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# Almost simplicial polytopes: the lower and upper bound theorems

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**Abstract.** This is an extended abstract of the full version. We study  $n$ -vertex  $d$ -dimensional polytopes with at most one nonsimplex facet with, say,  $d + s$  vertices, called *almost simplicial polytopes*. We provide tight lower and upper bounds for the face numbers of these polytopes as functions of  $d, n$  and  $s$ , thus generalizing the classical Lower Bound Theorem by Barnette and Upper Bound Theorem by McMullen, which treat the case  $s = 0$ . We characterize the minimizers and provide examples of maximizers, for any  $d$ .

**Résumé.** Ceci est un résumé étendu d'une version plus complète. Nous étudions les polytopes de dimension  $d$  à  $n$  sommets dont au plus une facette n'est pas un simplexe et contient par exemple  $d + s$  sommets. Nous appelons de tels polytopes des polytopes presque simpliciaux. Nous établissons des bornes inférieures et supérieures exactes pour le nombre de faces de ces polytopes en fonction de  $d, n$  et  $s$ , généralisant ainsi les résultats classiques de Barnette sur la borne inférieure et de McMullen sur la borne supérieure dans le cas où  $s = 0$ . Nous caractérisons les polytopes possédant un nombre de faces minimales et donnons des exemples de polytopes avec un nombre de faces maximal.

**Keywords.** polytope,  $f$ -vector, LBT, UBT, graph rigidity, moment curve

## 1 Introduction

In 1970 McMullen [14] proved the Upper Bound Theorem (UBT) for *simplicial polytopes*, polytopes with each facet being a simplex, while between 1971 and 1973 Barnette [2, 3] proved the Lower Bound Theorem (LBT) for the same polytopes. Both results are major achievements in the combinatorial theory of polytopes; see, e.g., the books [10, 19] for further details and discussion.

These results can be phrased as follows: let  $C(d, n)$  (resp.  $S(d, n)$ ) denote a cyclic (resp. stacked)  $d$ -polytope on  $n$  vertices, and for a polytope  $P$  let  $f_i(P)$  denote the number of its  $i$ -dimensional faces. Then the classical LBT and UBT read as follows.

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**Theorem 1.1 (Classical LBT and UBT)** For any simplicial  $d$ -polytope on  $n$  vertices, and any  $0 \leq i \leq d - 1$ ,

$$f_i(S(d, n)) \leq f_i(P) \leq f_i(C(d, n)).$$

The numbers  $f_i(S(d, n))$  and  $f_i(C(d, n))$  are explicit known functions of  $(d, n, i)$ , to be discussed later.

We generalize the UBT and LBT to the following context: consider a pair  $(P, F)$  where  $P$  is a polytope,  $F$  is a facet of  $P$ , and all facets of  $P$  different from  $F$  are simplices. We call such a polytope  $P$  an *almost simplicial polytope* (ASP) and a pair  $(P, F)$  an ASP-pair. We will be interested only in the combinatorics of  $P$ , thus the ASP-pair  $(P, F)$  is equivalent to specifying a regular triangulation of  $F$  admitting a lifting of its vertices that leaves the vertices of  $F$  fixed; we are interested in the simplicial ball  $P' := \partial P - \{F\}$ .

Let  $\mathcal{P}(d, n, s)$  denote the family of  $d$ -polytopes  $P$  on  $n$ -vertices such that  $(P, F)$  is an ASP-pair, where  $F$  has  $d + s$  vertices ( $s \geq 0$ ). Note that  $\mathcal{P}(d, n, 0)$  consists of the simplicial  $d$ -polytopes on  $n$  vertices. In this paper, we define certain polytopes  $C(d, n, s), S(d, n, s) \in \mathcal{P}(d, n, s)$ , explicitly compute their face numbers, and show the following.

**Theorem 1.2 (LBT and UBT for ASP)** For any  $d, n, s$ , any polytope  $P \in \mathcal{P}(d, n, s)$ , and any  $0 \leq i \leq d - 1$ ,

$$f_i(S(d, n, s)) \leq f_i(P) \leq f_i(C(d, n, s)).$$

Further, the polytopes  $P \in \mathcal{P}(d, n, s)$  with  $f_i(P) = f_i(S(d, n, s))$  for some  $0 \leq i \leq d - 1$  are characterized combinatorially, and satisfy the above equality for all  $0 \leq i \leq d - 1$ .

The characterization of the equality case above generalizes Kalai's result [11] that equality in the classical LBT holds for some  $1 \leq i \leq d - 1$  iff  $P$  is stacked. The polytopes  $C(d, n, s)$  form an ASP analog of cyclic polytopes and satisfy a combinatorial Gale-evenness type description of their facets.

Billera and Lee [5] considered the notion of polytope pairs. In particular, their results give tight upper and lower bound theorems for the face numbers of simplicial  $(d - 1)$ -dimensional balls of the “*polytope-antistar*” form; that is, balls of the form  $\partial Q - v$ , where  $Q$  is a simplicial  $d$ -polytope and  $v$  is a vertex of  $Q$  that is deleted. These bounds are given as functions of  $d, f_0(\partial Q - v), f_0(Q/v)$ , where  $Q/v$  denotes the vertex figure of  $v$  in  $Q$ . For an ASP-pair  $(P, F)$ , let  $Q$  be obtained from  $P$  by stacking a pyramid over  $F$  with a new vertex  $v$ . Then  $F \cong Q/v$  and  $P' = \partial P - \{F\} = \partial Q - v$ . Thus, our balls  $P'$  form a subfamily of the balls  $\partial Q - v$  considered in [5]. The bounds we obtain in Theorem 1.2 are strictly stronger than those of [5] which apply to all polytope-antistar balls.

