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Yang-Baxter basis of Hecke algebra and Casselman's problem (extended abstract)

Maki Nakasuji[1†] and Hiroshi Naruse[2‡]

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Abstract. We generalize the definition of Yang-Baxter basis of type A Hecke algebra introduced by A.Lascoux, B.Leclerc and J.Y.Thibon (Letters in Math. Phys., 40 (1997), 75–90) to all the Lie types and prove their duality. As an application we give a solution to Casselman's problem on Iwahori fixed vectors of principal series representation of p-adic groups.

Résumé. Nous généralisons la définition de la base de l'algèbre de Hecke de Yang-Baxter de type A introduit par A.Lascoux, B.Leclerc and J.Y.Thibon (Letters in Math. Phys., 40 (1997), 75–90) à tous les types de Lie et prouvons la dualité. Comme application nous donnons une solution du problème de Casselman sur les vecteurs fixés par le sous-groupe d'Iwahori de la série principale des groupes p-adiques.

Keywords. Yang-Baxter basis, Iwahori fixed vector, Casselman's problem, Demazure-Lusztig operator

1 Introduction

Yang-Baxter basis of Hecke algebra of type A was defined in the paper of Lascoux-Leclerc-Thibon [LLT]. There is also a modified version in [Las]. First we generalize the latter version to all the Lie types. Then we will solve the Casselman's problem on the basis of Iwahori fixed vectors using Yang-Baxter basis and Demazure-Lusztig type operator. This paper is an extended abstract and the detailed proofs will appear in [NN].

2 Generic Hecke algebra

2.1 Root system, Weyl group and generic Hecke algebra

Let $\mathcal{R} = (\Lambda, \Lambda^*, R, R^*)$ be a (reduced) semisimple root data cf. [Dem]. More precisely $\Lambda \simeq \mathbb{Z}^r$ is a weight lattice with rank $\Lambda = r$. There is a pairing $< , >: \Lambda^* \times \Lambda \to \mathbb{Z}$. $R \subset \Lambda$ is a root system with simple roots $\{\alpha_i\}_{1 \leq i \leq r}$ and positive roots R^+. $R^* \subset \Lambda^*$ is the set of coroots, and there is a bijection $R \to R^*$, $\alpha \mapsto \alpha^*$. We also denote the coroot $\alpha^* = h_\alpha$. The Weyl group W of $\mathcal{R}$ is generated by

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simple reflections $S = \{s_i\}_{1\leq i\leq r}$. The action of W on Λ is given by $s_i(\lambda) = \lambda - <\alpha_i^*, \lambda> \alpha_i$ for $\lambda \in \Lambda$. We define generic Hecke algebra $H_{t_1,t_2}(W)$ over $\mathbb{Z}[t_1, t_2]$ with two parameters t_1, t_2 as follows. Generators are $h_i = h_{s_i}$, with relations $(h_i - t_1)(h_i - t_2) = 0$ for $1 \leq i \leq r$ and the braid relations $\underbrace{h_i h_j \cdots}_{m_{i,j}} = \underbrace{h_j h_i \cdots}_{m_{i,j}}$, where $m_{i,j}$ is the order of $s_i s_j$ for $1 \leq i < j \leq r$. We need to extend the coefficients to the quotient field of the group algebra $\mathbb{Z}[\Lambda]$. An element of $\mathbb{Z}[\Lambda]$ is denoted as $\sum_{\lambda\in\Lambda} c_\lambda e^\lambda$. The Weyl group acts on $\mathbb{Z}[\Lambda]$ by $w(e^\lambda) = e^{w\lambda}$. We extend the coefficient ring $\mathbb{Z}[t_1, t_2]$ of $H_{t_1,t_2}(W)$ to

$$Q_{t_1,t_2}(\Lambda) := \mathbb{Z}[t_1, t_2] \otimes Q(\mathbb{Z}[\Lambda])$$

where $Q(\mathbb{Z}[\Lambda])$ is the quotient field of $\mathbb{Z}[\Lambda]$.

$$H_{t_1,t_2}^{Q(\Lambda)}(W) := Q_{t_1,t_2}(\Lambda) \otimes_{\mathbb{Z}[t_1,t_2]} H_{t_1,t_2}(W).$$

For $w \in W$, an expression of $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ with minimal number of generators $s_{i_k} \in S$ is called a reduced expression in which case we write $\ell(w) = \ell$ and call it the length of w. Then $h_w = h_{i_1} h_{i_2} \cdots h_{i_\ell}$ is well defined and $\{h_w\}_{w\in W}$ forms a $Q_{t_1,t_2}(\Lambda)$-basis of $H_{t_1,t_2}^{Q(\Lambda)}(W)$.

2.2 Yang-Baxter basis and its properties

Yang-Baxter basis was introduced in the paper [LLT] to investigate the relation with Schubert calculus. There is also a variant in [Las] for type A case. We generalize that results to all Lie types.

For $\lambda \in \Lambda$, we define $E(\lambda) = e^{-\lambda} - 1$. Then $E(\lambda + \nu) = E(\lambda) + E(\nu) + E(\lambda)E(\nu)$. In particuar, if $\lambda \neq 0$, $\frac{1}{E(\lambda)} + \frac{1}{E(-\lambda)} = -1$.

