



**HAL**  
open science

## Online graph coloring with bichromatic exchanges

Sylvain Gravier, Marc Heinrich

► **To cite this version:**

Sylvain Gravier, Marc Heinrich. Online graph coloring with bichromatic exchanges. 2019. hal-02167055

**HAL Id: hal-02167055**

**<https://hal.science/hal-02167055>**

Preprint submitted on 27 Jun 2019

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Online graph coloring with bichromatic exchanges

Sylvain Gravier<sup>1</sup> and Marc Heinrich<sup>\*2</sup>

<sup>1</sup>CNRS - Institut Fourier, Maths à Modeler, Grenoble, France.

<sup>2</sup>Université Lyon 1, LIRIS, France

June 27, 2019

## Abstract

Greedy algorithms for the graph coloring problem require a large number of colors, even for very simple classes of graphs. For example, any greedy algorithm coloring trees requires  $\Omega(\log n)$  colors in the worst case. We consider a variation of greedy algorithms in which the algorithm is allowed to make modifications to previously colored vertices by performing local bichromatic exchanges. We show that such algorithms can be used to find an optimal coloring in the case of bipartite graphs, chordal graphs and outerplanar graphs. We also show that it can find colorings of general planar graphs with  $O(\log \Delta)$  colors, where  $\Delta$  is the maximum degree of the graph. The question of whether planar graphs can be colored by an online algorithm with bichromatic exchanges using only a constant number of colors is still open.

**Keywords:** online algorithms, graph coloring, bichromatic exchange, kempe chain.

## 1 Introduction

Online algorithms are a class of algorithms reading their input sequentially. In the case of graph problems, this usually means that the vertices of the graph arrive one by one. As the formal definition of greedy, online, and sequential algorithms is not completely fixed, we start by precisizing the convention we use here. In an online algorithm, for each new vertex, the algorithm must adapt a partial solution of the problem on the graph without the new vertex into a solution for the whole graph. For the graph coloring problem, this means that, if  $G$  is the graph and  $v$  the newly added vertex, the algorithm must transform a coloring of  $G - v$  into a coloring of  $G$ .

A special class of online coloring algorithms which received a lot of attention is the class of greedy coloring algorithms, where the algorithm is not allowed to change his previous choices. The algorithm must assign a color to each new vertex that is different from the colors of its neighbors. Vertices colored at an earlier step cannot be recolored. The most common greedy algorithm for graph coloring is FIRST-FIT, where the color of the new vertex is taken to be the smallest color not already present in its neighborhood. These greedy algorithms are usually studied in a setting where the order on the vertices of the graph is arbitrary. The performance of these algorithms, is measured by the number of colors used for the worst case ordering.

A similar type of algorithms are sequential algorithms. These algorithms first decide on an ordering of the vertices, and then, apply an online algorithm to color the graph according to this ordering. In this case the ordering is chosen by the algorithm.

The greedy graph coloring problem is a widely studied subject. For general graphs, a randomized greedy algorithm finding an  $O(\frac{n}{\log n})$  approximation was devised in [Hal97] while there is a lower bound of  $\Omega(\frac{n}{\log^2 n})$  on the approximation ratio of any greedy algorithm. An important effort has been directed at studying the performance on usual graph classes. A greedy algorithm with an approximation ratio of  $O(\log n)$  was shown for trees [GL88], bipartite graphs [LST89], planar and chordal graphs [Ira94]. On the other hand, this approximation ratio was shown to be optimal for these graph classes [Bea76, AS17, LST89]. The lower bounds even holds for randomized algorithms, algorithms using a small reordering

---

\*Supported by the ANR-14-CE25-0006 project of the French National Research Agency

buffer, or algorithm allowed to look at a few future inputs before making a choice [AS17]. Online algorithms with constant approximation ratio exists for other classes of graphs e.g. interval graphs [KT81, LV98, Smi10, NB08] and disk intersection graphs [CFKP07, EF02, AS17].

Since the approximation ratio of greedy algorithms is quite large even for some simple classes of graphs, it is natural to look at more general algorithms. In this article, we consider online algorithms which are allowed to change the color of previous vertices only by making local bichromatic exchanges. A bichromatic exchange (also called Kempe change in the litterature) consists in swapping the colors of the vertices of a maximal connected 2-colored subgraph of  $G$ . Applying this transformation creates a new proper coloring of the graph. We call these algorithms online algorithm with bichromatic exchanges. For each new vertex  $v$ , the algorithm can perform bichromatic exchanges to remove one color from the neighbors of  $v$ , and use this color for  $v$ .

The choice of bichromatic exchanges as an operation to recolor the graph is quite natural. It was first considered in Kempe's failed attempt at proving the four color theorem, and was used later to prove Vizing's theorem [Viz64]. More recently, sequential algorithm using bichromatic exchange were considered in [MP99] to color a special subclass of perfect graphs, and in [HG96, HGM98, HM97, Tuc87] using more complex recolorings. In this context, bichromatic exchanges are used to locally modify an existing coloring in order to color a new vertex.

Finally, bichromatic exchanges were considered in the context of graph coloring reconfiguration. In this case the problem is, given two colorings of the same graph, to decide whether one can be transformed into the other by applying a sequence of basic operations. The case where the allowed transformation is to recolor a single vertex (a particular case of bichromatic exchange) has attracted a lot of attention. Some results were found for specific graph classes, for example chordal graphs [BJL<sup>+</sup>14] and graphs with bounded treewidth [BB13, BB14]. Questions about the connexity of the reconfiguration graph [CVdHJ09, CVDHJ11] and some complexity aspects of the problem [BC09, CVDHJ11, JKK<sup>+</sup>14] have also been considered. The case where the recoloring operations are bichromatic exchanges has been considered for planar graphs [Mey78, Moh06],  $K_5$ -minor free graphs [LVM81] and  $d$ -regular graphs [FJP15, BBFJ15].

This last problem corresponds to a more global setting where the bichromatic exchanges can be applied anywhere in the graph. In our case, we are interested in local modifications of the coloring. This means that, as in sequential algorithms, the bichromatic exchanges are restricted to the neighborhood of the newly added vertex.

**Overview.** The paper is organized as follows. In the next section, we give standard notations, and a formal definitions of online algorithm with bichromatic exchanges. Then, we describe such algorithms for several classes of graph: bipartite graphs in Section 2.1, chordal graphs in Section 3, outerplanar and planar graphs in Section 4. An extension of the algorithm on planar graphs to the case of graphs with bounded genus is presented in Section 5. Apart from the cases of planar graph and graphs with bounded genus, these algorithm are optimal and robust: they either find an optimal coloring or exhibit a forbidden substructure for this class of graphs. In the case of planar graph, the algorithm only provides a  $\log(\Delta)$  coloring, where  $\Delta$  is the maximum degree of the graph. The question of whether we can achieve a constant number of colors is still open. Finally, in Section 6 we give an example of a family of 3-colorable graph for which any online algorithm using bichromatic exchanges (with some additional restrictions) performs badly.

## 2 Definitions and notations

We start with some standard definitions and notations of usual concepts from graph theory. A graph is a pair  $(V, E)$  where  $V$  is the set of vertices, and  $E$ , the set of edges, is a subset of unordered pairs of distinct vertices. The graphs that we consider are simple, and undirected. Given a graph  $G = (V, E)$  and a vertex  $v$  of  $G$ , we denote by  $N_G(v)$  the neighborhood of  $v$ , i.e.  $N_G(v) = \{u \in V, \{u, v\} \in E\}$ , and  $\Delta(G)$  the maximum degree of the graph  $G$ . When the graph  $G$  is clear from the context, we will omit the subscript and just write  $N(v)$  the neighborhood of  $v$ , and  $\Delta$  the maximum degree. If  $S$  is a subset of vertices, we denote by  $G[S]$  the subgraph induced by all the vertices of  $S$ .

A  $k$ -coloring of a graph  $G = (V, E)$  is a function  $c : V \mapsto \{1, \dots, k\}$  such that there is no monochromatic edges: for every edge  $\{u, v\} \in E$ ,  $c(u) \neq c(v)$ . Given a graph  $G = (V, E)$ , a coloring  $c$  of  $G$ , and a

color  $i$ , we denote by  $c_i$  the set of vertices colored  $i$ . In particular, for any two colors  $i$  and  $j$ ,  $G[c_i]$  is a stable set, and  $G[c_i \cup c_j]$  is the bipartite graph induced by vertices colored  $i$  or  $j$ .