Let  $f(P) = (1, f_0(P), f_1(P), \dots, f_{d-1}(P))$  denote the  $f$ -vector of  $P$ , a vector recording the face numbers of  $P$ . The following problem naturally arises.

**Problem 1.3** Characterize the pairs of  $f$ -vectors  $(f(P), f(F))$  for ASP-pairs  $(P, F)$ .

A solution would generalize the well known  $g$ -theorem characterizing the face numbers of simplicial polytopes, conjectured by McMullen [15] and proved by Billera-Lee [4] (sufficiency) and Stanley [17] (necessity). We leave this general problem to a future study.

The proof of the LBT for ASP and the characterization of the equality cases are based on framework-rigidity arguments (cf. Kalai [11]) and on an adaptation of the well known McMullen-Perles-Walkup reduction (MPW) [11, Sec. 5] to ASP; see Section 3.

The numerical bounds obtained in the UBT for ASP are a special case of a recent result of Adiprasito and Sanyal [1, Thm. 3.10], who proved the bounds for homology balls whose boundary is an induced subcomplex. While their proof relies on machinery from commutative algebra, we provide an elementary

proof based on a suitable shelling of  $P$ . Further, our construction of maximizers  $C(d, n, s)$  is a generalization of cyclic polytopes, based on a suitable variation of the moment curve, and is of independent interest; see Section 4.

## 2 Preliminaries

For undefined terminology and notation, see [19] for polytopes and complexes, or [11, Sec. 2] for framework rigidity.

### 2.1 Polytopes and simplicial complexes

The  $k$ -dimensional faces of a polyhedral complex  $\Delta$  are called  $k$ -faces, where the empty face has dimension  $-1$ . For a simplicial complex  $\Delta$  of dimension  $d - 1$ , the number  $f_k(\Delta)$  is then related to the  $h$ -numbers  $h_k(\Delta) := \sum_{i=0}^k (-1)^{k-i} \binom{d-i}{k-i} f_{i-1}(\Delta)$  by

$$f_{k-1}(\Delta) = \sum_{i=0}^k \binom{d-i}{k-i} h_i(\Delta). \tag{1}$$

The  $h$ -vector of  $\Delta$ ,  $(\dots, h_k, h_{k+1}, \dots)$ , can be considered as an infinite sequence if we let  $h_k(\Delta) = 0$  for  $k > d$  and  $k < 0$ . The  $g$ -numbers are defined by  $g_k(\Delta) = h_k(\Delta) - h_{k-1}(\Delta)$ .

For an ASP pair  $(P, F)$ , where  $P$  is  $d$ -dimensional, the following version of the Dehn-Somerville equations applies to the complex  $P' = \partial P - \{F\}$ .

**Proposition 2.1 ([9, Thm. 18.3.6], Dehn-Somerville Equations for  $P'$ )** *The  $h$ -vector of the simplicial  $(d - 1)$ -ball  $P'$  with boundary  $\partial F$  satisfies for  $k = 0, \dots, d$*

$$h_k(P') = h_{d-k}(P') + g_k(\partial F). \tag{2}$$

Note that  $h_k(P') = 0$  and  $h_k(\partial F) = 0$  for  $k \geq d$  and  $h_{d-1}(\partial F) = 1$ .

Note that  $P'$  is shellable, by a Bruggesser-Mani line shelling.

The *link* of a face  $F$  in  $\Delta$  is  $\text{link}_\Delta(F) := \{T \in \Delta : T \cap F = \emptyset, F \cup T \in \Delta\}$ , and its *star*,  $\text{star}_\Delta(F)$  is the complex  $\cup_{F \subset T} 2^T$ . Thus, using the *join* operator on simplicial complexes, we obtain  $2^F * \text{link}_\Delta(F) = \text{star}_\Delta(F)$ . The definition of the star extends to polyhedral complexes. For a vertex  $v$  in a polytope  $Q$ , its *vertex figure*  $Q/v$  is a codimension 1 polytope obtained by intersecting  $Q$  with a hyperplane  $H$  a bit *below*  $v$ , so that  $v$  is on one side of  $H$  and the other vertices of  $Q$  are on the other side. If  $\text{star}_Q(v)$  is simplicial then the boundary complex of  $Q/v$  coincides with  $\text{link}_Q(v)$ .

A subcomplex  $K$  of  $\Delta$  is *induced* if it contains all the faces in  $\Delta$  which only involve vertices in  $K$ . Note that, for an ASP-pair  $(P, F)$ ,  $\partial F$  is an induced subcomplex of  $P'$ , by convexity.

A polytope is *k-neighborly* if each subset of at most  $k$  vertices forms the vertex set of a face. A  $\lfloor d/2 \rfloor$ -neighborly  $d$ -polytope is simply called *neighborly*. A polytope is *k-simplicial* if each  $k$ -face is a simplex.

