0}} \frac{|f(\underline{n})|}{K^q + \|\underline{n}\|_2^q} < +\infty \right\}. \quad (3.7)$$

We have the following result for time-averages of function in \mathcal{F}_K .

Theorem 3.4. *For all $K > K_0$, $f \in \mathcal{F}_{K,q}$, $T > 0$, and $\underline{n} \in \mathbb{Z}_+^d$, we have*

$$\begin{aligned} & \left| \mathbb{E}_{\underline{n}} [S_f(T, K)] - \nu_K(f) \right| \leq \mathcal{O}(1) \|f\|_{\kappa,q} \left(K^q + \|\underline{n}\|_2^q \right) \\ & \quad \times \left(\frac{1}{T} + e^{-\delta \left(\zeta \frac{\|\underline{n}\|_1}{K} \wedge \alpha \right) K} + T e^{-\alpha \delta K} + \frac{\log K}{aT} + (1 - e^{-\lambda_0(K)})^{\frac{1}{2}} \right) \end{aligned}$$

where α, δ and ζ are defined in Lemma 3.2, and $\lambda_0(K)$ is defined in (2.5).

Remark 3.1. *One can check that if one modifies slightly the definition of the time average (3.1) by integrating from 1 to $T+1$, then one can remove the term $\|\underline{n}\|_2^q$ from the previous estimate.*

Proof. For $f \in \mathcal{F}_{K,q}$, Corollary 2.5 gives

$$|\nu_K(f)| \leq \mathcal{O}(1) K^q \|f\|_{\kappa,q}.$$

By Proposition 2.3 we have

$$\left| \frac{1}{T} \mathbb{E}_{\underline{n}} \left[\int_0^{1 \wedge T} f(\underline{N}^K(s)) \, ds \right] \right| \leq \mathcal{O}(1) \|f\|_{\kappa,q} \frac{K^q + \|\underline{n}\|_2^q}{T}.$$

Hence for $T \leq 1$ we get

$$\left| \mathbb{E}_{\underline{n}} [S_f(T, K)] - \nu_K(f) \right| \leq \mathcal{O}(1) \|f\|_{\kappa,q} \left(K^q + \|\underline{n}\|_2^q \right) \left(\frac{1}{T} + 1 \right).$$

For $T > 1$, we have by the Markov property that

$$\begin{aligned} \frac{1}{T} \mathbb{E}_{\underline{n}} \left[\int_1^T f(\underline{N}^K(s)) \, ds \right] &= \frac{1}{T} \int_1^T \mathbb{E}_{\underline{n}} \left[\mathbb{E}_{\underline{N}^K(s-1)} [f(\underline{N}^K(1))] \right] \, ds \\ &= \frac{1}{T} \int_0^{T-1} \mathbb{E}_{\underline{n}} [g(\underline{N}^K(s))] \, ds \end{aligned}$$

where

$$g(\underline{m}) := \mathbb{E}_{\underline{m}} [f(\underline{N}^K(1))]. \quad (3.8)$$

By Corollary 2.2, the function g is bounded and

$$\|g\|_{\infty} \leq \mathcal{O}(1) \|f\|_{\kappa,q} K^q. \quad (3.9)$$

Applying Proposition 3.3 to g thus gives

$$\begin{aligned} &|\mathbb{E}_{\underline{n}} [g(\underline{N}^K(s))] - \nu_K(g)| \\ &\leq \mathcal{O}(1) \|f\|_{\kappa,q} K^q \left(e^{-\delta(\zeta \frac{\|\underline{n}\|_1}{K} \wedge \alpha)K} + s e^{-\alpha\delta K} + e^{-\frac{\alpha s}{\log K}} \right). \end{aligned}$$

Integrating over $s \in [0, T-1]$ yields

$$\begin{aligned} &\left| \frac{1}{T} \int_0^{T-1} \mathbb{E}_{\underline{n}} [g(\underline{N}^K(s))] \, ds - \frac{T-1}{T} \nu_K(g) \right| \\ &\leq \mathcal{O}(1) \|f\|_{\kappa,q} \frac{K^q}{T} \left((T-1) e^{-\delta(\zeta \frac{\|\underline{n}\|_1}{K} \wedge \alpha)K} + (T-1)^2 e^{-\alpha\delta K} + \frac{\log K}{a} \right). \end{aligned}$$

Using Lemma 3.5 (stated and proved right after this proof), we finally obtain

$$\begin{aligned} &|\mathbb{E}_{\underline{n}} [S_f(T, K)] - \nu_K(f)| \\ &\leq \mathcal{O}(1) \|f\|_{\kappa,q} \frac{K^q + \|\underline{n}\|_2^q}{T} \\ &\quad + \mathcal{O}(1) \|f\|_{\kappa,q} \frac{K^q}{T} \left((T-1) e^{-\delta(\zeta \frac{\|\underline{n}\|_1}{K} \wedge \alpha)K} + (T-1)^2 e^{-\alpha\delta K} + \frac{\log K}{a} \right) \\ &\quad + \mathcal{O}(1) \|f\|_{\kappa,q} K^q (1 - e^{-\lambda_0(K)})^{\frac{1}{2}} + \frac{1}{T} \nu_K(g) + \nu_K(f) \mathbf{1}_{\{T \leq 1\}} \\ &\leq \mathcal{O}(1) \|f\|_{\kappa,q} (K^q + \|\underline{n}\|_2^q) \left(\frac{1}{T} \left(2 + \frac{\log K}{a} \right) + e^{-\delta(\zeta \frac{\|\underline{n}\|_1}{K} \wedge \alpha)K} + T e^{-\delta \alpha K} \right. \\ &\quad \left. + (1 - e^{-\lambda_0(K)})^{\frac{1}{2}} + \mathbf{1}_{\{T \leq 1\}} \right). \end{aligned}$$

This finishes the proof of the theorem. \square

We use the following lemma in the previous proof.

Lemma 3.5. *For $f \in \mathcal{F}_{\kappa,q}$ and g defined in (3.8) we have*

$$|\nu_K(g) - \nu_K(f)| \leq \mathcal{O}(1) K^q \|f\|_{\kappa,q} (1 - e^{-\lambda_0(K)})^{\frac{1}{2}}.$$

Proof. We write

$$\nu_K(g) = \mathbb{E}_{\nu_K} [f(\underline{N}^K(1))\mathbb{1}_{\{T_0 > 1\}}] + \mathbb{E}_{\nu_K} [f(\underline{N}^K(1))\mathbb{1}_{\{T_0 \leq 1\}}].$$

Since ν_K is a qsd, using Cauchy-Schwarz inequality, it follows that

$$\begin{aligned} & |\nu_K(g) - \nu_K(f)| \\ & \leq (1 - e^{-\lambda_0(K)})|\nu_K(f)| + (\mathbb{E}_{\nu_K} [f^2(\underline{N}^K(1))])^{\frac{1}{2}} \left(\mathbb{E}_{\nu_K} [\mathbb{1}_{\{T_0 \leq 1\}}] \right)^{\frac{1}{2}} \\ & \leq \mathcal{O}(1)K^q \|f\|_{K,q} (1 - e^{-\lambda_0(K)})^{\frac{1}{2}} \end{aligned}$$

where we used Corollaries 2.2 and 2.5 and the fact that under ν_K the law of T_0 is exponential with parameter $\lambda_0(K)$. The lemma is proved. \square

4 Fluctuation and correlation relations

4.1 Proof of Theorem 1.1

Let

$$\tilde{\Sigma}_{i,j}^K(t) = \mathbb{E}_{\nu_K} [(N_i^K(t) - n_i^*)(N_j^K(0) - n_j^*)], \quad i, j = 1, \dots, d.$$

For $1 \leq i \leq d$, let $f_i(\underline{n}) = \langle \underline{n} - \underline{n}^*, \underline{e}^{(i)} \rangle$. We have, since $B_i(\underline{x}^*) = D_i(\underline{x}^*)$, $1 \leq i \leq d$, and $\underline{n}^*/K - \underline{x}^* = \mathcal{O}(1)/K$,

$$\begin{aligned} & \frac{d}{dt} \tilde{\Sigma}_{i,j}^K(t) \\ & = \mathbb{E}_{\nu_K} [\mathcal{L}_K f_i(\underline{N}^K(t))(N_j^K(0) - n_j^*)] \\ & = K \mathbb{E}_{\nu_K} \left[B_i \left(\frac{N_j^K(t)}{K} \right) (N_j^K(0) - n_j^*) \right] - K \mathbb{E}_{\nu_K} \left[D_i \left(\frac{N_j^K(t)}{K} \right) (N_j^K(0) - n_j^*) \right] \\ & = K \mathbb{E}_{\nu_K} \left[\left(B_i \left(\frac{N_j^K(t)}{K} \right) - B_i \left(\frac{n_j^*}{K} \right) \right) (N_j^K(0) - n_j^*) \right] \\ & \quad - K \mathbb{E}_{\nu_K} \left[\left(D_i \left(\frac{N_j^K(t)}{K} \right) - D_i \left(\frac{n_j^*}{K} \right) \right) (N_j^K(0) - n_j^*) \right] + \mathcal{O}(1). \end{aligned}$$

As in the previous proof, we split the integrals according to whether $\underline{N}^K(t) \in \mathcal{D}$ or $\underline{N}^K(t) \in \mathcal{D}^c$. Using Cauchy-Schwarz inequality, Corollary 2.5, and the fact that ν_K is a qsd, the second contribution is exponentially small in K . In the first contribution, we use Taylor expansion around \underline{x}^* . The error terms are bounded by

$$\frac{\mathcal{O}(1)}{K} \mathbb{E}_{\nu_K} [\|\underline{N}^K(t) - K\underline{x}^*\|_2^2 \|\underline{N}^K(0) - K\underline{x}^*\|_2] + \mathcal{O}(1).$$

