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Matching long and short distances in the form factors for $K \to \pi\ell^+\ell^-$

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Abstract

At order $O(\alpha G_{\rm F})$, the amplitudes for the decays $K \to \pi\ell^+\ell^-$ involve a form factor given by the matrix element of the time-ordered product of the electromagnetic current with the four-quark operators describing weak non-leptonic neutral-current transitions between a kaon and a pion. The short-distance behaviour of this time-ordered product, when considered at order $O(\alpha_s)$ in the perturbative expansion of QCD, involves terms linear and quadratic in the logarithm of the Euclidean momentum transfer squared. It is shown how one can exactly match these short-distance features using a dispersive representation of the form factor, with an absorptive part given by an infinite sum of zero-width resonances following a Regge-type spectrum. Some phenomenology-related issues are briefly discussed.

1. Introduction

The amplitude for a $\Delta S = +1$ neutral-current transition $K(k) \to \pi(p)\ell^+(p_+)\ell^-(p_-)$, $\ell = e, \mu$, where $(K, \pi) \in \{(K^+, \pi^+), (K^0, \pi^0)\}$, takes the form [1–3]

$$\begin{aligned}
\mathcal{A}^{K\to\pi\ell^+\ell^-}(s) &= -e^2 \times \bar{\mathrm{u}}(p_-)\gamma_\rho \mathrm{v}(p_+) \times \frac{1}{s} \\
&\quad \times \Big\{ i \int d^4x \, \langle \pi(p)|T\{j^\rho(0)\mathcal{L}^{\Delta S=1}_{\text{non-lept}}(x)\}|K(k)\rangle \\
&\quad + \left(-\frac{G_{\rm F}}{\sqrt{2}} V_{us}V_{ud}\right) \times \frac{C_{7V}(\nu)}{4\pi\alpha} \times s \\
&\quad \times \langle \pi(p)|(\bar{s}\gamma^\rho d)(0)|K(k)\rangle \Big\} \qquad (1.1) \\
&= -e^2 \times \bar{\mathrm{u}}(p_-)(\not{k} + \not{p})\mathrm{v}(p_+) \times \frac{W_{K\pi}(z)}{16\pi^2 M_K^2}.
\end{aligned}$$

Here $s = (k-p)^2 = (p_+ + p_-)^2 = zM_K^2$ is the square of the di-lepton invariant mass, M_K denotes the kaon mass [for our present purposes, it is not necessary to distinguish between the masses of charged and neutral kaons], e is the electric charge, $G_{\rm F}$ the Fermi constant, and V_{ud}, V_{us} are elements of the CKM matrix. Each matrix element occurring in the first expression has to be evaluated in QCD with three active flavours. In particular, the electromagnetic current j^ρ is made up from the contributions of the u, d and s quarks,

$$j^\rho(x) = \sum_{q=u,d,s} e_q(\bar{q}\gamma^\rho q)(x), \; e_u = +\frac{2}{3}, \; e_d = e_s = -\frac{1}{3}. \qquad (1.2)$$

Furthermore, $\mathcal{L}^{\Delta S=1}_{\text{non-lept}}(x)$ represents the effective Lagrangian for $\Delta S = +1$ transitions [4–9], and involves the two current-current four-quark operators, as well as the QCD penguin operators, modulated by the appropriate Wilson coefficients

$$\mathcal{L}^{\Delta S=1}_{\text{non-lept}}(x) = \left(-\frac{G_{\rm F}}{\sqrt{2}} V_{us}V_{ud}\right) \times \sum_{I=1}^{6} C_I(\nu) Q_I(x;\nu). \qquad (1.3)$$

The above representation of $\mathcal{A}^{K\to\pi\ell^+\ell^-}$ holds at order $O(\alpha G_{\rm F})$, $\alpha = e^2/4\pi$ denoting the fine-structure constant, and with a corresponding form factor for each (K, π) channel, that is for the CP conserving transitions $K^+ \to \pi^+\ell^+\ell^-$ and $K_S \to \pi^0\ell^+\ell^-$, but also for the direct-CP violating part of the amplitude for the decay $K_L \to \pi^0\ell^+\ell^-$. The purpose of the perhaps somewhat unfamiliar first expression of $\mathcal{A}^{K\to\pi\ell^+\ell^-}$ given in Eq. (1.1) above is to explicitly display the two main components of the weak transition form factor defined by the second expression in Eq. (1.1),

$$W_{K\pi}(z) = W^{\rm LD}_{K\pi}(z;\nu) + W^{\rm SD}_{K\pi}(z;\nu). \qquad (1.4)$$

The first part is dominated by long-distance contributions, the second one is generated at short distances, at the electroweak scale or beyond in case new physics sets in at even higher energies. Notice the appearance of an ultraviolet subtraction scale ν in each of the two terms in Eq. (1.4). Their sum $W_{K\pi}(z)$ appears in the amplitude $\mathcal{A}^{K\to\pi\ell^+\ell^-}$ entering the physical decay rate and should of course not depend on ν anymore. Explicitly, one has

$$\begin{aligned}
&\left[s(k+p)_\rho - (M_K^2 - M_\pi^2)(k-p)_\rho\right] \times \frac{W^{\rm LD}_{K\pi}(z;\nu)}{16\pi^2 M_K^2} \\
&\qquad = i \int d^4x \, \langle \pi(p)|T\{j_\rho(0)\mathcal{L}^{\Delta S=1}_{\text{non-lept}}(x)\}|K(k)\rangle \qquad (1.5)
\end{aligned}$$

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for the long-distance dominated part, whereas the contribution from short distances, which arises from a local term [7, 10–12]

$$\mathcal{L}_{\rm lept}^{\Delta S=1}(x;\nu) = -\frac{G_{\rm F}}{\sqrt{2}} V_{us}^* V_{ud} C_{7V}(\nu) Q_{7V}(x), \qquad (1.6)$$

with $Q_{7V} = (\bar{s}^i d_i)_{V-A}(\bar{\ell}\ell)_V$, reads

$$\frac{W_{K\pi}^{\rm SD}(z;\nu)}{16\pi^2 M_K^2} = -\left(-\frac{G_{\rm F}}{\sqrt{2}} V_{us}V_{ud}\right) \times \frac{C_{7V}(\nu)}{4\pi\alpha} \times C_{K\pi} f_+(s) \qquad (1.7)$$

in terms one of the form factor $f_+(s)$ describing the matrix element of the $\Delta S = +1$ neutral current

$$\langle \pi(p)| \left(\bar{s}\,\gamma_\rho d\right)(0)|K(k)\rangle = C_{K\pi}\left[(k+p)_\rho f_+^{K\pi}(s) + (k-p)_\rho f_-^{K\pi}(s)\right]. \qquad (1.8)$$

In these last formulas, $C_{K\pi}$ denotes a Clebsch-Gordan coefficient, chosen such that $f_+(0) = 1$ in the flavour-$SU(3)$ limit. In both Eqs. (1.5) and (1.8) the terms proportional to $(k-p)_\rho$ do not contribute to the amplitude $\mathcal{A}^{K\to\pi\ell^+\ell^-}$ due to the conservation of the leptonic current, $(k-p)_\rho\, \bar{\rm u}(p_-)\gamma^\rho {\rm v}(p_+) = 0$. For the sake of completeness, let us recall that in the standard model there is also a contribution to the short-distance part of the amplitude $\mathcal{A}^{K\to\pi\ell^+\ell^-}$ coming from a term proportional to $C_{7A}Q_{7A}(x)$, with $Q_{7A} = (\bar{s}^i d_i)_{V-A}(\bar{\ell}\ell)_A$. This term does not play any role in the present discussion, as it does not involve the short-distance scale ν and is anyway CKM suppressed. We will therefore not mention it any further.