Proposition 1 *For $\lambda \in \Lambda$, if $\lambda \neq 0$, let $h_i(\lambda) := h_i + \frac{t_1+t_2}{E(\lambda)}$. Then these satisfy the* **Yang-Baxter relations***, i.e. if we write $[p, q] := p\lambda + q\nu$ for fixed $\lambda, \nu \in \Lambda$, the following equations hold. We assume all appearance of $[p, q]$ is nonzero.*

$$\begin{aligned}
&h_i([1,0])h_j([0,1]) &&= h_j([0,1])h_i([1,0]) && \text{if } m_{i,j} = 2\\
&h_i([1,0])h_j([1,1])h_i([0,1]) &&= h_j([0,1])h_i([1,1])h_j([1,0]) && \text{if } m_{i,j} = 3\\
&h_i([1,0])h_j([1,1])h_i([1,2])h_j([0,1]) &&= h_j([0,1])h_i([1,2])h_j([1,1])h_i([1,0]) && \text{if } m_{i,j} = 4\\
&h_i([1,0])h_j([1,1])h_i([2,3]) && h_j([0,1])h_i([1,3])h_j([1,2]) &&\\
&\quad \times h_j([1,2])h_i([1,3])h_j([0,1]) &&\quad \times h_i([2,3])h_j([1,1])h_i([1,0]) && \text{if } m_{i,j} = 6
\end{aligned}$$

Proof: We can prove these equations by direct calculations. □

Remark 1 *In [Che] I. Cherednik treated Yang-Baxter relation in more general setting. There is also a related work [Kat] by S. Kato and the proof of Theorem 2.4 in [Kat] suggests a uniform way to prove Yang-Baxter relations without direct calculations.*

We use the Bruhat order $x \leq y$ on elements $x, y \in W$ (cf.[Hum]). Following [Las] we define the Yang-Baxter basis Y_w for $w \in W$ recursively as follows.

$$Y_e := 1, Y_w := Y_{w'}(h_i + \tfrac{t_1+t_2}{w'E(\alpha_i)}) \text{ if } w = w's_i > w'.$$

Using the Yang-Baxter relation above it is easy to see that Y_w does not depend on a reduced expression of w. As the leading term of Y_w with respect to the Bruhat order is h_w, they also form a $Q_{t_1,t_2}(\Lambda)$-basis $\{Y_w\}_{w\in W}$ of $H^{Q(\Lambda)}_{t_1,t_2}(W)$. We are interested in the transition coefficients $p(w,v)$ and $\tilde{p}(w,v) \in Q_{t_1,t_2}(\Lambda)$ between the two basis $\{Y_w\}_{w\in W}$ and $\{h_w\}_{w\in W}$, i.e.

$$Y_v = \sum_{w\le v} p(w,v)h_w, \text{ and } h_v = \sum_{w\le v} \tilde{p}(w,v)Y_w.$$

Take a reduced expression of v e.g. $v = s_{i_1}\cdots s_{i_\ell}$ where $\ell = \ell(v)$ is the length of v (cf. [Hum]). Then Y_v is expressed as follows.

$$Y_v = \prod_{j=1}^{\ell}\left(h_{i_j} + \frac{t_1+t_2}{E(\beta_j)}\right)$$

where $\beta_j := s_{i_1}\cdots s_{i_{j-1}}(\alpha_{i_j})$ for $j = 1,\dots,\ell$. The set $R(v) := \{\beta_1,\dots,\beta_\ell\} \subset R^+$ is independent of the reduced expression of v. The Yang-Baxter basis defined in [LLT] is normalized as follows.

$$Y_v^{LLT} := \left(\prod_{j=1}^{\ell}\frac{E(\beta_j)}{t_1+t_2}\right)Y_v = \prod_{j=1}^{\ell}\left(\frac{E(\beta_j)}{t_1+t_2}h_{i_j}+1\right).$$

Remark 2 *The relation to K-theory Schubert calculus is as follows. If we set $t_1 = 0, t_2 = -1$ and replacing α_i by $-\alpha_i$. Then the coefficient of h_w in Y_v^{LLT} is the localization $\psi^w(v)$ at v of the equivariant K-theory Schubert class ψ^w (cf. [LSS]).*

Let w_0 be the longest element in W. Define $Q_{t_1,t_2}(\Lambda)$-algebra homomorphism $\Omega : H^{Q(\Lambda)}_{t_1,t_2} \to H^{Q(\Lambda)}_{t_1,t_2}$ by $\Omega(h_w) = h_{w_0ww_0}$. Let $\star$ be the ring homomorphism on $\mathbb{Z}[\Lambda]$ induced by $\star(e^\lambda) = e^{-\lambda}$ and extend to $Q_{t_1,t_2}(\Lambda)$.

Proposition 2 *(Lascoux [Las] Lemma 1.8.1 for type A case) For $v \in W$,*

$$\Omega(Y_{w_0vw_0}) = \star[w_0(Y_v)]$$

where W acts only on the coefficients.

