# Quasi-Stationary Distributions and Resilience: What to get from a sample? 

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# Quasi-Stationary Distributions and Resilience: What to get from a sample? 

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#### Abstract

We study a class of multi-species birth-and-death processes going almost surely to extinction and admitting a unique quasi-stationary distribution (qsd for short). When rescaled by $K$ and in the limit $K \rightarrow+\infty$, the realizations of such processes get close, in any fixed finite-time window, to the trajectories of a dynamical system whose vector field is defined by the birth and death rates. Assuming that this dynamical has a unique attracting fixed point, we analyzed in a previous work what happens for large but finite $K$, especially the different time scales showing up. In the present work, we are mainly interested in the following question: Observing a realization of the process, can we determine the so-called engineering resilience? To answer this question, we establish two relations which intermingle the resilience, which is a macroscopic quantity defined for the dynamical system, and the fluctuations of the process, which are microscopic quantities. Analogous relations are well known in nonequilibrium statistical mechanics. To exploit these relations, we need to introduce several estimators which we control for times between $\log K$ (time scale to converge to the qsd) and $\exp (K)$ (time scale of mean time to extinction). We also provide variance estimates. Along the way, we prove moment estimates of independent interest for the process started either from an arbitrary state or from the qsd. We also obtain weak convergence of the rescaled qsd to a Gaussian measure.


[^0]Keywords: birth-and-death process; dynamical system; engineering resilience; quasi-stationary distribution; fluctuation-dissipation relation; empirical estimators.

## Contents

1 Introduction and main results ..... 3
1.1 Context ..... 3
1.2 Resilience and related statistics ..... 4
1.3 Organization of the paper ..... 7
2 Standing assumptions ..... 7
3 Time evolution of moments starting from anywhere ..... 9
4 Properties of the qsd ..... 12
4.1 Moments estimates for the qsd ..... 13
4.2 Gaussian limit for the rescaled qsd ..... 18
5 Controlling time averages ..... 21
6 Fluctuation and correlation relations ..... 27
6.1 Proof of Relation (1.5) ..... 27
6.2 Proof of Relation (1.6) ..... 28
7 Variance estimates of the statistics ..... 29
A Proof of the two variance estimates ..... 34
A. 1 Starting from anywhere: proof of Proposition 7.1 ..... 34
A. 2 Starting from the qsd: proof of Corollary 7.2 ..... 39
B Counting the number of births ..... 40
C A numerical example ..... 41

## 1 Introduction and main results

### 1.1 Context

The ability of an ecosystem to return to its reference state after a perturbation stress is given by its resilience, a concept pioneered by Holling. Resilience has several faces and multiple definitions [6]. In the traditional theoretical setting of dynamical systems, that is, coupled differential equations, one of them is the so-called engineering resilience. It is concerned with what happens in the vicinity of a fixed point (equilibrium state) of the system, and is given by minus the real part of the dominant eigenvalue of the Jacobian matrix evaluated at the fixed point. It can also be defined as the reciprocal of the characteristic return time to the fixed point after a (small) perturbation.

In this paper, we are interested in how one can determine the engineering resilience from the data. As a first step, we address this problem in the framework of birth-and-death processes. We consider a population made of $d$ species interacting with one another. If the state of the process $N^{K}(t)=$ $\left(N_{1}^{K}(t), \ldots, N_{d}^{K}(t)\right)$ at some time $t$ is $\underline{n}=\left(n_{1}, \ldots, n_{i} \ldots, n_{d}\right) \in \mathbb{Z}_{+}^{d}$, where $n_{i}$ is the number of individuals of the $i$ th species, then the rate at which the population increases (respectively decreases) by one individual of the $j$ th species is $K B_{j}(\underline{n} / K)$ (respectively $K D_{j}(\underline{n} / K)$ ), where $K$ is a scaling parameter. When $K$ tends to $+\infty$, the process renormalized by $K$ is well described by a dynamical system in any given finite-time window. More precisely, given any $0<\bar{t}<+\infty$ and any $\epsilon>0$, we have

$$
\lim _{K \rightarrow+\infty} \mathbb{P}\left(\sup _{0 \leq t \leq \bar{t}}\left\|\frac{N^{K}(t)}{K}-\underline{x}(t)\right\|>\epsilon\right)=0
$$

where $\underline{x}(t)$ is the solution of the differential equation in $\mathbb{R}_{+}^{d}$

$$
\begin{equation*}
\frac{\mathrm{d} \underline{x}}{\mathrm{~d} t}=\underline{B}(\underline{x})-\underline{D}(\underline{x}) \tag{1.1}
\end{equation*}
$$

with initial condition $\underline{x}_{0}=\underline{N}^{K}(0) / K$. See [4, Chapter 11] for a proof. We use the notations $\underline{x}=\left(x_{1}, \ldots, x_{d}\right), \underline{B}(\underline{x})=\left(B_{1}(\underline{x}), \ldots, B_{d}(\underline{x})\right)$, etc. We make further assumptions (see Section 2) on the birth and death rates ensuring that we are in the following situation. The vector field $\underline{B}-\underline{D}$ has a unique attracting fixed point $\underline{x}^{*}$ (lying in the interior of $\mathbb{R}_{+}^{d}$ ). For each $K$, the process $\underline{N}^{K}(t)$ goes almost surely to $\underline{0}$ (extinction), there is a unique stationary distribution which is the Dirac measure at $\underline{0}$, and there is a unique quasi-stationary distribution (qsd) $\nu_{K}$ which, roughly speaking, describes the statistics of the process conditioned not to be extinct.

In a previous paper [3], we proved a number of properties describing the behavior of $\underline{N}^{K}(t)$ when $K$ is large, but finite. We established a bound
for the total variation distance between the process conditioned to nonextinction before time $t$ and $\nu_{K}$. This bound is exponentially small in $t$ for $t \gg \log K$, and as a by-product, we obtained an estimate for the mean time to extinction in the qsd which behaves like $\exp (\mathcal{O}(1) K)$. We also quantified how close is the law of the process (not conditioned to non-extinction) either to the Dirac measure at $\underline{0}$ or $\nu_{K}$, for times much larger than $\log K$ and much smaller than the mean time to extinction. We will rely on these results that will be recalled along the way, and we will also prove some further properties.

### 1.2 Resilience and related statistics

Let

$$
\underline{X}=\underline{B}-\underline{D}
$$

be the vector field of the dynamical system (1.1), and $M^{*}$ its differential evaluated at the fixed point $\underline{x}^{*}$, namely

$$
\begin{equation*}
M^{*}=D \underline{X}\left(\underline{x}^{*}\right) \tag{1.2}
\end{equation*}
$$

We then define the (engineering) resilience as

$$
\begin{equation*}
\rho^{*}=-\sup \left\{\operatorname{Re}(z): z \in \operatorname{Sp}\left(M^{*}\right)\right\} \tag{1.3}
\end{equation*}
$$

where $\operatorname{Sp}\left(M^{*}\right)$ denotes the spectrum (set of eigenvalues) of the matrix $M^{*}$. Under our assumptions, we have $\rho^{*}>0$. Engineering resilience is useful for two major purposes:

1. It gives the exponential rate at which the system goes to the fixed point if the initial condition is taken nearby (the inverse of $\rho^{*}$ has dimension of time);
2. It allows the estimation of the change of the fixed point after a (small) perturbation of the system.

Our goal is to estimate the number $\rho^{*}$ from a long enough realization of the stochastic process $\left(\underline{N}^{K}(t), t \leq T\right)$, and for $K$ large, but finite. For this purpose, we will establish two matrix relations. The first one is a "fluctuation-dissipation relation". We need the following notations. Let $\mathcal{D}^{K}$ be the $d \times d$ diagonal matrix with entries

$$
\mathcal{D}_{p, p}^{K}=K B_{p}\left(\underline{x}^{*}\right)=K D_{p}\left(\underline{x}^{*}\right)
$$

and

$$
\Sigma_{p, q}^{K}=\int\left(n_{p}-\mu_{p}^{K}\right)\left(n_{q}-\mu_{q}^{K}\right) \mathrm{d} \nu_{K}(\underline{n})
$$

be the $d \times d$ covariance matrix around the vector $\underline{\mu}^{K}=\left(\mu_{1}^{K}, \ldots, \mu_{d}^{K}\right)$ of average populations sizes of the different species which is defined by

$$
\begin{equation*}
\mu_{p}^{K}=\int n_{p} \mathrm{~d} \nu_{K}(\underline{n}), p=1, \ldots, d \tag{1.4}
\end{equation*}
$$

We will show later on that $\underline{\mu}^{K}$ and $\Sigma^{K}$ are of order $K$.

Theorem 1.1. We have

$$
\begin{equation*}
M^{*} \Sigma^{K}+\Sigma^{K}{ }^{\mathrm{t}} M^{*}+2 \mathcal{D}^{K}=\mathcal{O}(\sqrt{K}) \tag{1.5}
\end{equation*}
$$

(The transpose of a matrix $M$ is denoted by ${ }^{\mathrm{t}} M$.) This relation will be proved in Section 6.1. For background on fluctuation-dissipation relations in Statistical Physics, we refer to $\left[8\right.$, sections 2-3]. Note that the matrix $\Sigma^{K}$ is symmetrical, but in general the matrix $M^{*}$ is not (see [3]). Note also that each term in the left hand side of (1.5) is of order $K$, as we will see below.

Remark 1.1. Equation (1.5) may be surprising since our process is not reversible. However, it is proved in [5] that a d-dimensional general OrnsteinUhlenbeck process $\mathrm{d} X=-A X \mathrm{~d} t+\sigma \mathrm{d} W_{t}$ has a unique stationary Gaussian measure whose covariance matrix $\Sigma$ satisfies

$$
A \Sigma+\Sigma^{\mathrm{t}} A=\sigma^{\mathrm{t}} \sigma
$$

provided $A$ is a stable matrix, that is, $\operatorname{Sp}(A) \subset\{\operatorname{Re}(z)<0\}$, which is not necessarily symmetric for $d>1$, and $\sigma$ is a nonsingular matrix.

If $\Sigma^{K}$ and $\mathcal{D}^{K}$ are known, then we have a Sylvester transpose equation whose solution is not unique, except for $d=1$ (see for example [9]). We can overcome this non-uniqueness issue by using another relation involving $M^{*}$. For $\tau \geq 0$, define

$$
\Sigma_{p, q}^{K}(\tau)=\mathbb{E}_{\nu_{K}}\left[\left(N_{p}^{K}(\tau)-\mu_{p}^{K}\right)\left(N_{q}^{K}(0)-\mu_{q}^{K}\right)\right] .
$$

Theorem 1.2. For all $\tau \geq 0$ we have

$$
\begin{equation*}
\Sigma^{K}(\tau)=\mathrm{e}^{\tau M^{*}} \Sigma^{K}(0)+\mathcal{O}(\sqrt{K}) \tag{1.6}
\end{equation*}
$$

This theorem will be proved in Section 6.2. Relation (1.6) allows to determine $M^{*}$. Indeed, we have

$$
\mathrm{e}^{\tau M^{*}}=\Sigma^{K}(\tau) \Sigma^{K}(0)^{-1}+O\left(\frac{1}{\sqrt{K}}\right)
$$

Obviously, $\Sigma^{K}(0)=\Sigma^{K}$, and as mentioned above, $\Sigma^{K}$ is of order $K$. Therefore the relevant range for $\tau$ is $\mathcal{O}(1) \leq \tau \ll \mathcal{O}(1) \log K$.

Remark $1.2(d=1)$. Where there is only one population, (1.5) easily gives the resilience since it becomes a scalar equation:

$$
\rho^{*}=\frac{K\left(B\left(\underline{x}^{*}\right)+D\left(\underline{x}^{*}\right)\right)}{2 \Sigma^{K}}+O\left(\frac{1}{\sqrt{K}}\right) .
$$

Note that $K\left(B\left(\underline{x}^{*}\right)+D\left(\underline{x}^{*}\right)\right)$ is the average total jump rate $K \nu_{K}(B(\underline{n} / K)+$ $D(\underline{n} / K))$ up to $O(1)$. This follows from a Taylor expansion of $B(\underline{n} / K)+$
$D(\underline{n} / K)$ around $\underline{x}^{*}$, Theorem 4.3 and Proposition 4.4.
Relation (1.6) gives another way to compute the resilience in this case, namely for all $\tau \geq 0$,

$$
\rho^{*}=-\frac{1}{\tau} \log \frac{\Sigma^{K}(\tau)}{\Sigma^{K}(0)}+O\left(\frac{1}{\sqrt{K}}\right) .
$$

Given a realization of $\left(\underline{N}^{K}(t), 0 \leq t \leq T\right)$ up to some time $T$, one can define several statistics, which are vectors or matrices, to estimate the above quantities.

Definition 1.1. For $T>0$ and $0<\tau<T, p=1, \ldots, d$, let
$S_{\bar{p}}^{\mu}(T, K)=\frac{1}{T} \int_{0}^{T} N_{p}^{K}(s) \mathrm{d} s$
$S_{p, q}^{\Sigma}(T, K)=\frac{1}{T} \int_{0}^{T}\left(N_{p}^{K}(s)-S_{p}^{\mu}(T, K)\right)\left(N_{q}^{K}(s)-S_{q}^{\mu}(T, K)\right) \mathrm{d} s$
$S_{p}^{\perp}(T, K)=\frac{1}{T} \#\{$ birth of species $p$ for $0 \leq t \leq T\}$
$S_{p, q}^{C}(T, \tau, K)=\frac{1}{T-\tau} \int_{0}^{T-\tau}\left(N_{p}^{K}(s+\tau)-S_{p}^{\mu}(T, K)\right)\left(N_{q}^{K}(s)-S_{q}^{\mu}(T, K)\right) \mathrm{d} s$.
The statistics $S \underline{\underline{\mu}}$ allow us to infer $\underline{\mu}^{K}$ under suitable conditions on $\underline{n}, K$ and $T$. Indeed, we will prove an estimate of the form

$$
\left|\mathbb{E}_{\underline{n}}\left[S_{\bar{p}}^{\mu}(T, K)\right]-\mu_{p}^{K}\right| \leq C\left(K+\|\underline{n}\|_{1}\right)\left(\frac{1+\log K}{T}+\mathrm{e}^{-c\left(\|\underline{n}\|_{1} \wedge K\right)}+T \mathrm{e}^{-c^{\prime} K}\right)
$$

for every $\underline{n} \in \mathbb{Z}_{+}^{d}, p=1, \ldots, d$, where $C, c, c^{\prime}$ are positive constants, and $\|\underline{n}\|_{1}=\sum_{i=1}^{d} n_{i}$. Let us make some comments on this bound. For $K$ large, the right-hand side is small when $\log K \ll\|\underline{n}\|_{1} \leq \mathcal{O}(1) K, T \gg K \log K$, and $T \ll \exp (\mathcal{O}(1) K)$.
When $\log T \gg K$, we have $\mathbb{E}_{\underline{n}}\left[S_{\bar{p}}^{\mu}(T, K)\right] \approx 0$ because with high probability the process is absorbed at $\underline{0}$, hence $\left|\mathbb{E}_{\underline{n}}\left[S_{\bar{p}}^{\mu}(T, K)\right]-\mu_{p}^{K}\right|$ is not small. This is also the manifestation of the fact that the only stationary (ergodic) distribution is the Dirac measure at $\underline{0}$.

We will use $S^{\Sigma}$ to infer $\Sigma^{K}, S^{\mathcal{D}}$ to infer $\mathcal{D}^{K}$, and $S^{C}(\tau)$ to infer $\Sigma^{K}(\tau)$. Estimates similar as the one given above for $S_{\bar{p}}^{\mu}(T, K)$ will be proved in Section 7.

We can now define an empirical matrix $M_{\text {emp }}^{*}(T, \tau, K)$ by

$$
\mathrm{e}^{\tau M_{\text {emp }}^{*}(T, \tau, K)}=S^{C}(T, \tau, K) S^{\Sigma}(T, K)^{-1}
$$

and an empirical resilience by

$$
\rho_{\mathrm{emp}}^{*}(T, \tau, K)=-\sup \left\{\operatorname{Re}(z): z \in \operatorname{Sp}\left(M_{\mathrm{emp}}^{*}(T, \tau, K)\right)\right\}
$$

From the above results one can derive various statistical estimates for the difference between $\rho_{\text {emp }}^{*}(T, \tau, K)$ and $\rho^{*}$. We will prove (Propostion 7.7) that for $\tau$ of order one, $T \ll \exp (\mathcal{O}(1) K)$ and $K$ large enough, we have

$$
\left|\rho_{\mathrm{emp}}^{*}(T, \tau, K)-\rho^{*}\right| \leq \mathcal{O}(1)\left(\frac{K^{2}}{\sqrt{T}}+\frac{1}{\sqrt{K}}\right)
$$

with a probability larger than $1-1 / K$. In particular, if $T \gg K^{5}$, we have

$$
\left|\rho_{\mathrm{emp}}^{*}(T, \tau, K)-\rho^{*}\right| \leq \mathcal{O}(1) / \sqrt{K}
$$

### 1.3 Organization of the paper

In Section 2, we will give the precise hypotheses. In Section 3, we will study the time evolution of the moments of the process. In Section 4, we will prove detailed properties of the qsd, namely moment estimates and weak convergence of the rescaled qsd to a Gaussian measure. In Section 5, we will obtain control on the large time behavior of averages for the process. In Section 6, we will prove the relations (1.5) and (1.6). In Section 7, we will obtain variance estimates for estimators of the quantities appearing in Definition 1.1, starting in the qsd or from an initial condition of order $K$.

## 2 Standing assumptions

Two (regular) vector fields $\underline{B}(\underline{x})$ and $\underline{D}(\underline{x})$ are given in $\mathbb{R}_{+}^{d}$. We assume that their components have second partial derivatives which are polynomially bounded. Obviously, we suppose that $B_{j}(\underline{x}) \geq 0$ and $B_{j}(\underline{x}) \geq 0$ for all $j=1, \ldots, d$ and $\underline{x} \in \mathbb{Z}_{+}^{d}$. A dynamical system in $\mathbb{R}_{+}^{d}$ is defined by the vector field $\underline{X}(\underline{x})=\underline{B}(\underline{x})-\underline{D}(\underline{x})$, namely

$$
\frac{\mathrm{d} \underline{x}}{\mathrm{~d} t}=\underline{B}(\underline{x})-\underline{D}(\underline{x})=\underline{X}(\underline{x}) .
$$

For $\underline{x} \in \mathbb{R}_{+}^{d}$, we use the following standard norms:

$$
\|\underline{x}\|_{1}=\sum_{j=1}^{d} x_{j},\|\underline{x}\|_{2}=\left(\sum_{j=1}^{d} x_{j}^{2}\right)^{\frac{1}{2}}
$$

We now state our hypotheses.
H. 1 The vectors fields $\underline{B}$ and $\underline{D}$ vanish only at $\underline{0}$.
H. 2 There exists $\underline{x}^{*}$ belonging to the interior of $\mathbb{R}_{+}^{d}$ (fixed point of $\underline{X}$ ) such that

$$
\underline{B}\left(\underline{x}^{*}\right)-\underline{D}\left(\underline{x}^{*}\right)=\underline{X}\left(\underline{x}^{*}\right)=\underline{0} .
$$

H. 3 Attracting fixed point: there exit $\beta>0$ and $R>0$ such that $\left\|\underline{x}^{*}\right\|_{2}<$ $R$, and for all $\underline{x} \in \mathbb{R}_{+}^{d}$ with $\|\underline{x}\|_{2}<R$,

$$
\begin{equation*}
\left\langle\underline{X}(\underline{x}),\left(\underline{x}-\underline{x}^{*}\right)\right\rangle \leq-\beta\|\underline{x}\|_{2}\left\|\underline{x}-\underline{x}^{*}\right\|_{2}^{2} . \tag{2.1}
\end{equation*}
$$

H. 4 The fixed point $\underline{0}$ of the vector field $\underline{X}$ is repelling (locally unstable). Moreover, on the boundary of $\mathbb{R}_{+}^{d}$, the vector field $\underline{X}$ points toward the interior (except at $\underline{0}$ ).
H. 5 Define

$$
\widehat{B}(y)=\sup _{\|\underline{x}\|_{1}=y} \sum_{j=1}^{d} B_{j}(\underline{x}), \widehat{D}(y)=\inf _{\|\underline{x}\|_{1}=y} \sum_{j=1}^{d} D_{j}(\underline{x})
$$

and for $y>0$, let

$$
F(y)=\frac{\widehat{B}(y)}{\widehat{D}(y)}
$$

We assume that there exists $0<L<R$ such that $\sup _{y>L} F(y)<1$ and $\lim _{y \rightarrow \infty} F(y)=0$.
H. 6 There exists $y_{0}>0$ such that $\int_{y_{0}}^{\infty} \frac{1}{\widehat{D}(y)} \mathrm{d} y<+\infty$ and $y \mapsto \widehat{D}(y)$ is increasing on $\left[y_{0},+\infty[\right.$.
H. 7 There exists $\xi>0$ such that

$$
\begin{equation*}
\inf _{\underline{x} \in \mathbb{R}_{+}^{d}} \inf _{1 \leq j \leq d} \frac{D_{j}(\underline{x})}{\sup _{1 \leq \ell \leq d} x_{\ell}}>\xi>0 \tag{H7}
\end{equation*}
$$

H. 8 Finally, we assume that

$$
\begin{equation*}
\inf _{1 \leq j \leq d} \partial_{x_{j}} B_{j}(\underline{0})>0 \tag{H8}
\end{equation*}
$$

(By $\partial_{x_{j}}$ we mean $\frac{\partial}{\partial_{x_{j}}}$.)
Assumptions H. 5 and H. 6 ensure that the time for "coming down from infinity" for the deterministic dynamical system is finite. Together with H.3, this also implies that $\underline{x}^{*}$ is a globally attracting stable fixed point. More comments on these assumptions can be found in [3]. For the interested reader, the arXiv version ${ }^{1}$ of this paper contains a supplementary section in which we study a concrete example of competition between two species for which the above assumptions hold.