In $W_{K\pi}^{\rm SD}(z;\nu)$ the scale dependence is entirely carried by the Wilson coefficient $C_{7V}(\nu)$,

$$\nu\frac{dC_{7V}(\nu)}{d\nu} = \frac{\alpha}{\alpha_s(\nu)} \sum_{J=1}^{6} \gamma_{J,7V}(\alpha_s)\, C_J(\nu). \qquad (1.9)$$

The anomalous dimensions $\gamma_{J,7V}(\alpha_s)$ are known to leading [10–13] and to next-to-leading [14] orders,

$$\gamma_{J,7V}(\alpha_s) = \gamma_{J,7V}^{(0)} \frac{\alpha_s}{4\pi} + \gamma_{J,7V}^{(1)} \left(\frac{\alpha_s}{4\pi}\right)^2 + \cdots. \qquad (1.10)$$

In $W_{K\pi}^{\rm LD}(z;\nu)$, the scale dependence arises from the singular structure, at short distances, of the time-ordered product of the electromagnetic current with the $\Delta S = +1$ effective Lagrangian [both j^μ and $\mathcal{L}_{\rm non\text{-}lept}^{\Delta S=+1}$ are finite operators]. This short-distance singularity can be studied perturbatively within the operator-product expansion (OPE) [15, 16]. After renormalization [we use dimensional regularization in the $\overline{\rm MS}$ subtraction scheme, and below q^μ denotes a Euclidian momentum, with $q^2 = s < 0$, whose components become simultaneously large] one obtains the general structure, cf. Ref. [3],

$$\lim_{q\to\infty} i\int d^4x\, e^{iq\cdot x} T\{j^\mu(x)\mathcal{L}_{\rm non\text{-}lept}^{\Delta S=+1}(0)\} = [q^\mu q^\rho - q^2\eta^{\mu\rho}]\times \bar{s}\gamma_\rho(1-\gamma_5)d \times\left(-\frac{G_{\rm F}}{\sqrt{2}} V_{us}V_{ud}\right)\times\frac{1}{4\pi}\sum_{I=1}^{6} C_I(\nu)\xi_I(\alpha_s\,;\nu^2/q^2) + O(q), \qquad (1.11)$$

with

$$\xi_I(\alpha_s\,;\nu^2/s) = \sum_{p\geq 0}\sum_{r=0}^{p+1} \xi_{pr}^I \alpha_s^p(\nu)\ln^r(-\nu^2/s). \qquad (1.12)$$

The subleading terms in the OPE have finite coefficients, and do therefore not depend on the renormalization scale ν. Consequently [recall that $z \equiv s/M_K^2$],

$$\nu\frac{dW_{K\pi}^{\rm LD}(z;\nu)}{d\nu} = -16\pi^2 M_K^2 C_{K\pi} f_+(s)\times\left(-\frac{G_{\rm F}}{\sqrt{2}} V_{us}V_{ud}\right) \times\frac{1}{4\pi}\nu\frac{d}{d\nu}\sum_{I=1}^{6} C_I(\nu)\xi_I(\alpha_s\,;\nu^2/s). \qquad (1.13)$$

From Eqs. (1.11) and (1.12) one infers that in the Eucliean region $z \to -\infty$ the form factor $W_{K\pi}^{\rm LD}(z;\nu)$ behaves asymptotically like powers of $\ln(-z)$ times powers of the strong coupling α_s.

In the present Letter, we wish to discuss how the above short-distance behaviour can be reproduced, up to the order $O(\alpha_s)$, by a model involving an infinite number of equally-spaced [in mass squared] zero-width resonances. Models of this kind were considered in various contexts in the past, see for instance the articles [17–19] and references therein. Such Regge-type models find their justification in the properties of the QCD spectrum in the limit of a large number of colours [20–22]. More recently quite efficient methods, based on the Converse Mapping Theorem [23] and the notion of harmonic sums, were developed in order to handle such models [24–26]. In the case at hand, the general idea put forward in Ref. [3] consists in decomposing the long-distance dominated part of the form factor as a sum

$$W_{K\pi}^{\rm LD}(z) = W_{K\pi}^{\pi\pi}(z) + W_{K\pi}^{\rm res}(z;\nu). \qquad (1.14)$$

The first term describes the contribution from the resonant P-wave two-pion intermediate state to $W_{K\pi}(z)$. It is constructed upon assuming that it is given by an unsubtracted dispersion integral. The absorptive part consists of the two-pion spectral density $\rho_{K\pi}^{\pi\pi}(s)$, and is obtained upon inserting a two-pion intermediate state in the representation of the form factor given in Eq. (1.1). This contribution is not relevant for the discussion of the short-distance properties, and we refer the interested reader to Ref. [3] for details. The second term describes the contributions from the intermediate states with higher thresholds as a sum of zero-width resonances. The weight with which each resonance contributes must be chosen such as to reproduce the short-distance behaviour given in Eqs. (1.11) and (1.12). In Ref. [3], this matching has been achieved at order $O(\alpha_s^0)$, which involves only a constant term and a term linear in $\ln(-z)$. Here we wish to extend this matching to the order $O(\alpha_s)$, where we also encounter a term quadratic in $\ln(-z)$.

The remainder of this Letter is thus organized as follows. In section 2 we determine the required coefficients

ξ^I_{01}, ξ^I_{11} and ξ^I_{12}, relying on a renormalization-group argument given in Ref. [3]. For paedagogical reasons, we then review (Section 3) the matching at order $O(\alpha_s^0)$, making the discussion of Ref. [3] simpler and, hopefully, also more intuitive. The matching at order $O(\alpha_s)$ is then presented in Section 4. Finally, we discuss some consequences and features of our results (Section 5).

2. Determination of the short-distance coefficients

From the perspective of perturbative QCD, the matrix element that defines the long-distance dominated part $W^{\rm LD}_{K\pi}(z;\nu)$ is described by the Feyman diagrams of the type shown on Fig. 1. This is obviously not a realistic description of $W^{\rm LD}_{K\pi}(z;\nu)$, except at short distances, when the momentum transfered to the electromagnetic current becomes large in the space-like or Euclidean domain, $s \to -\infty$. The leading contribution to the OPE of the electromagnetic current and the four-quark operators describing $\Delta S = +1$ transitions involves the neutral-current operator $\bar{s}\gamma_\rho(1-\gamma_5)d$, and corresponds to the diagram (a) of Fig. 1. The calculation with a bare four-quark operator Q_I is straightforward, see Ref. [3], and gives

$$\begin{aligned}
&\lim_{q\to\infty} i\int d^4x\, e^{iq\cdot x} T\{j^\mu(x)\mathcal{L}^{\Delta S=+1}_{\rm non\text{-}lept}(0)\}\\
&= [q^\mu q^\rho - q^2\eta^{\mu\rho}]\times[\bar{s}\gamma_\rho(1-\gamma_5)d](0)\times\left(-\frac{G_{\rm F}}{\sqrt{2}}V_{us}V_{ud}\right)\\
&\times\frac{1}{4\pi}\sum_{I=1}^{6} C_I\left\{\xi^I_{00}-\xi^I_{01}\left[\frac{2}{D-4}+\ln\left(\frac{-q^2}{\nu^2}\right)\right]+O(\alpha_s)\right\}\\
&+O(q). \qquad (2.1)
\end{aligned}$$

with

$$\begin{aligned}
\xi^I_{01} &= \frac{1}{4\pi}\,\frac{8}{9}\times(N_c\,,\,1\,,\,-1\,,\,-N_c\,,\,0\,,\,0)\,,\\
\xi^I_{00} &= \frac{\xi^I_{01}}{3}\times\begin{cases} 2 & \text{NDR}\\ 5 & \text{HV}\end{cases}\,, \qquad (2.2)
\end{aligned}$$

The result for ξ^I_{00} depends on the scheme used to handle the Dirac matrices in D dimensions, here either naive dimensional regularization (NDR) [27] or the 't Hooft-Veltman (HV) scheme [28, 29]. The divergent part is removed through the renormalization of the (bare) coupling C_{7V}. Notice also that at this order the Wilson coefficients are not yet running. In the presence of QCD corrections, the four-quark operators and the corresponding Wilson coefficients are renormalized. But this does not take care of all the divergences, since two external lines of the Q_I operators are closed into a loop with the insertion of the electromagnetic current, see Fig. 1. These remaining divergences are again absorbed through the higher-order renormalization of C_{7V}. To all orders in the powers of α_s, and after renormalization through minimal subtraction in the $\overline{\rm MS}$ scheme, the leading term in the OPE then takes the form given in Eqs. (1.11) and (1.12). From the scale dependence of the Wilson coefficients $C_{7V}(\nu)$, given in Eq. (1.9), and $C_I(\nu)$,

$$\nu\frac{dC_I(\nu)}{d\nu} = \sum_{J=1}^{6}\gamma_{J,I}(\alpha_s)C_J(\nu), \qquad (2.3)$$

one infers that the total form factor in Eq. (1.4) will be scale independent provided the equation