Given a graph  $G$  and two vertices  $u$  and  $v$ , a path between  $u$  and  $v$  is a sequence of distinct vertices  $w_0 = u, w_1, \dots, w_k = v$  such that for all  $0 \leq i < k$ ,  $w_i$  and  $w_{i+1}$  are adjacent in  $G$ . Given a coloring  $c$  of  $G$ , and two colors  $i$  and  $j$ , a path is a bichromatic  $(i, j)$ -path if every vertex of the path is colored either  $i$  or  $j$ .

We look at a particular recoloring procedure called bichromatic exchange. A bichromatic exchange is a transformation that changes a coloring into a new one. Informally this recoloring is done by selecting a vertex, changing its color to a new color, and propagating the change to its neighbors to prevent the creation of monochromatic edges. We give a formal definition below:

**Definition 1.** Let  $G$  be a graph,  $v$  a vertex of  $G$ , and  $j$  a color. We denote by  $\langle v, j \rangle$  the bichromatic exchange that transforms a coloring  $c$  of  $G$  into a coloring  $c' = \langle v, j \rangle(c)$  defined by:

$$c'(x) = \begin{cases} c(x) & \text{if } x \notin X \\ c(v) & \text{if } x \in X \text{ and } c(x) = j \\ j & \text{if } x \in X \text{ and } c(x) = c(v) \end{cases}$$

where  $X$  is the connected component of  $G[c_{c(v)} \cup c_j]$  containing  $v$ .

It is easy to check that this transformation does not create any monochromatic edges. From the definition, we can see that the color of a vertex  $u$  is changed by the bichromatic exchange  $\langle v, i \rangle$  if and only if there is a bichromatic  $(i, c(v))$ -path between  $u$  and  $v$ . Let  $S = \langle v_1, i_1 \rangle, \dots, \langle v_k, i_k \rangle$  be an ordered sequence of bichromatic exchanges, and  $c$  be a coloring. We denote by  $\langle S \rangle(c) = \langle v_k, i_k \rangle \circ \dots \circ \langle v_1, i_1 \rangle(c)$  the coloring obtained after applying successively the bichromatic exchanges of  $S$ . In general, two bichromatic exchanges do not commute, i.e. the order of application of several bichromatic exchanges is important. However, here are a few simple cases where they do commute:

**Remark 1.** Let  $G$  be a graph,  $c$  be a coloring of  $G$ , and  $\langle x, i \rangle, \langle y, j \rangle$  be two bichromatic exchanges.

- If the colors  $i, j, c(x), c(y)$  are all different, then  $\langle x, i \rangle \circ \langle y, j \rangle(c) = \langle y, j \rangle \circ \langle x, i \rangle(c)$ .
- If  $i = j$  and  $c(x) = c(y)$ , then  $\langle x, i \rangle \circ \langle y, j \rangle(c) = \langle y, j \rangle \circ \langle x, i \rangle(c)$ .

The algorithms that we consider are a special case of online algorithms where the algorithm is allowed to recolor previously colored vertices only by performing bichromatic exchanges. Allowing the algorithm to perform any bichromatic exchange would be too powerful as it would allow, in many cases, to recolor the whole graph. To prevent this, the algorithm is only allowed to perform bichromatic exchanges which are local to the newly added vertex in the following sense:

**Definition 2.** Let  $G$  be a graph,  $c$  a coloring of  $G$ , and  $v, x$  two vertices of  $G$ . The bichromatic exchange  $\langle i, x \rangle$  is local to  $v$  if  $x \in N(v)$ .

We will always consider bichromatic exchanges which are local to  $v$ , the (not yet colored) vertex that was just added to the graph. Consequently, we will omit to specify  $v$ , and just write that the bichromatic exchanges are local as a shorthand for local to  $v$ . Note that vertices outside the neighborhood of  $v$  might be recolored by a local bichromatic exchange. Indeed, we only require that the starting point of a local bichromatic exchange is in the neighborhood of  $v$ , but this transformation can still recolor a large part of the graph. We can now give a formal definition of online algorithms with bichromatic exchanges.

**Definition 3.** An online coloring algorithm with bichromatic exchanges is an online algorithms such that, for each new vertex  $v$ :

- it applies a sequence of local bichromatic exchanges,
- and then selects a color for  $v$  not used by any vertex in its neighborhood.

This definition is quite general, however the algorithms that we will consider are more simple: they only perform bichromatic exchanges when necessary. If a color is already available to the new vertex without any recoloring, then the algorithm immediately selects that colors. Additionally, they always choose the smallest color available, and make no assumption on eventual properties of the existing

coloring. In other words, with these restrictions an algorithm with bichromatic exchanges is given by a procedure taking as input a graph  $G$  with a vertex  $v$  and any coloring  $c$  of  $G - v$ , and finding a sequence of bichromatic exchanges such that applying these transformations removes one color from the neighborhood of  $v$ .

In terms of graph coloring reconfiguration, the question this type of algorithms try to answer can be formulated in the following way. Given a graph  $G$ , a vertex  $v$ , and a  $k$ -coloring  $c$  of  $G - v$ , is there a transformation of  $c$  using local bichromatic exchanges into a coloring  $c'$  for which  $N(v)$  is  $(k - 1)$ -colored. Thus, the problem is similar to a coloring reconfiguration problem with two main variations:

- the bichromatic exchanges must be local,
- the target coloring is not given, but can be any coloring with the desired properties.

An other important parameter for this kind of algorithm is the length of the reconfiguration path: the number of bichromatic exchanges that must be applied for each new vertex in the worst case. Since we are interested in only making local changes to the coloring, we want this number to be polynomial in  $\Delta$ .

The question we are interested in is to determine for which classes of graphs does such an algorithm exists and how many colors it requires. In the following, we will show several algorithms for different classes of graphs. Moreover, the algorithms that we will describe are robust in the following sense. For any graph  $G$ , if we run the algorithm  $A$  on  $G$ , then:

- either  $G \in \mathcal{C}$ , and  $A$  finds a coloring of  $G$ ,
- or,  $A$  exhibits a proof that  $G \notin \mathcal{C}$ .

## 2.1 Bipartite graphs

The first simple class of graph that we might look at are bipartite graphs. We prove the following:

**Theorem 2.** *There is an online coloring algorithm with bichromatic exchange finding a 2-coloring of bipartite graphs using at most  $\Delta$  operations at each step.*

*Proof.* Let  $G$  be the graph,  $v$  a vertex of  $G$ , and  $c$  the coloring of  $G - v$  obtained from the previous steps of the algorithm. If one of the two colors is not present in the neighborhood of  $v$ , then this color can be used to color  $v$ . Otherwise, the recoloring procedure is the following: while there is a vertex  $u \in N(v)$  colored 2, apply the bichromatic exchange  $\langle u, 1 \rangle$ . None of the vertices in  $N(v)$  colored 1 can change color back to 2. Indeed, assume by contradiction that at some step a vertex  $w \in N(v)$  changes color from 1 to 2 during the bichromatic exchange  $\langle u, 1 \rangle$ . This implies that there is a path  $p$  of odd length from  $u$  to  $w$  in  $G - v$ . Consequently,  $p \cup \{v\}$  is an odd cycle, a contradiction of the assumption that  $G$  is bipartite.

When the procedure ends, after at most  $\Delta$  bichromatic exchanges, all the vertices in  $N(v)$  are colored 1, and  $v$  can be colored 2.  $\square$

In particular, this algorithm colors optimally any tree, while the best greedy online algorithm needs  $\Omega(\log n)$  colors in the worst case.

## 3 Chordal graphs

In this section, we will exhibit an online algorithm with bichromatic exchanges that can color optimally any chordal graph. A chordal graph is a graph with no induced cycles of length larger than 3. They have the following property:

**Theorem 3** (Dirac, 1961). *A graph  $G$  is chordal if and only if there is an ordering  $u_1, \dots, u_n$  of its vertices such that for all  $i \geq 1$ ,  $N(u_i) \cap \{u_1, \dots, u_{i-1}\}$  is a clique. Such an ordering is called an elimination ordering.*

Such ordering can be computed in polynomial time. Chordal graphs are an important subclass of perfect graphs for which the chromatic number is equal to the largest clique in the graph. The idea of the algorithm is the following. Let  $v$  be the new vertex that was added to the graph. The algorithm will recolor  $N(v)$  using one less color. Since the subgraph induced by  $N(v)$  is also chordal, we can find

an elimination ordering of the vertices in  $N(v)$ , and recolor these vertices one by one, according to this ordering. The property that the whole graph is chordal will ensure that the colors of previously recolored vertices do not change during the successive operations. We first prove the following lemma:

**Lemma 4.** *Let  $G$  be a graph,  $v$  a vertex of  $G$ , and  $c$  a coloring of  $G - v$ . Assume that there are two vertices  $u_1, u_2 \in N(v)$ , such that there is a bichromatic  $(i, j)$ -path between  $u_1$  and  $u_2$  in  $G - v$ . If  $p$  is the shortest such path, then  $p \subseteq N(v)$ .*

*Proof.* Let  $u_1$  and  $u_2$  be two neighbors of  $v$  such that there is a bichromatic  $(i, j)$ -path between  $u_1$  and  $u_2$  in  $G - v$ . Take  $p$  to be the shortest bichromatic  $(i, j)$ -path from  $u_1$  to  $u_2$  for some colors  $i$  and  $j$ . We will show that  $p \subseteq N(v)$ . Assume by contradiction that this is not the case, and there is at least one vertex on the path  $p$  which is not a neighbor of  $v$ . We can find a subpath  $p'$  of  $p$  such that  $p'$  has length at least 2, and the only vertices of  $p'$  in  $N(v)$  are its two endpoints.