[^1]
## 3 Time evolution of moments starting from anywhere

The generator $\mathscr{L}_{K}$ of the birth and death process $\underline{N}^{K}=\left(\underline{N}^{K}(t), t \geq 0\right)$ is defined by

$$
\begin{align*}
& \mathscr{L}_{K} f(\underline{n})=  \tag{3.1}\\
& K \sum_{\ell=1}^{d} B_{\ell}\left(\frac{n}{K}\right)\left(f\left(\underline{n}+\underline{e}^{(\ell)}\right)-f(\underline{n})\right)+K \sum_{\ell=1}^{d} D_{\ell}\left(\frac{n}{K}\right)\left(f\left(\underline{n}-\underline{e}^{(\ell)}\right)-f(\underline{n})\right)
\end{align*}
$$

where $\underline{e}^{(\ell)}=(0, \ldots, 0,1,0, \ldots, 0)$, the 1 being at the $\ell$-th position, and $f: \mathbb{Z}_{+}^{d} \rightarrow \mathbb{R}$ is a function with bounded support. We denote by $\left(S_{t}^{K}, t \geq 0\right)$ the semigroup of the process $\underline{N}^{K}$ acting on bounded functions, that is, for $f: \mathbb{Z}_{+}^{d} \rightarrow \mathbb{R}$, we have

$$
S_{t}^{K} f(\underline{n})=\mathbb{E}\left[f\left(\underline{N}^{K}(t)\right) \mid \underline{N}^{K}(0)=\underline{n}\right]=\mathbb{E}_{\underline{n}}\left[f\left(\underline{N}^{K}(t)\right)\right] .
$$

For $A>1$, let

$$
\begin{equation*}
\mathcal{T}_{A}=\inf \left\{t>0:\left\|\underline{N}^{K}(t)\right\|_{1}>A\right\} . \tag{3.2}
\end{equation*}
$$

Notice that we will use either $\|\cdot\|_{1}$ or $\|\cdot\|_{2}$ (which are of course equivalent). We have the following result.

Theorem 3.1. There exists a constant $C_{(3.1)}>0$ such that for $K$ large enough, the operator group $S_{1}^{K}$ extends to exponentially bounded functions and

$$
\sup _{\underline{n} \in \mathbb{Z}_{+}^{d}} S_{1}^{K}\left(\mathrm{e}^{\|\cdot\|_{1}}\right)(\underline{n}) \leq \mathrm{e}^{C_{(3.1)} K}
$$

Proof. Let us first introduce the function $G_{K}$ defined on $\left[y_{0},+\infty\right)$ by

$$
G_{K}(y)=\int_{y}^{\infty} \frac{\mathrm{d} z}{\widehat{D}(z)}+\frac{1}{K \widehat{D}(y)}
$$

Assumption H. 6 implies that $G_{K}$ is well defined and decreasing. We can define its inverse function on $\left(0, s_{0}\right]$ for $s_{0}>0$ small enough (independent of $K)$. Take $0<\eta \leq s_{0} \wedge \frac{1-\mathrm{e}^{-1}}{4}$. Then there is a unique positive function $y_{K}$ defined by

$$
\begin{equation*}
y_{K}(s)=G_{K}^{-1}(\eta s), s \in(0,1] . \tag{3.3}
\end{equation*}
$$

Note that $y_{K}(s) \geq y_{0}$ and $\lim _{s \downarrow 0} y_{K}(s)=+\infty$. Let

$$
\varphi_{K}(s)=\frac{\mathrm{e}^{-K y_{K}(s)}}{K \widehat{D}\left(y_{K}(s)\right)}
$$

Note that

$$
\lim _{s \downarrow 0} \varphi_{K}(s)=0
$$

Using the Lipschitz continuity of $\widehat{D}$ (and then its differentiability almost everywhere) and (3.3), we obtain
$\dot{\varphi_{K}}(s)=\frac{\mathrm{d} \varphi_{K}}{\mathrm{~d} s}(s)=-\left(\frac{\mathrm{e}^{-K y_{K}(s)}}{\widehat{D}\left(y_{K}(s)\right)}+\frac{\mathrm{e}^{-K y_{K}(s)} \widehat{D}^{\prime}\left(y_{K}(s)\right)}{K \widehat{D}\left(y_{K}(s)\right)^{2}}\right) \frac{\mathrm{d} y_{K}}{\mathrm{~d} s}(s)=\eta \mathrm{e}^{-K y_{K}(s)}$.
We now consider the function

$$
f_{K}(t, \underline{n})=\varphi_{K}(t) \mathrm{e}^{\|\underline{n}\|_{1}}
$$

to which we apply Itô's formula to $f_{K}$ at time $t \wedge \mathcal{T}_{A}$. We get

$$
\mathbb{E}_{\underline{n}}\left[\varphi_{K}\left(t \wedge \mathcal{T}_{A}\right) \mathrm{e}^{\left\|\underline{N}^{K}\left(t \wedge \mathcal{T}_{A}\right)\right\|_{1}}\right]=\mathbb{E}_{\underline{n}}\left[\int_{0}^{t \wedge \mathcal{T}_{A}}\left(\partial_{t} f_{K}+\mathscr{L}_{K} f_{K}\right)\left(s, \underline{N}^{K}(s)\right) \mathrm{d} s\right]
$$

We have

$$
\begin{aligned}
& \partial_{t} f_{K}(t, \underline{n})+\mathscr{L}_{K} f_{K}(t, \underline{n})=\dot{\varphi}_{K}(t) \mathrm{e}^{\|\underline{n}\|_{1}} \\
& \quad+K \varphi_{K}(t) \mathrm{e}^{\|\underline{n}\|_{1}}\left((\mathrm{e}-1) \sum_{\ell=1}^{d} B_{\ell}\left(\frac{\underline{n}}{K}\right)+\left(\mathrm{e}^{-1}-1\right) \sum_{\ell=1}^{d} D_{\ell}\left(\frac{n}{K}\right)\right) .
\end{aligned}
$$

Note that

$$
\begin{aligned}
& \partial_{t} f_{K}(t, \underline{n})+\mathscr{L}_{K} f_{K}(t, \underline{n}) \\
& \leq \mathrm{e}^{\|\underline{n}\|_{1}}\left(\dot{\varphi}_{K}(t)+K \varphi_{K}(t)\left((\mathrm{e}-1) \widehat{B}\left(\frac{\|\underline{n}\|_{1}}{K}\right)-\left(1-\mathrm{e}^{-1}\right) \widehat{D}\left(\frac{\|\underline{n}\|_{1}}{K}\right)\right)\right) \\
& \leq \mathrm{e}^{\|\underline{n}\|_{1}}\left(\dot{\varphi}_{K}(t)-K \varphi_{K}(t)\left(1-\mathrm{e}^{-1}\right) \widehat{D}\left(\frac{\|\underline{n}\|_{1}}{K}\right)\left(1-\mathrm{e} F\left(\frac{\|\underline{n}\|_{1}}{K}\right)\right)\right) .
\end{aligned}
$$

It follows from H. 5 that there exists a number $\zeta>y_{0}$ such that if $y>\zeta$, then $F(y)<(2 \mathrm{e})^{-1}$.
If $\|\underline{n}\|_{1}<\zeta K$ we get

$$
\left|\partial_{t} f_{K}(t, \underline{n})+\mathscr{L}_{K} f_{K}(t, \underline{n})\right| \leq \mathcal{O}(1) \mathrm{e}^{\zeta K}\left(\dot{\varphi}_{K}(t)+K \varphi_{K}(t)\right)
$$

For $\|\underline{n}\|_{1} \geq K\left(\zeta \vee y_{K}(t)\right)$ we have

$$
\partial_{t} f_{K}(t, \underline{n})+\mathscr{L}_{K} f_{K}(t, \underline{n}) \leq 0
$$

since $\dot{\varphi}_{K}(t)=\eta K \widehat{D}\left(y_{K}(t)\right) \varphi_{K}(t)$ and $\widehat{D}\left(\|\underline{n}\|_{1} / K\right) \geq \widehat{D}\left(y_{K}(t)\right)$.
Finally, for $\zeta K \leq\|\underline{n}\|_{1}<K y_{K}(t)$ we get

$$
\left|\partial_{t} f_{K}(t, \underline{n})+\mathscr{L}_{K} f_{K}(t, \underline{n})\right| \leq \mathrm{e}^{K y_{K}(t)} \dot{\varphi}_{K}(t)=\eta
$$

We deduce that

$$
\mathbb{E}_{\underline{n}}\left[\varphi_{K}\left(1 \wedge \mathcal{T}_{A}\right) \mathrm{e}^{\left\|\underline{N}^{K}\left(1 \wedge \mathcal{T}_{A}\right)\right\|_{1}}\right] \leq \mathcal{O}(1) \mathrm{e}^{\zeta K}
$$

The result follows by letting $A$ tend to infinity by monotonicity.

We deduce moment estimates for the process which are uniform in the starting state, and in time, for times larger than 1.

Corollary 3.2. For all $t \geq 1$, the semi-group $S_{t}$ maps functions of polynomially bounded modulus in bounded functions. In particular, for all $q \in \mathbb{N}$, we have

$$
\begin{equation*}
\sup _{t \geq 1} \sup _{\underline{n} \in \mathbb{Z}_{+}^{d}} \mathbb{E}_{\underline{n}}\left[\left\|\underline{N}^{K}(t)\right\|_{1}^{q}\right] \leq q^{q} \mathrm{e}^{-q} K^{q} \mathrm{e}^{C_{(3,1)}} . \tag{3.4}
\end{equation*}
$$

Proof. We have

$$
\begin{aligned}
\mathbb{E}_{\underline{n}}\left[\left\|\underline{N}^{K}(1)\right\|_{1}^{q}\right] & =K^{q} \mathbb{E}_{\underline{n}}\left[\frac{\left\|\underline{N}^{K}(1)\right\|_{1}^{q}}{K^{q}} \mathrm{e}^{-\frac{\left\|N^{K}(1)\right\|_{1}}{K}} \mathrm{e}^{\frac{\left\|\underline{N}^{K}(1)\right\|_{1}}{K}}\right] \\
& \leq K^{q} q^{q} \mathrm{e}^{-q} \mathbb{E}_{\underline{n}}\left[\mathrm{e}^{\frac{\left\|N^{K}(1)\right\|_{1}}{K}}\right]
\end{aligned}
$$

since for all $x \geq 0, x^{q} \mathrm{e}^{-x} \leq q^{q} \mathrm{e}^{-q}$. Inequality (3.4) follows from Hölder's inequality and Theorem 3.1.

Let us now consider $t>1$. From the Markov property and by using the previous inequality, we deduce that

$$
\mathbb{E}_{\underline{n}}\left[\left\|\underline{N}^{K}(t)\right\|_{1}^{q}\right]=\mathbb{E}_{\underline{n}}\left[\mathbb{E}_{\underline{N}^{K}(t-1)}\left[\left\|\underline{N}^{K}(1)\right\|_{1}^{q}\right]\right] \leq q^{q} \mathrm{e}^{-q} K^{q} \mathrm{e}^{C_{(3.1)}} .
$$

The proof is finished.
For time $t$ less than 1, the moment estimates will depend on the initial state.

Proposition 3.3. For each integer $q$, there exists a constant $c_{q}>0$ such that for all $K>1, t \geq 0$ and $\underline{n} \in \mathbb{Z}_{+}^{d}$

$$
\mathbb{E}_{\underline{n}}\left[\left\|\underline{N}^{K}(t)\right\|_{2}^{q}\right] \leq c_{q} K^{q}+\|\underline{n}\|_{2}^{q} \mathbb{1}_{\{t<1\}} .
$$

Proof. We have only to study the case $t<1$, the other case being given in (3.4). We prove the result for $q$ even, namely $q=2 q^{\prime}$. The general for $q$ odd follows from Cauchy-Schwarz inequality. Letting

$$
f_{q^{\prime}}(\underline{n})=\|\underline{n}\|_{2}^{2 q^{\prime}}
$$

we have

$$
\begin{aligned}
\mathscr{L}_{K} f_{q^{\prime}}(\underline{n})= & \left.K \sum_{\ell=1}^{d} B_{\ell}\left(\frac{\underline{n}}{K}\right)\left(\left(\|\underline{n}\|_{2}^{2}+2 \underline{n}_{\ell}+1\right)^{q^{\prime}}-\|\underline{n}\|_{2}^{2 q^{\prime}}\right)\right) \\
& \left.+K \sum_{\ell=1}^{d} D_{\ell}\left(\frac{\underline{n}}{K}\right)\left(\left(\|\underline{n}\|_{2}^{2}-2 \underline{n}_{\ell}+1\right)^{q^{\prime}}-\| \underline{n}_{2}^{2_{2}^{\prime}}\right)\right) .
\end{aligned}
$$

Using H. 5 and the equivalence of norms, we see that there exists a constant $c_{q^{\prime}}>0$ such that if $\|\underline{n}\|_{2}>c_{q^{\prime}} K$

$$
\mathscr{L}_{K} f_{q^{\prime}}(\underline{n})<0 .
$$

We can moreover choose $c_{q^{\prime}}$ larger such that for all $\underline{n}$

$$
\mathscr{L}_{K} f_{q^{\prime}}(\underline{n}) \leq c_{q^{\prime}} K^{2 q^{\prime}}
$$

Applying Itô's formula to $f_{q^{\prime}}$ we get as in the proof of Theorem 3.1

$$
\begin{aligned}
\mathbb{E}_{\underline{n}}\left[\left\|\underline{N}^{K}\left(t \wedge \mathcal{T}_{A}\right)\right\|_{2}^{2 q^{\prime}}\right] & \leq\|\underline{n}\|_{2}^{2 q^{\prime}}+\mathbb{E}_{\underline{n}}\left[\int_{0}^{t \wedge \mathcal{T}_{A}} c_{q^{\prime}} K^{2 q^{\prime}} \mathrm{d} s\right] \\
& \leq\|\underline{n}\|_{2}^{2 q^{\prime}}+t c_{q^{\prime}} K^{2 q^{\prime}}
\end{aligned}
$$

(Recall that $\mathcal{T}_{A}$ is defined in (3.2).) The result follows by letting $A$ tend to infinity.

## 4 Properties of the qsd

It is proved in [3] that, under the assumptions of Section 2, there exists a unique qsd $\nu_{K}$ with support $\mathbb{Z}_{+}^{d} \backslash\{\underline{0}\}$. Recall that starting from the qsd, the extinction time is distributed according to an exponential law with parameter $\lambda_{0}(K)$ satisfying (cf Theorem 3.2 in [3])

$$
\begin{equation*}
\mathrm{e}^{-d_{1} K} \leq \lambda_{0}(K) \leq \mathrm{e}^{-d_{2} K} \tag{4.1}
\end{equation*}
$$

where $d_{1}>d_{2}>0$ are constants independent of $K$. Recall also that for all $t>0$,

$$
\begin{equation*}
\mathbb{P}_{\nu_{K}}\left(\underline{N}^{K}(t) \in \cdot, T_{\underline{0}}>t\right)=\mathrm{e}^{-\lambda_{0}(K) t} \nu_{K}(\cdot) \tag{4.2}
\end{equation*}
$$

where

$$
T_{0}=\inf \left\{t>0: \underline{N}^{K}(t)=\underline{0}\right\}
$$

Finally, for all $f$ in the domain of the generator

$$
\begin{equation*}
\mathscr{L}_{K}^{\dagger} \nu_{K}(f)=\nu_{K}\left(\mathscr{L}_{K} f\right)=-\lambda_{0}(K) \nu_{K}(f) \tag{4.3}
\end{equation*}
$$

We use the notation

$$
\nu_{K}(f)=\int f(\underline{n}) \mathrm{d} \nu_{K}(\underline{n})
$$

### 4.1 Moments estimates for the qsd

We use several notations from [3]. Let

$$
\underline{n}^{*}=\left\lfloor K \underline{x}^{*}\right\rfloor .
$$

For $\underline{x} \in \mathbb{R}_{+}^{d}$ and $r>0, \mathcal{B}(\underline{x}, r)$ is the ball of center $\underline{x}$ and radius $r$. We consider the sets

$$
\begin{equation*}
\Delta=\mathcal{B}\left(\underline{n}^{*}, \rho \sqrt{K}\right), \mathcal{D}=\mathcal{B}\left(\underline{n}^{*}, \frac{\min _{j} n_{j}}{2}\right) \cap \mathbb{Z}_{+}^{d} \tag{4.4}
\end{equation*}
$$

where $\rho>0$ is a constant defined in [3, Corollary 4.2]. Note that since $\underline{n}^{*}$ is of order $K$, we have $\Delta \subset \mathcal{D}$ for $K$ large enough. The first entrance time in $\Delta($ resp. $\mathcal{D})$ will be denoted by $T_{\Delta}\left(\right.$ resp. $\left.T_{\mathcal{D}}\right)$.

Proposition 4.1. There exists a constant $c_{(4.1)}>0$ such that for all $K$ large enough

$$
\nu_{K}\left(\mathcal{D}^{c}\right) \leq \mathrm{e}^{-c_{(4.1)} K} .
$$

Proof. We first recall two results from [3].
From Lemma 5.1 in [3], there exist $\gamma>0$ and $\delta \in(0,1)$ such that for all $K$ large enough

$$
\begin{equation*}
\sup _{\underline{n} \in \Delta^{c} \backslash \underline{0}} \mathbb{P}_{\underline{n}}\left(T_{\Delta}>\gamma \log K, T_{\underline{0}}>T_{\Delta}\right) \leq \delta . \tag{4.5}
\end{equation*}
$$

By Sublemma 5.8 in [3], there exist two constants $C>0$ and $c>0$ such that for all $K$ large enough, and for all $t>0$

$$
\begin{equation*}
\sup _{\underline{n} \in \Delta} \mathbb{P}_{\underline{n}}\left(T_{\mathcal{D}^{c}}<t\right) \leq C(1+t) \mathrm{e}^{-c K} . \tag{4.6}
\end{equation*}
$$

For $q \in \mathbb{N} \backslash\{0\}$ define

$$
t_{q}=q \gamma \log K .
$$

We will first estimate $\sup _{\underline{n}} \mathbb{P}_{\underline{n}}\left(\underline{N}^{K}\left(t_{q}\right) \in \mathcal{D}^{c}, T_{0}>t_{q}\right)$.
Note that $\underline{N}^{K}\left(t_{q}\right) \in \mathcal{D}^{\bar{c}}$ implies $T_{\mathcal{D}^{c}} \leq t_{q}$. We distinguish two cases.

1. Let $\underline{n} \in \Delta$. It follows from (4.6) that

$$
\mathbb{P}_{\underline{n}}\left(\underline{N}^{K}\left(t_{q}\right) \in \mathcal{D}^{c}\right) \leq C\left(1+t_{q}\right) \mathrm{e}^{-c K} .
$$

2. Let $\underline{n} \in \Delta^{c} \backslash\{\underline{0}\}$. We have

$$
\begin{aligned}
& \mathbb{P}_{\underline{n}}\left(\underline{N}^{K}\left(t_{q}\right) \in \mathcal{D}^{c} \backslash\{\underline{0}\}\right)= \\
& \quad \mathbb{P}_{\underline{n}}\left(\underline{N}^{K}\left(t_{q}\right) \in \mathcal{D}^{c} \backslash\{\underline{0}\}, T_{\Delta} \leq t_{q}\right)+\mathbb{P}_{\underline{n}}\left(\underline{N}^{K}\left(t_{q}\right) \in \mathcal{D}^{c} \backslash\{\underline{0}\}, T_{\Delta}>t_{q}\right) .
\end{aligned}
$$

Using the strong Markov property at time $T_{\Delta}$ and (4.6) we obtain

$$
\begin{aligned}
& \mathbb{P}_{\underline{n}}\left(\underline{N}^{K}\left(t_{q}\right) \in \mathcal{D}^{c} \backslash\{\underline{0}\}, T_{\Delta} \leq t_{q}\right) \\
& \quad=\mathbb{E}_{\underline{n}}\left[\mathbb{1}_{\left\{T_{\Delta} \leq t_{q}\right\}} \mathbb{P}_{\underline{N}^{K}\left(T_{\Delta}\right)}\left(\underline{N}^{K}\left(t_{q}-T_{\Delta}\right) \in \mathcal{D}^{c} \backslash\{\underline{0}\}\right)\right] \\
& \quad \leq C\left(1+t_{q}\right) \mathrm{e}^{-c K} .
\end{aligned}
$$

We bound the second term recursively in $q$.

$$
\begin{aligned}
& \mathbb{P}_{\underline{n}}\left(T_{\Delta}>t_{q}, T_{\underline{0}}>T_{\Delta}\right) \\
& \quad=\mathbb{E}_{\underline{n}}\left[\mathbb{1}_{\left\{T_{\Delta}>t_{q-1}\right\}} \mathbb{1}_{\left\{T_{\underline{0}}>T_{\Delta}\right\}} \mathbb{P}_{\underline{N}^{K}\left(t_{q-1}\right)}\left(T_{\Delta}>t_{1}, T_{\underline{0}}>T_{\Delta}\right)\right] \\
& \quad \leq \delta \sup _{\underline{n} \in \Delta \Delta^{c} \backslash\{\underline{0}\}} \mathbb{P}_{\underline{n}}\left(T_{\Delta}>t_{q-1}, T_{\underline{0}}>T_{\Delta}\right)
\end{aligned}
$$

where we used the strong Markov property at time $t_{q-1}$ and (4.5). This implies

$$
\begin{aligned}
& \sup _{\underline{n} \in \Delta^{c} \backslash\{\{0\}} \mathbb{P}_{\underline{n}}\left(\underline{N}^{K}\left(t_{q}\right) \in \mathcal{D}^{c} \backslash\{\underline{0}\}, T_{\Delta}>t_{q}\right) \\
& \quad \leq \sup _{\underline{n} \in \Delta^{c} \backslash\{\underline{0}\}} \mathbb{P}_{\underline{n}}\left(T_{\Delta}>t_{q}, T_{\underline{0}}>T_{\Delta}\right) \leq \delta^{q} .
\end{aligned}
$$