$$\nu\frac{d\xi_I(\alpha_s\,;\nu^2/s)}{d\nu}+\sum_{J=1}^{6}\gamma_{I,J}(\alpha_s)\xi_J(\alpha_s\,;\nu^2/s) = -\frac{\gamma_{I,7V}(\alpha_s)}{\alpha_s(\nu)} \qquad (2.4)$$

holds. At order $O(\alpha_s^0)$, this allows to recover the values of the coefficients ξ^I_{01} from the known one-loop anomalous dimension matrices. Including first-order QCD corrections, one obtains, after renormalization,

$$\begin{aligned}
&\lim_{q\to\infty} i\int d^4x\, e^{iq\cdot x} T\{j^\mu(x)\mathcal{L}^{\Delta S=+1}_{\rm non\text{-}lept}(0)\} \qquad (2.5)\\
&= [q^\mu q^\rho - q^2\eta^{\mu\rho}]\times\bar{s}\gamma_\rho(1-\gamma_5)d\times\left(-\frac{G_{\rm F}}{\sqrt{2}}V_{us}V_{ud}\right)\\
&\times\frac{1}{4\pi}\sum_{I=1}^{6}C_I(\nu)\left\{\xi^I_{00}-\xi^I_{01}\ln\left(\frac{-q^2}{\nu^2}\right)+\alpha_s(\nu)\left[\xi^I_{10}\right.\right.\\
&\left.\left.-\xi^I_{11}\ln\left(\frac{-q^2}{\nu^2}\right)+\xi^I_{12}\ln^2\left(\frac{-q^2}{\nu^2}\right)\right]+O(\alpha_s^2)\right\}+O(q).
\end{aligned}$$

From the above renormalization-group argument and the known two-loop anomalous dimension matrices [30–34], one infers the relations

$$\xi^I_{12} = \frac{1}{(4\pi)^2}\,\frac{4}{27}\left(N_c-\frac{1}{N_c}\right)\times\big(0\,,\,-8\,,\,+11\,,\,N_f\,,\,0\,,\,N_f\big), \qquad (2.6)$$

and

$$\begin{aligned}
\xi^I_{11} &= \frac{1}{(4\pi)^2}\,\frac{8}{3}\left(N_c-\frac{1}{N_c}\right)\\
&\times\begin{cases}\left(\frac{N_c}{2}\,,\,-\frac{19}{18}\,,\,\frac{17}{9}\,,\,\frac{7}{6}-\frac{N_c}{2}\,,\,0\,,\,\frac{7}{6}\right) & \text{NDR}\\ \left(\frac{N_c}{2}\,,\,-\frac{5}{18}\,,\,\frac{13}{9}\,,\,\frac{7}{6}-\frac{N_c}{2}\,,\,0\,,\,\frac{7}{6}\right) & \text{HV}\end{cases}. \qquad (2.7)
\end{aligned}$$

The coefficients ξ^I_{10} cannot be obtained this way, and their determination would require a full two-loop calculation, which we will however not need to attempt here.

3. The matching at order $O(\alpha_s^0)$

It is convenient to represent the resonance contribution to the form factor in Eq. (1.14) as a dispersive integral

$$W^{\rm res}_{K\pi}(z;D) = \frac{C_{K\pi}f_+^{K\pi}(zM_K^2)}{4\pi}\int dx\,\frac{\rho^{\rm res}_{K\pi}(x;D)}{x-zM_K^2-i0}, \qquad (3.1)$$

where the spectral density is constructed order by order in the expansion in powers of the strong coupling α_s,

$$\rho^{\rm res}_{K\pi}(x;D) = \rho^{{\rm res};0}_{K\pi}(x;D)+\rho^{{\rm res};1}_{K\pi}(x;D)+\cdots, \qquad (3.2)$$

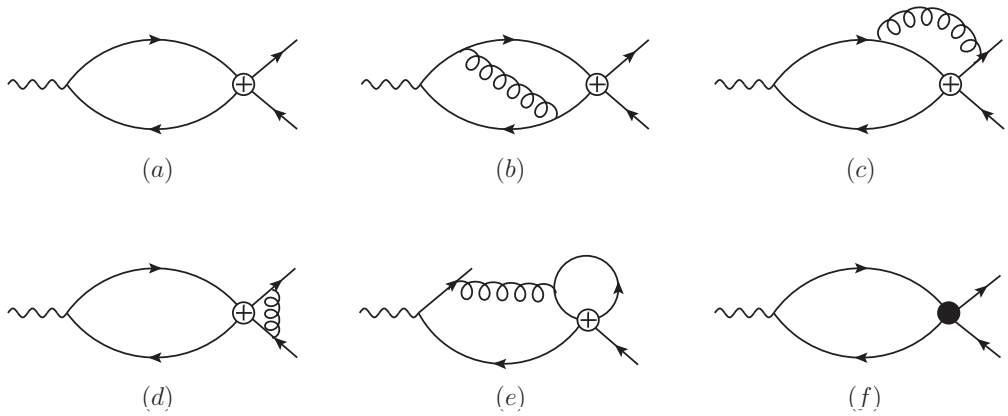

Figure 1: Diagrams contributing to the leading short-distance behaviour of the time-ordered product of the electromagnetic current (materialized by the wiggly line on the left) with one of the four-quark operators, whose insertion is shown by the crossed-circle (⊕) vertex on the right of diagram (*a*), of order $O(\alpha_s^0)$, and of the diagrams (*b*) to (*e*). The latter also form a representative subset of the diagrams contributing to the gluonic corrections at order $O(\alpha_s)$. The vertex represented by a circular black blob (●) on the right of diagram (*f*) corresponds to the insertion of an order $O(\alpha_s)$ countertem into the lowest-order diagram (*a*). The other external lines represent an incoming d quark and an outgoing s quark, while the loop consists of a $\bar{q}$-q pair, where $q = u, d, s$.

and is defined in D dimensions. This last point requires some explanation. Indeed, by power counting the naive superficial degree of divergence of the diagrams in Fig. 1 is two, and becomes actually zero once the Ward identity following from the conservation of the electromagnetic current is implemented. Therefore, in four dimensions the form factor satisfies a once-subtracted dispersion relation, which would thus constitute an appropriate representation to start with. However, the information from short distances at our disposal, and summarized in the preceding section, comes from calculations done within a dimensional renormalization scheme with minimal subtraction and not within a momentum subtraction one. Since we want to make direct use of this information without transforming it first into a different scheme, we choose instead to start from an unsubtracted dispersion relation in D dimensions. As will hopefully become clear in the sequel, far from being an unnecessary complication, this choice even presents some advantages in actually guiding our intuition in the process of constructing an appropriate ansatz for the spectral density $\rho_{K\pi}^{\mathrm{res};0}(x;D)$ or $\rho_{K\pi}^{\mathrm{res};1}(x;D)$.

At the one-loop level, i.e. order $O(\alpha_s^0)$, the resonance representation we are looking for should match, at large negative values of s, the behaviour in the same limit of the unrenormalized QCD diagram (*a*) of Fig. 1. In order to reproduce a logarithmic behaviour at short distances, an infinite set of resonances is required [22]. We will consider a simple Regge-type description of the resonance spectrum in terms of equally spaced [in mass squared] zero-width states. Accordingly, the general structure of $\rho_{K\pi}^{\mathrm{res};0}(x;D)$ is given by

$$\rho_{K\pi}^{\mathrm{res};0}(x;D) = A_0(D)(4\pi)^{2-\frac{D}{2}}\left(\frac{M^2}{\nu_{\mathrm{MS}}^2}\right)^{\frac{D}{2}-2}\Gamma\left(2-\frac{D}{2}\right) \times\sum_{n\geq 1} M^2\mu_n^{(0)}(D)\delta(x-nM^2), \tag{3.3}$$

with so-far unspecified functions $A_0(D)$ and $\mu_n^{(0)}(D)$. The additional prefactors simply account for the structure of diagram (*a*) of Fig. 1 in D dimensions. Since the one-loop divergence is already contained in the factor $\Gamma(2-D/2)$, $A_0(D)$ has to be regular at $D = 4$. It accounts for the scheme dependence, for instance in handling Dirac matrices in D dimensions, see Section 2. M denotes the mass of the lowest-lying resonance and ν_{MS} denotes the renormalization scale in the minimal subtraction scheme. Then, with $w \equiv -s/M^2$,

$$\int dx\,\frac{\rho_{K\pi}^{\mathrm{res};0}(x;D)}{x+wM^2} = A_0(D)(4\pi)^{2-\frac{D}{2}}\left(\frac{M^2}{\nu_{\mathrm{MS}}^2}\right)^{\frac{d}{2}-2} \times\Gamma\left(2-\frac{D}{2}\right)\sum_{n\geq 1}\frac{\mu_n^{(0)}(D)}{n+w}. \tag{3.4}$$