Now we use Cauchy-Schwarz inequality, Theorem 2.6 and that ν_K is a qsd to obtain

$$\begin{aligned} & \frac{d}{dt} \tilde{\Sigma}_{i,j}^K(t) \\ &= \sum_{\ell=1}^d (\partial_\ell B_i(\underline{x}^*) - \partial_\ell D_i(\underline{x}^*)) \mathbb{E}_{\nu_K}[(N_\ell^K(t) - n_\ell^*)(N_j^K(0) - n_j^*)] + \mathcal{O}(\sqrt{K}) \\ &= \sum_{\ell=1}^d M_{i,\ell}^* \tilde{\Sigma}_{\ell,j}^K(t) + \mathcal{O}(\sqrt{K}) \end{aligned}$$

Since M^* has a spectrum contained in the open left half-plane by **H.3**, we integrate the equation

$$\frac{d}{dt} \tilde{\Sigma}^K(t) = M^* \tilde{\Sigma}^K(t) + \mathcal{O}(\sqrt{K})$$

from 0 to τ to get

$$\tilde{\Sigma}^K(\tau) = e^{\tau M^*} \tilde{\Sigma}^K(0) + \mathcal{O}(\sqrt{K}).$$

We arrive at the desired relation by using (2.16).

4.2 Proof of Theorem 1.2

Recall that

$$\Sigma_{p,q}^K = \Sigma_{p,q}^K(0) = \int (n_p - \mu_p^K)(n_q - \mu_q^K) d\nu_K(\underline{n}).$$

We will first write the proof with the following matrix instead of Σ^K :

$$\tilde{\Sigma}_{i,j}^K = \int (n_i - n_i^*)(n_j - n_j^*) d\nu_K(\underline{n}).$$

On the one hand we have by (2.7)

$$\langle \mathcal{L}_K^\dagger \nu_K, (n_i - n_i^*)(n_j - n_j^*) \rangle = -\lambda_0(K) \langle \nu_K, (n_i - n_i^*)(n_j - n_j^*) \rangle. \quad (4.1)$$

By Theorem 2.6 and (2.5) the right-hand side of this equation is exponentially small in K . On the other hand we have using formula (2.1)

$$\begin{aligned}
\langle \mathcal{L}_K^\dagger \nu_K, (n_i - n_i^*)(n_j - n_j^*) \rangle &= \langle \nu_K, \mathcal{L}_K((n_i - n_i^*)(n_j - n_j^*)) \rangle \\
&= K \sum_{\ell=1}^d \left\langle \nu_K, B_\ell \left(\frac{\underline{n}}{K} \right) \left((n_j - n_j^*) \delta_{i,\ell} + (n_i - n_i^*) \delta_{j,\ell} + \delta_{i,\ell} \delta_{j,\ell} \right) \right\rangle \\
&\quad + K \sum_{\ell=1}^d \left\langle \nu_K, D_\ell \left(\frac{\underline{n}}{K} \right) \left(-(n_j - n_j^*) \delta_{i,\ell} - (n_i - n_i^*) \delta_{j,\ell} + \delta_{i,\ell} \delta_{j,\ell} \right) \right\rangle \\
&= K \left\langle \nu_K, \left(B_i \left(\frac{\underline{n}}{K} \right) - D_i \left(\frac{\underline{n}}{K} \right) \right) (n_j - n_j^*) \right\rangle \\
&\quad + K \left\langle \nu_K, \left(B_j \left(\frac{\underline{n}}{K} \right) - D_j \left(\frac{\underline{n}}{K} \right) \right) (n_i - n_i^*) \right\rangle \\
&\quad + K \left\langle \nu_K, B_i \left(\frac{\underline{n}}{K} \right) + D_i \left(\frac{\underline{n}}{K} \right) \right\rangle \delta_{i,j}.
\end{aligned}$$

We split each integral by separating integration over \mathcal{D} (defined in (2.8)) and integration over \mathcal{D}^c . Inside \mathcal{D}^c , we apply Corollary 2.5 and use the assumption that B and D are polynomially bounded. Inside \mathcal{D} , we use Taylor's formula around \underline{x}^* for the functions $B_i(\underline{n}/K) - D_i(\underline{n}/K)$, and $B_i(\underline{n}/K) + D_i(\underline{n}/K)$. We also use that $B_i(\underline{x}^*) = D_i(\underline{x}^*)$, $1 \leq i \leq d$, and $\underline{n}^*/K - \underline{x}^* = \mathcal{O}(1)/K$. The error terms are bounded by

$$\frac{\mathcal{O}(1)}{K} \int \|\underline{n} - K\underline{x}^*\|_2^3 d\nu_K(\underline{n}) \quad \text{and} \quad \mathcal{O}(1) \int \|\underline{n} - K\underline{x}^*\|_2 d\nu_K(\underline{n})$$

respectively. Using Theorem 2.6, both bounds are of order \sqrt{K} . We obtain

$$\sum_{\ell=1}^d M_{i,\ell}^* \tilde{\Sigma}_{\ell,j}^K + \sum_{\ell=1}^d M_{j,\ell}^* \tilde{\Sigma}_{\ell,i}^K + 2K B_i(\underline{x}^*) \delta_{i,j} = \mathcal{O}(\sqrt{K})$$

which can be written in the more compact form

$$M^* \tilde{\Sigma}^K + \tilde{\Sigma}^K M^{*\top} + 2\mathcal{D}^K = \mathcal{O}(\sqrt{K}) \tag{4.2}$$

where \mathcal{D}^K is the diagonal matrix of averages birth (or death) rates. To finish the proof, it remains to replace $\tilde{\Sigma}^K$ by Σ^K . This is done by using (2.17).

Remark 4.1. *Note that each term on the left hand side is of order K , see Corollary 2.8.*

Remark 4.2. *Dividing out (4.2) by $2K$ and taking the limit $K \rightarrow \infty$, we recover (C.1), as expected.*

5 Variance estimates for the estimators

It is straightforward to apply Theorem 3.4 to $S^\mu(T, K)$, $S^C(T, \tau, K)$, $S^\mathfrak{D}(T, K)$, and $S^\Sigma(T, K)$, which are defined respectively in (1.9), (1.10), (1.16), and (1.17). This gives the bound (1.11) on $S^\mu(T, K)$ announced in the Section 1. The bounds for the other estimators all have the same structure. We will not state them.

In this section we prove two variance estimates for any time average $S_f(T, K)$ with $f \in \mathcal{F}_{K,q}$. In the first one, one starts from anywhere in \mathbb{Z}_+^d , while in the second one the starting distribution is the qsd. Recall that $S^\Sigma(T, K) = S^C(T, 0, K)$. We will only give the proofs of these estimates for $S^\Sigma(T, K)$ since manipulating $S^C(T, \tau, K)$ is cumbersome but otherwise the proofs are the same.

Proposition 5.1. *There exist positive constants $\delta', \zeta', \alpha', \theta', C'$ and $K_0 \geq 2$ such that, for all $K \geq K_0$, $f \in \mathcal{F}_{K,q}$ (see Definition 3.7), $T \geq 0$, and $\underline{n} \neq \underline{0}$, we have*

$$\begin{aligned} \mathbb{E}_{\underline{n}} \left[(S_f(T, K) - \nu_K(f))^2 \right] &\leq C' \|f\|_{K,q}^2 (c_q \|\underline{n}\|_1^q + K^q) \\ &\times \left(\frac{\|\underline{n}\|_1^q + K^q \log K}{T \vee 1} + K^q e^{-\delta' (\zeta' \frac{\|\underline{n}\|_1}{K} \wedge \alpha') K} + T K^q e^{-\theta' K} \right) \end{aligned}$$

where c_q is defined in Proposition 2.3.

One can use Chebyshev inequality to bound $\mathbb{P}_{\underline{n}}(|S_f(T, K) - \nu_K(f)| > \delta)$, for any $\delta > 0$.

The proof of Proposition 5.1 is postponed to Appendix A. The previous estimate and all the estimates we will give below have the same behaviour in their dependence in K , \underline{n} and T . They display the qualitative behaviour that we met several times:

1. The bounds are not useful for K too small.
2. If K is large, the bounds are not useful if \underline{n} is small (order one) because the process can be absorbed at $\underline{0}$ in a time of order one with a sizeable probability.
3. Finally, for K large and \underline{n} of order K , the time T must be large enough (polynomial in K in our bounds) but not too large (less than an exponential in K because the process can reach the origin with high probability in such large times).

Integrating the previous estimate with respect to the qsd, we get the following control.

Corollary 5.2. *There exist two positive constants $C'' > 0$ and θ'' such that for all $K \geq K_0$, for all $f \in \mathcal{F}_{K,q}$ and for all $T \geq 0$, we have*

$$\begin{aligned} \mathbb{E}_{\nu_K} \left[(S_f(T, K) - \mathbb{E}_{\nu_K}(f))^2 \right] &\leq C'' \|f\|_{K,q}^2 K^{2q} \\ &\times \left((1 + C_{2q})(1 + c_q) \frac{\log K}{T \vee 1} + (1 + C_q)(1 + T) e^{-\theta'' K} \right) \end{aligned}$$

where K_0 is as in the previous proposition, c_q is defined in Proposition 2.3, and C_q is defined in Corollary 2.5.