Therefore

$$
\sup _{\underline{n} \neq \underline{0}} \mathbb{P}_{\underline{n}}\left(\underline{N}^{K}\left(t_{q}\right) \in \mathcal{D}^{c} \backslash\{\underline{0}\}\right) \leq C\left(1+t_{q}\right) \mathrm{e}^{-c K}+\delta^{q}
$$

By taking $q=[K]$ we conclude that there exists a constant $c^{\prime}>0$ such that for $K$ large enough

$$
\sup _{\underline{n} \neq \underline{0}} \mathbb{P}_{\underline{n}}\left(\underline{N}^{K}\left(t_{[K]}\right) \in \mathcal{D}^{c} \backslash\{\underline{0}\}\right) \leq \mathrm{e}^{-c^{\prime} K}
$$

This implies

$$
\mathbb{P}_{\nu_{K}}\left(\underline{N}^{K}\left(t_{[K]}\right) \in \mathcal{D}^{c}, T_{\underline{0}}>t_{[K]}\right) \leq \mathrm{e}^{-c^{\prime} K}
$$

but

$$
\mathbb{P}_{\nu_{K}}\left(\underline{N}^{K}\left(t_{[K]}\right) \in \mathcal{D}^{c}, T_{\underline{0}}>t_{[K]}\right)=\mathrm{e}^{-\lambda_{0}(K) t_{[K]}} \nu_{K}\left(\mathcal{D}^{c}\right)
$$

by (4.2) and the result follows from (4.1).
Corollary 4.2. For each $q \in \mathbb{N}$, there exists a constant $C_{q}$ such that for all $K$ large enough

$$
\int_{\mathcal{D}^{c}}\|\underline{n}\|_{1}^{q} \mathrm{~d} \nu_{K}(\underline{n}) \leq C_{q} K^{q} \mathrm{e}^{-c_{(4.1)} K} \quad \text { and } \quad \int\|\underline{n}\|_{1}^{q} \mathrm{~d} \nu_{K}(\underline{n}) \leq C_{q} K^{q} .
$$

Proof. It follows at once from (4.2) (at time 1) and Theorem 3.1 that

$$
\begin{equation*}
\int \mathrm{e}^{\|\underline{n}\|_{1}} \mathrm{~d} \nu_{K}(\underline{n}) \leq \mathrm{e}^{\lambda_{0}(K)} \mathrm{e}^{C_{(3,1)} K} \leq 2 \mathrm{e}^{C_{(3,1)} K} \tag{4.7}
\end{equation*}
$$

for $K$ large enough. We have

$$
\begin{aligned}
\int_{\mathcal{D}^{c}}\|\underline{n}\|_{1}^{q} \mathrm{~d} \nu_{K}(\underline{n}) & =K^{q} \int_{\mathcal{D}^{c}}\left(\frac{\|\underline{n}\|_{1}}{K}\right)^{q} \mathrm{e}^{-\frac{\|n\|_{1}}{K}} \mathrm{e}^{\frac{\|n\|_{1}}{K}} \mathrm{~d} \nu_{K}(\underline{n}) \\
& \leq K^{q} q^{q} \mathrm{e}^{-q} \int \mathrm{e}^{\frac{\|n\|_{1}}{K}} \mathbb{1}_{\mathcal{D}^{c}(\underline{n})} \mathrm{d} \nu_{K}(\underline{n}) .
\end{aligned}
$$

We use Hölder inequality to get

$$
\int_{\mathcal{D}^{c}}\|\underline{n}\|_{1}^{q} \mathrm{~d} \nu_{K}(\underline{n}) \leq K^{q} q^{q} \mathrm{e}^{-q}\left(\int \mathrm{e}^{\|\underline{n}\|_{1}} \mathrm{~d} \nu_{K}(\underline{n})\right)^{\frac{1}{K}}\left(\int \mathbb{1}_{\mathcal{D}^{c}(\underline{n})} \mathrm{d} \nu_{K}(\underline{n})\right)^{1-\frac{1}{K}} .
$$

The first result follows from (4.7) and Proposition 4.1. The second estimate follows from the first one and the bound $\sup _{\underline{n} \in \mathcal{D}}\|\underline{n}\|_{1} \leq \mathcal{O}(1) K$.

We now estimate centered moments.
Theorem 4.3. For each $q \in \mathbb{Z}_{+}$, there exists a constant $C_{q}>0$ such that for all $K$ large enough

$$
\int\left\|\underline{n}-K \underline{x}^{*}\right\|_{2}^{2 q} \mathrm{~d} \nu_{K}(\underline{n}) \leq C_{q} K^{q} .
$$

Proof. The proof is a recursion over $q$. The bound is trivial for $q=0$. For $q \in \mathbb{N}$ define the function

$$
f_{q}(\underline{n})=\left\|\underline{n}-K \underline{x}^{*}\right\|_{2}^{2 q} \mathbb{1}_{\mathcal{D}_{1}}(\underline{n})
$$

where

$$
\mathcal{D}_{1}=\mathcal{B}\left(K \underline{x}^{*}, \frac{2 K}{3} \min _{j} x_{j}^{*}\right) \cap \mathbb{Z}_{+}^{d} .
$$

Recall that $\underline{e}^{(j)}$ is the vector with 1 at the $j$ th coordinate and 0 elsewhere. From the trivial identity

$$
\begin{equation*}
\left\|\underline{n}-K \underline{x}^{*} \pm \underline{e}^{(j)}\right\|_{2}^{2}=\left\|\underline{n}-K \underline{x}^{*}\right\|_{2}^{2} \pm 2\left(n_{j}-K \underline{x}_{j}^{*}\right)+1 \tag{4.8}
\end{equation*}
$$

it follows that

$$
\begin{aligned}
& \left|\left\|\underline{n}-K \underline{x}^{*} \pm \underline{e}^{(j)}\right\|_{2}^{2 q}-\left\|\underline{n}-K \underline{x}^{*}\right\|_{2}^{2 q} \pm 2 q\left(n_{j}-K \underline{x}_{j}^{*}\right)\left\|\underline{n}-K \underline{x}^{*}\right\|_{2}^{2 q-2}\right| \\
& \quad \leq 3^{q} 2^{q}\left(1+\left\|\underline{n}-K \underline{x}^{*}\right\|_{2}^{2 q-2}\right) .
\end{aligned}
$$

Indeed, applying the trinomial expansion to (4.8), we obtain

$$
\begin{aligned}
& \left|\left\|\underline{n}-K \underline{x}^{*} \pm \underline{e}^{(j)}\right\|_{2}^{2 q}-\left\|\underline{n}-K \underline{x}^{*}\right\|_{2}^{2 q} \pm 2 q\left(n_{j}-K \underline{x}_{j}^{*}\right)\left\|\underline{n}-K \underline{x}^{*}\right\|_{2}^{2 q-2}\right| \\
& \leq q!\sum_{\substack{p_{1} \leq q-2 \\
p_{1}+p_{2}+p_{3}=q}} \frac{\left\|\underline{n}-K \underline{x}^{*}\right\|_{2}^{2 p_{1}}\left(2\left\|\underline{n}-K \underline{x}^{*}\right\|_{2}\right)^{p_{2}}}{p_{1}!p_{2}!p_{3}!}+q\left\|\underline{n}-K \underline{x}^{*}\right\|_{2}^{2 q-2}
\end{aligned}
$$

Observe that if $p_{1} \leq q-2, p_{1}+p_{2}+p_{3}=q$ and $p_{1} \leq q-2$, then $2 p_{1}+p_{2}=$ $p_{1}+q-p_{3} \leq 2 q-2-p_{3} \leq 2 q-2$ since $p_{3} \geq 0$. This implies that

$$
\left\|\underline{n}-K \underline{x}^{*}\right\|_{2}^{2 p_{1}}\left(2\left\|\underline{n}-K \underline{x}^{*}\right\|_{2}\right)^{p_{2}} \leq 2^{q}\left(1+\left\|\underline{n}-K \underline{x}^{*}\right\|_{2}^{2 q-2}\right)
$$

It follows that

$$
\begin{equation*}
\mathscr{L}_{K} f_{q}(\underline{n})=2 q K \sum_{j=1}^{d} X_{j}\left(\frac{\underline{n}}{K}\right)\left(n_{j}-K x_{j}^{*}\right)\left\|\underline{n}-K \underline{x}^{*}\right\|_{2}^{2 q-2} \mathbb{1}_{\mathcal{D}_{1}}(\underline{n})+R_{q}(\underline{n}) \tag{4.9}
\end{equation*}
$$

where

$$
\begin{equation*}
\left|R_{q}(\underline{n})\right| \leq \mathcal{O}(1) K 6^{q}\left(1+\left\|\underline{n}-K \underline{x}^{*}\right\|_{2}^{2 q-2}\right) \mathbb{1}_{\mathcal{D}_{1}}(\underline{n})+\mathcal{O}(1) q K^{2 q+1} \mathbb{1}_{\mathcal{D}^{c}}(\underline{n}) \tag{4.10}
\end{equation*}
$$

where we used the fact that

$$
\sup _{j=1, \ldots, d}\left|\mathbb{1}_{\mathcal{D}_{1}}\left(\underline{n} \pm \underline{e}^{(j)}\right)-\mathbb{1}_{\mathcal{D}_{1}}(\underline{n})\right| \leq \mathbb{1}_{\mathcal{D}^{c}}(\underline{n})
$$

Using (2.1) we get

$$
\begin{gather*}
K \sum_{j=1}^{d} X_{j}\left(\frac{\underline{n}}{K}\right)\left(n_{j}-K x_{j}^{*}\right)\left\|\underline{n}-K \underline{x}^{*}\right\|_{2}^{2 q-2} \mathbb{1}_{\mathcal{D}_{1}}(\underline{n}) \\
\quad \leq-\beta^{\prime}\left\|\underline{n}-K \underline{x}^{*}\right\|_{2}^{2 q} \mathbb{1}_{\mathcal{D}_{1}}(\underline{n})=-\beta^{\prime} f_{q}(\underline{n}) \tag{4.11}
\end{gather*}
$$

where

$$
\beta^{\prime}=\frac{\beta}{3} \min _{j} x_{j}^{*}
$$

Integrating the equation (4.9) with respect to $\nu_{K}$ and using (4.3), (4.10), (4.11) and Proposition 4.1 we obtain

$$
\begin{aligned}
\left(2 q \beta^{\prime}\right. & \left.-\lambda_{0}(K)\right) \nu_{K}\left(f_{q}\right) \\
& \leq \mathcal{O}(1) K 6^{q}\left(1+\nu_{K}\left(f_{q-1}\right)\right)+\mathcal{O}(1) 6^{q} K^{2 q+1} \mathrm{e}^{-c_{(4.1)} K}
\end{aligned}
$$

Observing that $\nu_{K}\left(f_{0}\right) \leq 1$, it follows by recursion over $q$ that for each integer $q$ there exists $C_{q}^{\prime}>0$ such that for all $K$ large enough $\nu_{K}\left(f_{q}\right) \leq C_{q}^{\prime} K^{q}$. Finally we have

$$
\begin{aligned}
& \nu_{K}\left(\left\|\underline{n}-K \underline{x}^{*}\right\|_{2}^{2 q}\right)=\nu_{K}\left(f_{q}\right)+\nu_{K}\left(\left\|\underline{n}-K \underline{x}^{*}\right\|_{2}^{2 q-2} \mathbb{1}_{\mathcal{D}_{1}^{c}}(\underline{n})\right) \\
& \quad \leq \nu_{K}\left(f_{q}\right)+\nu_{K}\left(\left\|\underline{n}-K \underline{x}^{*}\right\|_{2}^{2 q-2} \mathbb{1}_{\mathcal{D}^{c}}(\underline{n})\right)
\end{aligned}
$$

The result follows using the previous estimate and Corollary 4.2.

The next result gives a more precise estimate for the average of $\underline{n}$ (instead of an error of order $\sqrt{K}$ ). (Recall that $\underline{\mu}^{K}$ is defined in (1.4).)

Proposition 4.4. We have

$$
\underline{\mu}^{K}-K \underline{x}^{*}=\mathcal{O}(1)
$$

where $\underline{\mu}^{K}$ is defined in (1.4). Moreover, since $\left\|\underline{n}^{*}-K \underline{x}^{*}\right\|_{2}=\mathcal{O}(1)$, we have

$$
\begin{equation*}
\underline{\mu}^{K}-\underline{n}^{*}=\mathcal{O}(1) . \tag{4.12}
\end{equation*}
$$

Proof. We define the functions

$$
g_{j}(\underline{n})=\left\langle\underline{n}-K \underline{x}^{*}, \underline{e}^{(j)}\right\rangle, 1 \leq j \leq d .
$$

By Taylor expansion and the polynomial bounds on $\underline{B}$ and $\underline{D}$ we get

$$
\begin{aligned}
& \mathscr{L}_{K} g_{j}(\underline{n})=K\left(B_{j}(\underline{n} / K)-D_{j}(\underline{n} / K)\right) \\
& =\sum_{m=1}^{d}\left(\partial_{m} B_{j}\left(\underline{x}^{*}\right)-\partial_{m} D_{j}\left(\underline{x}^{*}\right)\right) g_{m}(\underline{n}) \mathbb{1}_{\mathcal{D}}(\underline{n})+\mathcal{O}(1) \frac{\left\|\underline{n}-K \underline{x}^{*}\right\|_{2}^{2}}{K} \mathbb{1}_{\mathcal{D}}(\underline{n}) \\
& \quad+\mathcal{O}(1)\left(K^{p}+\|\underline{n}\|_{2}^{p}\right) \mathbb{1}_{\mathcal{D} c}(\underline{n})
\end{aligned}
$$

for some positive integer $p$ independent of $K$. Using Cauchy-Schwarz inequality, identity (4.3), Corollary 4.2 and Proposition 4.1 we get

$$
\int\left(1+\|\underline{n}\|_{2}^{p}\right) \mathbb{1}_{\mathcal{D}^{c}}(\underline{n}) \mathrm{d} \nu_{K}(\underline{n})=o(1) .
$$

From Proposition 4.1, Theorem 4.3 and (4.1) we get

$$
\sum_{m=1}^{d}\left(\partial_{m} B_{j}\left(\underline{x}^{*}\right)-\partial_{m} D_{j}\left(\underline{x}^{*}\right)\right) \nu_{K}\left(g_{m}\right)=\mathcal{O}(1) .
$$

The result follows from the invertibility of the $d \times d$ matrix $\left(\partial_{m} B_{j}\left(\underline{x}^{*}\right)-\right.$ $\partial_{m} D_{j}\left(\underline{x}^{*}\right)$ ) which follows from H.3. The other inequalities follow immediately.

Corollary 4.5. We have

$$
\left\|\Sigma^{K}\right\| \leq \int\left\|\underline{n}-\underline{\mu}^{K}\right\|_{2}^{2} \mathrm{~d} \nu_{K}(\underline{n})=\int\left\|\underline{n}-K \underline{x}^{*}\right\|_{2}^{2} \mathrm{~d} \nu_{K}(\underline{n})+\mathcal{O}(1) \leq \mathcal{O}(1) K
$$

Proof. Combine Proposition 4.4 and Theorem 4.3.
We now show that $\Sigma^{K}$ is indeed of order $K$.

Proposition 4.6. For $K$ large enough, the matrix $\Sigma^{K}$ satisfies

$$
\Sigma^{K} \geq \mathcal{O}(1) K \operatorname{Id}
$$

for the order among positive definite matrices, Id being the identity matrix. In particular,

$$
\int\left\|\underline{n}-\underline{\mu}^{K}\right\|_{2}^{2} \mathrm{~d} \nu_{K}(\underline{n}) \geq \mathcal{O}(1) K
$$

Proof. We denote by $\widetilde{\Sigma}^{K}$ the positive definite matrix

$$
\widetilde{\Sigma}_{p, q}^{K}=\int\left(n_{p}-n_{p}^{*}\right)\left(n_{q}-n_{q}^{*}\right) \mathrm{d} \nu_{K}(\underline{n})
$$

By (4.12) we have

$$
\begin{equation*}
\left\|\widetilde{\Sigma}^{K}-\Sigma^{K}\right\|=\mathcal{O}(1) \tag{4.13}
\end{equation*}
$$

Let $\underline{v}$ be a unit vector in $\mathbb{R}^{d}$. We have

$$
\left\langle\underline{v}, \widetilde{\Sigma}^{K} \underline{v}\right\rangle=\int\left\langle\underline{v},\left(\underline{n}-\underline{n}^{*}\right)\right\rangle^{2} \mathrm{~d} \nu_{K}(\underline{n}) \geq \int_{\Delta}\left\langle\underline{v},\left(\underline{n}-\underline{n}^{*}\right)\right\rangle^{2} \mathrm{~d} \nu_{K}(\underline{n}) .
$$

From Lemma 5.3 in [3] there exists a constant $c>0$ such that for all $K$ large enough and all $\underline{n} \in \Delta$,

$$
\nu_{K}(\{\underline{n}\}) \geq c U_{\Delta}(\{\underline{n}\})
$$

where $U_{\Delta}$ is the uniform distribution on $\Delta$. Therefore

$$
\left\langle\underline{v}, \widetilde{\Sigma}^{K} \underline{v}\right\rangle \geq c \int_{\Delta}\left\langle\underline{v},\left(\underline{n}-\underline{n}^{*}\right)\right\rangle^{2} \mathrm{~d} U_{\Delta}(\underline{n})
$$

and we get

$$
\left\langle\underline{v}, \tilde{\Sigma}^{K} \underline{v}\right\rangle \geq \mathcal{O}(1) K\|\underline{v}\|_{2}^{2}
$$

The result follows.

### 4.2 Gaussian limit for the rescaled qsd

Recall that $\underline{n}^{*}=\left\lfloor K \underline{x}^{*}\right\rfloor$. We have the following theorem which partially generalizes a result obtained in [2] for models involving a single species ( $d=1$ ).

Theorem 4.7. For all $K>1$, define the measure $\mathfrak{a}_{K}$ on the Borel $\sigma$-algebra of $\mathbb{R}^{d}$ by

$$
\mathfrak{a}_{K}(\cdot)=\nu_{K}\left(\left\{\underline{n} \in \mathbb{Z}_{+}^{d}: \frac{\underline{n}-\underline{n}^{*}}{\sqrt{K}} \in \cdot\right\}\right)
$$

Then $\left(\mathfrak{a}_{K}\right)_{K}$ converges weakly to the centered Gaussian measure with covariance matrix

$$
\mathcal{S}=\int_{0}^{\infty} \mathrm{e}^{\tau M^{*}} \mathcal{B}^{*} \mathrm{e}^{\tau^{\mathrm{t}} M^{*}} \mathrm{~d} \tau
$$

where $\mathcal{B}^{*}$ is the diagonal matrix with entries $B_{\ell}\left(\underline{x}^{*}\right)=D_{\ell}\left(\underline{x}^{*}\right)$. The matrix $\mathcal{S}$ is also the unique symmetric solution of the (Lyapunov) equation (fluctuation-dissipation relation)

$$
\begin{equation*}
M^{*} \mathcal{S}+\mathcal{S}^{\mathrm{t}} M^{*}=-\mathcal{B}^{*} . \tag{4.14}
\end{equation*}
$$

Remark 4.1. We have

$$
\lim _{K \rightarrow+\infty} \frac{\Sigma^{K}}{K}=\delta
$$

This follows by dividing out equation (1.5) by $K$, letting $K$ tend to infinity, and using the uniqueness of the (symmetric) solution of (4.14).