The value of the above dispersive integral at $w = 0$ reproduces the divergent part of the diagram (*a*) of Fig. 1

$$\int \frac{dx}{x}\,\rho_{K\pi}^{\mathrm{res};0}(x;D) = A_0(D)(4\pi)^{2-\frac{D}{2}}\left(\frac{M^2}{\nu_{\mathrm{MS}}^2}\right)^{\frac{d}{2}-2} \times\Gamma\left(2-\frac{D}{2}\right)\sum_{n\geq 1}\frac{\mu_n^{(0)}(D)}{n}. \tag{3.5}$$

$A_0(4)$ being so far a free parameter, one may, without loss of generality, require that

$$\sum_{n\geq 1}\frac{\mu_n^{(0)}(D)}{n} = 1 + (D-4)\mu^{(0)} + O((D-4)^2), \tag{3.6}$$

with $\mu^{(0)}$ a constant that remains unspecified for the time being. Then

$$\int \frac{dx}{x}\,\rho_{K\pi}^{\mathrm{res};0}(x;D) = A_0(4)\left[-\frac{2}{D-4}+\ln\frac{\nu^2}{M^2} -2\frac{A_0'(4)}{A_0(4)}-2\mu^{(0)}\right]+O(D-4), \tag{3.7}$$

with $\nu \equiv \nu_{\rm MS} e^{-\gamma_E/2}\sqrt{4\pi}$ the subtraction scale in the $\overline{\rm MS}$ scheme. The constant $\mu^{(0)}$ can actually be absorbed without loss of generality into $A_0'(4)$, which has also not been specified so far. The remaining, subtracted, dispersive integral

$$wM^2\int\frac{dx}{x}\,\frac{\rho_{K\pi}^{\rm res;0}(x;D)}{x+wM^2} = A_0(D)(4\pi)^{2-\frac{D}{2}}\left(\frac{M^2}{\nu_{\rm MS}^2}\right)^{\frac{d}{2}-2} \times\Gamma\left(2-\frac{D}{2}\right)\sum_{n\geq1}\frac{w\mu_n^{(0)}(D)}{n(n+w)} \tag{3.8}$$

should then be finite as $D\to4$. This in turn will be the case if we require

$$\mu_n^{(0)}(D) = (D-4)\bar\mu_n^{(0)} + O((D-4)^2) \tag{3.9}$$

and

$$\xi(w) = \sum_{n\geq1}\frac{\bar\mu_n^{(0)}}{n(n+w)}\ \text{ converges.} \tag{3.10}$$

Consequently

$$wM^2\int\frac{dx}{x}\,\frac{\rho_{K\pi}^{\rm res;0}(x;D)}{x+wM^2} = A_0(4)(-2w)\xi(w) + O(D-4). \tag{3.11}$$

Finally, in order to reproduce the correct matching with the short-distance behaviour, one must also require

$$-2w\xi(w) = \ln w + \text{Cst} + \cdots \tag{3.12}$$

as $w\to+\infty$. At this stage, one may observe that $\mu_n^{(0)}\propto(D-4)\bar\mu^{(0)}$, with $\bar\mu^{(0)}$ a constant, would provide a convergent series $\xi(w)$, endowed with an asymptotic logarithmic behaviour. But it would fail to satisfy the condition (3.6). As we now show, this defect can be easily repaired. Intuitively, the task will consist in providing a convergence factor for the sum (3.6) when $D<4$, but which is no longer operative for $D=4$, otherwise the sum $\xi(w)$ would converge too quickly, and would no longer exhibit a logarithmic behaviour for large values of w. Instead, it would rather behave as a constant or an inverse power of w. Consider therefore the ansatz

$$\mu_n^{(0)}(D) = \mathrm{a}^{(0)}(D)n^{\frac{D}{2}-2}. \tag{3.13}$$

Then one has

$$\sum_{n\geq1}\frac{\mu_n^{(0)}(D)}{n} = \mathrm{a}^{(0)}(D)\zeta\left(3-\frac{D}{2}\right), \tag{3.14}$$

where $\zeta(s)$ denotes Riemann's zeta-function, see [35, section 25.2]. Since, as $D\to4$, $\zeta(3-D/2)\sim-2/(D-4)+\gamma_E$, where γ_E is the Euler constant, any choice of the form

$$\mathrm{a}^{(0)}(D) = \frac{1}{\zeta\left(3-\frac{D}{2}\right)+f(D)}, \tag{3.15}$$

where $f(D)$ is an arbitrary function regular at $D=4$, will lead to

$$\sum_{n\geq1}\frac{\mu_n^{(0)}(D)}{n} = 1+\frac{f(4)}{2}(D-4)+O((D-4)^2) \tag{3.16}$$

and thus satisfy the condition (3.6) with $\mu^{(0)}=f(4)/2$. The condition (3.9) is then also satisfied, with $\bar\mu_n=-1/2$, so that

$$\Gamma\left(2-\frac{D}{2}\right)\mu_n(D) = 1+O(D-4). \tag{3.17}$$

The condition (3.10) is met as well, with

$$-2w\xi(w) = \sum_{n\geq1}\frac{w}{n(n+w)} = \gamma_E+\psi(1+w). \tag{3.18}$$

Putting everything together, one ends up with

$$\int dx\,\frac{\rho_{K\pi}^{\rm res;0}(x;D)}{x+wM^2} = A_0(4)\Big\{-\frac{2}{D-4}+\ln\frac{\nu^2}{M^2}-2\frac{A_0'(4)}{A_0(4)} -2\mu^{(0)}-\left[\gamma_E+\psi(1+w)\right]\Big\}+O(D-4). \tag{3.19}$$

In this last formula, the di-gamma function ψ resums the dispersive integral

$$\psi(1+w) = -\gamma_E+M^2w\int\frac{dx}{x}\,\frac{1}{x+M^2w}\sum_{n\geq1}M^2\delta(x-nM^2). \tag{3.20}$$

Finally, for large positive w the di-gamma function behaves as $\psi(1+w)\sim\ln w=\ln(-s/M^2)$, so that the correct short-distance behaviour is also recovered, provided one takes

$$A_0(4) = -16\pi^2M_K^2\left(-\frac{G_{\rm F}}{\sqrt2}V_{us}V_{ud}\right)\times\sum_{I=1}^{6}C_I(\nu)\xi_{01}^I \tag{3.21}$$

and

$$A_0'(4)+\left(\mu^{(0)}+\frac{\gamma_E}{2}\right)A_0(4) = +16\pi^2M_K^2\left(-\frac{G_{\rm F}}{\sqrt2}V_{us}V_{ud}\right) \times\sum_{I=1}^{6}C_I(\nu)\frac{\xi_{00}^I}{2}. \tag{3.22}$$

Minimal subtraction of the divergence leads to the renormalized dispersion relation

$$\int dx\,\frac{\rho_{K\pi}^{\rm res;0}(x;D)}{x+wM^2} = -16\pi^2M_K^2\left(-\frac{G_{\rm F}}{\sqrt2}V_{us}V_{ud}\right)\times\sum_{I=1}^{6}C_I(\nu) \times\left[\xi_{01}^I\left(\ln\frac{\nu^2}{M^2}-\psi(1+w)\right)+\xi_{00}^I\right] +O(D-4), \tag{3.23}$$

with

$$\rho_{K\pi}^{\rm res;0}(x;D) = -16\pi^2M_K^2\left(-\frac{G_{\rm F}}{\sqrt2}V_{us}V_{ud}\right)\times\sum_{I=1}^{6}C_I(\nu) \times\Big\{\xi_{01}^I-\frac{D-4}{2}\left[\xi_{00}^I+(f(4)+2\gamma_E)\xi_{01}^I\right]\Big\} \times\left(\frac{M^2}{\nu^2}\right)^{\frac{D}{2}-2}\Gamma\left(2-\frac{D}{2}\right) \times\sum_{n\geq1}\frac{M^2n^{\frac{D}{2}-2}}{\zeta\left(3-\frac{D}{2}\right)+f(D)}\,\delta(x-nM^2). \tag{3.24}$$