Since  $p$  is the shortest bichromatic  $(i, j)$ -path from  $u_1$  to  $u_2$ , the graph induced by  $p'$  is a path, and consequently,  $G[v \cup p']$  is a cycle of length at least 4. This is a contradiction of the assumption that  $G$  is chordal.  $\square$

**Theorem 5.** *There is an online coloring algorithm with bichromatic exchange that colors optimally every chordal graph. The algorithm performs at most  $\Delta$  bichromatic exchanges for each new vertex.*

*Proof.* Given a graph  $H$ , we denote by  $\omega(H)$  the size of the largest clique in  $H$ . Let  $G$  be a graph,  $v$  a vertex of  $G$ , and  $c$  a coloring of  $G - v$  using  $\omega(G - v)$  colors. If  $\omega(G) = \omega(G - v) + 1$ , then we can directly color  $v$  using the new additional color. Otherwise, let  $\omega = \omega(G)$ . We will describe an algorithm finding a sequence of bichromatic exchanges such that the resulting coloring uses at most  $\omega - 1$  colors on  $N(v)$ . The transformation is made using at most  $\Delta$  bichromatic exchanges.

Since  $G$  is chordal, the induced subgraph  $G[N(v)]$  is also chordal. Let  $k = |N(v)|$  be the number of neighbors of  $v$ , and let  $v_1, \dots, v_k$  be an elimination ordering of the vertices of  $N(v)$ . We write  $G_i$  the subgraph induced by the vertices  $v_1, \dots, v_i$ . The recoloring procedure is, for each  $i$  from 1 to  $k$ , if  $v_i$  is colored  $\omega$ , we apply the bichromatic exchange  $\langle v_i, x_i \rangle$  where  $x_i$  is the smallest color not present in  $N_{G_i}(v_i)$ . There is always such a color  $x_i$ . Indeed, by definition of the order of the vertices,  $N_{G_i}(v_i)$  is a clique, and  $N_{G_i}(v_i) \cup \{v_i, v\}$  is also a clique of  $G$ . Consequently,  $|N_{G_i}(v_i)| \leq \omega - 2$ .

During step  $i$ , none of the vertices  $v_j$  with  $j < i$  is recolored. Indeed, suppose by contradiction that this is not the case, and let  $i$  be the first step at which a vertex  $v_j$ , with  $j < i$  is recolored. Before the exchange, we have  $c(v_j) \neq \omega$ . After the exchange, the color of  $v_j$  changes to  $\omega$ . Consequently, there is bichromatic  $(\omega, x_i)$ -path from  $v_i$  to  $v_j$  in  $G - v$ . By Lemma 4, this implies that there is a bichromatic  $(\omega, x_i)$ -path  $p \subseteq N(v)$  from  $v_i$  to  $v_j$ .

Let  $v_k$  be the vertex of the path  $p$  with the largest index  $k$ . By choice of the color  $x_i$ , we know that  $v_k \neq v_i$ , since the (only) neighbor of  $v_i$  in  $p$  has an index larger than  $i$ . Consequently, since  $j < i$ , the vertex  $v_k$  has two neighbors  $v_a$  and  $v_b$  in  $p$ . Since  $p$  is 2-colored, both  $v_a$  and  $v_b$  have the same color. Additionally, by choice of  $k$ , both  $a$  and  $b$  are smaller than  $k$ . Since the ordering of the vertices is an elimination ordering, the neighbors of  $v_k$  in  $G_{k+1}$  form a clique. In particular, there is an edge between  $v_a$  and  $v_b$ , and this edge is monochromatic, a contradiction.

After the recoloring is done, none of the vertices of  $N(v)$  is colored  $\omega$ , and the color  $\omega$  can be assigned to  $v$ .  $\square$

## 4 Planar graphs

The goal of this section is to show two online algorithm with bichromatic exchanges. The first one allows to color outer-planar graphs using at most 3 colors. The second one can color any planar graph, but only computes a  $O(\log \Delta)$  coloring. We start by the algorithm on outer-planar graphs.

### 4.1 Outer-planar graphs

An outerplanar graph is a graph which admits a planar drawing such that there is a face adjacent to all the vertices of the graph. Such a drawing of an outerplanar graph can be computed in linear time [Bre77]. The face containing all the vertices of the graph is the outer-face. Given an outerplanar graph  $G$ , and an outer-planar drawing of  $G$ , we can consider the sequence of vertices in the order they appear on the

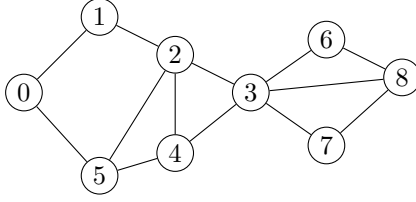


Figure 1: Example of outer-planar graph. The vertex 3 appears twice on the outer face of the graph.

outer face. Note that the same vertex can appear several times in this sequence as in the example in Figure 1. We will prove the following Theorem:

**Theorem 6.** *There is an online coloring algorithm with bichromatic exchange that colors optimally any outer-planar graph. The algorithm performs at most  $\Delta$  bichromatic exchanges at each round.*

*Proof.* Let  $G$  be an outer-planar graph,  $v$  a vertex of  $G$ , and  $c$  an optimal coloring of  $G - v$ . If  $G$  is bipartite, then we can use the procedure from Theorem 2 to remove one color from  $N(v)$ , and obtain a 2-coloring of  $G$ . Otherwise, we can assume that the colors 1, 2 and 3 are present in  $N(v)$ .

We will describe a procedure recoloring  $N(v)$  with only two colors using at most  $\Delta$  local bichromatic exchanges.

Let  $k = |N(v)|$ , and let  $v_0 = v, v_1, \dots, v_k$  be an ordering of the vertices of  $N(v) \cup v$ , in the order they appear on the outer face in an outerplanar drawing of  $G$ . Note that there is no ambiguity in the choice of this ordering. Indeed, suppose that there is a vertex, say  $v_i$ , that appears several times on the outer-face. Then none of the vertices that appear between the first and last occurrence of  $v_i$  on the outer face can be neighbors of  $v$ , since it would contradict the fact that  $G$  is outer-planar. We will recolor the vertices  $v_1, \dots, v_k$  successively using the colors 1 and 2.

The algorithm recolors the vertices  $v_1, \dots, v_k$  in this order by doing the following. For  $i$  from 1 to  $k$ , if  $c(v_i) = 3$ , let  $x \in \{1, 2\} \setminus \{c(v_{i-1})\}$  (if  $i = 1$ ,  $x$  can be either 1 or 2). We apply the bichromatic exchange  $\langle v_i, x \rangle$ .

During step  $i$ , none of the vertices  $v_1, \dots, v_{i-1}$  are recolored. Indeed, since  $G$  is outerplanar, any path in  $G - v$  from  $v_i$  to some vertex  $v_j$  with  $j < i$  necessarily goes through  $v_{i-1}$ . Additionally, the color of  $v_{i-1}$  is different from  $x$  and 3 by choice of  $x$ . Consequently,  $v_{i-1}$  is not contained in any bichromatic  $(x, 3)$ -path, and there is no bichromatic  $(x, 3)$ -path between  $v_i$  and  $v_j$ . At the end of this procedure,  $N(v)$  is colored with colors 1 and 2, and  $v$  can be colored 3.  $\square$

## 4.2 General planar graphs

The case of general planar graph is more complicated, and we will only prove the following weaker theorem.