The underlying set  $|\mathcal{C}|$  of a polyhedral complex  $\mathcal{C}$  is the point set  $\cup_{Q \in \mathcal{C}} Q$  of its geometric realization. A *refinement* (or subdivision) of  $\mathcal{C}$  is another polyhedral complex  $\mathcal{D}$  such that  $|\mathcal{D}| = |\mathcal{C}|$  and for any face  $F \in \mathcal{D}$  there exists a face  $T \in \mathcal{C}$  such that  $|F| \subseteq |T|$ .

Let  $G$  be a proper face of a polytope  $Q$ . A point  $w$  is *beyond*  $G$  (with respect to  $Q$ ) if (i)  $w$  is not on any hyperplane supporting a facet of  $Q$ , (ii)  $w$  and the interior of  $Q$  lie on different sides of any hyperplane supporting a facet containing  $G$ , but (iii) on the same side of every other facet-defining hyperplane which

does not contain  $G$ . For an ASP-pair  $(P, F)$  we will consider the simplicial polytope  $Q$  obtained as the convex hull of  $P$  and a vertex  $y$  beyond  $F$ .

A simplicial complex  $\Delta$  is a *homology sphere* (over a fixed field  $\mathbf{k}$ ) if for any face  $F \in \Delta$ , the homology groups  $H_i(\text{link}_\Delta(F); \mathbf{k}) \cong H_i(S^{\dim \Delta - \dim F - 1}; \mathbf{k})$  for all  $i$ , where  $S^j$  is the  $j$ -dimensional sphere. Say  $\Delta$  is a *homology ball* if  $H_i(\text{link}_\Delta(F); \mathbf{k})$  vanishes for  $i < \dim \Delta - \dim F - 1$  and is isomorphic to either  $0$  or  $\mathbf{k}$  for  $i = \dim \Delta - \dim F - 1$ . Furthermore, the boundary complex  $\partial\Delta$  of  $\Delta$ , consisting of all faces  $F$  for which  $H_{\dim \Delta - \dim F - 1}(\text{link}_\Delta(F); \mathbf{k}) = 0$ , is a homology sphere (of codimension 1). In particular, simplicial spheres (resp. balls) are homology spheres (resp. balls).

A polytope is *stacked* if it can be obtained from a simplex by repeatedly taking the convex hull with a vertex beyond some facet. A homology sphere is *stacked* if it is combinatorially isomorphic to the boundary complex of a stacked polytope.

## 2.2 Rigidity

We mostly follow the presentation in Kalai’s [11]. Let  $G = (V, E)$  be a graph, and  $d(a, b)$  denote Euclidean distance between points  $a$  and  $b$  in Euclidean space. A  $d$ -embedding  $f : V \rightarrow \mathbf{R}^d$  is called *rigid* if there exists an  $\epsilon > 0$  such that if  $g : V \rightarrow \mathbf{R}^d$  satisfies  $d(f(v), g(v)) < \epsilon$  for every  $v \in V$  and  $d(g(u), g(w)) = d(f(u), f(w))$  for every  $\{u, w\} \in E$ , then  $d(g(u), g(w)) = d(f(u), f(w))$  for every  $u, w \in V$ .  $G$  is called *generically  $d$ -rigid* if the set of its rigid  $d$ -embeddings is open and dense in the topological vector space of all of its  $d$ -embeddings. Given a  $d$ -embedding  $f : V \rightarrow \mathbf{R}^d$ , a *stress* of  $f$  is a function  $w : E \rightarrow \mathbf{R}$  such that for every vertex  $v \in V$

$$\sum_{u:\{v,u\} \in E} w(\{v, u\})(f(v) - f(u)) = 0.$$

The stresses of  $f$  form a vector space, called the *stress space*. Its dimension is the same for generic  $d$ -embeddings (namely, for an open and dense set in the space of all  $d$ -embeddings of  $G$ ). A graph  $G$  is called *generically  $d$ -stress free* if this dimension is zero.

If a generic  $f : V \rightarrow \mathbf{R}^d$  is rigid, then  $f_1(G) \geq df_0(G) - \binom{d+1}{2}$ . Thus, if  $\Delta$  is a simplicial complex of dimension  $d - 1$  whose 1-skeleton is generically  $d$ -rigid, then  $f_1(\Delta) \geq df_0(\Delta) - \binom{d+1}{2}$ , and  $g_2(\Delta)$  is the dimension of the stress space of any generic embedding. Based on these observations for  $\Delta$  the boundary of a simplicial  $d$ -polytope with  $d \geq 3$ , and more general complexes, Kalai [11] extended the LBT and characterized the minimizers.

For a  $d$ -polytope  $P$  with a simplicial 2-skeleton, the toric  $g_2(P)$  equals  $g_2(\partial P) := f_1(P) - df_0(P) + \binom{d+1}{2}$ , and by a result of Alexandrov (cf. Whiteley [18]), it equals the dimension of the stress space of the 1-skeleton of  $P$ . For our LBT for ASP, we will need the following very special case of Kalai’s monotonicity<sup>(i)</sup>, which Kalai proved using rigidity arguments.