We may now, in some sense, reverse-engineer the whole construction, starting directly from the representation of $\rho_{K\pi}^{\mathrm{res};0}(x;D)$ in D dimensions given in Eq. (3.24) above and showing that it satisfies the required properties. This will be useful in Section 4, where we will only sketch the construction of the spectral density $\rho_{K\pi}^{\mathrm{res};0}(x;D)$, give the result, and show that it indeed exhibits the appropriate features. Using the Mellin representation

$$\frac{1}{1+\frac{w}{n}} = \int_{c_1-i\infty}^{c_1+i\infty} \frac{du}{2\pi i}\left(\frac{w}{n}\right)^{-u}\Gamma(u)\Gamma(1-u), \tag{3.25}$$

where $0 < c_1 < 1$, one has

$$\begin{aligned}
&\int dx\,\frac{\rho_{K\pi}^{\mathrm{res};0}(x;D)}{x+wM^2}\\
&= -16\pi^2 M_K^2\left(-\frac{G_{\mathrm{F}}}{\sqrt{2}}V_{us}V_{ud}\right)\times\sum_{I=1}^{6}C_I(\nu)\Big\{\xi_{01}^I - \frac{D-4}{2}\\
&\quad\times\left[\xi_{00}^I + (f(4)+2\gamma_E)\xi_{01}^I\right]\Big\}\times\left(\frac{M^2}{\nu^2}\right)^{\frac{D}{2}-2}\\
&\quad\times\frac{\Gamma\left(2-\frac{D}{2}\right)}{\zeta\left(3-\frac{D}{2}\right)+f(D)}\times\int_{c_1-i\infty}^{c_1+i\infty}\frac{du}{2\pi i}w^{-u}\Gamma(u)\Gamma(1-u)\\
&\quad\times\zeta\left(3-\frac{D}{2}-u\right).
\end{aligned} \tag{3.26}$$

The first singularity of the integrand lying on the left of the fundamental band $c_1 = \mathrm{Re}\,u \in]0, 2-D/2[$ is a simple pole at $u = 0$, coming from the factor $\Gamma(u)$. According to the Converse Mapping Theorem [23] one therefore has

$$\begin{aligned}
&\int dx\,\frac{\rho_{K\pi}^{\mathrm{res};0}(x;D)}{x+wM^2}\\
&\underset{w\to 0}{=} -16\pi^2 M_K^2\left(-\frac{G_{\mathrm{F}}}{\sqrt{2}}V_{us}V_{ud}\right)\times\sum_{I=1}^{6}C_I(\nu)\Big\{\xi_{01}^I - \frac{D-4}{2}\\
&\quad\times\left[\xi_{00}^I + (f(4)+2\gamma_E)\xi_{01}^I\right]\Big\}\times\left(\frac{M^2}{\nu^2}\right)^{\frac{D}{2}-2}\\
&\quad\times\frac{\Gamma\left(2-\frac{D}{2}\right)\zeta\left(3-\frac{D}{2}\right)}{\zeta\left(3-\frac{D}{2}\right)+f(D)}\\
&\underset{D\to 4}{=} -16\pi^2 M_K^2\left(-\frac{G_{\mathrm{F}}}{\sqrt{2}}V_{us}V_{ud}\right)\times\sum_{I=1}^{6}C_I(\nu)\Big\{\frac{-2}{D-4}\xi_{01}^I\\
&\quad+\xi_{00}^I + \xi_{01}^I\gamma_E + \xi_{01}^I\ln\frac{M^2}{\nu^2}\Big\} + O(D-4).
\end{aligned} \tag{3.27}$$

The first singularity of the integrand lying on the right of the fundamental band $c_1 = \mathrm{Re}\,u \in]0, 2-D/2[$ is a single pole occurring at $u = 2-D/2$, with $\zeta\left(3-\frac{D}{2}-u\right) \sim -1/(u-2+D/2)$, so that, according to the Converse Mapping Theorem,

$$\begin{aligned}
&\int dx\,\frac{\rho_{K\pi}^{\mathrm{res};0}(x;D)}{x+wM^2}\\
&\underset{w\to+\infty}{\sim} -16\pi^2 M_K^2\left(-\frac{G_{\mathrm{F}}}{\sqrt{2}}V_{us}V_{ud}\right)\times\sum_{I=1}^{6}C_I(\nu)\Big\{\xi_{01}^I\\
&\quad-\frac{D-4}{2}\left[\xi_{00}^I + (f(4)+2\gamma_E)\xi_{01}^I\right]\Big\}\times\left(\frac{M^2}{\nu^2}\right)^{\frac{D}{2}-2}\\
&\quad\times\frac{\Gamma\left(2-\frac{D}{2}\right)}{\zeta\left(3-\frac{D}{2}\right)+f(D)}\times\Gamma\left(2-\frac{D}{2}\right)\Gamma\left(\frac{D}{2}-1\right)\\
&\quad\times\left[w^{\frac{D}{2}-2} + O\left(w^{-1}\right)\right].
\end{aligned} \tag{3.28}$$

Considering next the limit $D \to 4$, one indeed recovers the expected result:

$$\begin{aligned}
&\int dx\,\frac{\rho_{K\pi}^{\mathrm{res};0}(x;D)}{x+wM^2}\\
&\underset{w\to+\infty}{\sim} -16\pi^2 M_K^2\left(-\frac{G_{\mathrm{F}}}{\sqrt{2}}V_{us}V_{ud}\right)\times\sum_{I=1}^{6}C_I(\nu)\\
&\quad\times\left\{\frac{-2}{D-4}\xi_{01}^I + \xi_{00}^I + \xi_{01}^I\ln\frac{\nu^2}{-s}\right\} + O(D-4).
\end{aligned} \tag{3.29}$$

Summarizing the preceding analysis, we find inded that the spectral density (3.24) reproduces the minimally subtracted dispersive integral in Eq. (3.23). Let us notice, at this stage, that the result (3.23) for $D = 4$ does no longer depend at all on $f(D)$. We will return to this point in Section 5 below.

4. The matching at order $O(\alpha_s)$

In order to include the $O(\alpha_s)$ corrections that arise at two loops, it is necessary to start with a somewhat more involved expression of the spectral density. We again let ourselves be guided by the perturbative structure of this contribution. On the basis of the order $O(\alpha_s)$ diagrams shown in Fig. 1, we are led to consider as a starting point the sum of two terms,

$$\rho_{K\pi}^{\mathrm{res};1}(x;D) = \rho_{K\pi}^{\mathrm{res};1\mathrm{a}}(x;D) + \rho_{K\pi}^{\mathrm{res};1\mathrm{b}}(x;D) \tag{4.1}$$

with

$$\begin{aligned}
\rho_{K\pi}^{\mathrm{res};1\mathrm{a}}(x;D) &= A_{1a}(D)(4\pi)^{4-D}\left(\frac{M^2}{\nu_{\mathrm{MS}}^2}\right)^{D-4}\left[\Gamma\left(2-\frac{D}{2}\right)\right]^2\\
&\quad\times\sum_{n\geq 1}M^2\mu_n^{(1a)}(D)\,\delta(x-nM^2),\\
\rho_{K\pi}^{\mathrm{res};1\mathrm{b}}(x;D) &= \frac{A_{1b}(D)}{D-4}(4\pi)^{2-\frac{D}{2}}\left(\frac{M^2}{\nu_{\mathrm{MS}}^2}\right)^{\frac{D}{2}-2}\Gamma\left(2-\frac{D}{2}\right)\\
&\quad\times\sum_{n\geq 1}M^2\mu_n^{(1b)}(D)\,\delta(x-nM^2),
\end{aligned} \tag{4.2}$$

where both functions $A_{1a}(D)$ and $A_{1b}(D)$ are regular at $D = 4$. The first term, $\rho_{K\pi}^{\mathrm{res};1\mathrm{a}}(x;D)$, corresponds to the

genuine two-loop diagrams, like the graphs (b) to (e) of Fig. 1. The second term, $\rho_{K\pi}^{\mathrm{res;1b}}(x;D)$, stands for the diagram (f), i.e. the one-loop graph (a) with the insertion of a one-loop counterterm proportional to $1/(D-4)$. Without loss of generality, one may impose a normalization condition like (3.6) for each set of coefficients separately, and from there proceed as in Section 3. We will not go through the details of this straightforward exercise, but rather state the final result and show that it satisfies the required properties. Before that, let us make a few useful remarks.