Proof: When $\ell(v) > 0$ there exists $s \in S$ such that $v = v's > v'$. Using the induction assumption on v', we get the formula for v. □

Taking the coefficient of h_w in the above equation, we get

Corollary 1

$$p(w_0ww_0, w_0vw_0) = \star[w_0p(w,v)].$$

2.3 Inner product and orthogonality

Define inner product $(\ ,\ )^H$ on $H^{Q(\Lambda)}_{t_1,t_2}(W)$ by $(f,g)^H :=$ the coefficient of h_{w_0} in $fg^\vee$, where $g^\vee = \sum c_w h_{w^{-1}}$ if $g = \sum c_w h_w$. It is easy to see that $(fh_s, g)^H = (f, gh_s)^H$ for $f, g \in H^{Q(\Lambda)}_{t_1,t_2}(W)$ and $s \in S$. There is an involution$\hat{}: H^{Q(\Lambda)}_{t_1,t_2} \to H^{Q(\Lambda)}_{t_1,t_2}$ defined by $\hat{h}_i = h_i - (t_1+t_2), \hat{t}_1 = -t_2, \hat{t}_2 = -t_1$. It is easy to see that $\hat{h}_s h_s = -t_1 t_2$ for $s \in S$.

The following proposition is due to A.Lascoux for the type A case [Las] P.33.

Proposition 3 *For all $v, w \in W$,*

$$(h_v, \hat{h}_{w_0 w})^H = \delta_{v,w}.$$

Proof: We can use induction on the length $\ell(v)$ of v to prove the equation.

□

We have another orthogonality between Y_v and $w_0(Y_{w_0 w})$.

Proposition 4 *(Type A case was due to [LLT] Theorem 5.1 , [Las] Theorem 1.8.4.)*
For all $v, w \in W$,

$$(Y_v, w_0(Y_{w_0 w}))^H = \delta_{v,w}.$$

Proof: We use induction on $\ell(v)$ and use the fact that if $s \in S$ and $u \in W$, then $Y_u h_s = aY_{us} + bY_s$ for some $a, b \in Q_{t_1,t_2}(\Lambda)$.

□

2.4 Duality between the transition coefficients

Recall that we have two transition coefficients $p(w,v), \tilde{p}(w,v) \in Q_{t_1,t_2}(\Lambda)$ defined by the following expansions.

$$Y_v = \sum_{w \le v} p(w,v) h_w$$

$$h_v = \sum_{w \le v} \tilde{p}(w,v) Y_w$$

Below gives a relation between them.

Theorem 1 *(Lascoux [Las] Corollary 1.8.5 for type A case) For $w, v \in W$,*

$$\tilde{p}(w,v) = (-1)^{\ell(v)-\ell(w)} p(vw_0, ww_0).$$

Proof: We will calculate $(h_v, w_0(Y_{w_0 w}))^H$ in two ways. As $h_v = \sum_{w \le v} \tilde{p}(w,v) Y_w$,

$$(h_v, w_0(Y_{w_0 w}))^H = \tilde{p}(w,v)$$

by the orthogonality on Y_v (Proposition 4). On the other hand, as $h_i + \frac{t_1+t_2}{E(\beta)} = \hat{h}_i - \frac{t_1+t_2}{E(-\beta)}$ for $\beta \in R$, we can expand Y_v in terms of $\hat{h}_w$ as follows.

$$Y_v = \sum_{w \le v} (-1)^{\ell(v)-\ell(w)} \star [p(w,v)] \hat{h}_w.$$

So we have

$$w_0(Y_{w_0 w}) = \sum_{w_0 v \le w_0 w} (-1)^{\ell(v)-\ell(w)} w_0[\star p(w_0 v, w_0 w)] \hat{h}_{w_0 v}.$$

Then using the orthogonality on h_v (Proposition 3) and Corollary 1,

$$(h_v, w_0(Y_{w_0 w}))^H = (-1)^{\ell(v)-\ell(w)} w_0[\star p(w_0 v, w_0 w)] = (-1)^{\ell(v)-\ell(w)} p(v w_0, w w_0).$$

The theorem is proved.

□

2.5 Recurrence relations

Here we give some recurrence relations on $p(w,v)$ and $\tilde{p}(w,v)$.

Proposition 5 *(left p) For $w \in W$ and $s \in S$, if $sv > v$ then*

$$p(w,sv) = \begin{cases} \frac{t_1+t_2}{E(\alpha_s)} s[p(w,v)] - t_1 t_2 s[p(sw,v)] & \text{if } sw > w \\ (t_1+t_2)(\frac{1}{E(\alpha_s)} + 1) s[p(w,v)] + s[p(sw,v)] & \text{if } sw < w. \end{cases}$$

Proof: By the definition we have $Y_{sv} = Y_s s[Y_v]$ from which we can deduce the recurrence formula. □

We note that by this recurrence we can identify $p(w,v)$ as a coefficient of transition between two bases of the space of Iwahori fixed vectors cf. Theorem 3 below.