**Theorem 7.** *There is an online algorithm with bichromatic exchanges coloring any planar graph with  $O(\log(\Delta))$  colors. The algorithm performs at most  $\frac{\Delta^2}{2}$  bichromatic exchanges for each new vertex.*

The question of whether there exists an algorithm using only a constant number of colors is still open. The algorithm is quite simple but the analysis is more complex than previous cases. The idea is to repeatedly apply a greedy procedure trying to remove one color from the neighborhood of the new vertex  $v$ . We will show that when this greedy procedure cannot be applied any more,  $N(v)$  is colored using at most  $O(\log \Delta)$  colors.

We start by a few definitions. Let  $G$  be a graph,  $v$  a vertex of  $G$ , and  $c$  a coloring of  $G - v$ . Given a color  $i$ , the *size* of  $i$  in the coloring  $c$  is the number of neighbors of  $v$  colored  $i$ :  $\text{size}_c(i) = |N(v) \cap c_i|$ . Let  $\langle u, j \rangle$  be a local bichromatic exchange, and let  $c' = \langle u, j \rangle(c)$  and  $i = c(u)$ . Suppose that  $\text{size}_c(i) \leq \text{size}_c(j)$ . The bichromatic exchange  $\langle u, j \rangle$  is said to *increase inequalities* in  $c$  if we have  $\text{size}_{c'}(i) < \text{size}_c(i)$  and  $\text{size}_{c'}(j) > \text{size}_c(j)$ . Intuitively, a bichromatic exchange increases inequalities if it decreases the size of colors with smaller sizes, and increases the size of color with larger sizes. The greedy recoloring procedure that we apply is the following.

**Procedure (GreedyRecolor).** *While there is a local bichromatic exchange  $\langle u, i \rangle$  that increases inequalities, apply this bichromatic exchange.*

To prove the theorem, we only need to prove two things: (i) the procedure **GreedyRecolor** ends in a polynomial in  $\Delta$  number of rounds, and (ii) if  $c$  is a coloring such that there is no local bichromatic exchange increasing inequalities, then  $N(v)$  is colored with at most  $O(\log \Delta)$  colors. The first point is proved in the following Lemma 8. The second point will be proved in Lemma 10 in the next subsection.

**Lemma 8.** *The procedure **GreedyRecolor** ends after at most  $\frac{\Delta^2}{2}$  rounds.*

*Proof.* To show this result, we will exhibit a potential function  $\Psi$  such that:

- $\Psi$  increases at every iteration of **GreedyRecolor**,
- $\Psi$  is upper bounded.

Let  $c$  be a  $K$ -coloring, we define the potential  $\Psi(c)$  by:

$$\Psi(c) = \sum_{\text{color } i} (\text{size}_c(i))^2.$$

We will show that  $\Psi$  increases by at least 2 at each iteration of **GreedyRecolor**. Let  $\langle u, j \rangle$  be a local bichromatic exchange increasing inequalities in  $c$ , and let  $i = c(u)$  and  $c' = \langle u, j \rangle(c)$ . By assumption on  $\langle u, j \rangle$ , we know that  $\text{size}_{c'}(i) \leq \text{size}_c(i)$ . Moreover, there is an integer  $x$  such that:

- $\text{size}_{c'}(i) = \text{size}_c(i) - x$
- $\text{size}_{c'}(j) = \text{size}_c(j) + x$

Since  $\langle u, j \rangle$  increases inequalities, we have  $x \geq 1$ . Additionally, the following holds:

$$\begin{aligned} \Psi(c') - \Psi(c) &= (\text{size}_{c'}(i))^2 + (\text{size}_{c'}(j))^2 - (\text{size}_c(i))^2 - (\text{size}_c(j))^2 \\ &= (\text{size}_c(i) - x)^2 + (\text{size}_c(j) + x)^2 - (\text{size}_c(i))^2 - (\text{size}_c(j))^2 \\ &= 2x^2 + 2x(\text{size}_c(j) - \text{size}_c(i)) \geq 2 \end{aligned}$$

Moreover  $\Psi(c)$  is upper bounded by  $\Delta^2$  for any coloring  $c$ . Indeed, since we know that for any coloring  $c$ ,  $\sum_i \text{size}_c(i) = \Delta$ ,  $\Psi(c)$  is maximum when  $\text{size}_c(i)$  is zero for all but one color. The potential  $\Psi$  is positive, upper bounded by  $\Delta^2$ , and increases by 2 at each iteration of **GreedyRecolor**. Consequently, the procedure **GreedyRecolor** must end after at most  $\frac{\Delta^2}{2}$  rounds.  $\square$

### 4.3 Grundy coloring of circular graphs

To prove the correctness of the algorithm described in previous subsection, we only need to show that if a coloring has no local bichromatic exchange that increases inequalities, then  $N(v)$  is colored using at most  $O(\log \Delta)$  colors. To prove this, we will use a result on a particular drawing of a graph which could be of independent interest. We will first describe this construction, and prove that any coloring satisfying some properties related to this drawing uses a small number of colors. This will allow us to complete the proof of Theorem 7.

Let  $G$  be a graph, with  $G$  not necessarily planar. A drawing of  $G$  is a function  $f$  that associates to each vertex  $v$  of the graph a point  $f(v)$  in the plane, and to each edge  $(u, v)$  of the graph a path from  $f(u)$  to  $f(v)$ . We call a *circular drawing* of  $G$  a drawing of  $G$  such that all the vertices are represented by points on a circle, and the paths representing the edges are straight line segments.

**Definition 4.** *Let  $G = (V, E)$  be a graph, with a drawing of  $G$ . A proper coloring  $c$  of  $G$  is intersection compatible if for any two crossing edges  $(u_1, v_1)$  and  $(u_2, v_2)$ , the two sets  $\{c(u_1), c(v_1)\}$  and  $\{c(u_2), c(v_2)\}$  intersect.*

An example of a circular drawing of a graph with an intersection compatible coloring is given in Figure 2.

Not all graphs  $G$  have a drawing with an intersection compatible coloring. Determining which graphs have one could be an interesting question on its own. For example, there is no intersection compatible coloring of  $K_5$ , the clique on 5 vertices, for any of its drawing on a plane. On the contrary, any 3-colorable



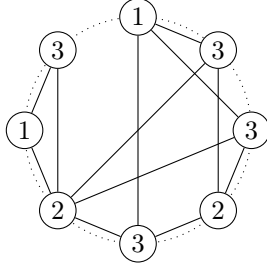


Figure 2: Example of circular drawing of a graph with an intersection compatible coloring.

graph has one. In fact any 3-coloring of a 3-colorable graph is intersection compatible for any drawing of the graph, since in this case any two edges share at least one color.

Given a graph  $G$  with a coloring  $c$ , the colored graph  $(G, c)$  will be called circular if there is a circular drawing of  $G$  that makes the coloring  $c$  intersection compatible. A  $k$ -coloring  $c$  of a graph is a *Grundy  $k$ -coloring* if for every vertex  $u$ , and for every color  $j < c(u)$ , there is a neighbor  $w \in N(u)$  such that  $c(w) = j$ , and at least one vertex is colored  $k$ . In other words, every vertex is assigned the smallest color that is not present in its neighborhood. The Grundy chromatic number of a graph  $G$  is the largest number of colors  $k$  such that  $G$  has a Grundy  $k$ -coloring. To put it differently, a coloring is a Grundy coloring if it can be obtained from the algorithm FIRST-FIT, with a certain ordering of the vertices. The Grundy chromatic number is the largest number of colors that the algorithm FIRST-FIT might need for the worst case ordering. We will prove the following result.

**Lemma 9.** *There is a constant  $K_0$ , such that for any circular colored graph  $(G, c)$  with  $n$  vertices, if  $c$  is a Grundy  $k$ -coloring of  $G$ , then  $k \leq K_0 \log n$ .*

Note that the lemma does not prove that the Grundy chromatic number of a circular graph is at most  $O(\log n)$ . Indeed, a circular graph could have a Grundy coloring using more colors. However, the lemma states that in this case, this coloring is not intersection compatible.

*Proof.* We denote by  $T(k)$  the smallest number of vertices of a circular graph  $G$  having an intersection compatible Grundy  $k$ -coloring  $c$ . To prove the lemma, it is enough to show that there are is a constant  $\alpha > 1$  such that  $T(k) \geq \alpha^{k-1}$  for any  $k \geq 1$ . We take  $\alpha = 2^{\frac{1}{4}}$ , and show this inequality by induction on  $k$ . If  $k \leq 4$ , we know that  $T(k) \geq k \geq \alpha^{k-1}$ .