With the normalizations set as in Eq. (3.6) for both $\rho_{K\pi}^{\mathrm{res;1a}}(x;D)$ and $\rho_{K\pi}^{\mathrm{res;1b}}(x;D)$, the corresponding dispersive integrals at $w = 0$ display double and single poles at $D = 4$. The single poles contain a contribution proportional to $\ln(M^2/\nu^2)$, which, on general grounds [36], is not allowed, and hence has to cancel in the sum of the two dispersive integrals. For this to happen, we need to impose the condition $A_{1b}(4) = 4A_{1a}(4)$.

Next, $\rho_{K\pi}^{\mathrm{res;1b}}(x;D)$ has to describe the same structure than $\rho_{K\pi}^{\mathrm{res;0}}(x;D)$ up to an additional factor $\propto 1/(D-4)$ coming from the inserted counterterm. It is thus natural to consider for it the ansatz

$$\mu_n^{(1b)}(D) = \frac{n^{\frac{D}{2}-2}}{\zeta\left(3-\frac{D}{2}\right)+\tilde{f}(D)}, \tag{4.3}$$

where $\tilde{f}(D)$ is an arbitrary function regular at $D = 4$. Then the resulting subtracted dispersive integral

$$wM^2 \int \frac{dx}{x}\,\frac{\rho_{K\pi}^{\mathrm{res;1b}}(x;D)}{x+wM^2} \tag{4.4}$$

still contains a term proportional to $\psi(1+w)/(D-4)$, which has eventually to be canceled by a similar contribution from the dispersive integral involving $\rho_{K\pi}^{\mathrm{res;1a}}(x;D)$.

Turning to the latter, we consider the ansatz

$$\mu_n^{(1a)}(D) = \frac{n^{y(D-4)}}{\zeta(1+4y-yD)+g(D)}, \tag{4.5}$$

where $g(D)$ is an arbitrary function regular at $D = 4$, and y is a so-far free parameter. Considering the dispersive integral

$$wM^2 \int \frac{dx}{x}\,\frac{\rho_{K\pi}^{\mathrm{res;1a}}(x;D)}{x+wM^2}, \tag{4.6}$$

we find that the required cancellation of the unwanted pole terms takes place for the choice $y = 1$.

Adjusting the remaining free parameters such as to reproduce the asymptotic behaviour of Eq. (2.5), we finally arrive at the result[1]

$$\begin{aligned}
&\rho_{K\pi}^{\mathrm{res;1}}(x;D)\\
&= -16\pi^2 M_K^2\left(-\frac{G_{\mathrm{F}}}{\sqrt{2}}V_{us}V_{ud}\right)\alpha_s(\nu)\sum_{I=1}^{6}C_I(\nu)\\
&\times\Bigg\{\left(\frac{M^2}{\nu^2}\right)^{D-4}\left[\Gamma\left(2-\frac{D}{2}\right)\right]^2 e^{\gamma_E(4-D)}\xi_{12}^I\\
&\qquad\times\sum_{n\geq 1}M^2\frac{n^{D-4}}{\zeta(5-D)}\,\delta(x-nM^2)\\
&+\left(\frac{M^2}{\nu^2}\right)^{\frac{D}{2}-2}\Gamma\left(2-\frac{D}{2}\right)\frac{e^{\gamma_E(4-D)/2}}{D-4}\\
&\qquad\times\Bigg\{4\xi_{12}^I+(D-4)\left[\xi_{11}^I+2\gamma_E\xi_{12}^I\right]\\
&\qquad-\frac{(D-4)^2}{2}\left[\xi_{10}^I+\xi_{11}^I\gamma_E-2\xi_{12}^I\left(\frac{\pi^2}{6}+\gamma_1\right)\right]\Bigg\}\\
&\qquad\times\sum_{n\geq 1}M^2\frac{n^{\frac{D}{2}-2}}{\zeta\left(3-\frac{D}{2}\right)}\,\delta(x-nM^2)\Bigg\}.
\end{aligned} \tag{4.7}$$

It is a straightforward exercise, making use of the techniques described in the second part of Section 3, to show that this spectral density leads to the desired properties. For instance, after minimal subtraction of the simple and double poles at $D = 4$, the renormalized dispersive integral reads

$$\begin{aligned}
&\int dx\,\frac{\rho_{K\pi}^{\mathrm{res;1}}(x;D)}{x+wM^2}\\
&= -16\pi^2 M_K^2\left(-\frac{G_{\mathrm{F}}}{\sqrt{2}}V_{us}V_{ud}\right)\alpha_s(\nu)\sum_{I=1}^{6}C_I(\nu)\Bigg\{\xi_{12}^I\widetilde{\psi}(w)\\
&+\left[2\xi_{12}^I\ln\frac{M^2}{\nu^2}-\xi_{11}^I\right]\psi(1+w)+\xi_{10}^I-2\left(\frac{\pi^2}{6}+\gamma_1\right)\xi_{12}^I\\
&+\left[\xi_{12}^I\ln\frac{M^2}{\nu^2}-\xi_{11}^I\right]\ln\frac{M^2}{\nu^2}\Bigg\}+O(D-4),
\end{aligned} \tag{4.8}$$

with

$$\begin{aligned}
\widetilde{\psi}(w) &= 2M^2 w\int\frac{dx}{x}\,\frac{1}{x+M^2w}\sum_{n\geq 1}M^2(\ln n)\delta(x-nM^2)\\
&= 2w\sum_{n\geq 1}\frac{\ln n}{n(n+w)}.
\end{aligned} \tag{4.9}$$

We now need to establish some properties of the function $\widetilde{\psi}(w)$, which is defined by an absolutely convergent sum as long as w is not equal to a negative integer. Thus, $\widetilde{\psi}(0) = 0$ and

$$\widetilde{\psi}'(0) = 2\sum_{n\geq 1}\frac{\ln n}{n^2} = -2\zeta'(2). \tag{4.10}$$

[1] Up to a factor $(-1)^n$, the Stieltjes constants γ_n give the coefficients of the Taylor expansion of the regular part of $\zeta(s)$ at $s = 1$, with $\gamma_0 = \gamma_E = 0.577216\cdots$, $\gamma_1 = -0.0728\cdots$, $\gamma_2 = -0.00969\cdots$; see [35, section 25.2].

Next, using the relation (3.25) and

$$\sum_{n\geq 1} n^{u-2}\ln n = -\zeta'(2-u), \tag{4.11}$$

one obtains the following Mellin representation of the function $\widetilde{\psi}(w)$

$$\widetilde{\psi}(w) = -2w\int_{c_1-i\infty}^{c_1+i\infty}\frac{du}{2\pi i}w^{-u}\Gamma(u)\Gamma(1-u)\times\zeta'(2-u). \tag{4.12}$$

The first singularity on the left of the fundamental strip $0 < c_1 < 1$ lies at $u = 0$. It consists of a simple pole coming from the factor $\Gamma(u)$ of the integrand. Therefore, the Converse Mapping Theorem allows to state that $\widetilde{\psi}(w) = -2w\zeta'(2) + \cdots$ as $w \to 0$, a property already established before. The first singularity one encounters on the right of the fundamental strip consists of a triple pole and a simple pole, both located at $u = +1$,

$$\Gamma(u)\Gamma(1-u)\zeta'(2-u) \underset{u\to+1}{\sim} \frac{1}{(u-1)^3}+\frac{\frac{\pi^2}{6}+\gamma_1}{u-1}+\gamma_2+O(u-1). \tag{4.13}$$

The Converse Mapping Theorem then allows to conclude that

$$\widetilde{\psi}(w) \underset{w\to+\infty}{\sim} \ln^2 w + 2\left(\frac{\pi^2}{6}+\gamma_1\right)+\cdots, \tag{4.14}$$

which is precisely what is required. The function $\widetilde{\psi}(w)$ does not seem to be related in an obvious way to the standard sets of functions that have been studied in the mathematical literature, see e.g. [35] and references therein. It is thus interesting to notice that the Mellin representation (4.12) of $\widetilde{\psi}(w)$, which holds for $\mathrm{Re}w > 0$, can be recast into [the integral is understood as its Cauchy principal value for w real and positive]

$$\begin{aligned}\widetilde{\psi}(w) = 2\int_0^\infty \frac{dx}{x}\frac{w}{x-w}\left[\psi(1+x)+\gamma_E\right]\\ +2[\psi(1+w)+\gamma_E]\ln w.\end{aligned} \tag{4.15}$$

This representation allows to extend the function $\widetilde{\psi}(w)$ to negative values of $\mathrm{Re}w$. The integral is then regular, while the second term reproduces the poles of $\widetilde{\psi}(w)$ when w equals a strictly negative integer. The determination of $\ln w$ is given by $w \to w - i\epsilon$, with $\epsilon > 0$ and infinitesimal, in agreement with the prescription $s \to s + i\epsilon$. This representation also proves quite useful for the numerical evaluation of $\widetilde{\psi}(w)$.