Proposition 6 *(right p) For $w \in W$ and $s \in S$, if $vs > v$ then*

$$p(w,vs) = \begin{cases} \frac{t_1+t_2}{v E(\alpha_s)} p(w,v) - t_1 t_2 p(ws,v) & \text{if } ws > w \\ (t_1+t_2)(\frac{1}{v E(\alpha_s)} + 1) p(w,v) + p(ws,v) & \text{if } ws < w. \end{cases}$$

Proof: We can use the equation $Y_{vs} = Y_v v[Y_s]$ and taking the coefficient of h_w, we get the formula.

□

Proposition 7 *(left $\tilde{p}$) For $w \in W$ and $s \in S$, if $sv > v$ then*

$$\tilde{p}(w,sv) = \begin{cases} -\frac{t_1+t_2}{E(\alpha_s)} \tilde{p}(w,v) + (1 + \frac{t_1+t_2}{E(\alpha_s)})(1 + \frac{t_1+t_2}{E(-\alpha_s)}) s[\tilde{p}(sw,v)] & \text{if } sw > w \\ -\frac{t_1+t_2}{E(\alpha_s)} \tilde{p}(w,v) + s[\tilde{p}(sw,v)] & \text{if } sw < w. \end{cases}$$

Proof: We can prove the recurrence relation using Corollary 2 below. □

Proposition 8 *(right $\tilde{p}$) For $w \in W$ and $s \in S$, if $vs > v$ then*

$$\tilde{p}(w,vs) = \begin{cases} -\frac{t_1+t_2}{w E(\alpha_s)} \tilde{p}(w,v) + (1 + \frac{t_1+t_2}{w E(\alpha_s)})(1 + \frac{t_1+t_2}{w E(-\alpha_s)}) \tilde{p}(ws,v) & \text{if } ws > w \\ -\frac{t_1+t_2}{w E(\alpha_s)} \tilde{p}(w,v) + \tilde{p}(ws,v) & \text{if } ws < w. \end{cases}$$

Proof: We can prove the recurrence relation using Corollary 2 below. □

3 Kostant-Kumar's twisted group algebra

Let $Q^{KK}_{t_1,t_2}(W) := Q_{t_1,t_2}(\Lambda)\#\mathbb{Z}[W]$ be the (generic) twisted group algebra of Kostant-Kumar. Its element is of the form $\sum_{w\in W} f_w\delta_w$ for $f_w \in Q_{t_1,t_2}(\Lambda)$ and the product is defined by

$$(\sum_{w\in W} f_w\delta_w)(\sum_{u\in W} g_u\delta_u) = \sum_{w,u\in W} f_w w(g_u)\delta_{wu}.$$

Define $y_i \in Q^{KK}_{t_1,t_2}(W)$ $(i = 1,\dots,r)$ by

$$y_i := A_i\delta_i + B_i \text{ where } A_i := \frac{t_1 + t_2e^{-\alpha_i}}{1-e^{\alpha_i}}, B_i := \frac{t_1+t_2}{1-e^{-\alpha_i}}.$$

Proposition 9 *We have the following equations.*
(1) $(y_i - t_1)(y_i - t_2) = 0$ *for* $i = 1,\dots,r$.
(2) $\underbrace{y_iy_j\cdots}_{m_{i,j}} = \underbrace{y_jy_i\cdots}_{m_{i,j}}$, *where* $m_{i,j}$ *is the order of* s_is_j.

Proof: These equations can be shown by direct calculations. □

By this proposition we can define $y_w := y_{i_1}\cdots y_{i_\ell}$ for a reduced expression $w = s_{i_1}\cdots s_{i_\ell}$. These $\{y_w\}_{w\in W}$ become a $Q_{t_1,t_2}(\Lambda)$-basis of $Q^{KK}_{t_1,t_2}(W)$.

Remark 3 *This operator* y_i *can be seen as a generic Demazure-Lusztig operator. When* $t_1 = -1, t_2 = q$, *it becomes* $y^q_{s_i}$ *in Kumar's book[Kum](12.2.E(9)). We can also set* A_i *which satisfies*

$$A_iA_{-i} = \frac{(t_1 + t_2e^{-\alpha_i})(t_1+t_2e^{\alpha_i})}{(1-e^{\alpha_i})(1-e^{-\alpha_i})}.$$

For example, if we set $A_i = \frac{t_1+t_2e^{\alpha_i}}{1-e^{\alpha_i}}$ *and* $t_1 = q, t_2 = -1$ *and replace* α_i *by* $-\alpha_i$, *it becomes Lusztig's* T_{s_i} *[Lu1]. If we set* $A_i = -\frac{t_1+t_2e^{\alpha_i}}{1-e^{-\alpha_i}}$ *and* $t_1 = -1, t_2 = v$ *and replace* α_i *by* $-\alpha_i$, *it becomes* $\mathcal{T}_i$ *in [BBL].*

We can define a $Q_{t_1,t_2}(\Lambda)$-module isomorphism $\Phi : Q^{KK}_{t_1,t_2}(W) \to H^{Q(\Lambda)}_{t_1,t_2}(W)$ by $\Phi(y_w) = h_w$. Let $\Delta_{s_i} := A_i\delta_i$. Define $A(w) := \prod_{\beta\in R(w)} \frac{t_1+t_2e^{-\beta}}{1-e^{\beta}}$ and $\Delta_w := A(w)\delta_w$. Then it becomes that $\Delta_{s_{i_1}}\cdots\Delta_{s_{i_\ell}} = A(w)\delta_w = \Delta_w$. In particular, Δ_{s_i}'s satisfy the braid relations. We can show below by induction on length $\ell(w)$.