**Theorem 2.2 (Kalai’s Monotonicity [12, Thm. 4.1], weak form)** *Let  $d \geq 4$ ,  $P$  a  $d$ -polytope with a simplicial 2-skeleton, and  $F$  a facet of  $P$ . Then*

$$g_2(P) \geq g_2(F).$$

*Equivalently,  $f_1(P) - f_1(F) \geq (df_0(P) - \binom{d+1}{2}) - ((d - 1)f_0(F) - \binom{d}{2})$ .*

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<sup>(i)</sup> Kalai’s monotonicity conjecture on the toric  $g$ -polynomials, asserting that  $g(P) \geq g(F)g(P/F)$  coefficientwise for any face  $F$  of  $P$ , was first proved for rational polytopes by Braden and MacPherson [7]. By the combinatorial intersection homology, it is now known to hold in full generality; cf. [6].

### 3 A lower bound theorem for almost simplicial polytopes

Recall that a simplicial  $d$ -polytope is called *stacked* if it can be obtained from a  $d$ -simplex by repeated *stacking*, namely, adding a vertex beyond a facet and taking the convex hull. While stacked  $d$ -polytopes on  $n$  vertices, denoted  $S(d, n)$ , may have different combinatorial structures, they all have the same  $f$ -vector, given by

$$f_k(S(d, n)) = \phi_k(d, n) := \begin{cases} \binom{d}{k}n - \binom{d+1}{k+1}k & \text{for } k = 1, \dots, d-2 \\ (d-1)n - (d+1)(d-2) & \text{for } k = d-1. \end{cases}$$

For any integers  $d \geq 3$ ,  $s \geq 0$  and  $n \geq d + s + 1$ , let  $F$  be a stacked  $(d-1)$ -polytope with  $d + s$  vertices. Construct a pyramid over  $F$  and then stack  $n - d - s - 1$  times over facets of the resulting polytope which are different from  $F$  to obtain a polytope  $S(d, n, s)$  in  $\mathcal{P}(d, n, s)$ . One easily computes the  $f$ -vector of  $S(d, n, s)$ , since refining  $F$  by its (unique) stacked triangulation refines the boundary complex of  $S(d, n, s)$  to a stacked simplicial sphere with  $f$ -vector  $f(S(d, n))$ . We obtain

$$f(S(d, n, s)) = f(S(d, n)) - (0, 0, \dots, 0, s, s).$$

Note that for  $n \geq s + 4$ , any  $P \in \mathcal{P}(3, n, s)$  has  $f$ -vector  $f(P) = (1, n, 3n - 6 - s, 2n - 4 - s) = f(S(3, n, s))$ . We are ready to state the LBT for ASP; its minimizers will be characterized later.

**Theorem 3.1 (LBT for ASP)** *Let  $d \geq 3$ ,  $s \geq 0$ ,  $n \geq d + s + 1$ . Then for any  $P \in \mathcal{P}(d, n, s)$  and  $1 \leq i \leq d - 1$  we have*

$$f_i(S(d, n, s)) \leq f_i(P).$$

**Proof:** We proceed by induction on  $d$ , the case  $d = 3$  was verified above. Let  $d \geq 4$ . By a result of Whiteley [18], the 1-skeleton of  $P$  is generically  $d$ -rigid, hence  $f_1(P) \geq \phi_1(d, n)$ , and by the MPW reduction,  $f_i(P) \geq \phi_i(d, n)$  for all  $2 \leq i \leq d - 3$  as well; see [11, Thm. 12.2]<sup>(ii)</sup>.

Denote by  $(P, F)$  the ASP-pair, and by  $\deg_P(v)$  the degree of a vertex  $v$  in the 1-skeleton of  $P$ . We now prove the inequality for the facets, by a variation of the MPW reduction. Note that the vertex figure  $P/v$  in  $P$  of any vertex  $v \in \text{vert } F$  is an ASP (with  $\deg_P(v)$  vertices), while for any vertex  $v \in \text{vert } P \setminus \text{vert } F$   $P/v$  is a simplicial polytope; cf. [8, Thm. 11.5]. Furthermore, for a vertex  $v \in \text{vert } F$ , letting  $s_v := \deg_F(v) - (d - 1) \geq 0$  gives  $P/v \in \mathcal{P}(d - 1, \deg_P(v), s_v)$ .

Double counting the number of pairs  $(v, A)$  for a vertex  $v$  in a facet  $A$  of  $P$ , we obtain the following inequalities:

$$\begin{aligned} d(f_{d-1}(P) - 1) + (d + s) &= \sum_{v \in \text{vert } P} f_{d-2}(\text{link}_P(v)) \\ &\geq \sum_{v \in \text{vert } P \setminus \text{vert } F} ((d - 2) \deg_P(v) - d(d - 3)) + \sum_{v \in \text{vert } F} ((d - 2) \deg_P(v) - d(d - 3) - s_v) \\ &= 2(d - 2)f_1(P) - d(d - 3)f_0(P) - 2f_1(F) + (d - 1)(d + s) \\ &\geq 2(d - 2) \left[ df_0(P) - \binom{d + 1}{2} \right] - d(d - 3)f_0(P) - 2 \left[ (d - 1)f_0(F) - \binom{d}{2} \right] + (d - 1)(d + s) \\ &= d(d - 1)f_0(P) - d(d + 1)(d - 2) - s(d - 1), \end{aligned}$$