Observe that the previous inequality is only useful in the range $0 \leq T \leq e^{\theta'' K}$. The proofs of the two previous estimates are postponed to the Appendix A.

We now apply the previous results to our estimators.

Proposition 5.3. *We have for all $1 \leq p \leq d$*

$$\begin{aligned} \mathbb{E}_{\underline{n}} \left[|S_p^\mu(T, K) - \mu_p^K|^2 \right] &\leq \mathcal{O}(1)(c_1 \|\underline{n}\|_1 + K) \\ &\times \left(\frac{\|\underline{n}\|_1 + K \log K}{T \vee 1} + K e^{-\delta' (\zeta' \frac{\|\underline{n}\|_1}{K} \wedge \beta')} + T K e^{-\theta K} \right) \end{aligned}$$

and

$$\mathbb{E}_{\nu_K} \left[|S_p^\mu(T, K) - \mu_p^K|^2 \right] \leq \mathcal{O}(1) K^2 \left(\frac{1 + \log K}{T \vee 1} + (1 + T) e^{-\theta'' K} \right).$$

Proof. The proof follows by applying Proposition 5.1 and Corollary 5.2 to the functions $f(\underline{n}) = n_j$, $1 \leq j \leq d$, which belong to $\mathcal{F}_{K,1}$. \square

Proposition 5.4. *For $1 \leq p, p' \leq d$ and for all $\underline{n} \neq \underline{0}$, we have*

$$\begin{aligned} \mathbb{E}_{\underline{n}} \left[(S_{p,p'}^\Sigma(T, K) - \Sigma_{p,p'}^K)^2 \right] &\leq \mathcal{O}(1)(c_2 \|\underline{n}\|_1^2 + K^2)^2 \\ &\times \left(\frac{1 + \log K}{T \vee 1} + e^{-\delta' (\zeta' \frac{\|\underline{n}\|_1}{K} \wedge \beta')} + T e^{-\theta K} \right) \end{aligned}$$

and

$$\mathbb{E}_{\nu_K} \left[(S_{p,p'}^\Sigma(T, K) - \Sigma_{p,p'}^K)^2 \right] \leq \mathcal{O}(1) K^4 \left(\frac{1 + \log K}{T \vee 1} + (1 + T) e^{-\theta'' K} \right).$$

Proof. The proof follows by applying Proposition 5.1 and Corollary 5.2 to the functions $f(\underline{n}) = n_p n_{p'}$, $1 \leq p, p' \leq d$, which belong to $\mathcal{F}_{K,2}$. \square

Proposition 5.5. *There exist positive constants $\tilde{C}, \tilde{\theta}, \tilde{\delta}, \tilde{\zeta}$ and $\tilde{\beta}$ such that for all $K \geq 2$, $T > 0$ and $1 \leq \ell \leq d$,*

$$\begin{aligned} \mathbb{E}_{\underline{n}} \left[(S_\ell^{\mathfrak{D}}(T, K) - K B_\ell(\underline{x}^*))^2 \right] &\leq \\ &\tilde{C} \left(K + \frac{A_\ell(1 + C_{q_\ell})K}{T} + K^{1-q_\ell} \frac{A_\ell}{T} (K + \|\underline{n}\|_1)^{q_\ell} \mathfrak{R}_\ell \right. \\ &\quad \left. + K^{2-2q_\ell} A_\ell^2 (K + \|\underline{n}\|_1)^{2q_\ell} (\mathfrak{R}_\ell^2 + \mathfrak{R}_\ell) \right) \end{aligned}$$

where

$$\mathfrak{R}_\ell = (1 + c_{q_\ell}) \left(\frac{1 + \log K}{T} + T e^{-\tilde{\theta}K} + e^{-\tilde{\delta}(\tilde{\zeta}_{\frac{n}{K}} \wedge \tilde{\beta})K} \right)$$

and $A_\ell > 0$, $q_\ell \in \mathbb{N}$, are such that, for all $\underline{x} \in \mathbb{R}_+^d$,

$$|B_\ell(\underline{x})| \leq A_\ell(1 + \|\underline{x}\|_1^{q_\ell}).$$

The existence of A_ℓ and q_ℓ follows from the assumptions on B . The constants C_{q_ℓ} and c_{q_ℓ} are defined in Corollary 2.5 and Lemma 2.3, respectively.

We also have

$$\begin{aligned} & \mathbb{E}_{\nu_K} \left[(S_\ell^{\mathcal{D}}(T, K) - K B_\ell(\underline{x}^*))^2 \right] \leq \\ & \tilde{C} \left(K + \frac{A_\ell(1 + C_{q_\ell})K}{T} + \frac{A_\ell}{T} K \tilde{\mathfrak{R}}_\ell + K^2 A_\ell^2 (\tilde{\mathfrak{R}}_\ell^2 + \tilde{\mathfrak{R}}_\ell) \right) \end{aligned}$$

where

$$\tilde{\mathfrak{R}}_\ell = (1 + C_{q_\ell}) \left((1 + C_{2q_\ell})(1 + c_{q_\ell}) \frac{\log K}{T} + (1 + T) e^{-\tilde{\theta}K} \right).$$

Proof. First observe that

$$S_\ell^{\mathcal{D}}(T, K) = \frac{1}{T} \mathcal{N}_\ell^K(0, T)$$

where $\mathcal{N}_\ell^K(0, T)$ is defined in Appendix B. By assumption, the function $f_\ell(\underline{n}) = K^{q_\ell} B_\ell\left(\frac{\underline{n}}{K}\right) \in \mathcal{F}_{\kappa, q_\ell}$. Let \mathbf{m} be any probability measure on \mathbb{Z}_+^d having all its moments finite. We apply Theorem 3.4 to the function f_ℓ , and then using integration against \mathbf{m} we get

$$\begin{aligned} & \left| \mathbb{E}_{\mathbf{m}}[S_{f_\ell}(T, K)] - \nu_\kappa(f_\ell) \right| \leq \mathcal{O}(1) \|f_\ell\|_{\kappa, q_\ell} \\ & \times \int \left((K + \|\underline{n}\|_2)^{q_\ell} \left(e^{-\delta(\zeta_{\frac{\|\underline{n}\|_1}{K}} \wedge \beta)K} + T e^{-\delta\beta K} + \frac{1 + \log K}{T} \right) \right) d\mathbf{m}(\underline{n}). \end{aligned}$$

We now apply the identity in Proposition B.1 and divide by $K^{q_\ell-1}$. We obtain

$$\begin{aligned} & \left| \mathbb{E}_{\mathbf{m}}[S_\ell^{\mathcal{D}}(T, K)] - \nu_\kappa \left(K B_\ell \left(\frac{\underline{n}}{K} \right) \right) \right| \leq \mathcal{O}(1) \|f_\ell\|_{\kappa, q_\ell} K^{1-q_\ell} \\ & \times \int \left((K + \|\underline{n}\|_2)^{q_\ell} \left(e^{-\delta(\zeta_{\frac{\|\underline{n}\|_1}{K}} \wedge \beta)K} + T e^{-\delta\beta K} + \frac{1 + \log K}{T} \right) \right) d\mathbf{m}(\underline{n}). \end{aligned} \quad (5.1)$$

We now estimate

$$\int B_\ell\left(\frac{\underline{n}}{K}\right) d\nu_\kappa(\underline{n}) = \int_{\mathcal{D}} B_\ell\left(\frac{\underline{n}}{K}\right) d\nu_\kappa(\underline{n}) + \int_{\mathcal{D}^c} B_\ell\left(\frac{\underline{n}}{K}\right) d\nu_\kappa(\underline{n}).$$

The second integral is bounded from above by $\mathcal{O}(1)/K$ using the polynomial bound on B_ℓ and the first estimate in Corollary 2.5. For the first integral

we use Taylor expansion around \underline{x}^* to first order, then Cauchy-Schwarz inequality, and finally Theorem 2.6 for $q = 1$. Therefore we obtain

$$\begin{aligned} & |\mathbb{E}_{\mathbf{m}}[S_{\ell}^{\mathcal{D}}(T, K)] - KB_{\ell}(\underline{x}^*)| \leq \mathcal{O}(1)\sqrt{K} + \mathcal{O}(1)\|f_{\ell}\|_{K, q_{\ell}}K^{1-q_{\ell}} \\ & \times \int (K + \|\underline{n}\|_2)^{q_{\ell}} \left(e^{-\delta(\zeta \frac{\|\underline{n}\|_1}{K} \wedge \beta)^K} + T e^{-\delta\beta K} + \frac{1 + \log K}{T} \right) d\mathbf{m}(\underline{n}). \end{aligned} \quad (5.2)$$

Now we apply the estimate in Proposition B.1 to obtain

$$\begin{aligned} & \mathbb{E}_{\mathbf{m}} \left[(S_{\ell}^{\mathcal{D}}(T, K) - \mathbb{E}_{\mathbf{m}}[S_{\ell}^{\mathcal{D}}(T, K)])^2 \right] \\ &= \frac{1}{T^2} \mathbb{E}_{\mathbf{m}} \left[(\mathcal{N}_{\ell}^K(0, T) - \mathbb{E}_{\mathbf{m}}[\mathcal{N}_{\ell}^K(0, T)])^2 \right] \\ &\leq \mathbb{E}_{\mathbf{m}} \left[\left(\frac{1}{T} \int_0^T K^{1-q_{\ell}} f_{\ell}(\underline{N}^K(s)) ds - \mathbb{E}_{\mathbf{m}} \left[\frac{\mathcal{N}_{\ell}^K(0, T)}{T} \right] \right)^2 \right] \\ &\quad + \frac{2}{T} \mathbb{E}_{\mathbf{m}} \left[\frac{1}{T} \int_0^T K^{1-q_{\ell}} f_{\ell}(\underline{N}^K(s)) ds \right] \\ &\leq 2 \mathbb{E}_{\mathbf{m}} \left[\left(\frac{1}{T} \int_0^T K^{1-q_{\ell}} f_{\ell}(\underline{N}^K(s)) ds - K^{1-q_{\ell}} \nu_K(f_{\ell}) \right)^2 \right] \\ &\quad + 2 \mathbb{E}_{\mathbf{m}} \left[\left(K^{1-q_{\ell}} \nu_K(f_{\ell}) - \mathbb{E}_{\mathbf{m}}[S_{\ell}^{\mathcal{D}}(T, K)] \right)^2 \right] \\ &\quad + \frac{2}{T} \mathbb{E}_{\mathbf{m}} \left[\frac{1}{T} \int_0^T K^{1-q_{\ell}} f_{\ell}(\underline{N}^K(s)) ds \right]. \end{aligned}$$

For the first term we use either Corollary 5.2 or Proposition 5.1. For the second term we use (5.1). For the third and last term we apply Theorem 3.4, integrate with respect to \mathbf{m} and use (5.2). To finish the proof, we replace \mathbf{m} by either $\delta_{\underline{n}}$ or ν_K . \square

Recall that $B_p(\underline{x}^*) = D_p(\underline{x}^*)$, $1 \leq p \leq d$.