Theorem 2 *For* $w \in W$, *we have*

$$\Phi(\Delta_w) = Y_w.$$

Proof: If $w = s_i$, $\Delta_{s_i} = A_i\delta_i = y_i - B_i$. Therefore $\Phi(\Delta_{s_i}) = h_i - B_i = h_i + \frac{t_1+t_2}{E(\alpha_i)} = Y_{s_i}$. If $s_iw > w$, by induction hypothesis we can assume $\Phi(\Delta_w) = Y_w = \sum_{u\le w} p(u,w)h_u$. As Φ is a $Q_{t_1,t_2}(\Lambda)$-isomorphism, it follows that $\Delta_w = \sum_{u\le w} p(u,w)y_u$. Then $\Delta_{s_iw} = \Delta_{s_i}\Delta_w = A_i\delta_i\sum_{u\le w} p(u,w)y_u =$

$\sum_{u\le w} s_i[p(u,w)]A_i\delta_i y_u = \sum_{u\le w} s_i[p(u,w)](y_i - B_i)y_u = \sum_{u\le s_i w} p(u,s_i w)y_u$. We used the recurrence relation (Proposition 5) for the last equality. Therefore $\Phi(\Delta_{s_i w}) = \sum_{u\le s_i w} p(u,s_i w)h_u = Y_{s_i w}$. The theorem is proved. □

Corollary 2 *(Explicit formula for $\tilde{p}(w,v)$)*
Let $v = s_{i_1}\cdots s_{i_\ell}$ be a reduced expression. Then we have

$$\tilde{p}(w,v) = \frac{1}{A(w)} \sum_{\epsilon=(\epsilon_1,\cdots,\epsilon_\ell)\in\{0,1\}^\ell, s_{i_1}^{\epsilon_1}\cdots s_{i_\ell}^{\epsilon_\ell}=w} \prod_{j=1}^{\ell} C_j(\epsilon)$$

where for $\epsilon = (\epsilon_1,\cdots,\epsilon_\ell)\in\{0,1\}^\ell$, $C_j(\epsilon) := s_{i_1}^{\epsilon_1}s_{i_2}^{\epsilon_2}\cdots s_{i_{j-1}}^{\epsilon_{j-1}}(\delta_{\epsilon_j,1}A_{i_j} + \delta_{\epsilon_j,0}B_{i_j})$.

Proof: Taking the inverse image of the map Φ, the equality $h_v = \sum_{w\le v}\tilde{p}(w,v)Y_w$ becomes

$$y_v = \sum_{w\le v}\tilde{p}(w,v)\Delta_w = \sum_{w\le v}\tilde{p}(w,v)A(w)\delta_w.$$

As $v = s_{i_1}\cdots s_{i_\ell}$ is a reduced expression, $y_v = y_{s_{i_1}}\cdots y_{s_{i_\ell}} = (A_{i_i}\delta_{i_1} + B_{i_1}\delta_e)\cdots(A_{i_\ell}\delta_{i_\ell} + B_{i_\ell}\delta_e)$. By expanding this we get the formula.

□

Remark 4 *Using Theorem 1, we also have a closed form for $p(w,v)$. We have another conjectural formula for $p(w,v)$ using λ-chain cf. [Nar].*

Example 1 *Type A_2. We use notation $A_{-1} = \star(A_1)$, $B_{-1} = \star(B_1)$, $B_{12} = \frac{t_1+t_2}{1-e^{-(\alpha_1+\alpha_2)}}$.*
When $v = s_1s_2s_1$, $w = s_1$, then $\epsilon = (1,0,0),(0,0,1)$ and

$$\tilde{p}(s_1, s_1s_2s_1) = (A_1B_{12}B_{-1} + B_1B_2A_1)/A_1 = B_{12}B_{-1} + B_1B_2 = B_2B_{12}.$$

When $v = s_1s_2s_1$, $w = s_2$, then $\epsilon = (0,1,0)$ and

$$\tilde{p}(s_2, s_1s_2s_1) = (B_1A_2B_{12})/A_2 = B_1B_{12}.$$

When $v = s_1s_2s_1$, $w = e$, then $\epsilon = (0,0,0),(1,0,1)$ and

$$\tilde{p}(e, s_1s_2s_1) = B_1B_2B_1 + A_1B_{12}A_{-1}.$$

4 Casselman's problem

In his paper [Cas] B. Casselman gave a problem concerning transition coefficients between two bases in the space of Iwahori fixed vectors of a principal series representation of a p-adic group. We relate the problem with the Yang-Baxter basis and give an answer to the problem.