Proof. By Theorem 4.3, the family of measures $\left(\mathfrak{a}_{K}\right)_{K}$ is tight. For $\underline{p} \in \mathbb{R}^{d}$ define

$$
H_{K}(\underline{p})=\int \mathrm{e}^{\mathrm{i} \frac{\left\langle\underline{p},\left(\underline{n}-\underline{n}^{*}\right)\right\rangle}{\sqrt{K}}} \mathrm{~d} \nu_{K}(\underline{n}) .
$$

It follows also from Theorem 4.3 that the family of functions $\left(H_{K}\right)$ is uniformly bounded in $C^{2}$. We will prove that for all $\underline{p} \in \mathbb{R}^{d}$,

$$
\begin{equation*}
\lim _{K \rightarrow \infty} H_{K}(\underline{p})=\mathrm{e}^{-\langle\underline{p}, \underline{s}\rangle} \tag{4.15}
\end{equation*}
$$

This will prove that there is only one weak accumulation point for $\left(\mathfrak{a}_{K}\right)_{K}$. The proof will be the consequence of Prokhorov Theorem [1]. We now prove (4.15). Using (4.3) and (4.1), we have

$$
\lim _{K \rightarrow \infty} \nu_{K}\left(\mathscr{L}_{K} \mathrm{e}^{\mathrm{i} \frac{\left\langle\underline{p},\left(-\underline{n}^{*}\right)\right\rangle}{\sqrt{K}}}\right)=0 .
$$

We also have

$$
\begin{aligned}
\nu_{K}\left(\mathscr{L}_{K} \mathrm{e}^{\mathrm{i} \frac{\left\langle\underline{p},\left(\cdot-\underline{\underline{n}}^{*}\right)\right\rangle}{\sqrt{K}}}\right)=K \sum_{\ell=1}^{d} \int \mathrm{~d} \nu_{K}(\underline{n}) \mathrm{e}^{\mathrm{i} \frac{\left\langle\underline{p},\left(\underline{n}-\underline{n}^{*}\right)\right\rangle}{\sqrt{K}}} \\
\quad \times\left(B_{\ell}\left(\frac{n}{K}\right)\left(\mathrm{e}^{\mathrm{i} \frac{p_{\ell}}{\sqrt{K}}}-1\right)+D_{\ell}\left(\frac{n}{K}\right)\left(\mathrm{e}^{-\mathrm{i} \frac{p_{\ell}}{\sqrt{K}}}-1\right)\right) .
\end{aligned}
$$

We use Taylor expansion and using the moments estimates and the polyno-
mial bounds on $B_{\ell}$ and $D_{\ell}\left(\right.$ and $\left.B_{\ell}\left(\underline{x}^{*}\right)=D_{\ell}\left(\underline{x}^{*}\right)\right)$ we obtain

$$
\begin{aligned}
& \nu_{K}\left(\mathscr{L}_{K} \mathrm{e}^{\mathrm{i} \frac{\left\langle\underline{p},\left(\underline{\left.\left.-\underline{n}^{*}\right)\right\rangle}\right.\right.}{\sqrt{K}}}\right) \\
&=- \sum_{\ell=1}^{d} B_{\ell}\left(\frac{\underline{n}^{*}}{K}\right) p_{\ell}^{2} H_{K}(\underline{p})+\mathrm{i} \sum_{\ell=1}^{d} p_{\ell} \sum_{j=1}^{d}\left(\partial_{j} B_{\ell}\left(\frac{\underline{n}^{*}}{K}\right)-\partial_{j} D_{\ell}\left(\frac{\underline{n}^{*}}{K}\right)\right) \\
& \times \int \mathrm{e}^{\mathrm{i} \frac{1}{\sqrt{K}}\left\langle\underline{p}\left(\underline{n}-\underline{n}^{*}\right)\right\rangle} \frac{n_{j}^{*}-n_{j}^{*}}{\sqrt{K}} \mathrm{~d} \nu_{K}(\underline{n})+\mathcal{O}\left(\frac{1}{\sqrt{K}}\right) \\
&=- \sum_{\ell=1}^{d} B_{\ell}\left(\frac{n^{*}}{K}\right) p_{\ell}^{2} H_{K}(\underline{p}) \\
&+\sum_{\ell=1}^{d} p_{\ell} \sum_{j=1}^{d}\left(\partial_{j} B_{\ell}\left(\frac{n^{*}}{K}\right)-\partial_{j} D_{\ell}\left(\frac{\underline{n}^{*}}{K}\right)\right) \partial_{p_{j}} H_{K}(\underline{p})+\mathcal{O}\left(\frac{1}{\sqrt{K}}\right) \\
&=- \sum_{\ell=1}^{d} B_{\ell}\left(\underline{x}^{*}\right) p_{\ell}^{2} H_{K}(\underline{p})+\sum_{\ell=1}^{d} p_{\ell} \sum_{j=1}^{d} M_{\ell, j}^{*} \partial_{p_{j}} H_{K}(\underline{p})+\mathcal{O}\left(\frac{1}{\sqrt{K}}\right) .
\end{aligned}
$$

We conclude that every accumulation point $\breve{H}$ of $\left(H_{K}\right)_{K}$ is bounded in $C^{1}$, satisfies $\breve{H}(\underline{0})=1$, and is a solution of the equation

$$
-\sum_{\ell=1}^{d} B_{\ell}\left(\underline{x}^{*}\right) p_{\ell}^{2} \breve{H}(\underline{p})+\sum_{\ell=1}^{d} p_{\ell} \sum_{j=1}^{d} M_{\ell, j}^{*} \partial_{p_{j}} \breve{H}(\underline{p})=0
$$

Then (4.15) follows from Lemma 4.8 (stated and proved right after this proof) with $A=M^{*}$.

Lemma 4.8. Let $\left(B_{j}\right)$ be $d$ strictly positive numbers and $A$ a real $d \times d$ matrix such that $\operatorname{Sp}(A) \subset\{z \in \mathbb{C}: \operatorname{Re}(z)<0\}$. Then there exists a unique $C^{1}\left(\mathbb{R}^{d}, \mathbb{R}\right)$ function $H$ satisfying $H(\underline{0})=1$ and

$$
\begin{equation*}
-\sum_{\ell=1}^{d} B_{\ell} p_{\ell}^{2} H(\underline{p})+\sum_{\ell=1}^{d} p_{\ell} \sum_{j=1}^{d} A_{\ell, j} \partial_{p_{j}} H(\underline{p})=0, \underline{p} \in \mathbb{R}^{d} \tag{4.16}
\end{equation*}
$$

This function is given by

$$
H(\underline{p})=\mathrm{e}^{-\langle\underline{p}, \underline{s} \underline{p}\rangle}
$$

where

$$
\mathcal{S}=\int_{0}^{\infty} \mathrm{e}^{\tau A} \mathcal{B} \mathrm{e}^{\tau^{\mathrm{t}} A} \mathrm{~d} \tau
$$

where $\mathcal{B}$ is the diagonal matrix with entries $\left(B_{j}\right)$. The matrix $S$ is also the unique symmetric solution of the equation

$$
A \mathcal{S}+\mathcal{S}^{\mathrm{t}} A=-\mathcal{B}
$$

Proof. We use the method of characteristics. For all $\underline{p} \in \mathbb{R}^{d}$, we define the function $\underline{p}(s), s \geq 0$ as the solution of

$$
\frac{\mathrm{d} \underline{p}}{\mathrm{~d} s}(s)=^{\mathrm{t}} A \underline{p}(s), \quad \underline{p}(0)=\underline{p} .
$$

Let

$$
\mathfrak{b}(s)=-\sum_{\ell=1}^{d} B_{\ell} \int_{0}^{s} p_{\ell}(\tau)^{2} \mathrm{~d} \tau .
$$

Let $H$ be a solution of (4.16). It is easy to check that for all $\underline{p} \in \mathbb{R}_{+}^{d}$ and $s \in \mathbb{R}$

$$
\frac{\mathrm{d}}{\mathrm{~d} s}\left(H(\underline{p}(s)) \mathrm{e}^{\mathfrak{b}(s)}\right)=0 .
$$

Integrating from 0 to $u$ yields

$$
H(\underline{p})=H(\underline{p}(u)) \mathrm{e}^{\mathfrak{b}(u)} .
$$

From the spectral properties of $A$ we get

$$
\lim _{u \rightarrow+\infty} H(\underline{p}(u))=H(\underline{0})=1 .
$$

Therefore

$$
H(\underline{p})=\mathrm{e}^{\mathfrak{b}(\infty)}
$$

and

$$
\mathfrak{b}(\infty)=-\int_{0}^{\infty}\left\langle\underline{p}, \mathrm{e}^{\tau A} \mathcal{B} \mathrm{e}^{\tau^{\mathrm{t} A}} \underline{p}\right\rangle \mathrm{d} \tau=-\langle\underline{p}, \delta \underline{p}\rangle .
$$

Finally we get from the spectral properties of $A$

$$
\begin{aligned}
A \mathcal{S}+\mathcal{S}^{\mathrm{t}} A & =\int_{0}^{\infty}\left(A \mathrm{e}^{\tau A} \mathcal{B} \mathrm{e}^{\tau^{\mathrm{t}} A}+\mathrm{e}^{\tau A} \mathcal{B} \mathrm{e}^{\tau^{\mathrm{t}} A} \mathrm{t} A\right) \mathrm{d} \tau \\
& =\int_{0}^{\infty} \frac{\mathrm{d}}{\mathrm{~d} \tau}\left(\mathrm{e}^{\tau A} \mathcal{B} \mathrm{e}^{\tau^{\mathrm{t} A}}\right) \mathrm{d} \tau=-\mathcal{B} .
\end{aligned}
$$

This finishes the proof of the lemma.

## 5 Controlling time averages

For $T>0$, we define the time average of a function $f: \mathbb{Z}_{+}^{d} \rightarrow \mathbb{R}$ by

$$
\begin{equation*}
S_{f}(T, K)=\frac{1}{T} \int_{0}^{T} f\left(\underline{N}^{K}(s)\right) \mathrm{d} s \tag{5.1}
\end{equation*}
$$

The goal of this section is to obtain a control of $\left|S_{f}(T, K)-\nu_{K}(f)\right|$ for a suitable class of functions.

We recall the following result from [3, Theorem 3.1].

Theorem 5.1 ([3]). There exist $a>0, K_{0}>1$ such that, for all $t \geq 0$ and for all $K \geq K_{0}$, we have

$$
\begin{equation*}
\sup _{\underline{n} \in \mathbb{Z}_{+}^{d} \backslash\{\underline{0}\}}\left\|\mathbb{P}_{\underline{n}}\left(\underline{N}^{K}(t) \in \cdot, t<T_{\underline{0}}\right)-\mathbb{P}_{\underline{n}}\left(t<T_{\underline{0}}\right) \nu_{K}(\cdot)\right\|_{\mathrm{TV}} \leq 2 \mathrm{e}^{-\frac{a t}{\log K}} \tag{5.2}
\end{equation*}
$$

It is also proved in [3] that, for a time much larger than $\log K$ and much smaller than the extinction time (which is of order $\exp (\mathcal{O}(1) K)$ ), the law of the process is close to the qsd with a positive probability. The accuracy of the approximation depends on the initial condition. This suggests to study the distance between the law of the process at time $s$ and the qsd as a function of the initial condition, $K$ and $s$. This will result from (5.2) if we can estimate $\mathbb{P}_{\underline{n}}\left(T_{\underline{0}} \leq t\right)$. In fact we prove a more general result.

Lemma 5.2. For $\gamma \geq 0$, define $\tau_{\gamma}=\inf \left\{t \geq 0:\left\|\underline{N}^{K}(t)\right\|_{1} \leq \gamma K\right\}$. There exist $\delta>0, \alpha>0$ and $C>0$ such that for all $\underline{n} \in \mathbb{Z}_{+}^{d}, K \geq 1,0 \leq \gamma \leq$ $1 \wedge \frac{\alpha}{\left\|\underline{x}^{*}\right\|_{1}}$ and $t \geq 0$, we have

$$
\begin{equation*}
\mathbb{P}_{\underline{n}}\left(\tau_{\gamma} \leq t\right) \leq C\left(\mathrm{e}^{-\delta\left(\zeta\left(\frac{\|\underline{n}\|_{1}}{K} \wedge \alpha\right)-\gamma\left\|\underline{x}^{*}\right\|_{1}\right) K}+t \mathrm{e}^{-\delta\left(\alpha-\gamma\left\|\underline{x}^{*}\right\|_{1}\right) K}\right) \tag{5.3}
\end{equation*}
$$

where

$$
\begin{equation*}
\zeta=\min _{1 \leq j \leq d} x_{j}^{*}>0 \tag{5.4}
\end{equation*}
$$

Taking $\gamma=0$ in (5.3), we get

$$
\begin{equation*}
\mathbb{P}_{\underline{n}}\left(T_{\underline{0}} \leq t\right) \leq C\left(\mathrm{e}^{-\delta\left(\zeta \frac{\|\underline{n}\|_{1}}{K} \wedge \alpha\right) K}+t \mathrm{e}^{-\alpha \delta K}\right) \tag{5.5}
\end{equation*}
$$

Proof. It follows from H. 1 and H. 3 (using Taylor's expansion of $\underline{X}(\underline{x})$ near $\underline{0}$ ) that there exists $\alpha_{0} \in(0, R)$ (where $R$ was introduced in Assumption H.3) such that for all $\underline{x} \in \mathbb{R}_{+}^{d}$ satisfying $\|\underline{x}\|_{2} \leq \alpha_{0}$ we have

$$
\begin{align*}
\left\langle\underline{X}(\underline{x}), \underline{x}^{*}\right\rangle & \geq \beta\left\|\underline{x}^{*}\right\|_{2}^{2}\|\underline{x}\|_{2}-2 \beta\|\underline{x}\|_{2}\left\langle\underline{x}, \underline{x}^{*}\right\rangle+\beta\|\underline{x}\|_{2}^{3}+\langle\underline{X}(\underline{x}), \underline{x}\rangle \\
& \geq \beta\left\|\underline{x}^{*}\right\|_{2}^{2}\|\underline{x}\|_{2}+\mathcal{O}(1)\|\underline{x}\|_{2}^{2} \geq \frac{\beta\left\|\underline{x}^{*}\right\|_{2}^{2}}{2}\|\underline{x}\|_{2} . \tag{5.6}
\end{align*}
$$

For $\alpha \in\left(0, \alpha_{0}\right]$ and $\delta>0$ to be chosen later on, we define

$$
\psi(\underline{n})=\mathrm{e}^{-\delta\left(\left\langle\underline{n}, \underline{x}^{*}\right\rangle \wedge \alpha K\right)} .
$$

It is easy to verify that if $\left\langle\underline{n}, \underline{x}^{*}\right\rangle>\alpha K+\left\|\underline{x}^{*}\right\|_{2}$ we have

$$
\mathscr{L}_{K} \psi=0 .
$$

If $\alpha K-\left\|\underline{x}^{*}\right\|_{2} \leq\left\langle\underline{n}, \underline{x}^{*}\right\rangle \leq \alpha K+\left\|\underline{x}^{*}\right\|_{2}$ we have

$$
\left|\mathscr{L}_{K} \psi\right| \leq \mathcal{O}(1) K \mathrm{e}^{-\alpha \delta K}
$$

For $\left\langle\underline{n}, \underline{x}^{*}\right\rangle \leq \alpha K-\left\|\underline{x}^{*}\right\|_{2}$, we have $\|\underline{n}\|_{1} \leq\left\langle\underline{n}, \underline{x}^{*}\right\rangle / \zeta \leq \alpha K / \zeta$, where $\zeta$ is defined in (5.4), and

$$
\mathscr{L}_{K} \psi(\underline{n})=K \mathfrak{g}\left(\delta, \frac{n}{K}\right) \mathrm{e}^{-\delta\left\langle\underline{n}, \underline{x}^{*}\right\rangle}
$$

where the function $\mathfrak{g}$ is defined by

$$
\mathfrak{g}(s, \underline{x})=\sum_{j=1}^{d} B_{j}(\underline{x})\left(\mathrm{e}^{-s x_{j}^{*}}-1\right)+\sum_{j=1}^{d} D_{j}(\underline{x})\left(\mathrm{e}^{s x_{j}^{*}}-1\right) .
$$

We have

$$
\begin{aligned}
\mathfrak{g}(s, \underline{x})=- & s \sum_{j=1}^{d}\left(B_{j}(\underline{x})-D_{j}(\underline{x})\right) x_{j}^{*} \\
& +\sum_{j=1}^{d} B_{j}(\underline{x})\left(\mathrm{e}^{-s x_{j}^{*}}-1+s x_{j}^{*}\right)+\sum_{j=1}^{d} D_{j}(\underline{x})\left(\mathrm{e}^{s x_{j}^{*}}-1-s x_{j}^{*}\right) .
\end{aligned}
$$

From the differentiability of the vector fields $\underline{B}$ and $\underline{D}$ and using (5.6), it follows that there exists a constant $\Gamma>0$ such that, for all $0 \leq s \leq 1$ and $\|\underline{x}\|_{2}<\alpha_{0}$ we have

$$
\begin{aligned}
\mathfrak{g}(s, \underline{x}) & =-s\left\langle\underline{X}(\underline{x}), \underline{x}^{*}\right\rangle+\mathcal{O}(1) s^{2}\|\underline{x}\|_{2} \\
& \leq-s \frac{\beta\left\|\underline{x}^{*}\right\|_{2}^{2}}{2}\|\underline{x}\|_{2}+\Gamma s^{2}\|\underline{x}\|_{2} .
\end{aligned}
$$

Therefore we can choose $\delta>0$ and $0<\alpha<\alpha_{0}$ such that

$$
\sup _{\|\underline{x}\|_{2} \leq \alpha} \mathfrak{g}(\delta, \underline{x})<0
$$

Therefore, for all $\underline{n}$

$$
\mathscr{L}_{K} \psi(\underline{n}) \leq \mathcal{O}(1) K \mathrm{e}^{-\alpha \delta K}
$$

For $\tilde{\gamma}>0$ (independent of $K$ ), we define

$$
\tilde{\tau}_{\tilde{\gamma}}=\inf \left\{t \geq 0:\left\langle\underline{N}^{K}(t), \underline{x}^{*}\right\rangle \leq \tilde{\gamma} K\right\}
$$

We apply Ito's formula to $\psi$ to get

$$
\mathbb{E}_{\underline{n}}\left[\psi\left(\underline{N}^{K}\left(t \wedge \tilde{\tau}_{\tilde{\gamma}}\right)\right)\right]=\psi(\underline{n})+\mathbb{E}_{\underline{n}}\left[\int_{0}^{t \wedge \tilde{\tau}_{\tilde{\gamma}}} \mathscr{L}_{K} \psi\left(\underline{N}^{K}(s)\right) \mathrm{d} s\right]
$$

We have

$$
\tilde{\gamma} K-\zeta \leq\left\langle\underline{N}^{K}\left(\tilde{\tau}_{\tilde{\gamma}}\right), \underline{x}^{*}\right\rangle \leq \tilde{\gamma} K
$$

hence

$$
\psi\left(\underline{N}^{K}\left(\widetilde{\tau}_{\tilde{\gamma}}\right)\right) \geq \mathrm{e}^{-\delta(\tilde{\gamma} \wedge \alpha) K} \mathrm{e}^{-\delta \zeta}
$$

We have

$$
\mathbb{E}_{\underline{n}}\left[\psi\left(\underline{N}^{K}\left(t \wedge \widetilde{\tau}_{\tilde{\gamma}}\right)\right)\right] \geq \mathbb{P}_{\underline{n}}\left(\tilde{\tau}_{\tilde{\gamma}} \leq t\right) \mathrm{e}^{-\delta(\tilde{\gamma} \wedge \alpha) K} \mathrm{e}^{-\delta \zeta}
$$

Therefore

$$
\mathbb{P}_{\underline{n}}\left(\tilde{\tau}_{\tilde{\gamma}} \leq t\right) \mathrm{e}^{-\delta(\tilde{\gamma} \wedge \alpha) K} \mathrm{e}^{-\delta \zeta} \leq \mathrm{e}^{-\delta\left(\left\langle n, \underline{x}^{*}\right\rangle \wedge \alpha K\right)}+t \mathcal{O}(1) K \mathrm{e}^{-\alpha \delta K}
$$

To conclude, observe that

$$
\mathbb{P}_{\underline{n}}\left(\tau_{\gamma} \leq t\right) \leq \mathbb{P}_{\underline{n}}\left(\tilde{\tau}_{\tilde{\gamma}} \leq t\right)
$$

for $\tilde{\gamma}=\gamma\left\|\underline{x}^{*}\right\|_{1}$ because for all $\underline{n} \in \mathbb{Z}_{+}^{d}$,

$$
0<\zeta\|\underline{n}\|_{1} \leq\left\langle\underline{n}, \underline{x}^{*}\right\rangle \leq\|\underline{n}\|_{1} \sup _{j=1, \ldots, d} x_{j}^{*} \leq\|\underline{n}\|_{1}\left\|\underline{x}^{*}\right\|_{1}
$$

and $\left\|\underline{N}^{K}\left(\tau_{\gamma}\right)\right\|_{1} \leq \gamma K$.
We have the following result.
Proposition 5.3. For all bounded functions $h: \mathbb{Z}_{+}^{d} \rightarrow \mathbb{R}, t \geq 0, \underline{n} \in \mathbb{Z}_{+}^{d}$, and $K>K_{0}$, we have

$$
\left|\mathbb{E}_{\underline{n}}\left[h\left(\underline{N}^{K}(t)\right)\right]-\nu_{K}(h)\right| \leq \mathcal{O}(1)\|h\|_{\infty}\left(\mathrm{e}^{-\delta\left(\zeta \frac{\|n\|_{1}}{K} \wedge \alpha\right) K}+t \mathrm{e}^{-\alpha \delta K}+\mathrm{e}^{-\frac{a t}{\log K}}\right)
$$

where $\alpha, \delta$ and $\zeta$ are defined in Lemma 5.2, and a and $K_{0}$ are defined in Theorem 5.1.

Proof. From the bound (5.2) we get

$$
\left|\mathbb{E}_{\underline{n}}\left[h\left(\underline{N}^{K}(t)\right) \mathbb{1}_{\left\{T_{\underline{0}}>t\right\}}\right]-\mathbb{P}_{\underline{n}}\left(t<T_{\underline{0}}\right) \nu_{K}(h)\right| \leq \mathcal{O}(1)\|h\|_{\infty} \mathrm{e}^{-\frac{a t}{\log K}} .
$$

This implies

$$
\begin{aligned}
\mid \mathbb{E}_{\underline{n}} & {\left[h\left(\underline{N}^{K}(t)\right)\right]-\nu_{K}(h) \mid } \\
& \leq\left|\mathbb{E}_{\underline{n}}\left[h\left(\underline{N}^{K}(t)\right) \mathbb{1}_{\left\{T_{\underline{0}} \leq s\right\}}\right]\right|+\mathbb{P}_{\underline{n}}\left(t \geq T_{\underline{0}}\right) \nu_{K}(h)+\mathcal{O}(1)\|h\|_{\infty} \mathrm{e}^{-\frac{a t}{\log K}} \\
& \leq \mathcal{O}(1)\|h\|_{\infty}\left(\mathbb{P}_{\underline{n}}\left(t \geq T_{\underline{0}}\right)+\mathrm{e}^{-\frac{a t}{\log K}}\right) \\
& \leq \mathcal{O}(1)\|h\|_{\infty}\left(\mathrm{e}^{-\delta\left(\zeta \frac{\|\underline{n}\|_{1}}{K} \wedge \alpha\right) K}+s \mathrm{e}^{-\alpha \delta K}+\mathrm{e}^{-\frac{a t}{\log K}}\right)
\end{aligned}
$$

using (5.5).
We now extend Proposition 5.3 to more general functions. For $q \in \mathbb{Z}_{+}$, we define the Banach space $\mathscr{F}_{K, q}$ by

$$
\begin{equation*}
\mathscr{F}_{K, q}=\left\{f: \mathbb{Z}_{+}^{d} \rightarrow \mathbb{R}:\|f\|_{K, q}:=\sup _{\underline{n} \neq \underline{0}} \frac{|f(\underline{n})|}{K^{q}+\|\underline{n}\|_{2}^{q}}<+\infty\right\} . \tag{5.7}
\end{equation*}
$$

We have the following result for time-averages of function in $\mathscr{F}_{K}$.