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<sup>(ii)</sup> Kalai's theorem contains a typo. It includes the case  $i = k$ , while it holds only for  $i < k$ , where  $P$  is  $k$ -simplicial. Our ASP  $P$  is  $(d - 2)$ -simplicial.

where the first inequality is by the induction hypothesis and the second inequality is by Kalai's monotonicity Theorem 2.2 and the LBT inequality for  $f_1(P)$ . Comparing the LHS with the RHS gives

$$f_{d-1}(P) \geq \phi_{d-1}(d, n) - s.$$

The inequality for  $f_{d-2}(P)$  follows from the inequality for  $f_{d-1}(P)$  by double counting. Since any ridge in  $P$  is contained in exactly two facets, counting the number of pairs  $(R, A)$  for a ridge  $R$  in a facet  $A$  of  $P$ , we obtain that

$$2f_{d-2}(P) = d(f_{d-1}(P) - 1) + f_{d-2}(F).$$

Applying the classical LBT to the simplicial polytope  $F$  with  $f_0(F) = d + s$ , we get

$$2f_{d-2}(P) \geq d(f_{d-1}(P) - 1) + (d - 2)(d + s) - d(d - 3),$$

and applying the lower bound for  $f_{d-1}(P)$  yields, after dividing both sides by 2, the desired lower bound  $f_{d-2}(P) \geq \phi_{d-2}(d, n) - s$ .  $\square$

We now turn our attention to characterizing the minimizers of Theorem 3.1. We start with some terminology and background.

A proper subset  $A$  of the vertices of a  $d$ -polytope  $P$  is called a *missing  $k$ -face* of  $P$  if the cardinality of  $A$  is  $k + 1$ , the simplex on  $A$  is not a face of  $P$ , but for any proper subset  $B$  of  $A$  the simplex on  $B$  is a face of  $P$ . If  $A$  is a missing  $(d - 1)$ -face of  $P$  then adding the simplex  $A$  cuts  $P$  into two  $d$ -polytopes  $P_1, P_2$ , glued along the simplex  $A$ . We denote this operation by  $P = P_1 \# P_2$ . Repeating this procedure on each  $P_i$  until no piece  $P_i$  contains a missing  $(d - 1)$ -face results in a decomposition  $P = P_1 \# P_2 \# \dots \# P_t$ , where intersections along missing  $(d - 1)$ -faces of  $P$  define a tree whose vertices are the  $P_i$ 's. Note that for  $d \geq 3$  a decomposition of  $P$  as above is uniquely defined; just insert all the missing  $(d - 1)$ -faces. Call such a decomposition the *prime decomposition* of  $P$ , and call each  $P_i$  a *prime factor* of  $P$ . Denote by  $\Delta_P$  the polyhedral complex defined by the prime decomposition of  $P$ . Then a simplicial  $d$ -polytope  $P$  is stacked iff all its prime factors are  $d$ -simplices. This definition immediately extends to polyhedral spheres where the operation  $\#$  corresponds to the topological connected sum.

We start with the characterization of the minimizers for the easier case  $d > 4$ .

**Theorem 3.2 (Characterization of minimizers for  $d > 4$ )** *Let  $d > 4$  and  $P \in \mathcal{P}(d, n, s)$ . Let  $\Delta_F$  be the polyhedral complex corresponding to the prime decomposition of  $F$ , and let  $\Delta$  be the refinement of the boundary complex  $\partial P$  of  $P$  obtained by refining  $F$  by  $\Delta_F$ . Assume there is some  $1 \leq i \leq d - 1$  for which  $f_i(P) = f_i(S(d, n, s))$ . Then, all prime factors in the prime decomposition of  $\Delta$  are  $d$ -simplices. In particular,  $f(P) = f(S(d, n, s))$ .*

**Proof:** By the MPW reduction and the variation of it we used in the proof of Theorem 3.1, it is enough to consider the case  $i = 1$ . From Kalai's monotonicity (Theorem 2.2) and our assumption  $g_2(P) = 0$ , it follows that  $g_2(F) = 0$ . As  $F$  is simplicial of dimension  $\geq 4$ , Kalai's [11, Thm. 1.1(ii)] says that  $F$  is stacked, thus  $\Delta$  is a simplicial  $(d - 1)$ -sphere. Since  $g_2(\Delta) = 0$ , by [11, Thm. 1.1(ii)] again,  $\Delta$  is stacked, as desired. In particular,  $f(P) = f(S(d, n, s))$ .  $\square$

For  $d = 4$ ,  $F$  need not be stacked. For example, the pyramid over any simplicial 3-polytope is a minimizer. We obtain the following characterization of minimizers.

**Theorem 3.3 (Characterization of minimizers for  $d = 4$ )** *Let  $P \in \mathcal{P}(4, n, s)$ , and keep the notation of Theorem 3.2. Assume there is some  $1 \leq i \leq d - 1$  for which  $f_i(P) = f_i(S(d, n, s))$ . Then, the prime factors in the prime decomposition of  $\Delta$  are either  $d$ -simplices with no facet contained in  $|F|$ , or pyramids over prime factors of  $F$ .*

In order to prove this theorem we first need to show generic  $d$ -rigidity for the 1-skeleton of a much larger class of complexes. Let  $\mathcal{C}_k$  be the family of homology  $k$ -balls  $\Delta$  such that:

- the induced subcomplex  $\Delta[I]$  on the set of internal vertices  $I$  has a connected 1-skeleton, and
- for any edge  $e$  in the boundary complex  $\partial\Delta$ , there exists a 2-simplex  $T$ ,  $e \subset T$ , such that  $T$  has a vertex in  $I$ .