Proposition 5.6. *Under the assumptions of Proposition 5.1 and Corollary 5.2, we have, for all $1 \leq p, p' \leq d$, and $\tau \geq 0$,*

$$\begin{aligned} \mathbb{E}_{\underline{n}} \left[(S_{p, p'}^C(T, \tau, K) - \Sigma_{p, p'}^K(\tau))^2 \right] &\leq \mathcal{O}(1)(c_2 \|\underline{n}\|_1^2 + K^2)^2 \times \\ &\quad \left(\frac{1 + \tau + \log K}{T \vee 1} + e^{-\delta'(\zeta' \frac{\|\underline{n}\|_1}{K} \wedge \beta')^K} + (T + \tau) e^{-\theta K} \right). \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}_{\nu_K} \left[(S_{p, p'}^C(T, \tau, K) - \Sigma_{p, p'}^K(\tau))^2 \right] \\ \leq \mathcal{O}(1)K^4 \left(\frac{1 + \tau + \log K}{T} + (1 + T + \tau) e^{-\theta'' K} \right) \end{aligned}$$

Proof. The proof requires some simple modifications of the proofs of Propositions 5.1 and 5.2. This is left to the reader. \square

Remark 5.1. *If one modifies slightly the definition of the estimator by integrating from time 1, then, in the four previous propositions, one can replace the factor $(\|\underline{n}\|_1 + K)$ by K , and the factor $(\|\underline{n}\|_1^2 + K^2)$ by K^2 .*

Recall that we defined in Section 1 an empirical matrix $M_{\text{emp}}^*(T, \tau, K)$ by

$$e^{\tau M_{\text{emp}}^*(T, \tau, K)} = S^C(T, \tau, K) S^\Sigma(T, K)^{-1}$$

and an empirical resilience by

$$\rho_{\text{emp}}^*(T, \tau, K) = -\sup\{\text{Re}(z) : z \in \text{Sp}(M_{\text{emp}}^*(T, \tau, K))\}.$$

From the above results one can derive various statistical estimates for the difference between $\rho_{\text{emp}}^*(T, \tau, K)$ and ρ^* . We have the following result which was stated in Section 1.

Theorem 5.7. *For τ of order one, $\|\underline{n}\|_1$ of order K , $\log T \ll K$ and K large enough, we have, with a probability higher than $1 - 1/K$,*

$$|\rho_{\text{emp}}^*(T, \tau, K) - \rho^*| \leq \mathcal{O}(1) \left(\frac{K^2}{\sqrt{T}} + \frac{1}{\sqrt{K}} \right).$$

In particular, if $T \gg K^5$, $|\rho_{\text{emp}}^(T, \tau, K) - \rho^*| \leq \mathcal{O}(1)/\sqrt{K}$.*

Proof. It follows from Propositions 5.4 and 5.6 and the standing assumptions that, with a probability higher than $1 - 1/K$, we have

$$\|S^C(T, \tau, K) - \Sigma^K(\tau)\| \leq \mathcal{O}(1) \frac{K^3}{\sqrt{T}}$$

and

$$\|S^\Sigma(T, K) - \Sigma^K\| \leq \mathcal{O}(1) \frac{K^3}{\sqrt{T}}.$$

We now use Theorem 1.1 and Proposition 2.9 to obtain

$$\left\| e^{\tau M_{\text{emp}}^*(T, \tau, K)} - e^{\tau M^*} \right\| \leq \mathcal{O}(1) \left(\frac{1}{\sqrt{K}} + \frac{K^2}{\sqrt{T}} \right).$$

The result follows since τ is of order one. \square

A Proof of the two variance estimates

A.1 Starting from anywhere: proof of Proposition 5.1

It is enough to prove the result for $\|f\|_{K,q} = 1$. We have

$$\mathbb{E}_{\underline{n}} \left[\left(\frac{1}{T} \int_0^T f(\underline{N}^K(t)) dt \right)^2 \right] = \frac{2}{T^2} \int_0^T dt_2 \int_0^{t_2} \mathbb{E}_{\underline{n}} [f(\underline{N}^K(t_1))f(\underline{N}^K(t_2))] dt_1.$$

Step 1 is to estimate the contribution of the range $0 \leq t_1 \leq t_2 \leq 1$. Using Cauchy-Schwarz inequality and Proposition 2.3 we get

$$\left| \int_0^1 dt_2 \int_0^{t_2} \mathbb{E}_{\underline{n}} [f(\underline{N}^K(t_1))f(\underline{N}^K(t_2))] dt_1 \right| \leq \mathcal{O}(1) \left(\|\underline{n}\|_1^q + K^q \right)^2.$$

Step 2 is to estimate the contribution in the range $0 \leq t_2 - 1 \leq t_1 \leq t_2$. This implied that $T > 1$. We have using again Proposition 2.3

$$\begin{aligned} & \left| \int_1^T dt_2 \int_{t_2-1}^{t_2} \mathbb{E}_{\underline{n}} [f(\underline{N}^K(t_1))f(\underline{N}^K(t_2))] dt_1 \right| \\ & \leq \int_1^T dt_2 \int_{t_2-1}^{t_2} \left(\mathbb{E}_{\underline{n}} [f(\underline{N}^K(t_1))^2] + \mathbb{E}_{\underline{n}} [f(\underline{N}^K(t_2))^2] \right) dt_1 \\ & \leq \mathcal{O}(1) T \left(\|\underline{n}\|_1^q + K^q \right)^2. \end{aligned}$$

Step 3

(1) Using the Markov property and the definition of g (see (3.8)) we have

$$\begin{aligned} & \int_1^T dt_2 \int_0^{t_2-1} \mathbb{E}_{\underline{n}} [f(\underline{N}^K(t_1))f(\underline{N}^K(t_2))] dt_1 \\ & = \int_0^{T-1} ds \int_0^s \mathbb{E}_{\underline{n}} [f(\underline{N}^K(t_1))g(\underline{N}^K(s))] dt_1 \\ & = \int_0^{T-1} ds \int_0^s \mathbb{E}_{\underline{n}} [f(\underline{N}^K(t_1))\mathbb{E}_{\underline{N}^K(t_1)}[g(\underline{N}^K(s-t_1))]] dt_1. \end{aligned}$$

Let us first write

$$\mathbb{E}_{\underline{n}} [f(\underline{N}^K(t_1))\mathbb{E}_{\underline{N}^K(t_1)}[g(\underline{N}^K(s-t_1))]]$$

as the sum of $J_1(\underline{n})$ and $J_2(\underline{n})$ where

$$J_1(\underline{n}) = \mathbb{E}_{\underline{n}} [f(\underline{N}^K(t_1))\mathbb{E}_{\underline{N}^K(t_1)}[\mathbf{1}_{\{T_0 > s-t_1\}}g(\underline{N}^K(s-t_1))]]$$

and

$$J_2(\underline{n}) = \mathbb{E}_{\underline{n}} [f(\underline{N}^K(t_1))\mathbb{E}_{\underline{N}^K(t_1)}[\mathbf{1}_{\{T_0 \leq s-t_1\}}g(\underline{N}^K(s-t_1))]].$$

We further decompose $J_1(\underline{n})$ as $J_{1,1}(\underline{n}) + J_{1,2}(\underline{n})$ where

$$J_{1,1}(\underline{n}) = \mathbb{E}_{\underline{n}} \left[f(\underline{N}^K(t_1)) \mathbf{1}_{\{T_0 \leq t_1\}} \mathbb{E}_{\underline{N}^K(t_1)} \left[\mathbf{1}_{\{T_0 > s - t_1\}} g(\underline{N}^K(s - t_1)) \right] \right]$$

and

$$J_{1,2}(\underline{n}) = \mathbb{E}_{\underline{n}} \left[f(\underline{N}^K(t_1)) \mathbf{1}_{\{T_0 > t_1\}} \mathbb{E}_{\underline{N}^K(t_1)} \left[\mathbf{1}_{\{T_0 > s - t_1\}} g(\underline{N}^K(s - t_1)) \right] \right].$$