4.1 Principal series representations of p-adic group and Iwahori fixed vector

We follow the notations of M.Reeder [Re1, Re2]. Let G be a connected reductive p-adic group over a non-archimedian local field F. For simplicity we restrict to the case of split semisimple G. Associated to F, there is the ring of integer $\mathcal{O}$, the prime ideal $\mathfrak{p}$ with a generator ϖ, and the residue field with $q = |\mathcal{O}/\mathfrak{p}|$ elements. Let P be a minimal parabolic subgroup (Borel) of G, and A be the maximal split torus of P so that $A \simeq (F^*)^r$ where r is the rank of G. For an unramified quasi-character τ of A, i.e. a group homomorphism $\tau : A \to \mathbb{C}^*$ which is trivial on $A_0 = A \cap K$, where $K = G(\mathcal{O})$ is a maximal compact subgtoup of G. Let $T = \mathbb{C}^* \otimes X^*(A)$ be the complex torus dual to A, where $X^*(A)$ is the group of rational characters on A, i.e. $X^*(A) = \{\lambda : A \to F^*,$ algebraic group homomorphism$\}$. We have a pairing $<,>: A/A_0 \times T \to \mathbb{C}^*$ given by $< a, z \otimes \lambda >= z^{\mathrm{val}(\lambda(a))}$. This gives an identification $T \simeq X^{nr}(A)$ of T with the set of unramified quasi-characters on A (cf. [Bum] Exercise 18,19).

Let $\Delta \subset X^*(A)$ be the set of roots of A in G, Δ^+ be the set of positive roots corresponding to P and $\Sigma \subset \Delta^+$ be the set of simple roots . For a root $\alpha \in \Delta$, we define $e_\alpha \in X^*(T)$ by

$$e_\alpha(\tau) =< h_\alpha(\varpi), \tau >$$

for $\tau \in T$ where $h_\alpha : F^* \to A$ is the one parameter subgroup (coroot) corresponding to α.

Remark 5 *As the definition shows, e_α is defined using the coroot $\alpha^* = h_\alpha$. So it should be parametrized by α^*, but for convenience we follow the notation of [Re1]. Later we will identify $e_\alpha(\alpha \in \Delta = R^*)$ with $e^\alpha(\alpha \in R = \Delta^*)$ by the map $* : \Delta \to R$ of root data.*

W acts on right of $X^{nr}(A)$ so that $\tau^w(a) = \tau(waw^{-1})$ for $a \in A$, $\tau \in T$ and $w \in W$. The action of W on $X^*(T)$ is given by $(we_\alpha)(\tau) = e_{w\alpha}(\tau) = e_\alpha(\tau^w)$ for $\alpha \in \Delta$, $\tau \in T$ and $w \in W$.

The principal series representation $I(\tau)$ of G associated to an unramified quasicharacter τ of A is defined as follows. As a vector space over $\mathbb{C}$ it consists of locally constant functions on G with values in $\mathbb{C}$ which satisfy the left relative invariance properties with respect to P where τ is extended to P with trivial value on the unipotent radical N of $P = AN$.

$$I(\tau) := \mathrm{Ind}_P^G(\tau) = \{f : G \to \mathbb{C} \text{ loc. const. function } | f(pg) = \tau\delta^{1/2}(p)f(g) \text{ for } \forall p \in P, \forall g \in G\}.$$

Here δ is the modulus of P. The action of G on $I(\tau)$ is defined by right translation, i.e. for $g \in G$ and $f \in I(\tau)$, $(\pi(g)f)(x) = f(xg)$.

Let B be the Iwahori subgroup which is the inverse image $\pi^{-1}(P(\mathbb{F}_q))$ of the Borel subgroup $P(\mathbb{F}_q)$ of $G(\mathbb{F}_q)$ by the projection $\pi : G(\mathcal{O}) \to G(\mathbb{F}_q)$. Then we define $I(\tau)^B$ to be the space of Iwahori fixed vectors in $I(\tau)$, i.e.

$$I(\tau)^B := \{f \in I(\tau) \mid f(gb) = f(g) \text{ for } \forall b \in B, \forall g \in G\}.$$

This space has a natural basis $\{\varphi_w^\tau\}_{w \in W}$. $\varphi_w^\tau \in I(\tau)^B$ is supported on PwB and satisfies

$$\varphi_w^\tau(pwb) = \tau\delta^{1/2}(p) \text{ for } \forall p \in P, \forall b \in B.$$