Note that any homology  $k$ -ball  $\Delta$  whose boundary  $\partial\Delta$  is an induced subcomplex is in  $\mathcal{C}_k$ . In particular, for  $P \in \mathcal{P}(d, n, s)$ , the simplicial complex  $P' = \partial P - \{F\}$  is in  $\mathcal{C}_{d-1}$ .

**Lemma 3.4** *Let  $d \geq 4$ . The 1-skeleton of any  $\Delta \in \mathcal{C}_{d-1}$  is generically  $d$ -rigid. Thus,  $f_1(\Delta) \geq df_0(\Delta) - \binom{d+1}{2}$ .*

The proof is similar to Kalai’s proof of the classical LBT [11] and is omitted.

**Proof of Theorem 3.3:** Consider a prime factor  $L$  of  $\Delta$ . Then  $L$  is a 4-polytope with a generically 4-rigid 1-skeleton. As  $g_2(P) = 0$ , the 1-skeleton of  $L$ , denoted by  $G$ , must be generically 4-stress free. Thus,  $g_2(L) = 0$ .

If  $L$  does not contain a facet in  $F$ , then  $L$  is simplicial, with  $g_2(L) = 0$ , hence is stacked by [11, Thm. 1.1]. Being also prime,  $L$  is a 4-simplex.

Assume then that  $L$  contains a facet  $F''$  contained in  $|F|$ , so  $(L, F'')$  is an ASP-pair. If  $L$  has a unique vertex outside  $F''$ , then  $L$  is a pyramid over a prime factor of  $F$  and we are done. Assume the contrary, so there is an edge  $vu \in G$  with  $v, u \notin F''$  (for concreteness, taking  $v, u$  to be the highest two vertices of  $L$  above the hyperplane of  $F$  works).

First we show that  $vu$  satisfies the link condition  $\text{link}_L(v) \cap \text{link}_L(u) = \text{link}_L(vu)$ , which guarantees that contracting the edge  $vu$  in the simplicial complex  $\partial L - \{F''\}$  results in  $\tilde{\Delta} \in \mathcal{C}_3$ ; see e.g.[16, Prop.2.4]<sup>(iii)</sup>. Indeed, if  $vu$  fails the link condition it means that  $vu$  is contained in a missing face  $M$ , with 3 or 4 vertices. Now,  $M$  cannot have 4 vertices as  $L$  is prime. If  $M = vuz$  then  $uz$  is an edge of  $L$  not in  $\text{link}_L(v)$ . Since  $\text{link}_L(v)$  is a homology 2-sphere (thus, a simplicial 2-sphere), its 1-skeleton is generically 3-rigid. Consequently, the 1-skeleton of  $\text{star}_L(v)$  is generically 4-rigid, and adding  $vu$  to it yields a 4-stress in  $G$ , a contradiction.

Let  $m$  be the number of vertices in the cycle  $\text{link}_L(vu)$ , then  $f_1(\tilde{\Delta}) = f_1(L) - m - 1$  and  $f_0(\tilde{\Delta}) = f_0(L) - 1$ , which implies that  $g_2(L) = g_2(\tilde{\Delta}) + (m - 3)$ .

If  $m > 3$ , then applying Lemma 3.4 to  $\tilde{\Delta}$  yields  $g_2(L) > 0$ , a contradiction. So assume  $m = 3$ .

Denote by  $x, y, z$  the vertices of  $\text{link}_L(vu)$ . If the triangle  $xyz \in L$ , then, as  $L$  is prime, both tetrahedra  $xyzv, xyzu$  are faces of  $L$ , so  $L$  is the 4-simplex  $xyzuv$ , a contradiction (as it has a facet  $F''$  in  $F$ ).

We are left to consider the case  $xyz \notin L$ . The argument here is inspired by Barnette [2, Thm. 2]. In this case, the 3-ball formed by the join  $vu * \partial(xyz)$  is an induced subcomplex of  $\partial L - \{F''\}$ . Now replace it by  $\partial(vu) * xyz$  (this is a bistellar move) to obtain from  $\partial L - \{F''\}$  the complex  $\Delta''$ . Clearly  $\Delta''$  is a

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<sup>(iii)</sup> To apply [16, Prop. 2.4], phrased for homology spheres, simply cone the boundary of the homology ball  $\Delta$  to obtain a homology sphere.



homology 3-ball, and any edge on its boundary is part of a 2-simplex with an internal vertex (just take the same one as in  $\partial L - \{F''\}$ ). To show  $\Delta'' \in \mathcal{C}_3$  we are left to show that the graph on the internal vertices  $I$  of  $\Delta''$  is connected. Assume not, namely removing the edge  $uv$  disconnects the induced graph on  $I$  in  $\partial L - \{F''\}$ . In particular,  $x, y, z \in F''$ . But  $xyz \notin L$ , so  $xyz$  is a missing face of  $F''$ , contradicting that  $F''$  is a prime factor of  $F$ .