Since $\underline{0}$ is an absorbing state, we have for all $\underline{n} \neq \underline{0}$ that

$$J_{1,1}(\underline{n}) = 0.$$

(2) We start by estimating $J_2(\underline{n})$. Since $\underline{0}$ is an absorbing state, we have

$$J_2(\underline{n}) = g(\underline{0}) \mathbb{E}_{\underline{n}} \left[f(\underline{N}^K(t_1)) \mathbb{P}_{\underline{N}^K(t_1)}(T_0 \leq s - t_1) \right].$$

Note that $g(\underline{0}) = \mathbb{E}_{\underline{0}}[f(\underline{N}^K(1))] = f(\underline{0})$. Since we are going to use Lemma 3.2, we write $J_2(\underline{n}) = J_{2,1}(\underline{n}) + J_{2,2}(\underline{n})$ where

$$J_{2,1}(\underline{n}) = f(\underline{0}) \mathbb{E}_{\underline{n}} \left[f(\underline{N}^K(t_1)) \mathbf{1}_{\{\|\underline{N}^K(t_1)\|_1 > K\alpha/\zeta\}} \mathbb{P}_{\underline{N}^K(t_1)}(T_0 \leq s - t_1) \right].$$

and

$$J_{2,2}(\underline{n}) = f(\underline{0}) \mathbb{E}_{\underline{n}} \left[f(\underline{N}^K(t_1)) \mathbf{1}_{\{\|\underline{N}^K(t_1)\|_1 \leq K\alpha/\zeta\}} \mathbb{P}_{\underline{N}^K(t_1)}(T_0 \leq s - t_1) \right].$$

We first estimate $J_{2,1}(\underline{n})$. Using (3.9), Lemma 3.2 with $\gamma = 0$, and since f belongs to $\mathcal{F}_{K,q}$ (see (3.7)), we have

$$\begin{aligned} |J_{2,1}(\underline{n})| &\leq \mathcal{O}(1) \mathbb{E}_{\underline{n}} [|f(\underline{N}^K(t_1))|] e^{-\alpha\delta K} (1 + C(s - t_1)) \\ &\leq \mathcal{O}(1) (\|\underline{n}\|_1^q + K^q) e^{-\alpha\delta K} (1 + C(s - t_1)) \end{aligned}$$

where we used Proposition 2.3 for the second inequality.

We now estimate $J_{2,2}(\underline{n})$ by splitting it as $J_{2,2,1}(\underline{n}) + J_{2,2,2}(\underline{n})$ where

$$\begin{aligned} J_{2,2,1}(\underline{n}) &= f(\underline{0}) \mathbb{E}_{\underline{n}} \left[f(\underline{N}^K(t_1)) \mathbf{1}_{\{\|\underline{N}^K(t_1)\|_1 \leq K\alpha/\zeta\}} \mathbf{1}_{\{\|\underline{N}^K(t_1)\|_1 > (\lfloor \frac{1}{2\|\underline{x}^*\|_1} (\zeta \frac{\|\underline{n}\|_1}{K} \wedge \alpha) \rfloor \wedge 1) K\}} \right. \\ &\quad \left. \mathbb{P}_{\underline{N}^K(t_1)}(T_0 \leq s - t_1) \right]. \end{aligned}$$

and

$$\begin{aligned} J_{2,2,2}(\underline{n}) &= f(\underline{0}) \mathbb{E}_{\underline{n}} \left[f(\underline{N}^K(t_1)) \mathbf{1}_{\{\|\underline{N}^K(t_1)\|_1 \leq K\alpha/\zeta\}} \mathbf{1}_{\{\|\underline{N}^K(t_1)\|_1 \leq (\lfloor \frac{1}{2\|\underline{x}^*\|_1} (\zeta \frac{\|\underline{n}\|_1}{K} \wedge \alpha) \rfloor \wedge 1) K\}} \right. \\ &\quad \left. \mathbb{P}_{\underline{N}^K(t_1)}(T_0 \leq s - t_1) \right]. \end{aligned}$$

Proceeding as before we get

$$\begin{aligned} |J_{2,2,1}(\underline{n})| &\leq \mathcal{O}(1)K^q \mathbb{E}_{\underline{n}} \left[\mathbb{1}_{\{\|\underline{N}^K(t_1)\|_1 > (\lfloor \frac{1}{2\|\underline{x}^*\|_1} (\zeta \frac{\|\underline{n}\|_1}{K} \wedge \alpha) \rfloor \wedge 1)K\}} \mathbb{P}_{\underline{N}^K(t_1)}(\mathbb{1}_{\{T_0 \leq s-t_1\}}) \right] \\ &\leq \mathcal{O}(1)K^q \left(e^{-\delta K} \left(\left(\lfloor \frac{1}{2\|\underline{x}^*\|_1} (\zeta \frac{\|\underline{n}\|_1}{K} \wedge \alpha) \rfloor \wedge 1 \right) \wedge \alpha \right) + (s-t_1) e^{-\alpha \delta K} \right). \end{aligned}$$

We used Lemma 3.2 with $\gamma = 0$.

We now handle $J_{2,2,2}(\underline{n})$.

Note that $\gamma \leq 1 \wedge \frac{\alpha}{\|\underline{x}^*\|_1}$. We proceed as before with f and g , and we use Lemma 3.2 with

$$\gamma = \left(\frac{1}{2\|\underline{x}^*\|_1} \left(\zeta \frac{\|\underline{n}\|_1}{K} \wedge \alpha \right) \right) \wedge 1.$$

to get

$$\begin{aligned} |J_{2,2,2}(\underline{n})| &\leq \mathcal{O}(1) K^q \mathbb{P}_{\underline{n}} \left(\|\underline{N}^K(t_1)\|_1 \leq \left(\left(\frac{1}{2\|\underline{x}^*\|_1} \left(\zeta \frac{\|\underline{n}\|_1}{K} \wedge \alpha \right) \right) \wedge 1 \right) K \right) \\ &\leq \mathcal{O}(1)K^q \left(e^{-\delta \left(\frac{1}{2} \left(\zeta \frac{\|\underline{n}\|_1}{K} \wedge \alpha \right) \wedge \|\underline{x}^*\|_1 \right) K} + C t_1 e^{-\frac{\alpha \delta K}{2}} \right). \end{aligned}$$

(3) Let us now estimate for all $\underline{n} \neq \underline{0}$

$$|J_{1,2}(\underline{n}) - \nu_K(f)^2|.$$

We have

$$\begin{aligned} |J_{1,2}(\underline{n}) - \nu_K(f)^2| &\leq \left| J_{1,2}(\underline{n}) - \nu_K(g) \mathbb{E}_{\underline{n}} [f(\underline{N}^K(t_1)) \mathbb{1}_{\{T_0 > t_1\}} \mathbb{P}_{\underline{N}^K(t_1)}(T_0 > s-t_1)] \right| \\ &+ \left| \nu_K(g) \mathbb{E}_{\underline{n}} \left(f(\underline{N}^K(t_1)) \mathbb{1}_{\{T_0 > t_1\}} \mathbb{P}_{\underline{N}^K(t_1)}(T_0 > s-t_1) \right) - \nu_K(g) \mathbb{E}_{\underline{n}} \left(f(\underline{N}^K(t_1)) \mathbb{1}_{\{T_0 > t_1\}} \right) \right| \\ &+ \left| \nu_K(g) \mathbb{E}_{\underline{n}} \left(f(\underline{N}^K(t_1)) \mathbb{1}_{\{T_0 > t_1\}} \right) - \nu_K(g) \nu_K(f) \right| + \left| \nu_K(g) \nu_K(f) - \nu_K(f)^2 \right| \\ &= W_1(\underline{n}) + W_2(\underline{n}) + W_3(\underline{n}) + W_4. \end{aligned}$$

(3)-(i) By Theorem 3.1 and since $\underline{N}^K(t_1) \neq \underline{0}$, we have

$$\left| \mathbb{E}_{\underline{N}^K(t_1)} [\mathbb{1}_{\{T_0 > s-t_1\}} g(\underline{N}^K(s-t_1))] - \mathbb{P}_{\underline{N}^K(t_1)}(T_0 > s-t_1) \nu(g) \right| \leq \mathcal{O}(1) K^q e^{-a \frac{(s-t_1)}{\log K}}.$$

Hence, using Proposition 2.3, we get for all $\underline{n} \neq \underline{0}$

$$W_1(\underline{n}) \leq \mathcal{O}(1) K^q (c_q \|\underline{n}\|_1^q + K^q) e^{-a \frac{(s-t_1)}{\log K}}.$$

(3)-(ii) We have

$$\begin{aligned} &\left| \nu_K(g) \mathbb{E}_{\underline{n}} \left[f(\underline{N}^K(t_1)) \mathbb{1}_{\{T_0 > t_1\}} \mathbb{P}_{\underline{N}^K(t_1)}(T_0 > s-t_1) \right] - \nu_K(g) \mathbb{E}_{\underline{n}} \left[f(\underline{N}^K(t_1)) \mathbb{1}_{\{T_0 > t_1\}} \right] \right| \\ &\leq \left| \nu_K(g) \right| \mathbb{E}_{\underline{n}} \left(\left| f(\underline{N}^K(t_1)) \right| \mathbb{1}_{\{T_0 > t_1\}} \mathbb{P}_{\underline{N}^K(t_1)}(T_0 \leq s-t_1) \right) \end{aligned}$$

Define $0 < \gamma' < \beta$ by

$$\gamma' = \gamma'(\underline{n}) = \frac{1}{2} \left(\frac{\zeta \|\underline{n}\|_1}{K} \wedge \alpha \right).$$