Let us assume that  $k > 4$ , and let  $(G, c)$  be a colored circular graph with  $T(k)$  vertices such that  $c$  is a Grundy  $k$ -coloring. Let  $u_0$  be a vertex colored  $k$ . Since  $c$  is a Grundy coloring, there are three vertices  $u_1, u_2$ , and  $u_3$  adjacent to  $u_0$  such that for each  $i \in \{1, 2, 3\}$ ,  $u_i$  is colored  $k - i$ . Let  $v_0 = u_0, v_1, v_2, v_3$  be the vertices  $\{u_0, u_1, u_2, u_3\}$  in the order they appear on the circle in the corresponding circular drawing of  $G$ . We denote by  $G'$  the subgraph induced by the vertices with a color at most  $k - 4$ . We will show that  $G'$  is composed of at least 2 connected components, each of which contains all the colors from 1 to  $k - 4$ . By applying the induction hypothesis on each of these components, this gives as needed:

$$T(k) = |G| \geq |G'| \geq 2 \cdot T(k - 4) \geq 2 \cdot \alpha^{k-5} = \alpha^{k-1}.$$

Thus, we only need to show that  $G'$  contains at least two connected components, each of which contains all the colors from 1 to  $k - 4$ . Since  $c$  is a Grundy coloring and  $c(v_1) > k - 4$ , there is a vertex  $w_1$  adjacent to  $v_1$  with  $c(w_1) = k - 4$ . Similarly, there is a vertex  $w_2$  adjacent to  $v_3$  with color  $c(w_2) = k - 4$ . See Figure 3 for an illustration. Let  $G_1$  and  $G_2$  be the connected components of  $G'$  containing  $w_1$  and  $w_2$  respectively. Then  $G_1 \neq G_2$ . Indeed, if  $w_1$  and  $w_2$  were in the same connected component of  $G'$ , then there would be a path from  $w_1$  to  $w_2$  using only colors less than or equal to  $k - 4$ . By adding the two edges  $(v_1, w_1)$  and  $(v_3, w_2)$ , we would obtain a path from  $u_1$  to  $u_3$  which does not use the colors  $k = c(v_0)$  and  $c(v_2)$ . However, this path must necessarily cross the edge  $(v_0, v_2)$ , a contradiction of the assumption that the coloring  $c$  is intersection compatible.

Consequently we have  $T(k) \geq \alpha^{k-1}$  which can be rewritten as  $k \leq 1 + \frac{\log(T(k))}{\log(\alpha)} \leq K_0 \log(n)$  with  $K_0 = 1 + \frac{1}{\log(\alpha)}$ .  $\square$

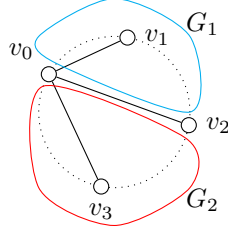


Figure 3: Any edge crossing the edge  $(v_0, v_2)$  must have either  $c(v_0)$  or  $c(v_2)$  as the color of one of its endpoints. Since  $G'$  contains no vertex with color  $c(v_0)$  or  $c(v_2)$ ,  $G'$  is split into two connected components, one on each side of the edge  $(v_0, v_2)$ .

We can now see how to use the previous result to complete the proof of the algorithm on planar graphs. The following lemma is the only remaining missing part to prove Theorem 7.

**Lemma 10.** *Let  $G$  be a planar graph,  $v$  a vertex of  $G$ , and  $c$  a coloring of  $G - v$ . If there is no local bichromatic exchange that increases inequalities,  $N(v)$  is colored using at most  $K_0 \log \Delta$  colors for some constant  $K_0$ .*

*Proof.* We build the graph  $G'$  as follows. The set of vertices of  $G'$  is  $N(v)$ , the neighbors of  $v$ . For every pair of vertices  $u_1, u_2 \in N(v)$  such that  $c(u_1) \neq c(u_2)$ , we add the edge  $(u_1, u_2)$  in  $G'$  if there is a bichromatic path from  $u_1$  to  $u_2$  in  $G$ . We consider a circular drawing of  $G'$  where the vertices appear on the circle in the order they appear around the vertex  $v$  in a planar drawing of  $G$ .

The coloring  $c$  of  $G$  induces a coloring  $c'$  of  $G'$  such that:

- The coloring  $c'$  is proper. Indeed, we only added edges in  $G'$  between vertices with different colors.
- The coloring  $c'$  is intersection compatible. Let  $(u_1, u_3)$  and  $(u_2, u_4)$  be two crossing edges in  $G'$ . Then, the vertices  $u_i$  appear with the order  $u_1, u_2, u_3, u_4$  around the vertex  $v$  in the planar drawing of  $G$ . Additionally, there is a bichromatic  $(c(u_1), c(u_3))$ -path  $p_1$  in  $G$  from  $u_1$  to  $u_3$ , and a bichromatic  $(c(u_2), c(u_4))$ -path  $p_2$  from  $u_2$  to  $u_4$ . Since  $G$  is planar,  $p_1$  and  $p_2$  must cross at some vertex  $w$ . This implies that  $c(w) \in \{c(u_1), c(u_3)\} \cap \{c(u_2), c(u_4)\}$ , and consequently, this intersection is not empty.
- Finally, assume that the colors are ordered  $1, \dots, k$  such that  $\text{size}_c(1) \geq \text{size}_c(2) \geq \dots \geq \text{size}_c(k)$ . Then  $c$  is a Grundy coloring of  $G'$  for this ordering of the colors. Indeed let  $u$  be a vertex of  $G'$ , and assume by contradiction that there is a vertex  $u$  and a color  $i < c'(u)$  such that  $u$  has no neighbor colored  $i$ . Then consider the bichromatic exchange  $\langle u, i \rangle$  in the graph  $G$ . This bichromatic exchange does not change the color of any neighbor of  $v$  colored  $i$ . And since we have  $i < c(u)$ , this implies  $\text{size}(i) \geq \text{size}(u)$ . Performing the bichromatic exchange  $\langle u, i \rangle$  would increase the size of  $i$ , and decrease the size of  $c(u)$ . Thus  $\langle u, i \rangle$  increases inequalities, a contradiction of the assumption that  $c$  did not contain any bichromatic exchange increasing inequalities.

By applying Lemma 9 on the graph  $G'$  with the coloring  $c'$ , we obtain that there is a constant  $K_0$  such that  $c'$  uses at most  $K_0 \log \Delta$  different colors. Consequently, the coloring  $c$  uses at most  $K_0 \log \Delta$  colors on  $N(v)$ .  $\square$

We now have all the ingredients to prove the theorem.

*Proof of Theorem 7.* Let  $K_0$  be the constant in Lemma 10. The algorithm uses  $K_0 \log \Delta + 1$  colors. For each new vertex  $v$ , the procedure **GreedyRecolor** is applied on  $N(v)$ . By Lemma 8, the procedure performs at most  $\frac{\Delta^2}{2}$  of bichromatic exchanges. After the procedure has ended, we obtain a coloring  $c'$  such that there is no local bichromatic exchange that increases inequalities. By Lemma 10, this implies that  $c'$  colors  $N(v)$  using at most  $K_0 \log \Delta$  colors. Consequently, the algorithm colors any planar graph with  $K_0 \log \Delta + 1$  colors.  $\square$

## 5 Graphs of bounded genus

In this section, we show that the algorithm we described above can be adapted to the case of graphs with bounded genus, by only paying a small additional number of colors. We start by some very quick notations. We will assume that the reader is familiar with basic notions of topology and surfaces with boundaries. An introduction to these notions can be found for example in chapter 4 from [Ada04].

Given a triangulation  $T$  of a surface  $S$ , we denote by  $V(T)$ ,  $E(T)$  and  $F(T)$  the set of vertices, edges and faces respectively of the triangulation  $T$ . Let  $S$  be a surface with a triangulation  $T$ . The *Euler characteristic* of  $S$  is the quantity  $|V(T)| - |E(T)| + |F(T)|$ . This quantity is an invariant of the surface  $S$  that is preserved by homeomorphism, and is independent of the chosen triangulation. The Euler characteristic can be negative, and is at most 2 for a connected closed surface, and at most 1 for a connected surface with boundary. For an orientable surface  $S$ , the genus  $g$  of  $S$  is related to its Euler characteristic  $h$  by the formula  $h = 2 - 2g$ .