Theorem 5.4. For all $K>K_{0}, f \in \mathscr{F}_{K, q}, T>0$, and $\underline{n} \in \mathbb{Z}_{+}^{d}$, we have

$$
\begin{aligned}
& \left|\mathbb{E}_{\underline{n}}\left[S_{f}(T, K)\right]-\nu_{K}(f)\right| \leq \mathcal{O}(1)\|f\|_{K, q}\left(K^{q}+\|\underline{n}\|_{2}^{q}\right) \\
& \quad \times\left(\frac{1}{T}+\mathrm{e}^{-\delta\left(\zeta \frac{\|\underline{n}\|_{1}}{K} \wedge \alpha\right) K}+T \mathrm{e}^{-\alpha \delta K}+\frac{\log K}{a T}+\left(1-\mathrm{e}^{-\lambda_{0}(K)}\right)^{\frac{1}{2}}\right)
\end{aligned}
$$

where $\alpha, \delta$ and $\zeta$ are defined in Lemma 5.2, and $\lambda_{0}(K)$ is defined in (4.1).
Remark 5.1. One can check that if one modifies slightly the definition of the time average (5.1) by integrating from 1 to $T+1$, then one can remove the term $\|\underline{n}\|_{2}^{q}$ from the previous estimate.

Proof. For $f \in \mathscr{F}_{K, q}$, Corollary 4.2 gives

$$
\left|\nu_{K}(f)\right| \leq \mathcal{O}(1) K^{q}\|f\|_{K, q} .
$$

By Proposition 3.3 we have

$$
\left|\frac{1}{T} \mathbb{E}_{\underline{n}}\left[\int_{0}^{1 \wedge T} f\left(\underline{N}^{K}(s)\right) \mathrm{d} s\right]\right| \leq \mathcal{O}(1)\|f\|_{K, q} \frac{K^{q}+\|\underline{n}\|_{2}^{q}}{T}
$$

Hence for $T \leq 1$ we get

$$
\left|\mathbb{E}_{\underline{n}}\left[S_{f}(T, K)\right]-\nu_{K}(f)\right| \leq \mathcal{O}(1)\|f\|_{K, q}\left(K^{q}+\|\underline{n}\|_{2}^{q}\right)\left(\frac{1}{T}+1\right)
$$

For $T>1$, we have by the Markov property that

$$
\begin{aligned}
\frac{1}{T} \mathbb{E}_{\underline{n}}\left[\int_{1}^{T} f\left(\underline{N}^{K}(s)\right) \mathrm{d} s\right] & =\frac{1}{T} \int_{1}^{T} \mathbb{E}_{\underline{n}}\left[\mathbb{E}_{\underline{N}^{K}(s-1)}\left[f\left(\underline{N}^{K}(1)\right)\right]\right] \mathrm{d} s \\
& =\frac{1}{T} \int_{0}^{T-1} \mathbb{E}_{\underline{n}}\left[g\left(\underline{N}^{K}(s)\right)\right] \mathrm{d} s
\end{aligned}
$$

where

$$
\begin{equation*}
g(\underline{m}):=\mathbb{E}_{\underline{m}}\left[f\left(\underline{N}^{K}(1)\right)\right] . \tag{5.8}
\end{equation*}
$$

By Corollary 3.2, the function $g$ is bounded and

$$
\begin{equation*}
\|g\|_{\infty} \leq \mathcal{O}(1)\|f\|_{K, q} K^{q} \tag{5.9}
\end{equation*}
$$

Applying Proposition 5.3 to $g$ thus gives

$$
\begin{aligned}
\mid \mathbb{E}_{\underline{n}} & {\left[g\left(\underline{N}^{K}(s)\right)\right]-\nu_{K}(g) \mid } \\
& \leq \mathcal{O}(1)\|f\|_{K, q} K^{q}\left(\mathrm{e}^{-\delta\left(\zeta \frac{\|\underline{n}\|_{1}}{K} \wedge \alpha\right) K}+s \mathrm{e}^{-\alpha \delta K}+\mathrm{e}^{-\frac{a s}{\log K}}\right) .
\end{aligned}
$$

Integrating over $s \in[0, T-1]$ yields

$$
\begin{aligned}
& \left|\frac{1}{T} \int_{0}^{T-1} \mathbb{E}_{\underline{n}}\left[g\left(\underline{N}^{K}(s)\right)\right] \mathrm{d} s-\frac{T-1}{T} \nu_{K}(g)\right| \\
& \quad \leq \mathcal{O}(1)\|f\|_{K, q} \frac{K^{q}}{T}\left((T-1) \mathrm{e}^{-\delta\left(\zeta \frac{\|\underline{n}\|_{1}}{K} \wedge \alpha\right) K}+(T-1)^{2} \mathrm{e}^{-\alpha \delta K}+\frac{\log K}{a}\right) .
\end{aligned}
$$

Using Lemma 5.5 (stated and proved right after this proof), we finally obtain

$$
\begin{aligned}
\mid \mathbb{E}_{\underline{n}}[ & \left.S_{f}(T, K)\right]-\nu_{K}(f) \mid \\
& \leq \mathcal{O}(1)\|f\|_{K, q} \frac{K^{q}+\|\underline{n}\|_{2}^{q}}{T} \\
& +\mathcal{O}(1)\|f\|_{K, q} \frac{K^{q}}{T}\left((T-1) \mathrm{e}^{-\delta\left(\zeta \frac{\|\underline{n}\|_{1}}{K} \wedge \alpha\right) K}+(T-1)^{2} \mathrm{e}^{-\alpha \delta K}+\frac{\log K}{a}\right) \\
& +\mathcal{O}(1)\|f\|_{K, q} K^{q}\left(1-\mathrm{e}^{-\lambda_{0}(K)}\right)^{\frac{1}{2}}+\frac{1}{T} \nu_{K}(g)+\nu_{K}(f) \mathbb{1}_{\{T \leq 1\}} \\
\leq & \mathcal{O}(1)\|f\|_{K, q}\left(K^{q}+\|\underline{n}\|_{2}^{q}\right)\left(\frac{1}{T}\left(2+\frac{\log K}{a}\right)+\mathrm{e}^{-\delta\left(\zeta \frac{\|n\|_{1}}{K} \wedge \alpha\right) K}+T \mathrm{e}^{-\delta \alpha K}\right. \\
& \left.+\left(1-\mathrm{e}^{-\lambda_{0}(K)}\right)^{\frac{1}{2}}+\mathbb{1}_{\{T \leq 1\}}\right)
\end{aligned}
$$

This finishes the proof of the theorem.
We use the following lemma in the previous proof.
Lemma 5.5. For $f \in \mathscr{F}_{K, q}$ and $g$ defined in (5.8) we have

$$
\left|\nu_{K}(g)-\nu_{K}(f)\right| \leq \mathcal{O}(1) K^{q}\|f\|_{K, q}\left(1-\mathrm{e}^{-\lambda_{0}(K)}\right)^{\frac{1}{2}} .
$$

Proof. We write

$$
\nu_{K}(g)=\mathbb{E}_{\nu_{K}}\left[f\left(\underline{N}^{K}(1)\right) \mathbb{1}_{\left\{T_{\underline{0}}>1\right\}}\right]+\mathbb{E}_{\nu_{K}}\left[f\left(\underline{N}^{K}(1)\right) \mathbb{1}_{\left\{T_{\underline{0}} \leq 1\right\}}\right] .
$$

Since $\nu_{K}$ is a qsd, using Cauchy-Schwarz inequality, it follows that

$$
\begin{aligned}
\mid \nu_{K}(g) & -\nu_{K}(f) \mid \\
& \leq\left(1-\mathrm{e}^{-\lambda_{0}(K)}\right)\left|\nu_{K}(f)\right|+\left(\mathbb{E}_{\nu_{K}}\left[f^{2}\left(\underline{N}^{K}(1)\right)\right]\right)^{\frac{1}{2}}\left(\mathbb{E}_{\nu_{K}}\left[\mathbb{1}_{\left\{T_{\underline{0}} \leq 1\right\}}\right]\right)^{\frac{1}{2}} \\
& \leq \mathcal{O}(1) K^{q}\|f\|_{K, q}\left(1-\mathrm{e}^{-\lambda_{0}(K)}\right)^{\frac{1}{2}}
\end{aligned}
$$

where we used Corollaries 3.2 and 4.2 and the fact that under $\nu_{K}$ the law of $T_{\underline{0}}$ is exponential with parameter $\lambda_{0}(K)$. The lemma is proved.

## 6 Fluctuation and correlation relations

### 6.1 Proof of Relation (1.5)

Let

$$
\widetilde{\Sigma}_{i, j}^{K}=\int\left(n_{i}-n_{i}^{*}\right)\left(n_{j}-n_{j}^{*}\right) \mathrm{d} \nu_{K}(\underline{n}) .
$$

On the one hand we have by (4.3)

$$
\begin{equation*}
\left\langle\mathscr{L}_{K}^{\dagger} \nu_{K},\left(n_{i}-n_{i}^{*}\right)\left(n_{j}-n_{j}^{*}\right)\right\rangle=-\lambda_{0}(K)\left\langle\nu_{K},\left(n_{i}-n_{i}^{*}\right)\left(n_{j}-n_{j}^{*}\right)\right\rangle . \tag{6.1}
\end{equation*}
$$

By Theorem 4.3 and (4.1) the right-hand side of this equation is exponentially small in $K$. On the other hand we have using formula (3.1)

$$
\begin{aligned}
&\left\langle\mathscr{L}_{K}^{\dagger} \nu_{K},\left(n_{i}-n_{i}^{*}\right)\left(n_{j}-n_{j}^{*}\right)\right\rangle=\left\langle\nu_{K}, \mathscr{L}_{K}\left(\left(n_{i}-n_{i}^{*}\right)\left(n_{j}-n_{j}^{*}\right)\right)\right\rangle \\
&= K \sum_{\ell=1}^{d}\left\langle\nu_{K}, B_{\ell}\left(\frac{n}{K}\right)\left(\left(n_{j}-n_{j}^{*}\right) \delta_{i, \ell}+\left(n_{i}-n_{i}^{*}\right) \delta_{j, \ell}+\delta_{i, \ell} \delta_{j, \ell}\right)\right\rangle \\
&+K \sum_{\ell=1}^{d}\left\langle\nu_{K}, D_{\ell}\left(\frac{n}{K}\right)\left(-\left(n_{j}-n_{j}^{*}\right) \delta_{i, \ell}-\left(n_{i}-n_{i}^{*}\right) \delta_{j, \ell}+\delta_{i, \ell} \delta_{j, \ell}\right)\right\rangle \\
&= K\left\langle\nu_{K},\left(B_{i}\left(\frac{n}{K}\right)-D_{i}\left(\frac{n}{K}\right)\right)\left(n_{j}-n_{j}^{*}\right)\right\rangle \\
&+K\left\langle\nu_{K},\left(B_{j}\left(\frac{n}{K}\right)-D_{j}\left(\frac{n}{K}\right)\right)\left(n_{i}-n_{i}^{*}\right)\right\rangle \\
&+K\left\langle\nu_{K}, B_{i}\left(\frac{n}{K}\right)+D_{i}\left(\frac{n}{K}\right)\right\rangle \delta_{i, j} .
\end{aligned}
$$

We split each integral by separating integration over $\mathcal{D}$ (defined in (4.4)) and integration over $\mathcal{D}^{c}$. Inside $\mathcal{D}^{c}$, we apply Corollary 4.2 and use the assumption that $B$ and $D$ are polynomially bounded. Inside $\mathcal{D}$, we use Taylor's formula around $\underline{x}^{*}$ for the functions $B_{i}(\underline{n} / K)-D_{i}(\underline{n} / K)$, and $B_{i}(\underline{n} / K)+D_{i}(\underline{n} / K)$. We also use that $B_{i}\left(\underline{x}^{*}\right)=D_{i}\left(\underline{x}^{*}\right), 1 \leq i \leq d$, and $\underline{n}^{*} / K-\underline{x}^{*}=\mathcal{O}(1) / K$. The error terms are bounded by

$$
\frac{\mathcal{O}(1)}{K} \int\left\|\underline{n}-K \underline{x}^{*}\right\|_{2}^{3} \mathrm{~d} \nu_{K}(\underline{n}) \quad \text { and } \quad \mathcal{O}(1) \int\left\|\underline{n}-K \underline{x}^{*}\right\|_{2} \mathrm{~d} \nu_{K}(\underline{n})
$$

respectively. Using Theorem 4.3, both bounds are of order $\sqrt{K}$. We obtain

$$
\sum_{\ell=1}^{d} M_{i, \ell}^{*} \widetilde{\Sigma}_{\ell, j}^{K}+\sum_{\ell=1}^{d} M_{j, \ell}^{*} \widetilde{\Sigma}_{\ell, i}^{K}+2 K B_{i}\left(\underline{x}^{*}\right) \delta_{i, j}=\mathcal{O}(\sqrt{K})
$$

which can be written in the more compact form

$$
\begin{equation*}
M^{*} \widetilde{\Sigma}^{K}+\widetilde{\Sigma}^{K}{ }^{\mathrm{t}} M^{*}+2 \mathcal{D}^{K}=\mathcal{O}(\sqrt{K}) \tag{6.2}
\end{equation*}
$$

where $\mathcal{D}^{K}$ is the diagonal matrix of averages birth (or death) rates. To finish the proof, use (4.13).

Remark 6.1. Note that each term on the left hand side is of order $K$, see Corollary 4.5.
Remark 6.2. Dividing out (6.2) by $2 K$ and taking the limit $K \rightarrow \infty$, we recover (4.14), as expected.

### 6.2 Proof of Relation (1.6)

Let

$$
\widetilde{\Sigma}_{i, j}^{K}(t)=\mathbb{E}_{\nu_{K}}\left[\left(N_{i}^{K}(t)-n_{i}^{*}\right)\left(N_{j}^{K}(0)-n_{j}^{*}\right)\right], i, j=1, \ldots, d
$$

For $1 \leq i \leq d$, let $f_{i}(\underline{n})=\left\langle\underline{n}-\underline{n}^{*}, \underline{e}^{(i)}\right\rangle$. We have, since $B_{i}\left(\underline{x}^{*}\right)=D_{i}\left(\underline{x}^{*}\right), 1 \leq$ $i \leq d$, and $\underline{n}^{*} / K-\underline{x}^{*}=\mathcal{O}(1) / K$,

$$
\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{~d} t} \widetilde{\Sigma}_{i, j}^{K}(t) \\
& =\mathbb{E}_{\nu_{K}}\left[\mathscr{L}_{K} f_{i}\left(\underline{N}^{K}(t)\right)\left(N_{j}^{K}(0)-n_{j}^{*}\right)\right] \\
& =K \mathbb{E}_{\nu_{K}}\left[B_{i}\left(\frac{\underline{N}^{K}(t)}{K}\right)\left(N_{j}^{K}(0)-n_{j}^{*}\right)\right]-K \mathbb{E}_{\nu_{K}}\left[D_{i}\left(\frac{N^{K}(t)}{K}\right)\left(N_{j}^{K}(0)-n_{j}^{*}\right)\right] \\
& =K \mathbb{E}_{\nu_{K}}\left[\left(B_{i}\left(\frac{\underline{N}^{K}(t)}{K}\right)-B_{i}\left(\frac{n^{*}}{K}\right)\right)\left(N_{j}^{K}(0)-n_{j}^{*}\right)\right] \\
& \quad-K \mathbb{E}_{\nu_{K}}\left[\left(D_{i}\left(\frac{N^{K}(t)}{K}\right)-D_{i}\left(\frac{n^{*}}{K}\right)\right)\left(N_{j}^{K}(0)-n_{j}^{*}\right)\right]+\mathcal{O}(1)
\end{aligned}
$$

As in the previous proof, we split the integrals according to whether $\underline{N}^{K}(t) \in$ $\mathcal{D}$ or $\underline{N}^{K}(t) \in \mathcal{D}^{c}$. Using Cauchy-Schwarz inequality, Corollary 4.2, and the fact that $\nu_{K}$ is a qsd, the second contribution is exponentially small in $K$. In the first contribution, we use Taylor expansion around $\underline{x}^{*}$. The error terms are bounded by

$$
\frac{\mathcal{O}(1)}{K} \mathbb{E}_{\nu_{K}}\left[\left\|\underline{N}^{K}(t)-K \underline{x}^{*}\right\|_{2}^{2}\left\|\underline{N}^{K}(0)-K \underline{x}^{*}\right\|_{2}\right]+\mathcal{O}(1)
$$

Now we use Cauchy-Schwarz inequality, Theorem 4.3 and that $\nu_{K}$ is a qsd to obtain

$$
\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{~d} t} \widetilde{\Sigma}_{i, j}^{K}(t) \\
& =\sum_{\ell=1}^{d}\left(\partial_{\ell} B_{i}\left(\underline{x}^{*}\right)-\partial_{\ell} D_{i}\left(\underline{x}^{*}\right)\right) \mathbb{E}_{\nu_{K}}\left[\left(N_{\ell}^{K}(t)-n_{\ell}^{*}\right)\left(N_{j}^{K}(0)-n_{j}^{*}\right)\right]+\mathcal{O}(\sqrt{K}) \\
& =\sum_{\ell=1}^{d} M_{i, \ell}^{*} \widetilde{\Sigma}_{\ell, j}^{K}(t)+\mathcal{O}(\sqrt{K})
\end{aligned}
$$

Since $M^{*}$ has a spectrum contained in the open left half-plane by H.3, we integrate the equation

$$
\frac{\mathrm{d}}{\mathrm{~d} t} \widetilde{\Sigma}^{K}(t)=M^{*} \widetilde{\Sigma}^{K}(t)+\mathcal{O}(\sqrt{K})
$$

from 0 to $\tau$ to get

$$
\widetilde{\Sigma}^{K}(\tau)=\mathrm{e}^{\tau M^{*}} \widetilde{\Sigma}^{K}(0)+\mathcal{O}(\sqrt{K})
$$

We arrive at the desired relation by using (4.12).

## 7 Variance estimates of the statistics

The matrices $\Sigma^{K}(\tau)$ and $\Sigma^{K}$ can be estimated using Theorem 5.4 with $S^{C}(\tau)$ to infer $\Sigma^{K}(\tau)$ and $S^{\Sigma}$, respectively.

Here we prove two variance estimates to complement this result. In the first one, one starts from anywhere in $\mathbb{Z}_{+}^{d}$, while in the second one the starting distribution is the qsd.

Proposition 7.1. There exist positive constants $\delta^{\prime}, \zeta^{\prime}, \alpha^{\prime}, \theta^{\prime}, C^{\prime}$ and $K_{0} \geq 2$ such that, for all $K \geq K_{0}, f \in \mathscr{F}_{K, q}$ (see Definition 5.7), $T \geq 0$, and $\underline{n} \neq \underline{0}$, we have

$$
\begin{aligned}
& \mathbb{E}_{\underline{n}}\left[\left(S_{f}(T, K)-\nu_{K}(f)\right)^{2}\right] \leq C^{\prime}\|f\|_{K, q}^{2}\left(c_{q}\|\underline{n}\|_{1}^{q}+K^{q}\right) \\
& \quad \times\left(\frac{\|\underline{n}\|_{1}^{q}+K^{q} \log K}{T \vee 1}+K^{q} \mathrm{e}^{-\delta^{\prime}\left(\zeta^{\prime} \frac{\|\underline{n}\|_{1}}{K} \wedge \alpha^{\prime}\right) K}+T K^{q} \mathrm{e}^{-\theta^{\prime} K}\right)
\end{aligned}
$$

where $c_{q}$ is defined in Proposition 3.3.
One can use Chebyshev inequality to bound $\mathbb{P}_{\underline{n}}\left(\left|S_{f}(T, K)-\nu_{K}(f)\right|>\delta\right)$, for any $\delta>0$.

The proof of Proposition 7.1 is postponed to Appendix A. The previous estimate and all the estimates below have the same behaviour in their dependence in $K, \underline{n}$ and $T$. They display the qualitative behaviour that we met several times, namely

1. The bounds are not useful for $K$ too small.
2. If $K$ is large, the bounds are not useful if $\underline{n}$ is small (order one) because the process can be absorbed at $\underline{0}$ in a time of order one with a sizeable probability.
3. Finally for $K$ large and $\underline{n}$ of order $K$, the time $T$ must be large enough (polynomial in $K$ in our bounds) but not too large (less than an exponential in $K$ because again the process can reach the origin in such large times).

Under these assumptions the statistics of the process are well approximated by the qsd.

Corollary 7.2. There exist two positive constants $C^{\prime \prime}>0$ and $\theta^{\prime \prime}$ such that for all $K \geq K_{0}$, for all $f \in \mathscr{F}_{K, q}$ and for all $T \geq 0$, we have

$$
\begin{aligned}
\mathbb{E}_{\nu_{K}} & {\left[\left(S_{f}(T, K)-\mathbb{E}_{\nu_{K}}(f)\right)^{2}\right] \leq C^{\prime \prime}\|f\|_{K, q}^{2} K^{2 q} } \\
& \times\left(\left(1+C_{2 q}\right)\left(1+c_{q}\right) \frac{\log K}{T \vee 1}+\left(1+C_{q}\right)(1+T) \mathrm{e}^{-\theta^{\prime \prime} K}\right)
\end{aligned}
$$

where $K_{0}$ is as in the previous proposition, $c_{q}$ is defined in Proposition 3.3, and $C_{q}$ is defined in Corollary 4.2.

Observe that the previous inequality is only useful in the range $0 \leq$ $T \leq \mathrm{e}^{\theta^{\prime \prime} K}$. The proofs of the two previous estimates are postponed to the Appendix.