We conclude that  $\Delta'' \in \mathcal{C}_3$ , thus, by Lemma 3.4,  $\Delta'' \cup \{vu\}$  has a nonzero 4-stress, so  $g_2(L) > 0$ , a contradiction. The proof is then complete.  $\square$

**Remark 3.5** *The above shows that, for any  $k \geq 3$ , the lower bounds of Theorem 3.1 are valid for any complex in  $\mathcal{C}_k$ , and the minimizers in  $\mathcal{C}_k$  are exactly the complexes  $\partial P - \{F\}$  described in Theorems 3.2 and 3.3.*

## 4 An upper bound theorem for almost simplicial polytopes

Throughout this section, we let  $P \in \mathcal{P}(d, n, s)$  denote an almost simplicial polytope,  $(P, F)$  the ASP-pair, and  $P' = \partial P \setminus \{F\}$  the corresponding (shellable) simplicial  $(d - 1)$ -ball.

### 4.1 ASP generalization of cyclic polytopes

The *moment curve* in  $\mathbf{R}^d$  is defined by  $t \mapsto (t, t^2, \dots, t^d)$  for  $t \in \mathbf{R}^d$ . We consider related curves  $x(t) = (t, t^2, \dots, t^{d-r}, p_1(t), \dots, p_r(t))$ , where  $p_i(t)$  are continuous functions in  $t$  for  $i = 1, \dots, r$ . Later, a special choice of the curve  $x(t)$  and points on it will give our maximizer polytope  $C(d, n, s)$ .

We let  $V(t_1, \dots, t_d)$  denote the *Vandermonde determinant* on variables  $t_1, \dots, t_d$ .

$$V(t_1, \dots, t_d) := \begin{vmatrix} 1 & 1 & \dots & 1 \\ t_1 & t_2 & \dots & t_d \\ t_1^2 & t_2^2 & \dots & t_d^2 \\ \vdots & \vdots & \dots & \vdots \\ t_1^{d-1} & t_2^{d-1} & \dots & t_d^{d-1} \end{vmatrix} = \prod_{1 \leq i < j \leq d} (t_j - t_i).$$

**Lemma 4.1** *Consider the curve  $x(t)$ . Then the following holds:*

1. *Any  $d - r + 1$  points on the curve  $x(t)$  are affinely independent.*
2. *For any  $n$  distinct numbers  $t_1, \dots, t_n$ , the polytope  $Q = \text{conv}(\{x(t_1), \dots, x(t_n)\})$  is  $(d - r - 1)$ -simplicial.*
3. *The polytope  $Q$  is  $\lfloor (d - r)/2 \rfloor$ -neighbourly.*

The proof is similar to [10, Sec. 4.7], and is omitted.

Consider the curve  $y(t) = (t, t^2, \dots, t^{d-1}, p(t))$ , where

$$p(t) := (n - 1)^{(t-1)(d-1)} t(t + 1) \dots (t + d + s - 1),$$

and  $n$  and  $s$  are fixed integers with  $n > d + s$  and  $s \geq 0$ . Let  $C(d, n, s) := \text{conv}(\{y(t_1), \dots, y(t_n)\})$ , where  $t_i = -s - d + i$  for  $i = 1, \dots, n$ . Also, let  $T = \{t_i : i = 1, \dots, n\}$ ,  $I = \{t_i : i = 1, \dots, d + s\}$  and  $y(S) := \{y(t_i) : t_i \in S\}$  for  $S \subset T$ .

The following proposition collects a number of properties of the  $d$ -polytope  $C(d, n, s)$ .

**Proposition 4.2** *The  $d$ -polytope  $C(d, n, s)$  ( $n > d + s$ ) satisfies the following properties.*

1.  $C(d, n, s) \in \mathcal{P}(d, n, s)$ .
2. **Gale's evenness condition:** *A  $d$ -subset  $S_d$  of  $\text{vert } C(d, n, s)$  such that  $S_d \not\subset I$  forms a simplex facet iff, for any two elements  $u, v \in T \setminus S_d$ , the number of elements of  $S_d$  between  $u$  and  $v$  on the curve  $y(t)$  is even.*

**Proof:** (1) We first show that the first  $d + s$  vertices span a facet  $F$ . Let  $y = (y_1, \dots, y_d) \in \mathbf{R}^d$  and let

$$D((t_1, t_2, \dots, t_d); y) := \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ t_1 & t_2 & \cdots & t_d & y_1 \\ t_1^2 & t_2^2 & \cdots & t_d^2 & y_2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ t_1^{d-1} & t_2^{d-1} & \cdots & t_d^{d-1} & y_{d-1} \\ p(t_1) & p(t_2) & \cdots & p(t_d) & y_d \end{vmatrix}.$$

Let  $D(y) := D((t_1, t_2, \dots, t_d); y)$  and consider the hyperplane  $H_D := \{y \in \mathbf{R}^d : D(y) = 0\}$ . The points  $y(t_i)$  ( $i = 1, \dots, d + s$ ) are all contained in  $H_D$ , by vanishing of the last row of  $D(y(t_i))$ . Let  $y(t^*) \in \text{vert } C(d, n, s) \setminus y(I)$ , then  $D(y(t^*)) = p(t^*)V(t_1, \dots, t_d) > 0$  since  $p(t^*) > 0$  and  $V(t_1, \dots, t_d) > 0$ . Thus,  $F$  is a facet of  $C(d, n, s)$ .