A drawing of a graph  $G$  on a surface  $S$  is a mapping  $f$  that associates to every vertex  $v$  of  $G$ , a point  $f(v)$  on  $S$ , and to every edge  $(u, v)$  of  $G$ , a path on  $S$  from  $f(u)$  to  $f(v)$ . An embedding of a graph  $G$  on  $S$  is a drawing  $f$  of  $G$  on the surface  $S$  such that for any two edges  $e_1$  and  $e_2$ , the paths  $f(e_1)$  and  $f(e_2)$  do not intersect. In the following, the Euler characteristic (resp. genus) of a graph  $G$  will denote the largest number  $h$  (resp. smallest number  $g$ ) such that there exists an embedding of  $G$  on a connected surface  $S$  with Euler characteristic  $h$  (resp. genus  $g$ ). Planar graphs have Euler characteristic 2 and genus 0. The goal of this section is to show the following theorem:

**Theorem 11.** *There is an online algorithm with bichromatic exchanges that can color any graph with Euler characteristic  $h$  using  $5 - 2h + O(\log \Delta)$  colors.*

The algorithm is exactly the same as for the planar case. To remove one color from the neighborhood of a vertex  $v$ , we just apply the procedure **GreedyRecolor**. Thus, to prove the correctness of the algorithm, we only need to show that, when there is no more bichromatic exchanges increasing inequalities, the neighborhood of the  $v$  is colored using at most  $4 - 2h + O(\log \Delta)$  colors.

To prove this result, we generalize the definition of circular graph from previous section to handle graphs drawn on surfaces with higher genus. Let  $S$  be a surface with boundary, and  $G$  be a graph. A boundary drawing of  $G$  on  $S$  is a drawing  $f$  of  $G$  on the surface  $S$  such that for every vertex  $v$ ,  $f(v)$  is on the boundary of  $S$ . In a boundary drawing, paths corresponding to different edges are allowed to intersect. In particular, if  $S$  is a disk, a boundary drawing of a graph  $G$  on  $S$  is a circular drawing of  $G$ .

The definition of intersection compatible coloring extends to boundary drawings in a natural way. If  $G$  is a graph with a boundary drawing  $f$  on a surface  $S$ , a coloring  $c$  of  $G$  is intersection compatible if for every pair of edges  $e = (u, v)$  and  $e' = (u', v')$ , if the paths  $f(e)$  and  $f(e')$  intersect, then the two edges have at least one color in common, i.e.,  $\{c(u), c(v)\} \cap \{c(u'), c(v')\} \neq \emptyset$ . We will need the following simple result describing how the Euler characteristic of a surface changes when we cut this surface along a path.

**Lemma 12.** *Let  $S$  be a surface with a boundary, and  $p$  be a simple path on  $S$  between two boundary points. Cutting the surface  $S$  along the path  $p$  yields a surface with Euler characteristic increased by 1.*

*Proof.* Let  $S$  be a surface with boundary, and  $h$  the Euler characteristic of  $S$ . Let  $p$  be a path between two boundary points of  $S$ , and  $S'$  be the surface obtained by cutting  $S$  along the path  $p$ . Let  $T$  be a triangulation of  $S$ . Without loss of generality, we can assume that the path  $p$  is along the edges of the triangulation  $T$ . Consider the triangulation  $T'$  where each edge and each vertex of the path  $p$  were duplicated as in Figure 4. Let  $n$  be the number of vertices in the path  $p$ . Then,  $T'$  is a triangulation for  $S'$ , and the Euler characteristic of  $S'$  is:

$$|V(T')| - |E(T')| + |F(T')| = (|V(T)| + n) - (|E(T)| + n - 1) + |F(T)| = h + 1$$

□

Note that in some cases, cutting a surface along a path can disconnect the surface, or increase the number of boundary components as in Figure 4. To prove the theorem, the key ingredient is to prove an extension of Lemma 9 for graphs with a boundary drawing on an arbitrary surface. This is done in the following lemma.

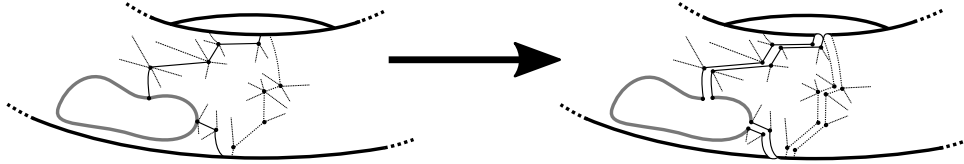


Figure 4: Cutting a surface (here a torus with a boundary) along a path. The bold grey line is the boundary of the surface.

**Lemma 13.** *There exists a constant  $K_0$  such that, if  $G$  is a graph with  $n$  vertices with a boundary drawing on a surface  $S$  with Euler characteristic  $h$ , and  $c$  is a Grundy  $k$ -coloring of  $G$  which is also intersection compatible, then  $k \leq K_0 \log n + 2(1 - h)$ .*

*Proof.* We show the result by induction on  $h$ , the Euler characteristic of the surface  $S$ . If  $h = 1$ , then the surface  $S$  is a disk, and the result follows from Lemma 9. Suppose that  $h < 1$ , and let  $G$  be a graph with a boundary drawing  $f$  on a surface  $S$  with Euler characteristic  $h$ . Let  $c$  be a Grundy coloring of  $G$  that is intersection compatible for this drawing. We distinguish two cases:

**Case 1.** There is an edge  $e$  of  $G$  such that cutting the surface  $S$  along the path  $f(e)$  leaves the surface connected. Let  $a$  and  $b$  be the two colors of the endpoints of this edge. We consider  $S'$ , the surface obtained by cutting  $S$  along  $f(e)$ , and  $G'$  the graph induced by the vertices with a color different from  $a$  or  $b$ . Then, the boundary drawing of  $G$  on  $S$  induces a boundary drawing of  $G'$  on  $S'$ . Indeed, if  $(u, v)$  is an edge of  $G'$ , then the path corresponding to  $(u, v)$  cannot cross  $f(e)$  since otherwise one of the endpoints  $u$  or  $v$  would be colored either  $a$  or  $b$  since the coloring is intersection compatible.

Moreover, by Lemma 12, the surface  $S'$  has Euler characteristic  $h + 1$ . Using the induction hypothesis on  $G'$ , with the coloring induced by  $c$  on  $G'$ , we know that  $c$  colors  $G'$  with at most  $K_0 \log n + 2(1 - (h + 1))$  colors. Since the vertices we removed were the ones colored  $a$  or  $b$ , this implies that the coloring  $c$  of  $G$  uses at most  $K_0 \log n + 2(1 - h)$  colors, which proves the induction step.

**Case 2.** For any edge  $e$ , cutting  $S$  along the path  $f(e)$  disconnects the surface. In this case, we claim that the graph  $G$  has a boundary drawing on a disk such that  $c$  is intersection compatible for this drawing. To prove this, we first remark that we can assume that there is only one boundary component on the surface  $S$ . Indeed, suppose that this is not the case. Consider the boundary that contains a vertex  $v$  with color  $k$ , the largest color used by  $c$ . Let  $G'$  be the graph induced by all the vertices on this boundary. Then all the colors are present in  $G'$ . Indeed,  $G'$  contains the vertex  $v$  with color  $k$ . Since  $c$  is a Grundy coloring, there are vertices  $v_1, \dots, v_{k-1}$  adjacent to  $v$ , with colors  $1, \dots, k - 1$  respectively. For all  $i$ , the vertex  $v_i$  is on the same boundary as  $v$  since otherwise cutting along the path corresponding to the edge  $(v, v_i)$  would leave the surface connected. Let  $c'$  be the coloring induced by  $c$  on  $G'$ . If the result holds for  $G'$  with coloring  $c'$ , it also holds for  $G$  with coloring  $c$  since  $c$  and  $c'$  use the same number of colors. So we now assume that  $S$  has one unique boundary.

Let  $u_1, \dots, u_n$  be the vertices of  $G$  in the order they appear on the boundary of  $S$ , and consider the circular drawing of  $G$  using the same ordering  $u_1, \dots, u_n$  around the circle. To prove the result, we only need to show that  $c$  is also intersection compatible for this new drawing. Indeed, if this is the case, then the result immediately follows from Lemma 9. Let  $e_1$  and  $e_2$  be two edges of  $G$  that intersect in the circular drawing of  $G$ . Without loss of generality, we can assume that  $e_1 = (u_1, u_i)$ , and  $e_2 = (u_j, u_l)$ , with  $1 < j < i$ , and  $i < l \leq n$ . Let  $p_1$  and  $p_2$  be the paths on  $S$  corresponding to these two edges. Assume by contradiction that in the drawing of  $G$  on  $S$ , the two paths  $p_1$  and  $p_2$  do not intersect, we will show that cutting along  $p_1$  does not disconnect the surface  $S$ . For this, it is enough to show that there is a path not intersecting  $p_1$  going from one side of  $p_1$  to the other side. We will construct a path from  $u_{i-1}$  to  $u_{i+1}$  that does not intersect  $p_1$ . The path is as follows. Go from  $u_{i-1}$  to  $u_j$  by following the border of  $S$ , then follow the path  $p_2$  from  $u_j$  to  $u_l$ , and finally go from  $u_l$  to  $u_{i+1}$  by following again the border of  $S$ . This path does not intersect  $p_1$ . Hence cutting along  $p_1$  leaves the surface  $S$  connected, a contradiction.