4.2 Intertwiner and Casselman's basis

From now on we always assume that τ is regular i.e. the stabilizer $W_\tau = \{w \in W \mid \tau^w = \tau\}$ is trivial. The intertwining operator $\mathcal{A}_w^\tau : I(\tau) \to I(\tau^w)$ is defined by

$$\mathcal{A}_w^\tau(f)(g) := \int_{N_w} f(wng)dn$$

where $N_w := N \cap w^{-1}N_-w$, with N_- being the unipotent radical of opposite parabolic P_- which corresponds to the negative roots Δ^-. The integral is convergent when $|e_\alpha(\tau)| < 1$ for all $\alpha \in \Delta^+$ such that $w\alpha \in \Delta^-$ (cf. [Bum] Proposition 63), and may be meromorphically continued to all τ. It has the property that for $x, y \in W$ with $\ell(xy) = \ell(x) + \ell(y)$, then

$$\mathcal{A}_y^{\tau^x}\mathcal{A}_x^\tau = \mathcal{A}_{xy}^\tau.$$

The Casselman's basis $\{f_w^\tau\}_{w\in W}$ of $I(\tau)^B$ is defined as follows. $f_w^\tau \in I(\tau)^B$ and

$$\mathcal{A}_y^\tau f_w^\tau(1) = \begin{cases} 1 & \text{if } y = w \\ 0 & \text{if } y \neq w. \end{cases}$$

M.Reeder characterizes this using the action of affine Hecke algebra (cf. [Re2] Section 2). The affine Hecke algebra $\mathcal{H} = \mathcal{H}(G, B)$ is the convolution algebra of B bi-invariant locally constant functions on G with values in $\mathbb{C}$. By the theorem of Iwahori-Matsumoto it can be described by generators and relations. The basis $\{T_w\}_{w\in\widetilde{W}_{aff}}$ consists of characteristic functions $T_w := ch_{BwB}$ of double coset BwB. Let $\mathcal{H}_W$ be the Hecke algebra of the finite Weyl group W generated by the simple reflections s_α for simple roots $\alpha \in \Sigma$. As a vector space $\mathcal{H}$ is the tensor product of two subalgebras $\mathcal{H} = \Theta \otimes \mathcal{H}_W$. The subalgebra Θ is commutative and isomorphic to the coordinate ring of the complex torus T with a basis $\{\theta_a \mid a \in A/A_0\}$, where θ_a is defined as follows (cf. [Lu2]). Define $A^- := \{a \in A \mid |\alpha(a)|_F \leq 1\ \forall \alpha \in \Sigma\}$. For $a \in A$, choose $a_1, a_2 \in A^-$ such that $a = a_1a_2^{-1}$. Then $\theta_a = q^{(\ell(a_1)-\ell(a_2))/2}T_{a_1}T_{a_2}^{-1}$ where for $x \in G$, $\ell(x)$ is the length function defined by $q^{\ell(x)} = [BxB : B]$ and $T_x \in \mathcal{H}$ is the characteristic function of BxB.

By Lemma (4.1) of [Re1], there exists a unique $f_w^\tau \in I(\tau)_w \cap I(\tau)^B$ for each $w \in W$ such that
(1)$f_w^\tau(w) = 1$ and
(2)$\pi(\theta_a)f_w^\tau = \tau^w(a)f_w^\tau$ for all $a \in A$.
Here $I(\tau)_w := \{f \in I(\tau) \mid$ support of f is contained in $\bigcup_{x\geq w} PxP\}$.

4.3 Transition coefficients

Let

$$f_w^\tau = \sum_{w\leq v} a_{w,v}(\tau)\varphi_v^\tau$$

and

$$\varphi_w^\tau = \sum_{w\leq v} b_{w,v}(\tau)f_v^\tau.$$

The Casselman's problem is to find an explicit formula for $a_{w,v}(\tau)$ and $b_{w,v}(\tau)$.

To relate the results in Sections 2 and 3 with the Casselman's problem, in this subsection we specialize the parameters $t_1 = -q^{-1}$, $t_2 = 1$ and take tensor product with the complex field $\mathbb{C}$. For example, the Yang-Baxter basis Y_w will become a $Q_{t_1,t_2}(\Lambda)\otimes\mathbb{C}$ basis in $H_{t_1,t_2}^{Q(\Lambda)}(W)_\mathbb{C} = H_{t_1,t_2}^{Q(\Lambda)}(W)\otimes\mathbb{C}$. The generic Demazure-Lusztig operator defined in Section 3 will become

$$y_i := A_i\delta_i + B_i \text{ where } A_i := \frac{-q^{-1}+e^{-\alpha_i}}{1-e^{\alpha_i}}, B_i := \frac{-q^{-1}+1}{1-e^{-\alpha_i}}.$$

Then $(y_i + q^{-1})(y_i - 1) = 0$.

Theorem 3 *We identify e^α with e_α (cf. Remark 5). Then,*

$$a_{w,v}(\tau) = \tilde{p}(w,v)(\tau)|_{t_1=-q^{-1},t_2=1}$$
$$b_{w,v}(\tau) = p(w,v)(\tau)|_{t_1=-q^{-1},t_2=1}.$$

Proof: $b_{w,v}$'s satisfy the same recurrence relation (Proposition 5 with $t_1 = -q^{-1}, t_2 = 1$) as $p(w,v)$'s (cf. [Re2] Proposition (2.2)). The initial condition $b_{w,w} = p(w,w) = 1$ leads to the second equation. The first equation then also holds. Note that the $b_{y,w}$ in [Re2] is our $b_{w,y}$.