We split the right hand side in two terms:

$$\begin{aligned} & \mathbb{E}_{\underline{n}} \left[|f(\underline{N}^K(t_1))| \mathbf{1}_{\{T_0 > t_1\}} \mathbb{P}_{\underline{N}^K(t_1)}(T_0 \leq s - t_1) \right] \\ &= \mathbb{E}_{\underline{n}} \left[\mathbf{1}_{\{\|\underline{N}^K(t_1)\|_1 \leq \gamma' K\}} |f(\underline{N}^K(t_1))| \mathbf{1}_{\{T_0 > t_1\}} \mathbb{P}_{\underline{N}^K(t_1)}(T_0 \leq s - t_1) \right] \\ & \quad + \mathbb{E}_{\underline{n}} \left[\mathbf{1}_{\{\|\underline{N}^K(t_1)\|_1 > \gamma' K\}} |f(\underline{N}^K(t_1))| \mathbf{1}_{\{T_0 > t_1\}} \mathbb{P}_{\underline{N}^K(t_1)}(T_0 \leq s - t_1) \right] \end{aligned}$$

The first term is estimated using the growth property of f , Lemma 3.2, and Cauchy-Schwarz inequality, namely

$$\begin{aligned} & \mathbb{E}_{\underline{n}} \left[\mathbf{1}_{\{\|\underline{N}^K(t_1)\|_1 \leq \gamma' K\}} |f(\underline{N}^K(t_1))| \mathbf{1}_{\{T_0 > t_1\}} \mathbb{P}_{\underline{N}^K(t_1)}(T_0 \leq s - t_1) \right] \\ & \leq \mathbb{E}_{\underline{n}} \left[\mathbf{1}_{\{\|\underline{N}^K(t_1)\|_1 \leq \gamma' K\}} |f(\underline{N}^K(t_1))|^2 \right]^{\frac{1}{2}} \mathbb{P}_{\underline{n}}(\|\underline{N}^K(t_1)\|_1 \leq \gamma' K)^{\frac{1}{2}} \\ & \leq \mathcal{O}(1) K^q \mathbb{P}_{\underline{n}}(\|\underline{N}^K(t_1)\|_1 \leq \gamma' K)^{\frac{1}{2}} \\ & \leq \mathcal{O}(1) K^q \left(e^{-\frac{\delta}{2} \left(\frac{\zeta \|\underline{n}\|_1}{K} \wedge \alpha \right) K} + \mathcal{O}(1) t_1 e^{-\frac{\alpha \delta K}{2}} \right)^{\frac{1}{2}}. \end{aligned}$$

To deal with the second term, we observe using Lemma 3.2 and Proposition 2.3 that, if $\|\underline{N}^K(t_1)\|_1 > \gamma' K$, then

$$\begin{aligned} \mathbb{P}_{\underline{N}^K(t_1)}(T_0 \leq s - t_1) & \leq e^{-\delta \left(\frac{\zeta \|\underline{N}^K(t_1)\|_1}{K} \wedge \alpha \right) K} + \mathcal{O}(1)(s - t_1) e^{-\alpha \delta K} \\ & \leq e^{-\delta(\zeta \gamma' \wedge \alpha) K} + \mathcal{O}(1)(s - t_1) e^{-\alpha \delta K} \\ & = e^{-\delta \left(\zeta \left(\frac{1}{2} \left(\frac{\zeta \|\underline{n}\|_1}{K} \wedge \alpha \right) \right) \wedge \alpha \right) K} + \mathcal{O}(1)(s - t_1) e^{-\alpha \delta K}. \end{aligned}$$

Now

$$\begin{aligned} & \mathbb{E}_{\underline{n}} \left[\mathbf{1}_{\{\|\underline{N}^K(t_1)\|_1 > \gamma' K\}} |f(\underline{N}^K(t_1))| \mathbf{1}_{\{T_0 > t_1\}} \mathbb{P}_{\underline{N}^K(t_1)}(T_0 \leq s - t_1) \right] \\ & \leq \mathcal{O}(1) \mathbb{E}_{\underline{n}} \left[\mathbf{1}_{\{\|\underline{N}^K(t_1)\|_1 > \gamma' K\}} |f(\underline{N}^K(t_1))| \mathbf{1}_{\{T_0 > t_1\}} \right] \\ & \quad \times \left(e^{-\delta \left(\zeta \left(\frac{1}{2} \left(\frac{\zeta \|\underline{n}\|_1}{K} \wedge \alpha \right) \right) \wedge \alpha \right) K} + \mathcal{O}(1)(s - t_1) e^{-\alpha \delta K} \right) \\ & \leq \mathcal{O}(1) \left(\|\underline{n}\|^q + K^q \right) \left(e^{-\delta \left(\zeta \left(\frac{1}{2} \left(\frac{\zeta \|\underline{n}\|_1}{K} \wedge \alpha \right) \right) \wedge \alpha \right) K} + \mathcal{O}(1)(s - t_1) e^{-\alpha \delta K} \right) \\ & \quad \times \left(e^{-\delta \frac{(1 \wedge \zeta)}{2} \left(\frac{\zeta^2 \|\underline{n}\|_1}{2K} \wedge \alpha \right) K} + \mathcal{O}(1)(s - t_1) e^{-\alpha \delta K} \right). \end{aligned}$$

(3)-(iii) Let us now prove that for all $\underline{n} \neq \underline{0}$,

$$\begin{aligned} W_3(\underline{n}) &= \left| \mathbb{E}_{\underline{n}} \left(f(\underline{N}^K(t_1)) \mathbf{1}_{\{T_{\underline{0}} > t_1\}} \right) - \nu_K(f) \right| \\ &\leq \mathcal{O}(1)(c_q \|\underline{n}\|_1^q + K^q) \left(e^{-\frac{a(t_1-1)}{\log K}} + e^{-\lambda_0(K)} e^{-\delta \left(\zeta \frac{\|\underline{n}\|_1}{K} \wedge \alpha \right) K} \right. \\ &\quad \left. + C(t_1 - 1) e^{-\alpha \delta K} + 1 - e^{-\lambda_0(K)} \right). \quad (\text{A.1}) \end{aligned}$$

For $0 \leq t_1 \leq 1$, using Proposition 2.3 we obtain

$$\left| \mathbb{E}_{\underline{n}} \left[f(\underline{N}^K(t_1)) \mathbf{1}_{\{T_{\underline{0}} > t_1\}} \right] \right| \leq \mathcal{O}(1)(c_q \|\underline{n}\|_1^q + K^q).$$

We now deal with $t_1 > 1$. By the Markov property one has

$$\begin{aligned} \mathbb{E}_{\underline{n}} \left[f(\underline{N}^K(t_1)) \mathbf{1}_{\{T_{\underline{0}} > t_1\}} \right] &= \mathbb{E}_{\underline{n}} \left[\mathbf{1}_{\{T_{\underline{0}} > t_1 - 1\}} \mathbb{E}_{\underline{N}^K(t_1 - 1)} \left[f(\underline{N}^K(1)) \mathbf{1}_{\{T_{\underline{0}} > 1\}} \right] \right] \\ &= \mathbb{E}_{\underline{n}} \left[\mathbf{1}_{\{T_{\underline{0}} > t_1 - 1\}} \tilde{g}(\underline{N}^K(t_1 - 1)) \right] \end{aligned}$$

where

$$\tilde{g}(\underline{n}) = \mathbb{E}_{\underline{n}} \left[\underline{N}^K(1) \mathbf{1}_{\{T_{\underline{0}} > 1\}} \right] \leq g(\underline{n})$$

is a function bounded by $\mathcal{O}(1)K^q$. For $\underline{n} \neq \underline{0}$, we use Theorem 3.1 and Corollary 2.2 to get

$$\begin{aligned} &\left| \mathbb{E}_{\underline{n}} \left[\mathbf{1}_{\{T_{\underline{0}} > t_1 - 1\}} \mathbb{E}_{\underline{N}^K(t_1 - 1)} \left[f(\underline{N}^K(1)) \mathbf{1}_{\{T_{\underline{0}} > 1\}} \right] \right] \right. \\ &\quad \left. - \mathbb{P}_{\underline{n}}(T_{\underline{0}} > t_1 - 1) \mathbb{E}_{\nu_K} \left[f(\underline{N}^K(1)) \mathbf{1}_{\{T_{\underline{0}} > 1\}} \right] \right| \\ &\leq \mathcal{O}(1) K^q e^{-a \frac{(t_1 - 1)}{\log K}}. \end{aligned}$$

Since ν_K is the qsd, we have

$$\mathbb{E}_{\nu_K} \left[f(\underline{N}^K(1)) \mathbf{1}_{\{T_{\underline{0}} > 1\}} \right] = e^{-\lambda_0(K)} \nu_K(f).$$

Using Corollary 2.5, Lemma 3.2 and the properties of f we obtain

$$\begin{aligned} &\left| \mathbb{P}_{\underline{n}}(T_{\underline{0}} > t_1 - 1) \mathbb{E}_{\nu_K} \left[f(\underline{N}^K(1)) \mathbf{1}_{\{T_{\underline{0}} > 1\}} \right] - \nu_K(f) \right| \\ &\leq \mathcal{O}(1) K^q \left(e^{-\lambda_0(K)} e^{-\delta \left(\zeta \frac{\|\underline{n}\|_1}{K} \wedge \alpha \right) K} + C(t_1 - 1) e^{-\beta \delta K} + 1 - e^{-\lambda_0(K)} \right) \end{aligned}$$

and (A.1) is proved.