Thus, we know that for every pair of edges  $e_1$  and  $e_2$ , if the segments corresponding to  $e_1$  and  $e_2$  in the circular drawing of  $G$  intersect, then the paths corresponding to these two edges on the drawing of

$G$  on  $S$  also intersect. Consequently,  $c$  is intersection compatible for this circular drawing of  $G$ , and the result follows from Lemma 9. □

We now have everything we need to prove the theorem.

(*proof of Theorem 11*). The proof follows the same argument as the proof of Theorem 7. Let  $G$  be a graph with Euler characteristic  $h$ ,  $v$  a vertex of  $G$ , and  $c$  a coloring of  $G - v$ . We want to show that after applying procedure **GreedyRecolor**, the neighborhood of  $v$  is colored using at most  $K_0 \log \Delta + 2(2 - h)$  colors. Let  $S$  be a closed surface with Euler characteristic  $h$  such that there is an embedding of  $G$  on  $S$ . Let  $G'$  be the graph whose vertices are the neighbors of  $v$ , and such that there is an edge between  $u_1$  and  $u_2$  if and only if  $c(u_1) \neq c(u_2)$  and there is a bichromatic path from  $u_1$  to  $u_2$ . Without loss of generality, we can assume that the colors are ordered by decreasing size:  $\text{size}_c(1) \geq \text{size}_c(2) \geq \dots$ . Then, as before,  $c$  is a Grundy coloring of  $G'$ . Indeed, if there is a vertex  $v$  and a color  $i < c(v)$  such that  $v$  has no neighbor colored  $i$ , then the bichromatic exchange  $\langle v, i \rangle$  increases inequality.

Moreover, if  $S'$  is the surface  $S$  where a small disk around vertex  $v$  was removed, then from the embedding of  $G$  on  $S$ , we can construct a boundary drawing of  $G'$  on  $S'$ . The coloring  $c$  induces an intersection compatible drawing of  $G'$  for this drawing. Since the Euler characteristic of  $S'$  is  $h - 1$ , by Lemma 13, at most  $K_0 \log \Delta + 2(2 - h)$  are used by  $c$  on  $G'$ . Consequently, at least one color is not present in the neighborhood of  $v$ . □

## 6 Counter example

The goal of this section is to exhibit concrete example of graphs with small chromatic number that need a large number of colors to be colored by an online algorithm using bichromatic exchanges. Unfortunately, this example does not work for general algorithms with bichromatic exchanges, but only to more special cases with the following restrictions on the algorithm:

- The algorithm is specified as input the number  $k$  of colors it is allowed to use to color the graph.
- If one of the  $k$  colors is available, the algorithm must select it, without performing any bichromatic exchange.
- The algorithm always chooses the smallest color available.

These assumptions might seem a bit restrictive, but all the algorithms we described above satisfy these constraints. Getting a counter example that would work in the general case is more complicated as it is difficult to quantify how much the graph can be recolored by bichromatic exchanges performed by the algorithm. We show the following theorem:

**Theorem 14.** *There is a sequence of graph  $(G_k)_{k \geq 0}$  such that for all  $k \geq 4$ ,  $G_k$  is 3 colorable, and no online algorithm with bichromatic exchanges with the restrictions above can color  $G_k$  with  $k$  colors or less.*

*Proof.* Let  $A$  be an online algorithm with bichromatic exchanges with the restrictions above. First observe that as long as one color is available, the algorithm  $A$  behaves exactly as FIRST-FIT: it assigns the smallest color available without changing the color of previously colored vertices. We denote by  $T_k$  the fibonacci tree with the following definition. The tree  $T_1$  is a single vertex, and for  $k > 0$ ,  $T_k$  is composed of a root vertex  $u$ , to which we attach the trees  $T_1, \dots, T_{k-1}$ .

It was proved in [Bea76], that if the vertices of  $T_k$  are presented starting from the leaves, and going up to the root, then FIRST-FIT needs  $k$  colors to color  $T_k$ , and the root of  $T_k$  is colored with color  $k$ . We build the graph  $H_k$  in the following way:

- for each  $1 \leq i \leq k$ , we add a copy of  $T_i$ , with vertex  $u_i$  as the root,
- for each  $i < j$ , and each  $i' \neq i$  and  $j' \neq j$  with  $i' \neq j'$ , we add one copy of  $T_{i'}$  and one copy of  $T_{j'}$  with roots  $w_{i,j,i',j'}^1$  and  $w_{i,j,i',j'}^2$  respectively,
- we add all the following edges:  $(u_i, w_{i,j,i',j'}^1)$ ,  $(u_j, w_{i,j,i',j'}^2)$ , and  $(w_{i,j,i',j'}^1, w_{i,j,i',j'}^2)$ ,

- finally, a vertex  $u$  is added, with  $u$  adjacent to all the  $u_i$ .

The graph  $H_k$  is presented starting from the leaves of all the copies of  $T_i$ , and going up to the roots. The vertex  $u$  is presented last. Since  $H_k$  is 2-degenerate, it is 3 colorable. We prove that the algorithm  $A$  can't color  $H_k$  if it is given exactly  $k$  colors. Indeed, since the algorithm always color immediately a vertex if one color is available, it will be able to color all the vertices except for  $u$ , by applying the rules of FIRST-FIT. When it tries to color  $u$ , all the colors are already present in the neighborhood of  $u$ , so the algorithm might try to remove one color using local bichromatic exchanges. However, we will show that at this point, no local bichromatic exchange can remove one color from the neighborhood of  $u$ . More precisely, we will show that any local bichromatic exchange will preserve the following invariants:

1. The vertices  $u_i$  all have different colors.
2. For every indices  $i$  and  $j$ , with  $i \neq j$ , and every pair of colors  $a \neq b$  with  $a \neq c(u_i)$ , and  $b \neq c(u_j)$ , there is a path on four vertices  $(u_i, w^1, w^2, u_j)$  such that  $c(w^1) = a$  and  $c(w^2) = b$ .

By construction, these invariants are satisfied before any bichromatic exchange is made. Suppose now that there is a coloring  $c$  that satisfies these invariants. Consider the bichromatic exchange  $\langle u_i, x \rangle$ , and let  $j \neq i$  such that  $x = c(u_j)$ . We will show that the coloring  $c' = \langle u_i, x \rangle(c)$  still satisfy the two invariants above.

Using the fact that  $c$  satisfies the second invariant, we know that there is a bichromatic path from  $u_i$  to  $u_j$ . Consequently, applying the bichromatic exchange  $\langle u_i, x \rangle$ , swaps the colors of the vertices  $u_i$  and  $u_j$ . Consequently,  $c'$  still satisfies the invariant 1. We only need to show that  $c'$  also satisfies the second invariant. Consider two indices  $i'$  and  $j'$ , and two colors  $a \neq c'(i')$  and  $b \neq c'(j')$ . We consider the following three cases:

- If  $i' \neq i$  and  $j' \neq j$ , then  $c'(u_{i'}) = c(u_{i'})$  and  $c'(u_{j'}) = c(u_{j'})$ . Since  $c$  satisfies the second property, there are two vertices  $w^1$  and  $w^2$  such that  $c(w^1) = a$  and  $c(w^2) = b$  such that  $(u_{i'}, w^1, w^2, u_{j'})$  is a path on four vertices. Since the colors of  $w^1$  and  $w^2$  also do not change during the bichromatic exchange, we also have  $c'(w^1) = a$  and  $c'(w^2) = b$ .
- If  $i' = i$  and  $j' \neq j$ , then  $c'(u_i) = c(u_j)$ , and  $c'(u_{j'}) = c(u_{j'})$ . If  $a \neq c(u_j)$ , then in the coloring  $c$  there was a path on four vertices with successive colors  $(c(u_i), a, b, c(u_{j'}))$ . This path now has colors  $(c(u_j), a, b, c(u_{j'}))$ . If  $a = c(u_i)$  and  $b \neq c(u_j)$ , then with the coloring  $c$  there was a path from  $u_i$  to  $u_{j'}$  with colors  $(c(u_i), c(u_j), b, c(u_{j'}))$ . This path is now colored  $(c(u_j), a = c(u_i), b, c(u_{j'}))$ . Finally, if  $a = c(u_i)$  and  $b = c(u_j)$ , the path with colors  $(c(u_i), c(u_j), c(u_i), c(u_{j'}))$  now has the necessary colors.