We now apply the previous results to our statistics (see Definition (1.1)).
Proposition 7.3. We have for all $1 \leq p \leq d$

$$
\begin{aligned}
\mathbb{E}_{\underline{n}}\left[\mid S_{\bar{p}}^{\mu}(T, K)\right. & \left.-\left.\mu_{p}^{K}\right|^{2}\right] \leq \mathcal{O}(1)\left(c_{1}\|\underline{n}\|_{1}+K\right) \\
& \times\left(\frac{\|\underline{n}\|_{1}+K \log K}{T \vee 1}+K \mathrm{e}^{-\delta^{\prime}\left(\zeta^{\prime} \frac{\|n\|_{1}}{K} \wedge \beta^{\prime}\right) K}+T K \mathrm{e}^{-\theta K}\right)
\end{aligned}
$$

and

$$
\mathbb{E}_{\nu_{K}}\left[\left|S_{\bar{p}}^{\mu}(T, K)-\mu_{p}^{K}\right|^{2}\right] \leq \mathcal{O}(1) K^{2}\left(\frac{1+\log K}{T \vee 1}+(1+T) \mathrm{e}^{-\theta^{\prime \prime} K}\right) .
$$

Proof. The proof follows by applying Proposition 7.1 and Corollary 7.2 to the functions $f(\underline{n})=n_{j}, 1 \leq j \leq d$, which belong to $\mathscr{F}_{K, 1}$.

Proposition 7.4. For $1 \leq p, p^{\prime} \leq d$ and for all $\underline{n} \neq \underline{0}$, we have

$$
\begin{aligned}
& \mathbb{E}_{\underline{n}}\left[\left(S_{p, p^{\prime}}^{\Sigma}(T, K)-\Sigma_{p, p^{\prime}}^{K}\right)^{2}\right] \leq \mathcal{O}(1)\left(c_{2}\|\underline{n}\|_{1}^{2}+K^{2}\right)^{2} \\
& \quad \times\left(\frac{1+\log K}{T \vee 1}+\mathrm{e}^{-\delta^{\prime}\left(\zeta^{\prime} \frac{\|n\|_{1}}{K} \wedge \beta^{\prime}\right) K}+T \mathrm{e}^{-\theta K}\right)
\end{aligned}
$$

and

$$
\mathbb{E}_{\nu_{K}}\left[\left(S_{p, p^{\prime}}^{\Sigma}(T, K)-\Sigma_{p, p^{\prime}}^{K}\right)^{2}\right] \leq \mathcal{O}(1) K^{4}\left(\frac{1+\log K}{T \vee 1}+(1+T) \mathrm{e}^{-\theta^{\prime \prime} K}\right)
$$

Proof. The proof follows by applying Proposition 7.1 and Corollary 7.2 to the functions $f(\underline{n})=n_{p} n_{p^{\prime}}, 1 \leq p, p^{\prime} \leq d$, which belong to $\mathscr{F}_{K, 2}$.
Proposition 7.5. There exist positive constants $\widetilde{C}, \tilde{\theta}, \tilde{\delta}, \tilde{\zeta}$ and $\tilde{\beta}$ such that for all $K \geq 2, T>0$ and $1 \leq \ell \leq d$,

$$
\begin{aligned}
& \mathbb{E}_{\underline{n}}\left[\left(S_{\ell}^{D}(T, K)-K B_{\ell}\left(\underline{x}^{*}\right)\right)^{2}\right] \leq \\
& \\
& \widetilde{C}\left(K+\frac{A_{\ell}\left(1+C_{q_{\ell}}\right) K}{T}+K^{1-q_{\ell}} \frac{A_{\ell}}{T}\left(K+\|\underline{n}\|_{1}\right)^{q_{\ell}} \mathfrak{R}_{\ell}\right. \\
& \left.\quad+K^{2-2 q_{\ell}} A_{\ell}^{2}\left(K+\|\underline{n}\|_{1}\right)^{2 q_{\ell}}\left(\mathfrak{R}_{\ell}^{2}+\mathfrak{R}_{\ell}\right)\right)
\end{aligned}
$$

where

$$
\mathfrak{R}_{\ell}=\left(1+c_{q_{\ell}}\right)\left(\frac{1+\log K}{T}+T \mathrm{e}^{-\tilde{\theta} K}+\mathrm{e}^{-\tilde{\delta}\left(\tilde{\zeta}_{\bar{K}} \wedge \tilde{\beta}\right) K}\right)
$$

and $A_{\ell}>0, q_{\ell} \in \mathbb{N}$, are such that, for all $\underline{x} \in \mathbb{R}_{+}^{d}$,

$$
\left|B_{\ell}(\underline{x})\right| \leq A_{\ell}\left(1+\|\underline{x}\|_{1}^{q_{\ell}}\right) .
$$

The existence of $A_{\ell}$ and $q_{\ell}$ follows from the assumptions on $B$. The constants $C_{q_{\ell}}$ and $c_{q_{\ell}}$ are defined in Corollary 4.2 and Lemma 3.3, respectively. We also have

$$
\begin{gathered}
\mathbb{E}_{\nu_{K}}\left[\left(S_{\ell}^{D}(T, K)-K B_{\ell}\left(\underline{x}^{*}\right)\right)^{2}\right] \leq \\
\widetilde{C}\left(K+\frac{A_{\ell}\left(1+C_{q_{\ell}}\right) K}{T}+\frac{A_{\ell}}{T} K \widetilde{\mathfrak{R}}_{\ell}+K^{2} A_{\ell}^{2}\left(\widetilde{\mathfrak{R}}_{\ell}^{2}+\widetilde{\mathfrak{R}}_{\ell}\right)\right)
\end{gathered}
$$

where

$$
\widetilde{\Re}_{\ell}=\left(1+C_{q_{\ell}}\right)\left(\left(1+C_{2 q_{\ell}}\right)\left(1+c_{q_{\ell}}\right) \frac{\log K}{T}+(1+T) \mathrm{e}^{-\tilde{\theta} K}\right) .
$$

Proof. First observe that

$$
S_{\ell}^{\perp}(T, K)=\frac{1}{T} \mathcal{N}_{\ell}^{K}(0, T)
$$

where $\mathcal{N}_{\ell}^{K}(0, T)$ is defined in Appendix B. By assumption, the function $f_{\ell}(\underline{n})=K^{q_{\ell}} B_{\ell}\left(\frac{n}{K}\right) \in \mathscr{F}_{K, q_{\ell}}$. Let $\mathfrak{m}$ be any probability measure on $\mathbb{Z}_{+}^{d}$ having all its moments finite. We apply Theorem 5.4 to the function $f_{\ell}$, and then using integration against $\mathfrak{m}$ we get

$$
\begin{gathered}
\left|\mathbb{E}_{\mathfrak{m}}\left[S_{f_{\ell}}(T, K)\right]-\nu_{K}\left(f_{\ell}\right)\right| \leq \mathcal{O}(1)\left\|f_{\ell}\right\|_{K, q_{\ell}} \\
\times \int\left(\left(K+\|\underline{n}\|_{2}\right)^{q_{\ell}}\left(\mathrm{e}^{-\delta\left(\zeta \frac{\|n\|_{1}}{K} \wedge \beta\right) K}+T \mathrm{e}^{-\delta \beta K}+\frac{1+\log K}{T}\right)\right) \mathrm{d} \mathfrak{m}(\underline{n}) .
\end{gathered}
$$

We now apply the identity in Proposition B. 1 and divide by $K^{q_{\ell}-1}$. We obtain

$$
\begin{align*}
& \left|\mathbb{E}_{\mathfrak{m}}\left[S_{\ell}^{\mathfrak{D}}(T, K)\right]-\nu_{K}\left(K B_{\ell}\left(\frac{n}{K}\right)\right)\right| \leq \mathcal{O}(1)\left\|f_{\ell}\right\|_{K, q_{\ell}} K^{1-q_{\ell}}  \tag{7.1}\\
& \quad \times \int\left(\left(K+\|\underline{n}\|_{2}\right)^{q_{\ell}}\left(\mathrm{e}^{-\delta\left(\zeta \frac{\|n\|_{1}}{K} \wedge \beta\right) K}+T \mathrm{e}^{-\delta \beta K}+\frac{1+\log K}{T}\right)\right) \mathrm{d} \mathfrak{m}(\underline{n}) .
\end{align*}
$$

We now estimate

$$
\int B_{\ell}\left(\frac{\underline{n}}{K}\right) \mathrm{d} \nu_{K}(\underline{n})=\int_{\mathcal{D}} B_{\ell}\left(\frac{\underline{n}}{K}\right) \mathrm{d} \nu_{K}(\underline{n})+\int_{\mathcal{D}^{c}} B_{\ell}\left(\frac{\underline{n}}{K}\right) \mathrm{d} \nu_{K}(\underline{n}) .
$$

The second integral is bounded from above by $\mathcal{O}(1) / K$ using the polynomial bound on $B_{\ell}$ and the first estimate in Corollary 4.2. For the first integral
we use Taylor expansion around $\underline{x}^{*}$ to first order, then Cauchy-Schwarz inequality, and finally Theorem 4.3 for $q=1$. Therefore we obtain

$$
\begin{align*}
& \left|\mathbb{E}_{\mathfrak{m}}\left[S_{\ell}^{D}(T, K)\right]-K B_{\ell}\left(\underline{x}^{*}\right)\right| \leq \mathcal{O}(1) \sqrt{K}+\mathcal{O}(1)\left\|f_{\ell}\right\|_{K, q_{\ell}} K^{1-q_{\ell}} \\
& \times \int\left(K+\|\underline{n}\|_{2}\right)^{q_{\ell}}\left(\mathrm{e}^{-\delta\left(\zeta \frac{\|n\|_{1}}{K} \wedge \beta\right) K}+T \mathrm{e}^{-\delta \beta K}+\frac{1+\log K}{T}\right) \mathrm{d} \mathfrak{m}(\underline{n}) . \tag{7.2}
\end{align*}
$$

Now we apply the estimate in Proposition B. 1 to obtain

$$
\begin{aligned}
& \mathbb{E}_{\mathfrak{m}} {\left[\left(S_{\ell}^{\mathbb{D}}(T, K)-\mathbb{E}_{\mathfrak{m}}\left[S_{\ell}^{D}(T, K)\right]\right)^{2}\right] } \\
&= \frac{1}{T^{2}} \mathbb{E}_{\mathfrak{m}}\left[\left(\mathcal{N}_{\ell}^{K}(0, T)-\mathbb{E}_{\mathfrak{m}}\left[\mathcal{N}_{\ell}^{K}(0, T)\right]\right)^{2}\right] \\
& \leq \mathbb{E}_{\mathfrak{m}}\left[\left(\frac{1}{T} \int_{0}^{T} K^{1-q_{\ell}} f_{\ell}\left(\underline{N}^{K}(s)\right) \mathrm{d} s-\mathbb{E}_{\mathfrak{m}}\left[\frac{\mathcal{N}_{\ell}^{K}(0, T)}{T}\right]\right)^{2}\right] \\
&+\frac{2}{T} \mathbb{E}_{\mathfrak{m}}\left[\frac{1}{T} \int_{0}^{T} K^{1-q_{\ell}} f_{\ell}\left(\underline{N}^{K}(s)\right) \mathrm{d} s\right] \\
& \leq 2 \mathbb{E}_{\mathfrak{m}}\left[\left(\frac{1}{T} \int_{0}^{T} K^{1-q_{\ell}} f_{\ell}\left(\underline{N}^{K}(s)\right) \mathrm{d} s-K^{1-q_{\ell}} \nu_{K}\left(f_{\ell}\right)\right)^{2}\right] \\
&+2 \mathbb{E}_{\mathfrak{m}}\left[\left(K^{\left.\left.1-q_{\ell} \nu_{K}\left(f_{\ell}\right)-\mathbb{E}_{\mathfrak{m}}\left[S_{\ell}^{\mathcal{D}}(T, K)\right]\right)^{2}\right]}\right.\right. \\
& \quad+\frac{2}{T} \mathbb{E}_{\mathfrak{m}}\left[\frac{1}{T} \int_{0}^{T} K^{1-q_{\ell}} f_{\ell}\left(\underline{N}^{K}(s)\right) \mathrm{d} s\right] .
\end{aligned}
$$

For the first term we use either Corollary 7.2 or Proposition 7.1. For the second term we use (7.1). For the third and last term we apply Theorem 5.4, integrate with respect to $\mathfrak{m}$ and use (7.2). To finish the proof, we replace $\mathfrak{m}$ by either $\delta_{\underline{n}}$ or $\nu_{K}$.

Recall that $B_{p}\left(\underline{x}^{*}\right)=D_{p}\left(\underline{x}^{*}\right), 1 \leq p \leq d$.
Proposition 7.6. Under the assumptions of Proposition 7.1 and Corollary 7.2, we have, for all $1 \leq p, p^{\prime} \leq d$, and $\tau \geq 0$,

$$
\begin{aligned}
\mathbb{E}_{\underline{n}} & {\left[\left(S_{p, p^{\prime}}^{C}(T, \tau, K)-\Sigma_{p, p^{\prime}}^{K}(\tau)\right)^{2}\right] \leq \mathcal{O}(1)\left(c_{2}\|\underline{n}\|_{1}^{2}+K^{2}\right)^{2} \times } \\
& \left(\frac{1+\tau+\log K}{T \vee 1}+\mathrm{e}^{-\delta^{\prime}}\left(\zeta^{\prime} \frac{\|n\|_{1}}{K} \wedge \beta^{\prime}\right) K\right. \\
& \left.(T+\tau) \mathrm{e}^{-\theta K}\right) .
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{E}_{\nu_{K}} & {\left[\left(S_{p, p^{\prime}}^{C}(T, \tau, K)-\Sigma_{p, p^{\prime}}^{K}(\tau)\right)^{2}\right] } \\
& \leq \mathcal{O}(1) K^{4}\left(\frac{1+\tau+\log K}{T}+(1+T+\tau) \mathrm{e}^{-\theta^{\prime \prime} K}\right)
\end{aligned}
$$

Proof. The proof requires some simple modifications of the proofs of Propostions 7.1 and 7.2. This is left to the reader.

Remark 7.1. One can check that if one modifies slightly the definition of the statistics in Definition 1.1 by integrating from time 1, then, in the four previous propositions, one can replace the factor $\left(\|\underline{n}\|_{1}+K\right)$ by $K$, and the factor $\left(\|\underline{n}\|_{1}^{2}+K^{2}\right)$ by $K^{2}$.

We now define an empirical matrix $M_{\text {emp }}^{*}(T, \tau, K)$ by

$$
\mathrm{e}^{\tau M_{\mathrm{emp}}^{*}(T, \tau, K)}=S^{C}(T, \tau, K) S^{\Sigma}(T, K)^{-1}
$$

and an empirical resilience by

$$
\rho_{\mathrm{emp}}^{*}(T, \tau, K)=-\sup \left\{\operatorname{Re}(z): z \in \operatorname{Sp}\left(M_{\mathrm{emp}}^{*}(T, \tau, K)\right)\right\} .
$$

From the above results one can derive various statistical estimates for the difference between $\rho_{\text {emp }}^{*}(T, \tau, K)$ and $\rho^{*}$.

Proposition 7.7. For $\tau$ of order one, $\|\underline{n}\|_{1}$ of order $K, \log T \ll K$ and $K$ large enough, we have, with a probability higher than $1-1 / K$,

$$
\left|\rho_{\text {emp }}^{*}(T, \tau, K)-\rho^{*}\right| \leq \mathcal{O}(1)\left(\frac{K^{2}}{\sqrt{T}}+\frac{1}{\sqrt{K}}\right) .
$$

In particular, if $T \gg K^{5},\left|\rho_{\text {emp }}^{*}(T, \tau, K)-\rho^{*}\right| \leq \mathcal{O}(1) / \sqrt{K}$.
Proof. It follows from Propositions 7.4 and 7.6 and the standing assumptions that, with a probability higher that $1-1 / K$, we have

$$
\left\|S^{C}(T, \tau, K)-\Sigma^{K}(\tau)\right\| \leq \mathcal{O}(1) \frac{K^{3}}{\sqrt{T}}
$$

and

$$
\left\|S^{\Sigma}(T, K)-\Sigma^{K}\right\| \leq \mathcal{O}(1) \frac{K^{3}}{\sqrt{T}}
$$

We now use Theorem 1.6 and Proposition 4.6 to obtain

$$
\left\|\mathrm{e}^{\tau M_{\mathrm{emp}}^{*}(T, \tau, K)}-\mathrm{e}^{\tau M^{*}}\right\| \leq \mathcal{O}(1)\left(\frac{1}{\sqrt{K}}+\frac{K^{2}}{\sqrt{T}}\right) .
$$

The result follows since $\tau$ is of order one.

## A Proof of the two variance estimates

## A. 1 Starting from anywhere: proof of Proposition 7.1

It is enough to prove the result for $\|f\|_{K, q}=1$. We have
$\mathbb{E}_{\underline{n}}\left[\left(\frac{1}{T} \int_{0}^{T} f\left(\underline{N}^{K}(t)\right) \mathrm{d} t\right)^{2}\right]=\frac{2}{T^{2}} \int_{0}^{T} \mathrm{~d} t_{2} \int_{0}^{t_{2}} \mathbb{E}_{\underline{n}}\left[f\left(\underline{N}^{K}\left(t_{1}\right)\right) f\left(\underline{N}^{K}\left(t_{2}\right)\right)\right] \mathrm{d} t_{1}$.
Step 1 is to estimate the contribution of the range $0 \leq t_{1} \leq t_{2} \leq 1$. Using Cauchy-Schwarz inequality and Proposition 3.3 we get

$$
\left|\int_{0}^{1} \mathrm{~d} t_{2} \int_{0}^{t_{2}} \mathbb{E}_{\underline{n}}\left[f\left(\underline{N}^{K}\left(t_{1}\right)\right) f\left(\underline{N}^{K}\left(t_{2}\right)\right)\right] \mathrm{d} t_{1}\right| \leq \mathcal{O}(1)\left(\|\underline{n}\|_{1}^{q}+K^{q}\right)^{2} .
$$

Step 2 is to estimate the contribution in the range $0 \leq t_{2}-1 \leq t_{1} \leq t_{2}$. This implied that $T>1$. We have using again Proposition 3.3

$$
\begin{aligned}
& \left|\int_{1}^{T} \mathrm{~d} t_{2} \int_{t_{2}-1}^{t_{2}} \mathbb{E}_{\underline{n}}\left[f\left(\underline{N}^{K}\left(t_{1}\right)\right) f\left(\underline{N}^{K}\left(t_{2}\right)\right)\right] \mathrm{d} t_{1}\right| \\
& \quad \leq \int_{1}^{T} \mathrm{~d} t_{2} \int_{t_{2}-1}^{t_{2}}\left(\mathbb{E}_{\underline{n}}\left[f\left(\underline{N}^{K}\left(t_{1}\right)\right)^{2}\right]+\mathbb{E}_{\underline{n}}\left[f\left(\underline{N}^{K}\left(t_{2}\right)\right)^{2}\right]\right) \mathrm{d} t_{1} \\
& \quad \leq \mathcal{O}(1) T\left(\|\underline{n}\|_{1}^{q}+K^{q}\right)^{2} .
\end{aligned}
$$

## Step 3

(1) Using the Markov property and the definition of $g$ (see (5.8)) we have

$$
\begin{array}{rl}
\int_{1}^{T} & \mathrm{~d} t_{2} \int_{0}^{t_{2}-1} \mathbb{E}_{\underline{n}}\left[f\left(\underline{N}^{K}\left(t_{1}\right)\right) f\left(\underline{N}^{K}\left(t_{2}\right)\right)\right] \mathrm{d} t_{1} \\
& =\int_{0}^{T-1} \mathrm{~d} s \int_{0}^{s} \mathbb{E}_{\underline{n}}\left[f\left(\underline{N}^{K}\left(t_{1}\right)\right) g\left(\underline{N}^{K}(s)\right)\right] \mathrm{d} t_{1} \\
& =\int_{0}^{T-1} \mathrm{~d} s \int_{0}^{s} \mathbb{E}_{\underline{n}}\left[f\left(\underline{N}^{K}\left(t_{1}\right)\right) \mathbb{E}_{\underline{N}^{K}\left(t_{1}\right)}\left[g\left(\underline{N}^{K}\left(s-t_{1}\right)\right)\right]\right] \mathrm{d} t_{1} .
\end{array}
$$

Let us first write

$$
\mathbb{E}_{\underline{n}}\left[f\left(\underline{N}^{K}\left(t_{1}\right)\right) \mathbb{E}_{\underline{N}^{K}\left(t_{1}\right)}\left[g\left(\underline{N}^{K}\left(s-t_{1}\right)\right)\right]\right]
$$

as the sum of $J_{1}(\underline{n})$ and $J_{2}(\underline{n})$ where

$$
J_{1}(\underline{n})=\mathbb{E}_{\underline{n}}\left[f\left(\underline{N}^{K}\left(t_{1}\right)\right) \mathbb{E}_{\underline{N}^{K}\left(t_{1}\right)}\left[\mathbb{1}_{\left\{T_{0}>s-t_{1}\right\}} g\left(\underline{N}^{K}\left(s-t_{1}\right)\right)\right]\right]
$$

and

$$
J_{2}(\underline{n})=\mathbb{E}_{\underline{n}}\left[f\left(\underline{N}^{K}\left(t_{1}\right)\right) \mathbb{E}_{\underline{N}^{K}\left(t_{1}\right)}\left[\mathbb{1}_{\left\{T_{\underline{0}} \leq s-t_{1}\right\}} g\left(\underline{N}^{K}\left(s-t_{1}\right)\right)\right]\right] .
$$