5. Discussion

We have shown that it is possible to construct a function $W^{\rm res}_{K\pi}(z;\nu)$ through an infinite sum of zero-width resonances with a Regge-type spectrum, such that its short-distance behaviour matches, at order $O(\alpha_s)$, the one obtained from the leading term in the OPE. After renormalization in the $\overline{\rm MS}$ scheme of dimensional regularisation, and up to the order $O(\alpha_s)$, this function reads

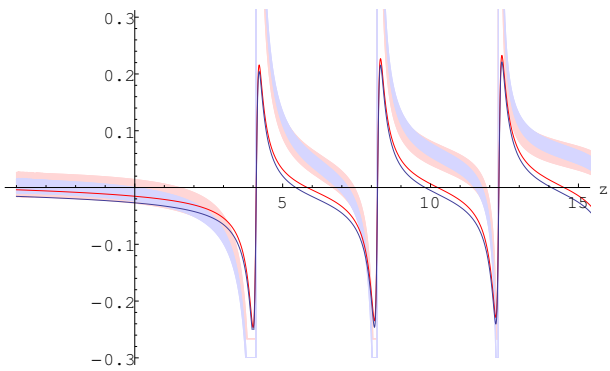

Figure 2: The function $\mathcal{W}^{\rm res}_{K\pi}(z;\nu)$ defined in Eq. (5.1) as a function of z for $\nu = 2$ GeV and $M = 1$ Gev. The light-red (purple) band corresponds to the NDR (HV) scheme. The red (blue) line coresponds to the truncation of $\mathcal{W}^{\rm res}_{K\pi}(z;\nu)$ to lowest order in the NDR (HV) scheme. For the ease of visualization, the location of the poles at $z = nM^2/M_K^2$ has been slightly shifted off the real-z axis.

$[w \equiv -zM_K^2/M^2]$

$$\begin{aligned}W^{\rm res}_{K\pi}(z;\nu) &= -16\pi^2 M_K^2\left(-\frac{G_{\rm F}}{\sqrt{2}}V_{us}V_{ud}\right)\frac{C_{K\pi}f_+^{K\pi}(zM_K^2)}{4\pi}\\ &\times\sum_{I=1}^{6}C_I(\nu)\Bigg\{\xi^I_{00}+\alpha_s(\nu)\xi^I_{10}\\ &+\ln\frac{\nu^2}{M^2}\left[\xi^I_{01}+\alpha_s(\nu)\left(\xi^I_{11}+\xi^I_{12}\ln\frac{\nu^2}{M^2}\right)\right]\\ &-\psi(1+w)\left[\xi^I_{01}+\alpha_s(\nu)\left(\xi^I_{11}+2\xi^I_{12}\ln\frac{\nu^2}{M^2}\right)\right]\\ &+\alpha_s(\nu)\,\xi^I_{12}\left(\widetilde{\psi}(w)-\frac{\pi^2}{3}-2\gamma_1\right)\Bigg\}\\ &\equiv -16\pi^2 M_K^2\left(-\frac{G_{\rm F}}{\sqrt{2}}V_{us}V_{ud}\right)\frac{C_{K\pi}f_+^{K\pi}(zM_K^2)}{4\pi}\\ &\times\mathcal{W}^{\rm res}_{K\pi}(z;\nu).\end{aligned} \tag{5.1}$$

This expression involves two scales. The first one, M, is the scale of the lowest resonance in the spectrum besides the $\rho(770)$, which is already taken into account by the contribution $W^{\pi\pi}_{K\pi}(z)$ in Eq. (1.14), see Ref. [3]. The first resonance appearing in $W^{\rm res}_{K\pi}(z;\nu)$ is thus the $K^*(892)$ or the $\phi(1020)$, hence $M \sim 1$ GeV. The second scale in Eq. (5.1) is the renormalization scale ν. It represents the onset of the perturbative regime of QCD. In practice, the description of the spectrum in terms of well-identified resonances extends to a few radial excitations of the resonances mentioned just above, for instance $\rho(1450)$, $\rho(1570)$, $\phi(1680)$..., before they merge into the continuum. This means $\nu \sim 2$ GeV.

The function $\mathcal{W}^{\rm res}_{K\pi}(z;\nu)$ is shown in Fig. 2. For positive values of z it displays the expected infinite series of equally-spaced poles, whereas for negative values of z the asymptotic regime sets in rapidly. In order to draw these plots, some knowledge of the coefficients ξ^I_{10}, which are not fixed by the renormalization-group constraint (2.4), is needed. For the sake of illustration, we have estimated

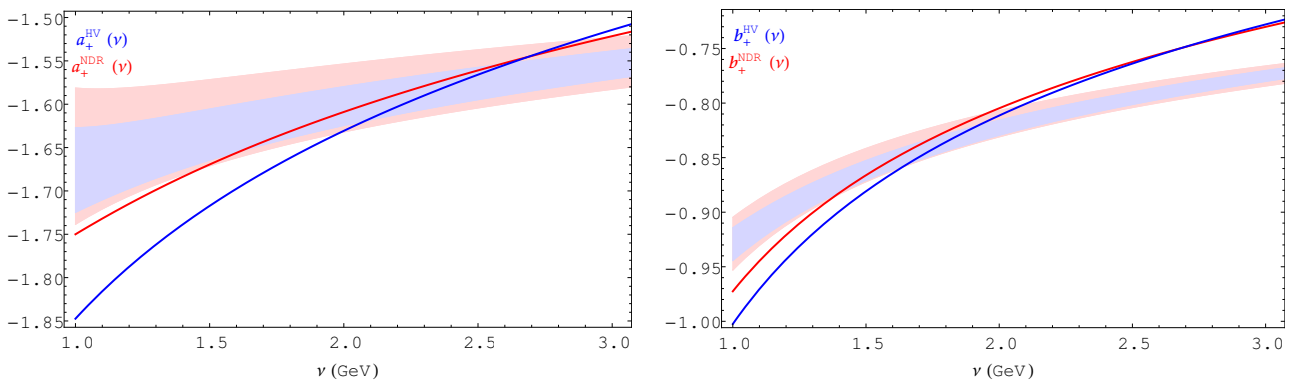

Figure 3: The variation of a_+ (left plot) and of b_+ (right plot) as a function of the short-distance renormalization scale ν. The scale dependence at lowest order corresponds to the red (blue) lines in the NDR (HV) scheme. The scale dependence after inclusion of $O(\alpha_s)$ corrections corresponds to the light-red (purple) bands in the NDR (HV) scheme.

these coefficients to vary in the range $-\xi^I_{11} \leq \xi^I_{10} \leq +\xi^I_{11}$, taking the relation (2.2) between the lowest-order coefficients ξ^I_{00} and ξ^I_{01} as a guide. The same choice also applies to Fig. 3.

In the remainder of the Letter, we wish to address in turn two issues that we think deserve to be given some consideration, namely: i) the size of the residual dependence with respect to the short-distance scale ν and ii) some features and properties of the resonance model that we have constructed, as well as possible improvements.

5.1. Residual scale dependence

By construction, the expression of $W^{\rm res}_{K\pi}(z;\nu)$ displayed in Eq. (5.1) provides a form factor that is independent of the renormalization scale ν at order $O(\alpha_s)$,

$$\nu\frac{d}{d\nu}\left[W^{\rm res}_{K\pi}(z;\nu) + W^{\rm SD}_{K\pi}(z;\nu)\right] = O(\alpha_s^2). \qquad (5.2)$$

We expect the residual scale dependence to be weaker that the one that results from the matching at lowest order only. We illustrate these changes in the case of the two constants a_+ and b_+ corresponding to the values of the form factor $W_{K\pi}(z)$, for $(K,\pi) = (K^+,\pi^+)$, and of its derivative at $z = 0$, respectively. For this purpose, we add to the sum of $W^{\rm res}_{K\pi}$ and $W^{\rm SD}_{K\pi}$ the contribution $W^{\pi\pi}_{K\pi}$ of the two-pion intermediate state evaluated in Ref. [3], see Eq. (1.14) above. This means $a_+|_{\pi\pi} = -1.58$ and $b_+|_{\pi\pi} = -0.76$. The improvement when going from lowest order to next-to-leading order can be appreciated from the plots shown in Fig. 3 [details on the numerical aspects can be found in the appendix]. We notice that indeed both the scale dependence and the scheme dependence become less pronounced when order $O(\alpha_s)$ corrections are included. Our ignorance of the coefficients ξ^I_{10} induces, under the conditions stated above, an uncertainty that amounts, at $\nu = 2$ GeV, to about 5% in both a_+ and b_+. We also observe that the corrections coming from the resonance and short-distance parts are quite small as compared to the contribution from the two-pion state.