(3)-(iv) Let us note that

$$W_4 \leq |\nu_K(f)| |\nu_K(f) - \nu_K(g)|.$$

Proposition 2.3 and Lemma 3.5 give

$$W_4 \leq \mathcal{O}(1) K^{2q} (1 - e^{-\lambda_0(K)})^{\frac{1}{2}}.$$

(3)-(v) Collecting the informations given in the four previous estimates, we obtain a precise estimation of $|J_{1,2}(\underline{n}) - \nu_K(f)^2|$ for all $\underline{n} \neq \underline{0}$.

(3)-(vi) We have

$$\begin{aligned} & \left| \mathbb{E}_{\underline{n}} \left[f(N^K(t_1)) \mathbb{E}_{N^K(t_1)} [g(N^K(s-t_1))] \right] - \nu_K(f)^2 \right| \\ & \leq |J_2(\underline{n})| + |J_{1,2}(\underline{n}) - \nu_K(f)^2|. \end{aligned}$$

Collecting the above relevant estimates we obtain that there exist $\delta', \zeta', \beta', \theta'$ (all being positive and independent of K) such that

$$\begin{aligned} & \left| \mathbb{E}_{\underline{n}} \left[f(N^K(t_1)) \mathbb{E}_{N^K(t_1)} [g(N^K(s-t_1))] \right] - \nu_K(f)^2 \right| \leq \mathcal{O}(1) K^q (c_q \|\underline{n}\|_1^q + K^q) \\ & \quad \times \left(\mathbf{1}_{\{t_1 \leq 1\}} + e^{-\delta' (\zeta' \frac{\|\underline{n}\|_1}{K} \wedge \beta')} K + (s+t_1+1) e^{-\theta' K} + e^{-\frac{\alpha(s-t_1)}{\log K}} + e^{-\frac{\alpha t_1}{\log K}} \right). \end{aligned}$$

Now we have

$$\begin{aligned} & \frac{2}{T^2} \left| \int_0^{T-1} ds \int_0^s \mathbb{E}_{\underline{n}} \left[f(N^K(t_1)) \mathbb{E}_{N^K(t_1)} [g(N^K(s-t_1))] \right] dt_1 - \nu_K(f)^2 \right| \\ & \leq \mathcal{O}(1) K^q (c_q \|\underline{n}\|_1^q + K^q) \left(\frac{1}{T} + e^{-\delta' (\zeta' \frac{\|\underline{n}\|_1}{K} \wedge \beta')} K + T e^{-\theta' K} + \frac{\log K}{T} \right). \end{aligned}$$

The final result for $T \geq 1$ follows by collecting all estimates. For $T < 1$ the bound follows directly from Proposition 2.3.

A.2 Starting from the qsd: proof of Corollary 5.2

The result follows from Proposition 5.1 by integrating over \underline{n} with respect to the qsd. More precisely, we have

$$\begin{aligned} \mathbb{E}_{\underline{n}} \left[|S_f(T, K) - \nu_K(f)|^2 \right] & \leq C' \|f\|_{K,q}^2 \left(\frac{(c_q \|\underline{n}\|_1^q + K^q) \|\underline{n}\|_1^q + K^q \log K}{T \vee 1} \right. \\ & \quad \left. + (c_q \|\underline{n}\|_1^q + K^q) K^q e^{-\delta' (\zeta' \frac{\|\underline{n}\|_1}{K} \wedge \beta')} K + (c_q \|\underline{n}\|_1^q + K^q) T K^q e^{-\theta' K} \right). \end{aligned}$$

The integrals of the first and third terms with respect to the q.s.d are estimated using Corollary 2.5. We deal with second term:

$$\begin{aligned} & \int (c_q \|\underline{n}\|_1^q + K^q) K^q e^{-\delta' (\zeta' \frac{\|\underline{n}\|_1}{K} \wedge \beta')} K \, d\nu_K(\underline{n}) = \\ & \quad \int \mathbf{1}_{\{\{\|\underline{n}\|_1 < \beta' K / \zeta'\} \cap \mathcal{D}\}} (c_q \|\underline{n}\|_1^q + K^q) K^q e^{-\delta' (\zeta' \frac{\|\underline{n}\|_1}{K} \wedge \beta')} K \, d\nu_K(\underline{n}) \\ & \quad + \int \mathbf{1}_{\{\{\|\underline{n}\|_1 < \beta' K / \zeta'\} \cap \mathcal{D}^c\}} (c_q \|\underline{n}\|_1^q + K^q) K^q e^{-\delta' (\zeta' \frac{\|\underline{n}\|_1}{K} \wedge \beta')} K \, d\nu_K(\underline{n}) \\ & \quad + \int \mathbf{1}_{\{\{\|\underline{n}\|_1 \geq \beta' K / \zeta'\}} (c_q \|\underline{n}\|_1^q + K^q) K^q e^{-\delta' (\zeta' \frac{\|\underline{n}\|_1}{K} \wedge \beta')} K \, d\nu_K(\underline{n}). \end{aligned}$$

The third integral is estimated using the fact that the integrand is exponentially small in K . The second integral is estimated using the first estimate in Corollary 2.5. We finally deal with the first integral. If $\underline{n} \in \mathcal{D}$ then $\|\underline{n}\|_1 \geq \|\underline{n}\|_2 \geq \|\underline{n}^*\|_2/2$. If $\{\|\underline{n}\|_1 < \beta'K/\zeta'\} \cap \mathcal{D} \neq \emptyset$, on this set we have $e^{-\delta'(\zeta' \frac{\|\underline{n}\|_1}{K} \wedge \beta')}K \leq e^{-\delta'\zeta'\|\underline{n}^*\|_2/2}$ (exponentially small in K). The estimate follows.

B Counting the number of births

Denote by $\mathcal{N}_\ell^K(t_1, t_2)$ the number of births of species of type ℓ between the times t_1 and t_2 ($1 \leq \ell \leq d$, $0 \leq t_1 \leq t_2$).

Proposition B.1. *For any probability measure \mathbf{m} on \mathbb{Z}_+^d , we have*

$$\mathbb{E}_\mathbf{m}[\mathcal{N}_\ell^K(t_1, t_2)] = K \int_{t_1}^{t_2} \mathbb{E}_\mathbf{m} \left[B_\ell \left(\frac{N^K(s)}{K} \right) \right] ds$$

and

$$\begin{aligned} \mathbb{E}_\mathbf{m} \left[\left(\mathcal{N}_\ell^K(t_1, t_2) - \mathbb{E}_\mathbf{m}[\mathcal{N}_\ell^K(t_1, t_2)] \right)^2 \right] &\leq 2K \mathbb{E}_\mathbf{m} \left[\int_{t_1}^{t_2} B_\ell \left(\frac{N^K(s)}{K} \right) ds \right] \\ &+ \mathbb{E}_\mathbf{m} \left[\left(\int_{t_1}^{t_2} K B_\ell \left(\frac{N^K(s)}{K} \right) ds - \mathbb{E}_\mathbf{m}[\mathcal{N}_\ell^K(t_1, t_2)] \right)^2 \right]. \end{aligned}$$

Proof. Recall that the generator of the process is given in (2.1). Let us now give a pathwise representation of the process. We introduce d independent point Poisson measures $M_\ell(ds, d\theta)$ on $\mathbb{R}_+ \times \mathbb{R}_+$ with intensity $ds d\theta$. We define the d -dimensional càd-làg process $(N_t, t \in \mathbb{R}_+)$

$$\begin{aligned} N_t &= N_0 + \sum_{\ell=1}^d \int_0^t \int M_\ell(ds, d\theta) \\ &\times \left(\mathbb{1}_{\left\{ \theta \leq K B_\ell \left(\frac{N^K(s)}{K} \right) \right\}} - \mathbb{1}_{\left\{ K B_\ell \left(\frac{N^K(s)}{K} \right) \leq \theta \leq K \left(B_\ell \left(\frac{N^K(s)}{K} \right) + D_\ell \left(\frac{N^K(s)}{K} \right) \right) \right\}} \right). \end{aligned}$$

Then the number of births of species between the times t_1 and t_2 is given by

$$\mathcal{N}_\ell^K(t_1, t_2) = \int_{t_1}^{t_2} \int \mathbb{1}_{\left\{ \theta \leq K B_\ell \left(\frac{N^K(s)}{K} \right) \right\}} M_\ell(ds, d\theta).$$

Using the Markov property we get at once the first identity.

We now establish the estimate. Indeed

$$\begin{aligned} & \mathbb{E}_m \left[\left(\mathcal{N}_\ell^K(t_1, t_2) - \mathbb{E}_m(\mathcal{N}_\ell^K(t_1, t_2)) \right)^2 \right] \\ & \leq 2\mathbb{E}_m \left[\left(\mathcal{N}_\ell^K(t_1, t_2) - \int_{t_1}^{t_2} KB_\ell \left(\frac{N^K(s)}{K} \right) ds \right)^2 \right] \\ & \quad + 2\mathbb{E}_m \left[\left(\int_{t_1}^{t_2} KB_\ell \left(\frac{N^K(s)}{K} \right) ds - \mathbb{E}_m[\mathcal{N}_\ell^K(t_1, t_2)] \right)^2 \right]. \end{aligned}$$

By the \mathbb{L}^2 -isometry for jump processes (see [6] Formula (3.9) p.62), we have

$$\begin{aligned} & \mathbb{E}_m \left[\left(\mathcal{N}_\ell^K(t_1, t_2) - \int_{t_1}^{t_2} KB_\ell \left(\frac{N^K(s)}{K} \right) ds \right)^2 \right] \\ & = \int_{t_1}^{t_2} \int \mathbb{E}_m \left(\left(\mathbf{1}_{\{\theta \leq KB_\ell \left(\frac{N^K(s)}{K} \right)\}} \right)^2 \right) ds d\theta \\ & = \int_{t_1}^{t_2} \mathbb{E}_m \left[KB_\ell \left(\frac{N^K(s)}{K} \right) \right] ds. \end{aligned}$$

This finishes the proof. \square

C Gaussian limit for the rescaled qsd

We have the following theorem of independent interest. A part of this theorem partially generalizes a result obtained in [2] for models involving a single species ($d = 1$). Recall that $\underline{n}^* = \lfloor K\underline{x}^* \rfloor$.