We further decompose $J_{1}(\underline{n})$ as $J_{1,1}(\underline{n})+J_{1,2}(\underline{n})$ where

$$
J_{1,1}(\underline{n})=\mathbb{E}_{\underline{n}}\left[f\left(\underline{N}^{K}\left(t_{1}\right)\right) \mathbb{1}_{\left\{T_{0} \leq t_{1}\right\}} \mathbb{E}_{\underline{N}^{K}\left(t_{1}\right)}\left[\mathbb{1}_{\left\{T_{\underline{0}}>s-t_{1}\right\}} g\left(\underline{N}^{K}\left(s-t_{1}\right)\right)\right]\right]
$$

and

$$
J_{1,2}(\underline{n})=\mathbb{E}_{\underline{n}}\left[f\left(\underline{N}^{K}\left(t_{1}\right)\right) \mathbb{1}_{\left\{T_{\underline{0}}>t_{1}\right\}} \mathbb{E}_{\underline{N}^{K}\left(t_{1}\right)}\left[\mathbb{1}_{\left\{T_{\underline{0}}>s-t_{1}\right\}} g\left(\underline{N}^{K}\left(s-t_{1}\right)\right)\right]\right] .
$$

Since $\underline{0}$ is an absorbing state, we have for all $\underline{n} \neq \underline{0}$ that

$$
J_{1,1}(\underline{n})=0 .
$$

(2) We start by estimating $J_{2}(\underline{n})$. Since $\underline{0}$ is an absorbing state, we have

$$
J_{2}(\underline{n})=g(\underline{0}) \mathbb{E}_{\underline{n}}\left[f\left(\underline{N}^{K}\left(t_{1}\right)\right) \mathbb{P}_{\underline{N}^{K}\left(t_{1}\right)}\left(T_{\underline{0}} \leq s-t_{1}\right)\right] .
$$

Note that $g(\underline{0})=\mathbb{E}_{\underline{0}}\left[f\left(\underline{N}^{K}(1)\right)\right]=f(\underline{0})$. Since we are going to use Lemma 5.2 , we write $J_{2}(\underline{n})=J_{2,1}(\underline{n})+J_{2,2}(\underline{n})$ where

$$
J_{2,1}(\underline{n})=f(\underline{0}) \mathbb{E}_{\underline{n}}\left[f\left(\underline{N}^{K}\left(t_{1}\right)\right) \mathbb{1}_{\left\{\left\|\underline{N}^{K}\left(t_{1}\right)\right\|_{1}>K \alpha / \zeta\right\}} \mathbb{P}_{\underline{N}^{K}\left(t_{1}\right)}\left(T_{\underline{0}} \leq s-t_{1}\right)\right] .
$$

and

$$
J_{2,2}(\underline{n})=f(\underline{0}) \mathbb{E}_{\underline{n}}\left[f\left(\underline{N}^{K}\left(t_{1}\right)\right) \mathbb{1}_{\left\{\left\|\underline{N}^{K}\left(t_{1}\right)\right\|_{1} \leq K \alpha / \zeta\right\}} \mathbb{P}_{\underline{N}^{K}\left(t_{1}\right)}\left(T_{\underline{0}} \leq s-t_{1}\right)\right] .
$$

We first estimate $J_{2,1}(\underline{n})$. Using (5.9), Lemma 5.2 with $\gamma=0$, and since $f$ belongs to $\mathscr{F}_{K, q}$ (see (5.7)), we have

$$
\begin{aligned}
\left|J_{2,1}(\underline{n})\right| & \leq \mathcal{O}(1) \mathbb{E}_{\underline{n}}\left[\left|f\left(\underline{N}^{K}\left(t_{1}\right)\right)\right|\right] \mathrm{e}^{-\alpha \delta K}\left(1+C\left(s-t_{1}\right)\right) \\
& \leq \mathcal{O}(1)\left(\|\underline{n}\|_{1}^{q}+K^{q}\right) \mathrm{e}^{-\alpha \delta K}\left(1+C\left(s-t_{1}\right)\right)
\end{aligned}
$$

where we used Proposition 3.3 for the second inequality.
We now estimate $J_{2,2}(\underline{n})$ by splitting it as $J_{2,2,1}(\underline{n})+J_{2,2,2}(\underline{n})$ where

$$
\begin{gathered}
J_{2,2,1}(\underline{n})=f(\underline{0}) \mathbb{E}_{\underline{n}}\left[f\left(\underline{N}^{K}\left(t_{1}\right)\right) \mathbb{1}_{\left\{\left\|\underline{N}^{K}\left(t_{1}\right)\right\|_{1} \leq K \alpha / \zeta\right\}} \mathbb{1}_{\left\{\left\|\underline{N}^{K}\left(t_{1}\right)\right\|_{1}>\right.}\left(\left[\frac{1}{2\left\|\underline{x}^{*}\right\|_{1}}\left(\zeta \frac{\|n\|_{1}}{K} \wedge \alpha\right)\right] \wedge 1\right) K\right\} \\
\left.\mathbb{P}_{\underline{N}^{K}\left(t_{1}\right)}\left(T_{\underline{0}} \leq s-t_{1}\right)\right] .
\end{gathered}
$$

and

$$
\begin{gathered}
J_{2,2,2}(\underline{n})=f(\underline{0}) \mathbb{E}_{\underline{n}}\left[f\left(\underline{N}^{K}\left(t_{1}\right)\right) \mathbb{1}_{\left\{\left\|\underline{N}^{K}\left(t_{1}\right)\right\|_{1} \leq K \alpha / \zeta\right\}} \mathbb{1}\left\{\left\|\underline{N}^{K}\left(t_{1}\right)\right\|_{1} \leq\left(\left[\frac{1}{2 \mid \underline{x}^{*} \|_{1}}\left(\zeta \frac{\|\underline{n}\|_{1}}{K} \wedge \alpha\right)\right] \wedge 1\right) K\right\}\right. \\
\left.\mathbb{P}_{\underline{N}^{K}\left(t_{1}\right)}\left(T_{\underline{0}} \leq s-t_{1}\right)\right] .
\end{gathered}
$$

Proceeding as before we get

$$
\begin{aligned}
\left|J_{2,2,1}(\underline{n})\right| & \leq \mathcal{O}(1) K^{q} \mathbb{E}_{\underline{n}}\left[\mathbb{1}_{\left\{\left\|\underline{N} K\left(t_{1}\right)\right\|_{1}>\left(\left[\frac{1}{2\left\|\underline{x}^{*}\right\|_{1}}\left(\zeta \frac{\|\underline{n}\|_{1}}{K} \wedge \alpha\right)\right] \wedge 1\right) K\right\}} \mathbb{P}_{\underline{N}^{K}\left(t_{1}\right)}\left(\mathbb{1}_{\left\{T_{\underline{0}} \leq s-t_{1}\right\}}\right)\right] \\
& \leq \mathcal{O}(1) K^{q}\left(\mathrm{e}^{-\delta K\left(\left(\left[\frac{1}{2\left\|\underline{x}^{*}\right\|_{1}}\left(\zeta \frac{\|\underline{n}\|_{1}}{K} \wedge \alpha\right)\right] \wedge 1\right) \wedge \alpha\right)}+\left(s-t_{1}\right) \mathrm{e}^{-\alpha \delta K}\right) .
\end{aligned}
$$

We used Lemma 5.2 with $\gamma=0$.
We now handle $J_{2,2,2}(\underline{n})$.
Note that $\gamma \leq 1 \wedge \frac{\alpha}{\left\|x^{*}\right\|_{1}}$. We proceed as before with $f$ and $g$, and we use Lemma 5.2 with

$$
\gamma=\left(\frac{1}{2\left\|\underline{x}^{*}\right\|_{1}}\left(\zeta \frac{\|\underline{n}\|_{1}}{K} \wedge \alpha\right)\right) \wedge 1
$$

to get

$$
\begin{aligned}
\left|J_{2,2,2}(\underline{n})\right| & \leq \mathcal{O}(1) K^{q} \mathbb{P}_{\underline{n}}\left(\left\|\underline{N}^{K}\left(t_{1}\right)\right\|_{1} \leq\left(\left(\frac{1}{2\left\|\underline{x}^{*}\right\|_{1}}\left(\zeta \frac{\|\underline{n}\|_{1}}{K} \wedge \alpha\right)\right) \wedge 1\right) K\right) \\
& \leq \mathcal{O}(1) K^{q}\left(\mathrm{e}^{-\delta\left(\frac{1}{2}\left(\zeta \frac{\|\underline{n}\|_{1}}{K} \wedge \alpha\right) \wedge\left\|\underline{x}^{*}\right\|_{1}\right) K}+C t_{1} \mathrm{e}^{-\frac{\alpha \delta K}{2}}\right)
\end{aligned}
$$

(3) Let us now estimate for all $\underline{n} \neq \underline{0}$

$$
\left|J_{1,2}(\underline{n})-\nu_{K}(f)^{2}\right|
$$

We have

$$
\begin{aligned}
& \left|J_{1,2}(\underline{n})-\nu_{K}(f)^{2}\right| \leq\left|J_{1,2}(\underline{n})-\nu_{K}(g) \mathbb{E}_{\underline{n}}\left[f\left(\underline{N}^{K}\left(t_{1}\right)\right) \mathbb{1}_{\left\{T_{\underline{0}}>t_{1}\right\}} \mathbb{P}_{\underline{N}^{K}\left(t_{1}\right)}\left(T_{\underline{0}}>s-t_{1}\right)\right]\right| \\
& +\left|\nu_{K}(g) \mathbb{E}_{\underline{n}}\left(f\left(\underline{N}^{K}\left(t_{1}\right)\right) \mathbb{1}_{\left\{T_{\underline{0}}>t_{1}\right\}} \mathbb{P}_{\underline{N}^{K}\left(t_{1}\right)}\left(T_{\underline{0}}>s-t_{1}\right)\right)-\nu_{K}(g) \mathbb{E}_{\underline{n}}\left(f\left(\underline{N}^{K}\left(t_{1}\right)\right) \mathbb{1}_{\left\{T_{\underline{0}}>t_{1}\right\}}\right)\right| \\
& +\left|\nu_{K}(g) \mathbb{E}_{\underline{n}}\left(f\left(\underline{N}^{K}\left(t_{1}\right)\right) \mathbb{1}_{\left\{T_{\underline{0}}>t_{1}\right\}}\right)-\nu_{K}(g) \nu_{K}(f)\right|+\left|\nu_{K}(g) \nu_{K}(f)-\nu_{K}(f)^{2}\right| \\
& =W_{1}(\underline{n})+W_{2}(\underline{n})+W_{3}(\underline{n})+W_{4} .
\end{aligned}
$$

(3)-(i) By Theorem 5.1 and since $\underline{N}^{K}\left(t_{1}\right) \neq \underline{0}$, we have

$$
\left|\mathbb{E}_{\underline{N}^{K}\left(t_{1}\right)}\left[\mathbb{1}_{\left\{T_{\underline{0}}>s-t_{1}\right\}} g\left(\underline{N}^{K}\left(s-t_{1}\right)\right)\right]-\mathbb{P}_{\underline{N}^{K}\left(t_{1}\right)}\left(T_{\underline{0}}>s-t_{1}\right) \nu(g)\right| \leq \mathcal{O}(1) K^{q} \mathrm{e}^{-a \frac{\left(s-t_{1}\right)}{\log K}}
$$

Hence, using Proposition 3.3, we get for all $\underline{n} \neq \underline{0}$

$$
W_{1}(\underline{n}) \leq \mathcal{O}(1) K^{q}\left(c_{q}\|n\|_{1}^{q}+K^{q}\right) \mathrm{e}^{-a \frac{\left(s-t_{1}\right)}{\log K}}
$$

(3)-(ii) We have

$$
\begin{aligned}
\mid \nu_{K}(g) \mathbb{E}_{\underline{n}} & {\left[f\left(\underline{N}^{K}\left(t_{1}\right)\right) \mathbb{1}_{\left\{T_{\underline{0}}>t_{1}\right\}} \mathbb{P}_{\underline{N}^{K}\left(t_{1}\right)}\left(T_{\underline{0}}>s-t_{1}\right)\right]-\nu_{K}(g) \mathbb{E}_{\underline{n}}\left[f\left(\underline{N}^{K}\left(t_{1}\right)\right) \mathbb{1}_{\left\{T_{\underline{0}}>t_{1}\right\}}\right] \mid } \\
& \leq\left|\nu_{K}(g)\right| \mathbb{E}_{\underline{n}}\left(\left|f\left(\underline{N}^{K}\left(t_{1}\right)\right)\right| \mathbb{1}_{\left\{T_{\underline{0}}>t_{1}\right\}} \mathbb{P}_{\underline{N}^{K}\left(t_{1}\right)}\left(T_{\underline{0}} \leq s-t_{1}\right)\right)
\end{aligned}
$$

Define $0<\gamma^{\prime}<\beta$ by

$$
\gamma^{\prime}=\gamma^{\prime}(\underline{n})=\frac{1}{2}\left(\frac{\zeta\|\underline{n}\|_{1}}{K} \wedge \alpha\right) .
$$

We split the right hand side in two terms:

$$
\begin{aligned}
\mathbb{E}_{\underline{n}} & \left.\left|f\left(\underline{N}^{K}\left(t_{1}\right)\right)\right| \mathbb{1}_{\left\{T_{\underline{0}}>t_{1}\right\}} \mathbb{P}_{\underline{N}^{K}\left(t_{1}\right)}\left(T_{\underline{0}} \leq s-t_{1}\right)\right] \\
= & \mathbb{E}_{\underline{n}}\left[\mathbb{1}_{\left\{\| \| \mathbf{N}^{K}\left(t_{1}\right) \|_{1} \leq \gamma^{\prime} K\right\}}\left|f\left(\underline{N}^{K}\left(t_{1}\right)\right)\right| \mathbb{1}_{\left\{T_{0}>t_{1}\right\}} \mathbb{P}_{\underline{N}^{K}\left(t_{1}\right)}\left(T_{\underline{0}} \leq s-t_{1}\right)\right] \\
& \quad+\mathbb{E}_{\underline{n}}\left[\mathbb{1}_{\left\{\left\|\underline{N}^{K}\left(t_{1}\right)\right\|_{1}>\gamma^{\prime} K\right\}}\left|f\left(\underline{N}^{K}\left(t_{1}\right)\right)\right| \mathbb{1}_{\left\{T_{0}>t_{1}\right\}} \mathbb{P}_{\underline{N}^{K}\left(t_{1}\right)}\left(T_{\underline{0}} \leq s-t_{1}\right)\right]
\end{aligned}
$$

The first term is estimated using the growth property of $f$, Lemma 5.2 , and Cauchy-Schwarz inequality, namely

$$
\begin{aligned}
\mathbb{E}_{\underline{n}} & \left.\mathbb{1}_{\left\{\left\|\underline{N}^{K}\left(t_{1}\right)\right\|_{1} \leq \gamma^{\prime} K\right\}}\left|f\left(\underline{N}^{K}\left(t_{1}\right)\right)\right| \mathbb{1}_{\left\{T_{0}>t_{1}\right\}} \mathbb{P}_{\underline{N}^{K}\left(t_{1}\right)}\left(T_{\underline{0}} \leq s-t_{1}\right)\right] \\
& \leq \mathbb{E}_{\underline{n}}\left[\mathbb{1}_{\left\{\left\|\underline{N}^{K}\left(t_{1}\right)\right\|_{1} \leq \gamma^{\prime} K\right\}}\left|f\left(\underline{N}^{K}\left(t_{1}\right)\right)\right|^{2}\right]^{\frac{1}{2}} \mathbb{P}_{\underline{n}}\left(\left\|\underline{N}^{K}\left(t_{1}\right)\right\|_{1} \leq \gamma^{\prime} K\right)^{\frac{1}{2}} \\
& \leq \mathcal{O}(1) K^{q} \mathbb{P}_{\underline{n}}\left(\left\|\underline{N}^{K}\left(t_{1}\right)\right\|_{1} \leq \gamma^{\prime} K\right)^{\frac{1}{2}} \\
& \leq \mathcal{O}(1) K^{q}\left(\mathrm{e}^{-\frac{\delta}{2}\left(\zeta \frac{\|\underline{n}\|_{1}}{K} \wedge \alpha\right) K}+\mathcal{O}(1) t_{1} \mathrm{e}^{-\frac{\alpha \delta K}{2}}\right)^{\frac{1}{2}} .
\end{aligned}
$$

To deal with the second term, we observe using Lemma 5.2 and Proposition 3.3 that, if $\left\|\underline{N}^{K}\left(t_{1}\right)\right\|_{1}>\gamma^{\prime} K$, then

$$
\begin{aligned}
\mathbb{P}_{\underline{N}^{K}\left(t_{1}\right)}\left(T_{\underline{0}} \leq s-t_{1}\right) & \leq \mathrm{e}^{-\delta\left(\zeta \frac{\left\|\underline{N}^{K}\left(t_{1}\right)\right\|}{K} \wedge \alpha\right) K}+\mathcal{O}(1)\left(s-t_{1}\right) \mathrm{e}^{-\alpha \delta K} \\
& \leq \mathrm{e}^{-\delta\left(\zeta \gamma^{\prime} \wedge \alpha\right) K}+\mathcal{O}(1)\left(s-t_{1}\right) \mathrm{e}^{-\alpha \delta K} \\
& =\mathrm{e}^{\left.-\delta\left(\zeta\left(\frac{1}{2}\right)\left(\frac{\zeta\|n\| \|_{1}}{K} \wedge \alpha\right)\right) \wedge \alpha\right) K}+\mathcal{O}(1)\left(s-t_{1}\right) \mathrm{e}^{-\alpha \delta K} .
\end{aligned}
$$

Now

$$
\begin{aligned}
\mathbb{E}_{\underline{n}}[ & \left.\mathbb{1}_{\left\{\left\|\underline{N}^{K}\left(t_{1}\right)\right\|_{1}>\gamma^{\prime} K\right\}}\left|f\left(\underline{N}^{K}\left(t_{1}\right)\right)\right| \mathbb{1}_{\left\{T_{\underline{0}}>t_{1}\right\}} \mathbb{P}_{\underline{N}^{K}\left(t_{1}\right)}\left(T_{\underline{0}} \leq s-t_{1}\right)\right] \\
\leq & \mathcal{O}(1) \mathbb{E}_{\underline{n}}\left[\mathbb{1}_{\left\{\left\|\underline{N}^{K}\left(t_{1}\right)\right\| 1>\gamma^{\prime} K\right\}}\left|f\left(\underline{N}^{K}\left(t_{1}\right)\right)\right| \mathbb{1}_{\left\{T_{\underline{0}}>t_{1}\right\}}\right] \\
& \times\left(\mathrm{e}^{-\delta\left(\zeta\left(\frac{1}{2}\left(\frac{\zeta\|n\|_{1}}{K} \wedge \alpha\right)\right) \wedge \alpha\right) K}+\mathcal{O}(1)\left(s-t_{1}\right) \mathrm{e}^{-\alpha \delta K}\right) \\
\leq & \mathcal{O}(1)\left(\|\underline{n}\|^{q}+K^{q}\right)\left(\mathrm{e}^{-\delta\left(\zeta\left(\frac{1}{2}\left(\frac{\zeta\|n\|_{1}}{K} \wedge \alpha\right)\right) \wedge \alpha\right) K}+\mathcal{O}(1)\left(s-t_{1}\right) \mathrm{e}^{-\alpha \delta K}\right) \\
& \times\left(\mathrm{e}^{-\delta \frac{(1 \wedge \zeta)}{2}\left(\frac{\zeta^{2}\|\underline{n}\|_{1}}{2 K} \wedge \alpha\right.}\right) K \\
& \left.\mathcal{O}(1)\left(s-t_{1}\right) \mathrm{e}^{-\alpha \delta K}\right) .
\end{aligned}
$$

(3)-(iii) Let us now prove that for all $\underline{n} \neq \underline{0}$,

$$
\begin{align*}
& W_{3}(\underline{n})=\left|\mathbb{E}_{\underline{n}}\left(f\left(\underline{N}^{K}\left(t_{1}\right)\right) \mathbb{1}_{\left\{T_{0}>t_{1}\right\}}\right)-\nu_{K}(f)\right| \\
& \leq \mathcal{O}(1)\left(c_{q}\|\underline{n}\|_{1}^{q}+K^{q}\right)\left(\mathrm{e}^{-\frac{a\left(t_{1}-1\right)}{\log K}}+\mathrm{e}^{-\lambda_{0}(K)} \mathrm{e}^{-\delta\left(\zeta \frac{\|n\|_{1}}{K} \wedge \alpha\right) K}\right. \\
&\left.\quad+C\left(t_{1}-1\right) \mathrm{e}^{-\alpha \delta K}+1-\mathrm{e}^{-\lambda_{0}(K)}\right) \tag{A.1}
\end{align*}
$$

For $0 \leq t_{1} \leq 1$, using Proposition 3.3 we obtain

$$
\left|\mathbb{E}_{\underline{n}}\left[f\left(\underline{N}^{K}\left(t_{1}\right)\right) \mathbb{1}_{\left\{T_{0}>t_{1}\right\}}\right]\right| \leq \mathcal{O}(1)\left(c_{q}\|\underline{n}\|_{1}^{q}+K^{q}\right) .
$$