□

Remark 6 *There is also a direct proof that does not use recurrence relation cf. [NN].*

Corollary 3 *We have a closed formula for $a_{w,v}(\tau)$ and $b_{w,v}(\tau)$ by Corollary 2 and Theorem1.*

Corollary 4 *For $v \in W$, we have*

$$\sum_{w \le v} b_{w,v} = \prod_{\beta \in R(v)} \frac{1 - q^{-1}e^\beta}{1 - e^\beta},$$

and

$$\sum_{w \le v} b_{w,v}(-q^{-1})^{\ell(w)} = \prod_{\beta \in R(v)} \frac{1 - q^{-1}}{1 - e^\beta}.$$

Proof: When $t_1 = -q^{-1}, t_2 = 1$, we can specialize h_i to 1 and we get the first equation from the definition of Y_v, since $1 + \frac{(1-q^{-1})e^\beta}{1-e^\beta} = \frac{1-q^{-1}e^\beta}{1-e^\beta}$. We can also specialize h_i to $-q^{-1}$ and $-q^{-1} + \frac{(1-q^{-1})e^\beta}{1-e^\beta} = \frac{1-q^{-1}}{1-e^\beta}$ gives the second equation. □

Remark 7 *The left hand side of the first equation in Corollary 4 is $m(e, v^{-1})$ in [BN]. So this gives another proof of Theorem 1.4 in [BN].*

4.4 Whittaker function

M.Reeder [Re2] specified a formula for the Whittaker function $\mathcal{W}_\tau(f_w^\tau)$ and using $b_{w,v}$, he got a formula for $\mathcal{W}_\tau(\varphi_w^\tau)$. For $a \in A$, let $\lambda_a \in X^*(T)$ be

$$\lambda_a(z \otimes \mu) = z^{val(\mu(a))} \text{ for } z \in \mathbb{C}^*, \mu \in X^*(A).$$

Formally the result of M.Reeder [Re2] Corollary (3.2) is written as follows. For $w \in W$ and $a \in A^-$,

$$\mathcal{W}(\varphi_w)(a) = \delta^{1/2}(a) \sum_{w \le y} b_{w,y}\ y \left[\lambda_a \prod_{\beta \in R^+ - R(y)} \frac{1 - q^{-1}e^\beta}{1 - e^{-\beta}} \right] \in \mathbb{C}[T].$$

Then using Corollary 3, we have an explicit formula of $\mathcal{W}(\varphi_w)(a)$.

4.5 Relation with Bump-Nakasuji's work

Now we explain the relation between this paper and Bump-Nakasuji [BN]. First of all, the notational conventions are slightly different. Especially in the published [BN] the natural base and intertwiner are differently parametrized. The natural basis ϕ_w in [BN] is our $\varphi_{w^{-1}}$. The intertwiner M_w in [BN] is our $\mathcal{A}_{w^{-1}}$ so that if $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$, $M_{w_1w_2} = M_{w_1} \circ M_{w_2}$ while $\mathcal{A}_{w_1w_2} = \mathcal{A}_{w_2}\mathcal{A}_{w_1}$.

In the paper [BN], another basis $\{\psi_w\}_{w\in W}$ for the space $I(\tau)^B$ was defined and comparerd with the Casselman's basis. They defined $\psi_w := \sum_{v\geq w} \varphi_v$ and expand this as $\psi_w = \sum_{v\geq w} m(w,v) f_v$ and conversely $f_w = \sum_{v\geq w} \tilde{m}(w,v)\psi_v$. They observed that the transition coefficients $m(w,v)$ and $\tilde{m}(w,v)$ factor under certain condition. Let $S(w,v) := \{\alpha \in R^+ | w \leq s_\alpha v < v\}$ and $S'(w,v) := \{\alpha \in R^+ | w < s_\alpha w \leq v\}$. Then the statements of the conjectures are as follows.

Conjecture 1 *([BN] Conjecture 1.2) Assume that the root system R is simply-laced. Suppose $w \leq v$ and $|S(w,v)| = \ell(v) - \ell(w)$, then*

$$m(w,v) = \prod_{\alpha\in S(w,v)} \frac{1-q^{-1}z^\alpha}{1-z^\alpha}.$$

Conjecture 2 *([BN] Conjecture 1.3) Assume that the root system R is simply-laced. Suppose $w \leq v$ and $|S'(w,v)| = \ell(v) - \ell(w)$, then*

$$\tilde{m}(w,v) = (-1)^{\ell(v)-\ell(w)} \prod_{\alpha\in S'(w,v)} \frac{1-q^{-1}z^\alpha}{1-z^\alpha}.$$

Proposition 10 *Conjecture 1.2 and Conjecture 1.3 in [BN] are equivalent.*

Proof: We can show $m(w,v) = \sum_{w\leq z\leq v} p(z,v)$ and $\tilde{m}(w,v) = \sum_{w\leq z\leq v} (-1)^{\ell(v)-\ell(z)} \tilde{p}(w,z)$. Then it follows by the Theorem 1 that $\tilde{m}(w,v) = (-1)^{\ell(v)-\ell(w)} m(vw_0, ww_0)$. As $S'(w,v) = S(vw_0, ww_0)$ we get the desiered conclusion. □

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