This improvement in the control over the scheme and scale deprendences allows us to refine somewhat our evaluation of the coefficients a_+ and b_+ made in Ref. [3]. We obtain

$$a_+ = -1.59(8), \quad b_+ = -0.82(6). \qquad (5.3)$$

These values are still at variance with the experimental determinations $a_+^{\rm exp} = -0.593(11)$ and $b_+^{\rm exp} = -0.675(43)$, see Ref. [3] for a detailed discussion, especially as far as a_+ is concerned. On the other hand, as discussed in Ref. [3], in order to reach a definite conclusion, the contributions $a_+|_{\pi\pi}$ and $b_+|_{\pi\pi}$ need to be evaluated within a tighter framework than the one adopted there, and other exclusive contributions, in particular from two-kaon states, should eventually be accounted for explicitly, see also the discussion below. Work in this direction is on its way [37].

5.2. Properties and features of the resonance model

The resonance model that we have constructed in Sections 3 and 4 is by far not unique. Discussing the arbitrariness of the construction in full generality represents a formidable, if not impossible, task, given the fact that the corresponding dispersive integrals are only constrained to reproduce the divergences of perturbative QCD at $z = 0$ and the leading asymptotic behaviour from the OPE at $z \to -\infty$. The determination of sub-leading terms in the OPE might be a way to constrain the resonance model further. We will, however, not address this possibility here, but rather state a few remarks concerning the model that we have constructed above.

Some of the arbitrariness of the construction is embodied in the functions $A_0(D)$ and $f(D)$ in Section 3, or $A_{1a}(D)$, $A_{1b}(D)$ $\tilde{f}(D)$ and $g(D)$ in Section 4. These functions do not appear anymore in the final expression (5.1), partly because they are absorbed in the matching to the coefficients ξ^I_{rp}, partly because they contribute only to the terms of order $O(D-4)$ in the renormalized dispersive integral.

These features certainly do not exhaust all the arbitrariness of the model. While the contribution of the resonances with higher masses will be constrained by the

short-distance behaviour, one might expect that the descritpion of the lower-lying resonances like $K^*(892)$ or $\phi(1020)$ [recall that the contribution from $\rho(770)$ is already taken care of by $W_{K\pi}^{\pi\pi}$] provided by this model may be less realistic from a phenomenological point of view. One way to circumvent this possible drawback would be to consider additional intermediate states in a more explicit way, i.e. by extending the decomposition in Eq. (1.14) to, for instance [3]

$$W_{K\pi}^{\rm LD}(z) = W_{K\pi}^{\pi\pi}(z) + W_{K\pi}^{K\pi}(z) + W_{K\pi}^{K\bar{K}}(z) + W_{K\pi}^{\rm res}(z;\nu). \quad (5.4)$$

We will, however, not pursue this matter in the present Letter, and leave the discussion of such an extension and of the corresponding multi-channel analysis it requires for future work.

Appendix

In this appendix, we gather some information and give the input values used in order to produce the plots shown in Fig. 3. The running of the Wilson coefficients is given in Eqs. (1.9) and (2.3) using the anomalous dimensions $\gamma_{I,7V}$ given in [14] and $\gamma_{I,J}$ given in [33, 34]. We have, however, restricted ourselves to the contributions from the two current-current operators $C_\pm = C_2 \pm C_1$, neglecting those from the QCD penguin operators C_I, $I = 3,4,5,6$. The Wilson coefficients at next-to-leading order are then given by

$$\frac{C_+(\nu)}{C_+(\nu_0)} = \left[1 + S_{++}\left(\frac{\alpha_s(\nu)}{4\pi} - \frac{\alpha_s(\nu_0)}{4\pi}\right)\right]\left(\frac{\alpha_s(\nu)}{\alpha_s(\nu_0)}\right)^{-2/9}, \quad (A.1)$$

$$\frac{C_-(\nu)}{C_-(\nu_0)} = \left[1 + S_{--}\left(\frac{\alpha_s(\nu)}{4\pi} - \frac{\alpha_s(\nu_0)}{4\pi}\right)\right]\left(\frac{\alpha_s(\nu)}{\alpha_s(\nu_0)}\right)^{+4/9},$$

and

$$\begin{aligned}
&\frac{C_{7V}(\nu)}{\alpha} - \frac{C_{7V}(\nu_0)}{\alpha} \quad (A.2)\\
&= \frac{16}{99}\frac{C_+(\nu_0)}{\alpha_s(\nu_0)}\left\{1 - \left[1 - S_{++}\left(\frac{\alpha_s(\nu_0)}{4\pi} - \frac{\alpha_s(\nu)}{4\pi}\right)\right]\right.\\
&\quad\left.\times\left(\frac{\alpha_s(\nu)}{\alpha_s(\nu_0)}\right)^{-11/9} - S_{7+}\frac{\alpha_s(\nu_0)}{4\pi}\left[\left(\frac{\alpha_s(\nu)}{\alpha_s(\nu_0)}\right)^{-2/9} - 1\right]\right\}\\
&\quad - \frac{8}{45}\frac{C_-(\nu_0)}{\alpha_s(\nu_0)}\left\{1 - \left[1 - S_{--}\left(\frac{\alpha_s(\nu_0)}{4\pi} - \frac{\alpha_s(\nu)}{4\pi}\right)\right]\right.\\
&\quad\left.\times\left(\frac{\alpha_s(\nu)}{\alpha_s(\nu_0)}\right)^{-5/9} - S_{7-}\frac{\alpha_s(\nu_0)}{4\pi}\left[\left(\frac{\alpha_s(\nu)}{\alpha_s(\nu_0)}\right)^{+4/9} - 1\right]\right\},
\end{aligned}$$

where

$$S_{++} = \frac{307}{162},\ S_{--} = -\frac{181}{81},\ S_{7+} = -\frac{2071}{108},\ S_{7-} = \frac{443}{108} \quad (A.3)$$

in the NDR scheme, and

$$S_{++} = -\frac{773}{162},\ S_{--} = -\frac{397}{81},\ S_{7+} = -\frac{3331}{108},\ S_{7-} = \frac{1991}{108} \quad (A.4)$$

in the HV scheme. These expressions for $C_\pm(\nu)$ and for $C_{7V}(\nu)$ hold at next-to-leading order, and their truncation to the lowest order is obtained upon taking all the coefficients S_{ij} equal to zero. The input values we have used are [14]

$$C_+(\nu_0) = 0.688,\ C_-(\nu_0) = 2.108,\ C_{7V}(\nu_0)/\alpha = -0.026 \quad (A.5)$$

at lowest-order. At next-to-leading order they become

$$C_+(\nu_0) = 0.771,\ C_-(\nu_0) = 1.737,\ C_{7V}(\nu_0)/\alpha = -0.037 \quad (A.6)$$

in the NDR scheme and

$$C_+(\nu_0) = 0.735,\ C_-(\nu_0) = 1.937,\ C_{7V}(\nu_0)/\alpha = 0.000 \quad (A.7)$$

in the HV scheme. In all three cases these values hold for $\nu_0 = 1$ GeV. The running of α_s is given by

$$\alpha_s(\nu) = \frac{2\pi}{9\ln(\nu/\Lambda)}\left[1 - \frac{32}{81}\frac{\ln\left(\ln\nu^2/\Lambda^2\right)}{\ln(\nu/\Lambda)}\right] \quad (A.8)$$

at next-to-leading order, and with the obvious truncation at lowest order. The QCD scale for three active flavours is taken as $\Lambda = 332$ MeV. Finally, the mass of the lowest resonance is set at $M = 1$ GeV.

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