The argument is symmetrical if  $i' \neq i$  and  $j = j'$ .

- If  $i' = i$  and  $j' = j$ , the same argument as above can be used to prove that there is still a path between  $u_i$  and  $u_j$  with colors  $(c'(u_i), a, b, c'(u_j))$ . The idea is that colors different than  $c(u_i)$  and  $c(u_j)$  are unchanged, and vertices colored  $c(u_i)$  or  $c(u_j)$  gets their color swapped.

Consequently, the invariant is preserved when we perform any local bichromatic exchange. Consequently, the algorithm  $A$  won't be able to color  $H_k$  if it is given exactly  $k$  colors. However, the argument relies on the fact that the algorithm does not perform bichromatic exchanges before considering the vertex  $u$ . Consequently, the algorithm might still manage to color  $H_k$  if it is given less than  $k$  colors. This issue can be solved by considering  $G_k = \bigcup_{i \leq k} H_i$ . For any  $k' \leq k$ , the algorithm  $A$  will fail to color  $H'_k$ , and consequently  $G_k$ , and the theorem holds.  $\square$

## 7 Conclusion

We studied a variation of online algorithms where the algorithm is allowed to recolor some of the previously colored vertices. We have shown algorithms for several classes of graphs. In the case of planar graphs, it remains open whether the  $O(\log \Delta)$  bound can be improved or not.

We introduced the notion of intersection compatible coloring which could be interesting to look at on its own. In particular, a relevant question could be to try to characterize the graphs which have a drawing on a given surface  $S$  such that any two intersecting edges share a common color.

Finally, from a reconfiguration point of view, we studied a recoloring problem with some locality constraints on the transformations that we are allowed to make. It could be interesting to look at standard reconfiguration problems when we add these constraints. An example of problems to look at could be, deciding if a given coloring can be transformed into an other using local bichromatic exchanges, or given two vertices  $u$  and  $v$ , deciding if there is a sequence of bichromatic exchanges local to  $u$  that change the color of  $v$ .

## References

- [Ada04] Colin Conrad Adams. *The knot book: an elementary introduction to the mathematical theory of knots*. American Mathematical Soc., 2004.
- [AS17] Susanne Albers and Sebastian Schraink. Tight Bounds for Online Coloring of Basic Graph Classes. In Kirk Pruhs and Christian Sohler, editors, *25th Annual European Symposium on Algorithms (ESA 2017)*, volume 87 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 7:1–7:14, Dagstuhl, Germany, 2017. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.
- [BB13] Marthe Bonamy and Nicolas Bousquet. Recoloring bounded treewidth graphs. *Electronic Notes in Discrete Mathematics*, 44:257–262, 2013.
- [BB14] Marthe Bonamy and Nicolas Bousquet. Recoloring graphs via tree decompositions. *ArXiv preprint arXiv:1403.6386*, 2014.
- [BBFJ15] Marthe Bonamy, Nicolas Bousquet, Carl Feghali, and Matthew Johnson. On a conjecture of mohar concerning kempe equivalence of regular graphs. *ArXiv preprint arXiv:1510.06964*, 2015.
- [BC09] Paul Bonsma and Luis Cereceda. Finding paths between graph colourings: Pspace-completeness and superpolynomial distances. *Theoretical Computer Science*, 410(50):5215–5226, 2009.
- [Bea76] Dwight R Bean. Effective coloration. *The Journal of Symbolic Logic*, 41(2):469–480, 1976.
- [BJL<sup>+</sup>14] Marthe Bonamy, Matthew Johnson, Ioannis Lignos, Viresh Patel, and Daniël Paulusma. Reconfiguration graphs for vertex colourings of chordal and chordal bipartite graphs. *Journal of Combinatorial Optimization*, 27(1):132–143, 2014.
- [Bre77] Wayne M Brehaut. An efficient outerplanarity algorithm. In *Proceedings of the 8th South-Eastern Conference on Combinatorics, Graph Theory, and Computing*, pages 99–113, 1977.
- [CFKP07] Ioannis Caragiannis, Aleksei V Fishkin, Christos Kaklamanis, and Evi Papaioannou. A tight bound for online colouring of disk graphs. *Theoretical Computer Science*, 384(2-3):152–160, 2007.
- [CVdHJ09] Luis Cereceda, Jan Van den Heuvel, and Matthew Johnson. Mixing 3-colourings in bipartite graphs. *European Journal of Combinatorics*, 30(7):1593–1606, 2009.
- [CVDHJ11] Luis Cereceda, Jan Van Den Heuvel, and Matthew Johnson. Finding paths between 3-colorings. *Journal of graph theory*, 67(1):69–82, 2011.
- [EF02] Thomas Erlebach and Jiri Fiala. On-line coloring of geometric intersection graphs. *Computational Geometry*, 23(2):243–255, 2002.
- [FJP15] Carl Feghali, Matthew Johnson, and Daniël Paulusma. Kempe equivalence of colourings of cubic graphs. *Electronic Notes in Discrete Mathematics*, 49:243–249, 2015.
- [GL88] András Gyárfás and Jenő Lehel. On-line and first fit colorings of graphs. *Journal of Graph theory*, 12(2):217–227, 1988.

- [Hal97] Magnús M Halldórsson. Parallel and on-line graph coloring. *Journal of Algorithms*, 23(2):265–280, 1997.
- [HG96] Hacène Ait Haddadène and Sylvain Gravier. On weakly diamond-free berge graphs. *Discrete Mathematics*, 159(1-3):237–240, 1996.
- [HGM98] Hacène Ait Haddadène, Sylvain Gravier, and Frédéric Maffray. An algorithm for coloring some perfect graphs. *Discrete mathematics*, 183(1-3):1–16, 1998.
- [HM97] Hacène Ait Haddadène and Frédéric Maffray. Coloring perfect degenerate graphs. *Discrete Mathematics*, 163(1-3):211–215, 1997.
- [Ira94] Sandy Irani. Coloring inductive graphs on-line. *Algorithmica*, 11(1):53–72, 1994.
- [JKK<sup>+</sup>14] Matthew Johnson, Dieter Kratsch, Stefan Kratsch, Viresh Patel, and Daniël Paulusma. Colouring reconfiguration is fixed-parameter tractable. *ArXiv preprint arXiv:1403.6347*, 2014.
- [KT81] HA Kierstead and WT Trotter. An extremal problem in recursive combinatorics,. *Congressus Numerantium*, 33:143–153, 1981.
- [LST89] László Lovász, Michael Saks, and William T Trotter. An on-line graph coloring algorithm with sublinear performance ratio. *Discrete Mathematics*, 75(1-3):319–325, 1989.
- [LV98] Stefano Leonardi and Andrea Vitaletti. Randomized lower bounds for online path coloring. In *RANDOM*, volume 98, pages 232–247. Springer, 1998.
- [LVM81] Michel Las Vergnas and Henri Meyniel. Kempe classes and the hadwiger conjecture. *Journal of Combinatorial Theory, Series B*, 31(1):95–104, 1981.
- [Mey78] Henry Meyniel. Les 5-colorations d’un graphe planaire forment une classe de commutation unique. *Journal of Combinatorial Theory, Series B*, 24(3):251–257, 1978.
- [Moh06] Bojan Mohar. Kempe equivalence of colorings. In *Graph Theory in Paris, Proceedings of a Conference in Memory of Claude Berge*, pages 287–297. Springer, 2006.
- [MP99] Frédéric Maffray and Myriam Preissmann. Sequential colorings and perfect graphs. *Discrete Applied Mathematics*, 94(1):287 – 296, 1999. Proceedings of the Third International Conference on Graphs and Optimization GO-III.
- [NB08] N. S. Narayanaswamy and R. Subhash Babu. A note on first-fit coloring of interval graphs. *Order*, 25(1):49–53, Feb 2008.
- [Smi10] David A. Smith. *The First-fit Algorithm Uses Many Colors on Some Interval Graphs*. PhD thesis, Tempe, AZ, USA, 2010. AAI3428197.
- [Tuc87] Alan Tucker. Coloring perfect  $(k_4 - e)$ -free graphs. *Journal of Combinatorial Theory, Series B*, 42(3):313–318, 1987.
- [Viz64] Vadim G Vizing. On an estimate of the chromatic class of a p-graph. *Diskret analiz*, 3:25–30, 1964.