Theorem C.1. *For all $K > 1$, define the measure \mathfrak{a}_K on the Borel σ -algebra of \mathbb{R}^d by*

$$\mathfrak{a}_K(\cdot) = \nu_K \left(\left\{ \underline{n} \in \mathbb{Z}_+^d : \frac{\underline{n} - \underline{n}^*}{\sqrt{K}} \in \cdot \right\} \right).$$

Then $(\mathfrak{a}_K)_K$ converges weakly to the centered Gaussian measure with covariance matrix

$$\mathcal{S} = \int_0^\infty e^{\tau M^*} \mathcal{B}^* e^{\tau M^{*\top}} d\tau.$$

where \mathcal{B}^ is the diagonal matrix with entries $B_\ell(\underline{x}^*) = D_\ell(\underline{x}^*)$. The matrix \mathcal{S} is also the unique symmetric solution of the (Lyapunov) equation (fluctuation-dissipation relation)*

$$M^* \mathcal{S} + \mathcal{S} M^{*\top} = -\mathcal{B}^*. \quad (\text{C.1})$$

Remark C.1. *We have*

$$\lim_{K \rightarrow +\infty} \frac{\Sigma^K}{K} = \mathcal{S}.$$

This follows by dividing out equation (1.15) by K , letting K tend to infinity, and using the uniqueness of the (symmetric) solution of (C.1).

Proof. By Theorem 2.6, the family of measures $(\mathbf{a}_K)_K$ is tight. For $\underline{p} \in \mathbb{R}^d$ define

$$H_K(\underline{p}) = \int e^{i \frac{\langle \underline{p}, (\underline{n} - \underline{n}^*) \rangle}{\sqrt{K}}} d\nu_K(\underline{n}).$$

It follows also from Theorem 2.6 that the family of functions (H_K) is uniformly bounded in C^2 . We will prove that for all $\underline{p} \in \mathbb{R}^d$,

$$\lim_{K \rightarrow \infty} H_K(\underline{p}) = e^{-\langle \underline{p}, \mathcal{S} \underline{p} \rangle} \quad (\text{C.2})$$

This will prove that there is only one weak accumulation point for $(\mathbf{a}_K)_K$. The proof will be the consequence of Prokhorov Theorem [1]. We now prove (C.2). Using (2.7) and (2.5), we have

$$\lim_{K \rightarrow \infty} \nu_K \left(\mathcal{L}_K e^{i \frac{\langle \underline{p}, (\cdot - \underline{n}^*) \rangle}{\sqrt{K}}} \right) = 0.$$

We also have

$$\begin{aligned} \nu_K \left(\mathcal{L}_K e^{i \frac{\langle \underline{p}, (\cdot - \underline{n}^*) \rangle}{\sqrt{K}}} \right) &= K \sum_{\ell=1}^d \int d\nu_K(\underline{n}) e^{i \frac{\langle \underline{p}, (\underline{n} - \underline{n}^*) \rangle}{\sqrt{K}}} \\ &\quad \times \left(B_\ell \left(\frac{\underline{n}}{K} \right) \left(e^{i \frac{p_\ell}{\sqrt{K}}} - 1 \right) + D_\ell \left(\frac{\underline{n}}{K} \right) \left(e^{-i \frac{p_\ell}{\sqrt{K}}} - 1 \right) \right). \end{aligned}$$

We use Taylor expansion and using the moments estimates and the polynomial bounds on B_ℓ and D_ℓ (and $B_\ell(\underline{x}^*) = D_\ell(\underline{x}^*)$) we obtain

$$\begin{aligned} &\nu_K \left(\mathcal{L}_K e^{i \frac{\langle \underline{p}, (\cdot - \underline{n}^*) \rangle}{\sqrt{K}}} \right) \\ &= - \sum_{\ell=1}^d B_\ell \left(\frac{\underline{n}^*}{K} \right) p_\ell^2 H_K(\underline{p}) + i \sum_{\ell=1}^d p_\ell \sum_{j=1}^d \left(\partial_j B_\ell \left(\frac{\underline{n}^*}{K} \right) - \partial_j D_\ell \left(\frac{\underline{n}^*}{K} \right) \right) \\ &\quad \times \int e^{i \frac{1}{\sqrt{K}} \langle \underline{p}, (\underline{n} - \underline{n}^*) \rangle} \frac{n_j^* - n_j}{\sqrt{K}} d\nu_K(\underline{n}) + \mathcal{O} \left(\frac{1}{\sqrt{K}} \right) \\ &= - \sum_{\ell=1}^d B_\ell \left(\frac{\underline{n}^*}{K} \right) p_\ell^2 H_K(\underline{p}) \\ &\quad + \sum_{\ell=1}^d p_\ell \sum_{j=1}^d \left(\partial_j B_\ell \left(\frac{\underline{n}^*}{K} \right) - \partial_j D_\ell \left(\frac{\underline{n}^*}{K} \right) \right) \partial_{p_j} H_K(\underline{p}) + \mathcal{O} \left(\frac{1}{\sqrt{K}} \right) \\ &= - \sum_{\ell=1}^d B_\ell(\underline{x}^*) p_\ell^2 H_K(\underline{p}) + \sum_{\ell=1}^d p_\ell \sum_{j=1}^d M_{\ell,j}^* \partial_{p_j} H_K(\underline{p}) + \mathcal{O} \left(\frac{1}{\sqrt{K}} \right). \end{aligned}$$

We conclude that every accumulation point \check{H} of $(H_K)_K$ is bounded in C^1 , satisfies $\check{H}(\underline{0}) = 1$, and is a solution of the equation

$$- \sum_{\ell=1}^d B_\ell(\underline{x}^*) p_\ell^2 \check{H}(\underline{p}) + \sum_{\ell=1}^d p_\ell \sum_{j=1}^d M_{\ell,j}^* \partial_{p_j} \check{H}(\underline{p}) = 0.$$

Then (C.2) follows from Lemma C.2 (stated and proved right after this proof) with $A = M^*$. \square

Lemma C.2. *Let (B_j) be d strictly positive numbers and A a real $d \times d$ matrix such that $\text{Sp}(A) \subset \{z \in \mathbb{C} : \text{Re}(z) < 0\}$. Then there exists a unique $C^1(\mathbb{R}^d, \mathbb{R})$ function H satisfying $H(\underline{0}) = 1$ and*

$$-\sum_{\ell=1}^d B_\ell p_\ell^2 H(\underline{p}) + \sum_{\ell=1}^d p_\ell \sum_{j=1}^d A_{\ell,j} \partial_{p_j} H(\underline{p}) = 0, \quad \underline{p} \in \mathbb{R}^d. \quad (\text{C.3})$$

This function is given by

$$H(\underline{p}) = e^{-\langle \underline{p}, \mathcal{S} \underline{p} \rangle}$$

where

$$\mathcal{S} = \int_0^\infty e^{\tau A} \mathcal{B} e^{\tau A^\top} d\tau$$

where \mathcal{B} is the diagonal matrix with entries (B_j) . The matrix \mathcal{S} is also the unique symmetric solution of the equation

$$A\mathcal{S} + \mathcal{S}A^\top = -\mathcal{B}.$$

Proof. We use the method of characteristics. For all $\underline{p} \in \mathbb{R}^d$, we define the function $\underline{p}(s), s \geq 0$ as the solution of

$$\frac{d\underline{p}}{ds}(s) = A^\top \underline{p}(s), \quad \underline{p}(0) = \underline{p}.$$

Let

$$\mathfrak{b}(s) = -\sum_{\ell=1}^d B_\ell \int_0^s p_\ell(\tau)^2 d\tau.$$

Let H be a solution of (C.3). It is easy to check that for all $\underline{p} \in \mathbb{R}_+^d$ and $s \in \mathbb{R}$

$$\frac{d}{ds} \left(H(\underline{p}(s)) e^{\mathfrak{b}(s)} \right) = 0.$$

Integrating from 0 to u yields

$$H(\underline{p}) = H(\underline{p}(u)) e^{\mathfrak{b}(u)}.$$

From the spectral properties of A we get

$$\lim_{u \rightarrow +\infty} H(\underline{p}(u)) = H(\underline{0}) = 1.$$

Therefore

$$H(\underline{p}) = e^{\mathfrak{b}(\infty)}$$

and

$$\mathbf{b}(\infty) = - \int_0^\infty \langle \underline{p}, e^{\tau A} \mathcal{B} e^{\tau A^\top} \underline{p} \rangle d\tau = -\langle \underline{p}, \mathcal{S}\underline{p} \rangle.$$

Finally we get from the spectral properties of A

$$\begin{aligned} A\mathcal{S} + \mathcal{S}A^\top &= \int_0^\infty (A e^{\tau A} \mathcal{B} e^{\tau A^\top} + e^{\tau A} \mathcal{B} e^{\tau A^\top} A^\top) d\tau \\ &= \int_0^\infty \frac{d}{d\tau} (e^{\tau A} \mathcal{B} e^{\tau A^\top}) d\tau = -\mathcal{B}. \end{aligned}$$

This finishes the proof of the lemma. \square

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