We now deal with $t_{1}>1$. By the Markov property one has

$$
\begin{aligned}
\mathbb{E}_{\underline{n}}\left[f\left(\underline{N}^{K}\left(t_{1}\right)\right) \mathbb{1}_{\left\{T_{\underline{0}}>t_{1}\right\}}\right] & =\mathbb{E}_{\underline{n}}\left[\mathbb{1}_{\left\{T_{\underline{0}}>t_{1}-1\right\}} \mathbb{E}_{\underline{N}^{K}\left(t_{1}-1\right)}\left[f\left(\underline{N}^{K}(1)\right) \mathbb{1}_{\left\{T_{\underline{0}}>1\right\}}\right]\right] \\
& =\mathbb{E}_{\underline{n}}\left[\mathbb{1}_{\left\{T_{\underline{0}}>t_{1}-1\right\}} \tilde{g}\left(\underline{N}^{K}\left(t_{1}-1\right)\right)\right]
\end{aligned}
$$

where

$$
\tilde{g}(\underline{n})=\mathbb{E}_{\underline{n}}\left[\underline{N}^{K}(1) \mathbb{1}_{\left\{T_{\underline{0}}>1\right\}}\right] \leq g(\underline{n})
$$

is a function bounded by $\mathcal{O}(1) K^{q}$. For $\underline{n} \neq \underline{0}$, we use Theorem 5.1 and Corollary 3.2 to get

$$
\begin{gathered}
\mid \mathbb{E}_{\underline{n}}\left[\mathbb{1}_{\left\{T_{\underline{0}}>t_{1}-1\right\}} \mathbb{E}_{\underline{N}^{K}\left(t_{1}-1\right)}\left[f\left(\underline{N}^{K}(1)\right) \mathbb{1}_{\left\{T_{\underline{0}}>1\right\}}\right]\right] \\
-\mathbb{P}_{\underline{n}}\left(T_{\underline{0}}>t_{1}-1\right) \mathbb{E}_{\nu_{K}}\left[f\left(\underline{N}^{K}(1)\right) \mathbb{1}_{\left\{T_{\underline{0}}>1\right\}}\right] \mid \\
\leq \mathcal{O}(1) K^{q} \mathrm{e}^{-a \frac{\left(t 1_{1}-1\right)}{\log K}} .
\end{gathered}
$$

Since $\nu_{K}$ is the qsd, we have

$$
\mathbb{E}_{\nu_{K}}\left[f\left(\underline{N}^{K}(1)\right) \mathbb{1}_{\left\{T_{\underline{0}}>1\right\}}\right]=\mathrm{e}^{-\lambda_{0}(K)} \nu_{K}(f) .
$$

Using Corollary 4.2, Lemma 5.2 and the properties of $f$ we obtain

$$
\begin{aligned}
& \left|\mathbb{P}_{\underline{n}}\left(T_{\underline{0}}>t_{1}-1\right) \mathbb{E}_{\nu_{K}}\left[f\left(\underline{N}^{K}(1)\right) \mathbb{1}_{\left\{T_{0}>1\right\}}\right]-\nu_{K}(f)\right| \\
& \quad \leq \mathcal{O}(1) K^{q}\left(\mathrm{e}^{-\lambda_{0}(K)} \mathrm{e}^{-\delta\left(\zeta \frac{\|n\|_{1}}{K} \wedge \alpha\right) K}+C\left(t_{1}-1\right) \mathrm{e}^{-\beta \delta K}+1-\mathrm{e}^{-\lambda_{0}(K)}\right)
\end{aligned}
$$

and (A.1) is proved.
(3)-(iv) Let us note that

$$
W_{4} \leq\left|\nu_{K}(f)\right|\left|\nu_{K}(f)-\nu_{K}(g)\right| .
$$

Proposition 3.3 and Lemma 5.5 give

$$
W_{4} \leq \mathcal{O}(1) K^{2 q}\left(1-\mathrm{e}^{-\lambda_{0}(K)}\right)^{\frac{1}{2}}
$$

(3)-(v) Collecting the informations given in the four previous estimates, we obtain a precise estimation of $\left|J_{1,2}(\underline{n})-\nu_{K}(f)^{2}\right|$ for all $\underline{n} \neq \underline{0}$.
(3)-(vi) We have

$$
\begin{aligned}
\mid \mathbb{E}_{\underline{n}} & {\left[f\left(\underline{N}^{K}\left(t_{1}\right)\right) \mathbb{E}_{\underline{N}^{K}\left(t_{1}\right)}\left[g\left(\underline{N}^{K}\left(s-t_{1}\right)\right)\right]\right]-\nu_{K}(f)^{2} \mid } \\
& \leq\left|J_{2}(\underline{n})\right|+\left|J_{1,2}(\underline{n})-\nu_{K}(f)^{2}\right| .
\end{aligned}
$$

Collecting the above relevant estimates we obtain that there exist $\delta^{\prime}, \zeta^{\prime}, \beta^{\prime}, \theta^{\prime}$ (all being positive and independent of $K$ ) such that

$$
\begin{aligned}
\mid \mathbb{E}_{\underline{n}} & {\left[f\left(\underline{N}^{K}\left(t_{1}\right)\right) \mathbb{E}_{\underline{N}^{K}\left(t_{1}\right)}\left[g\left(\underline{N}^{K}\left(s-t_{1}\right)\right)\right]\right]-\nu_{K}(f)^{2} \mid \leq \mathcal{O}(1) K^{q}\left(c_{q}\|\underline{n}\|_{1}^{q}+K^{q}\right) } \\
& \left.\times\left(\mathbb{1}_{\left\{t_{1} \leq 1\right\}}+\mathrm{e}^{-\delta^{\prime}\left(\zeta^{\prime} \frac{\| n}{k} \|_{1}\right.} \wedge^{K}\right) \beta^{\prime}\right) K \\
& \left.\left(s+t_{1}+1\right) \mathrm{e}^{-\theta^{\prime} K}+\mathrm{e}^{-\frac{a\left(s-t_{1}\right)}{\log K}}+\mathrm{e}^{-\frac{a t_{1}}{\log K}}\right) .
\end{aligned}
$$

Now we have

$$
\begin{aligned}
& \frac{2}{T^{2}}\left|\int_{0}^{T-1} \mathrm{~d} s \int_{0}^{s} \mathbb{E}_{\underline{n}}\left[f\left(\underline{N}^{K}\left(t_{1}\right)\right) \mathbb{E}_{\underline{N}^{K}\left(t_{1}\right)}\left[g\left(\underline{N}^{K}\left(s-t_{1}\right)\right)\right]\right] \mathrm{d} t_{1}-\nu_{K}(f)^{2}\right| \\
& \quad \leq \mathcal{O}(1) K^{q}\left(c_{q}\|\underline{n}\|_{1}^{q}+K^{q}\right)\left(\frac{1}{T}+\mathrm{e}^{-\delta^{\prime}\left(\zeta^{\prime} \frac{\|n\|_{1}}{K} \wedge \beta^{\prime}\right) K}+T \mathrm{e}^{-\theta^{\prime} K}+\frac{\log K}{T}\right) .
\end{aligned}
$$

The final result for $T \geq 1$ follows by collecting all estimates. For $T<1$ the bound follows directly from Proposition 3.3.

## A. 2 Starting from the qsd: proof of Corollary 7.2

The result follows from Proposition 7.1 by integrating over $\underline{n}$ with respect to the qsd. More precisely, we have

$$
\begin{aligned}
& \mathbb{E}_{\underline{n}} {\left[\left|S_{f}(T, K)-\nu_{K}(f)\right|^{2}\right] \leq C^{\prime}\|f\|_{K, q}^{2}\left(\frac{\left(c_{q}\|\underline{n}\|_{1}^{q}+K^{q}\right)\|\underline{n}\|_{1}^{q}+K^{q} \log K}{T \vee 1}\right.} \\
&\left.\quad+\left(c_{q}\|\underline{n}\|_{1}^{q}+K^{q}\right) K^{q} \mathrm{e}^{-\delta^{\prime}\left(\zeta^{\prime} \frac{\|n\|_{1}}{K} \wedge \beta^{\prime}\right) K}+\left(c_{q}\|\underline{n}\|_{1}^{q}+K^{q}\right) T K^{q} \mathrm{e}^{-\theta^{\prime} K}\right) .
\end{aligned}
$$

The integrals of the first and third terms with respect to the q.s.d are estimated using Corollary 4.2. We deal with second term:

$$
\begin{aligned}
& \int\left(c_{q}\|\underline{n}\|_{1}^{q}+K^{q}\right) K^{q} \mathrm{e}^{-\delta^{\prime}\left(\zeta^{\prime} \frac{\|n\|_{1}}{K} \wedge \beta^{\prime}\right) K} \mathrm{~d} \nu_{K}(\underline{n})= \\
& \quad \int \mathbb{1}_{\left.\left\{\left\{\|\underline{n}\|_{1}<\beta^{\prime} K / \zeta^{\prime}\right\}\right\} \cap \mathcal{D}\right\}}\left(c_{q}\|\underline{n}\|_{1}^{q}+K^{q}\right) K^{q} \mathrm{e}^{-\delta^{\prime}\left(\zeta^{\prime} \frac{\|n\|_{1}}{K} \wedge \beta^{\prime}\right) K} \mathrm{~d} \nu_{K}(\underline{n}) \\
& \quad+\int \mathbb{1}_{\left.\left\{\left\{\|\underline{n}\|_{1}<\beta^{\prime} K / \zeta^{\prime}\right\}\right\} \cap \mathcal{D}^{c}\right\}}\left(c_{q}\|\underline{n}\|_{1}^{q}+K^{q}\right) K^{q} \mathrm{e}^{-\delta^{\prime}\left(\zeta^{\prime} \frac{n n \|_{1}}{K} \wedge \beta^{\prime}\right) K} \mathrm{~d} \nu_{K}(\underline{n}) \\
& \quad+\int \mathbb{1}_{\left\{\|\underline{n}\|_{1} \geq \beta^{\prime} K / \zeta^{\prime}\right\}}\left(c_{q}\|\underline{n}\|_{1}^{q}+K^{q}\right) K^{q} \mathrm{e}^{-\delta^{\prime}\left(\zeta^{\prime} \frac{\|n\|_{1}}{K} \wedge \beta^{\prime}\right) K} \mathrm{~d} \nu_{K}(\underline{n}) .
\end{aligned}
$$

The third integral is estimated using the fact that the integrand is exponentially small in $K$. The second integral is estimated using the first estimate in Corollary 4.2. We finally deal with the first integral. If $\underline{n} \in \mathcal{D}$ then $\|\underline{n}\|_{1} \geq\|\underline{n}\|_{2} \geq\left\|\underline{n}^{*}\right\|_{2} / 2$. If $\left\{\|\underline{n}\|_{1}<\beta^{\prime} K / \zeta^{\prime}\right\} \cap \mathcal{D} \neq \emptyset$, on this set we have $\mathrm{e}^{-\delta^{\prime}\left(\zeta^{\prime} \frac{\|n\| \|_{1}}{K} \wedge \beta^{\prime}\right) K} \leq \mathrm{e}^{-\delta^{\prime} \zeta^{\prime}\left\|\underline{n}^{*}\right\|_{2} / 2}$ (exponentially small in $K$ ). The estimate follows.

## B Counting the number of births

Denote by $\mathcal{N}_{\ell}^{K}\left(t_{1}, t_{2}\right)$ the number of births of species of type $\ell$ between the times $t_{1}$ and $t_{2}\left(1 \leq \ell \leq d, 0 \leq t_{1} \leq t_{2}\right)$.

Proposition B.1. For any probability measure $\mathfrak{m}$ on $\mathbb{Z}_{+}^{d}$, we have

$$
\mathbb{E}_{\mathfrak{m}}\left[\mathcal{N}_{\ell}^{K}\left(t_{1}, t_{2}\right)\right]=K \int_{t_{1}}^{t_{2}} \mathbb{E}_{\mathfrak{m}}\left[B_{\ell}\left(\frac{\underline{N}^{K}(s)}{K}\right)\right] \mathrm{d} s
$$

and

$$
\begin{gathered}
\mathbb{E}_{\mathfrak{m}}\left[\left(\mathcal{N}_{\ell}^{K}\left(t_{1}, t_{2}\right)-\mathbb{E}_{\mathfrak{m}}\left[\mathcal{N}_{\ell}^{K}\left(t_{1}, t_{2}\right)\right]\right)^{2}\right] \leq 2 K \mathbb{E}_{\mathfrak{m}}\left[\int_{t_{1}}^{t_{2}} B_{\ell}\left(\frac{\underline{N}^{K}(s)}{K}\right) \mathrm{d} s\right] \\
+\mathbb{E}_{\mathfrak{m}}\left[\left(\int_{t_{1}}^{t_{2}} K B_{\ell}\left(\frac{\underline{N}^{K}(s)}{K}\right) \mathrm{d} s-\mathbb{E}_{\mathfrak{m}}\left[\mathcal{N}_{\ell}^{K}\left(t_{1}, t_{2}\right)\right]\right)^{2}\right]
\end{gathered}
$$

Proof. Recall that the generator of the process is given in (3.1). Let us now give a pathwise representation of the process. We introduce $d$ independent point Poisson measures $M_{\ell}(\mathrm{d} s, \mathrm{~d} \theta)$ on $\mathbb{R}_{+} \times \mathbb{R}_{+}$with intensity $\mathrm{d} s \mathrm{~d} \theta$. We define the $d$-dimensionnal càd-làg process $\left(N_{t}, t \in \mathbb{R}_{+}\right)$

$$
\begin{aligned}
N_{t}= & N_{0}+\sum_{\ell=1}^{d} \int_{0}^{t} \int M_{\ell}(\mathrm{d} s, \mathrm{~d} \theta) \\
& \times\left(\mathbb{1}_{\left.\left\{\theta \leq K B_{\ell}\left(\frac{N^{K}(s)}{K}\right)\right\}^{-1}\left\{K B_{\ell}\left(\frac{N^{K}(s)}{K}\right) \leq \theta \leq K\left(B_{\ell}\left(\frac{N^{K}(s)}{K}\right)+D_{\ell}\left(\frac{N^{K}(s)}{K}\right)\right)\right\}\right) .} .\right.
\end{aligned}
$$

Then the number of births of species between the times $t_{1}$ and $t_{2}$ is given by

$$
\mathcal{N}_{\ell}^{K}\left(t_{1}, t_{2}\right)=\int_{t_{1}}^{t_{2}} \int \mathbb{1}_{\left\{\theta \leq K B_{\ell}\left(\frac{N^{K}(s)}{K}\right)\right\}} M_{\ell}(\mathrm{d} s, \mathrm{~d} \theta)
$$

Using the Markov property we get at once the first identity.

We now establish the estimate. Indeed

$$
\begin{aligned}
& \mathbb{E}_{\mathfrak{m}}\left[\left(\mathcal{N}_{\ell}^{K}\left(t_{1}, t_{2}\right)-\mathbb{E}_{\mathfrak{m}}\left(\mathcal{N}_{\ell}^{K}\left(t_{1}, t_{2}\right)\right)^{2}\right]\right. \\
& \leq 2 \mathbb{E}_{\mathfrak{m}}\left[\left(\mathcal{N}_{\ell}^{K}\left(t_{1}, t_{2}\right)-\int_{t_{1}}^{t_{2}} K B_{\ell}\left(\frac{\underline{N}^{K}(s)}{K}\right) \mathrm{d} s\right)^{2}\right] \\
& \quad+2 \mathbb{E}_{\mathfrak{m}}\left[\left(\int_{t_{1}}^{t_{2}} K B_{\ell}\left(\frac{\underline{N}^{K}(s)}{K}\right) \mathrm{d} s-\mathbb{E}_{\mathfrak{m}}\left[\mathcal{N}_{\ell}^{K}\left(t_{1}, t_{2}\right)\right]\right)^{2}\right]
\end{aligned}
$$

By the $\mathbb{L}^{2}$-isometry for jump processes (see [7] Formula (3.9) p.62), we have

$$
\begin{aligned}
\mathbb{E}_{\mathfrak{m}} & {\left[\left(\mathcal{N}_{\ell}^{K}\left(t_{1}, t_{2}\right)-\int_{t_{1}}^{t_{2}} K B_{\ell}\left(\frac{\underline{N}^{K}(s)}{K}\right) \mathrm{d} s\right)^{2}\right] } \\
& =\int_{t_{1}}^{t_{2}} \int \mathbb{E}_{\mathfrak{m}}\left(\left(\mathbb{1}_{\left\{\theta \leq K B_{\ell}\left(\frac{N^{K}(s)}{K}\right)\right\}}\right)^{2}\right) \mathrm{d} s \mathrm{~d} \theta \\
& =\int_{t_{1}}^{t_{2}} \mathbb{E}_{\mathfrak{m}}\left[K B_{\ell}\left(\frac{N^{K}(s)}{K}\right)\right] \mathrm{d} s
\end{aligned}
$$

This finishes the proof.

## C A numerical example

We consider the two-dimensional vector fields

$$
\underline{B}\left(x_{1}, x_{2}\right)=\binom{a x_{1}+b x_{2}}{e x_{1}+f x_{2}}
$$

and

$$
\underline{D}\left(x_{1}, x_{2}\right)=\binom{x_{1}\left(c x_{1}+d x_{2}\right)}{x_{2}\left(g x_{1}+h x_{2}\right)}
$$

where all the coefficients are positive. This is a model of competition between two species of Lotka-Volterra type. We have taken

$$
\begin{aligned}
& a=0.4569, b=0.2959, e=0.5920, f=0.6449 \\
& c=0.9263, d=0.9157, g=0.9971, h=0.2905
\end{aligned}
$$

Assumptions H. 1 and $\mathbf{H .} 4$ are easily verified numerically. Assumptions H. 5 and H. 6 are true because $\widehat{B}(y) \leq a+b+e+f$ and $\widehat{D}(y) \geq(c \wedge h) y^{2} / 4$. Concerning H.2, we checked numerically that there is a unique fixed point inside the positive quadrant, namely $\underline{x}^{*}=(0.3567,1.4855)$. It remains to check H.3, namely that

$$
-\beta=\sup _{\underline{x} \in \mathbb{R}_{+}^{2}} R(\underline{x})<0
$$

where

$$
R(\underline{x})=\frac{\left\langle\underline{X}(\underline{x}),\left(\underline{x}-\underline{x}^{*}\right)\right\rangle}{\|\underline{x}\|_{2}\left\|\underline{x}-\underline{x}^{*}\right\|_{2}^{2}} .
$$

We first checked that the numerator $N(\underline{x})=\left\langle\underline{X}(\underline{x}),\left(\underline{x}-\underline{x}^{*}\right)\right\rangle$ is negative and vanishes only in $\underline{0}$ and $\underline{x}^{*}$. It is easy to check that $N(\underline{x})<0$ for $\|\underline{x}\|_{2}$ large enough. We have verified numerically that the only solutions of the equations $\partial_{x_{1}} N=\partial_{x_{2}} N=0$ in the closed positive quadrant are $\underline{x}^{*}$ and $\underline{z}=(0.1739,0.4361)$, with $N(\underline{z})=-0.2852$, thus this is negative local minimum. This implies that $N(\underline{x})<0$ in the closed positive quadrant, except at $\underline{0}$ and $\underline{x}^{*}$ where it vanishes. This implies that $R \leq 0$ in the closed positive quadrant. It is easy to check that

$$
\limsup _{\|\underline{x}\|_{2} \rightarrow+\infty} R(\underline{x}) \leq-(c \wedge h) / \sqrt{2} .
$$

This implies that $R<0$ except perhaps in $\underline{0}$ and $\underline{x}^{*}$. Near $\underline{0}$ we have by Taylor expansion

$$
R(\underline{x})=-\frac{\left\langle D \underline{X}(0) \underline{x}, \underline{x}^{*}\right\rangle}{\|\underline{x}\|_{2}\left\|\underline{x}^{*}\right\|_{2}^{2}}\left(1+\mathcal{O}\left(\|\underline{x}\|_{2}\right)\right)=-\frac{\left\langle\underline{x},{ }^{\mathrm{t}} D \underline{B}(0) \underline{x}^{*}\right\rangle}{\|\underline{x}\|_{2}\left\|\underline{x}^{*}\right\|_{2}^{2}}\left(1+\mathcal{O}\left(\|\underline{x}\|_{2}\right)\right)
$$

and, since the vector ${ }^{\mathrm{t}} D \underline{B}(0) \underline{x}^{*}$ has positive components, there exists $\varrho>0$ such that for all $\underline{x} \in \mathbb{R}_{+}^{2}$

$$
\left\langle\underline{x},{ }^{\mathrm{t}} D \underline{B}(0) \underline{x}^{*}\right\rangle \geq \varrho\|\underline{x}\|_{2} .
$$

If $\underline{y}=\underline{x}-\underline{x}^{*}$ is small, we have by Taylor expansion $\left(\right.$ since $\left.\underline{X}\left(\underline{x}^{*}\right)=\underline{0}\right)$

$$
R(\underline{x})=\frac{\left\langle M^{*} \underline{y}, \underline{y}\right\rangle}{\left\|\underline{x}^{*}\right\|_{2}\|\underline{y}\|_{2}^{2}}\left(1+\mathcal{O}\left(\|\underline{y}\|_{2}\right)\right)=\frac{\left.\left\langle\underline{y}, \frac{1}{2}\left({ }^{\mathrm{t}} M^{*}+M^{*}\right) \underline{y}\right)\right\rangle}{\left\|\underline{x}^{*}\right\|_{2}\|\underline{y}\|_{2}^{2}}\left(1+\mathcal{O}\left(\|\underline{y}\|_{2}\right)\right) .
$$

One can check numerically that the two real eigenvalues of the symmetric matrix

$$
{ }^{\mathrm{t}} M^{*}+M^{*}
$$

are strictly negative, the largest being numerically equal to -0.786 . This completes the verification of hypothesis H.3.

We have chosen $K=10^{5}$ and simulated a realization of the process with $T=100$ which contains about $5.10^{7}$ jumps. The resilience computed from the vector field is numerically equal to 0.547 . We have computed $\rho_{\text {emp }}^{*}\left(100,1,10^{5}\right)$. The relative error, that is $\left|\rho_{\text {emp }}^{*}\left(100,1,10^{5}\right)-\rho^{*}\right| / \rho^{*}$, is equal to 0.022 .

We have also checked the size of the error in (1.5). We have computed

$$
\varepsilon=\frac{1}{\sqrt{K}}\left\|M^{*} S^{\Sigma}(T, K)+S^{\Sigma}(T, K)^{\mathrm{t}} M^{*}+2 \mathcal{D}^{K}\right\|
$$

where $\|\cdot\|$ stands for Hilbert-Schmidt norm. For the above values of $T$ and $K, \varepsilon=68$, which corresponds to a rather large error term. However, we have also checked that this quantity depends strongly on $T$. Namely, for $K=2000$ and $T=2000$ (which has roughly the same computational cost), we have found $\varepsilon=3.19$.

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