We now show that every other facet is a simplex. Consider any  $(d + 1)$ -set  $\{t_{i_1} < \dots < t_{i_d} < t_{i_{d+1}} = t^*\} \subset T$  not contained in  $I$ . Thus,  $t^* \in T \setminus I$ . Consider the determinant  $E(y) := D((t_{i_1}, t_{i_2}, \dots, t_{i_d}); y)$ .

The hyperplane  $H_E := \{y \in \mathbf{R}^d : E(y) = 0\}$  contains all the points  $y(t_{i_j})$  ( $j = 1, \dots, d$ ). We need to show that  $E(y(t^*)) \neq 0$ .

Note that  $p(t) = 0$  for  $t \in I$  and  $p(t) > 0$  for  $t \in T \setminus I$ . Also, note that  $|t_a - t_b| \leq n - 1$  for  $t_a, t_b \in [-s - d + 1, -s - d + n]$ . For the sake of clarity assume  $d$  is odd; the case of even  $d$  is analogous. Computing  $E(y(t^*))$  by expanding w.r.t. the last row gives

$$\begin{aligned} & (p(t^*)V(t_{i_1}, \dots, t_{i_d}) - p(t_{i_d})V(t_{i_1}, \dots, t_{i_{d-1}}, t^*)) + \cdots \\ & + (p(t_{i_2})V(t_{i_1}, t_{i_3}, \dots, t^*) - p(t_{i_1})V(t_{i_2}, \dots, t^*)). \end{aligned}$$

The definition of  $p(t)$  implies that each pair-summand is nonnegative and the first pair-summand is positive, and so the determinant is positive. Indeed, for  $j > 1$ , if  $p(t_{i_j}) = 0$  then also  $p(t_{i_{j-1}}) = 0$  and the corresponding pair-summand vanishes. Otherwise, let  $V(j) := V(t_{i_1}, \dots, t_{i_{j-1}}, t_{i_{j+1}}, \dots, t_{i_{d+1}})$  for short. Then,

$$p(t_{i_j})V(j) \geq (n - 1)^{(d-1)(t_{i_j}-1)} \prod_{l=0}^{d+s-1} (t_{i_{j-1}} + l) \frac{V(j-1)}{(n-1)^{d-1}} \geq p(t_{i_{j-1}})V(j-1).$$

This completes the proof of the first assertion.

(2) Consider a set  $S_d = \{t_{i_1} < \dots < t_{i_d}\} \not\subset I$ . Let  $t^* \in T$ ,  $t_{i_{j-1}} < t^* < t_{i_j}$  (include also the cases  $t^* < t_{i_1}$  with  $j = 1$  and  $t_{i_d} < t^*$  where we put  $j = d + 1$ ). From the above reasoning we see that if the column  $y(t^*)$  in the determinant  $E(y(t^*))$  is placed between the columns  $y(t_{i_{j-1}})$  and

$y(t_{i_j})$  then the resulting determinant is positive. To achieve this, we swap  $d - j + 1$  times the column  $y(t^*)$ , which gives that the sign of  $E(y(t^*))$  is  $(-1)^{d-j+1}$ . Consequently, on the curve  $y(t)$ , between  $[-s - d + 1, -s - d + n]$ , the determinant  $E(y(t^*))$  changes sign whenever the variable passes through one of the values  $t_{i_j}$  ( $i = 1, \dots, d$ ), and we are done.  $\square$

A polytope  $C(d, n, s)$  will be called *almost cyclic*. Having established in 4.1 that  $C(d, n, s)$  is  $\lfloor (d - 1)/2 \rfloor$ -neighbourly, we can compute its  $h$ -vector, in steps. Recall that  $P' = \partial P \setminus \{F\}$ .

**Proposition 4.3** *Let  $P \in \mathcal{P}(d, n, s)$  be  $\lfloor (d - 1)/2 \rfloor$ -neighbourly, and  $(P, F)$  the ASP-pair. Then,*

$$\begin{aligned} h_k(P') &= \binom{n - d - 1 + k}{k}, & \text{if } 0 \leq k \leq \lfloor (d - 1)/2 \rfloor; \\ h_{d-k}(P') &= \binom{n - d - 1 + k}{k} - \binom{s + k - 1}{k}, & \text{if } 1 \leq k \leq \lfloor (d - 1)/2 \rfloor. \end{aligned}$$

The proof basically uses the fact that  $f_{k-1}(P') = \binom{n}{k}$  for  $k \leq \lfloor (d - 1)/2 \rfloor$ , and the Dehn-Sommerville relations (2); we omit the details.

Observe that, for even  $d$ , being  $\lfloor (d - 1)/2 \rfloor$ -neighbourly does not determine the value of  $h_{d/2}(P')$ . With the help of Gale's evenness condition we can compute the number of facets of  $C(d, n, s)$ , and together with 4.3 and (1), we can compute  $h_{d/2}(C(d, n, s))$  for any even  $d$  as well. Let  $C' := C(d, n, s) - \{F\}$ .

**Proposition 4.4** *For the ASP-pair  $(C(d, n, s), F)$  with  $d$  even, consider the simplicial ball  $C'$ . Then*

$$f_{d-1}(C') = \left( \binom{n - d/2 - 1}{d/2} + \sum_{i=0}^{d/2-1} 2 \binom{n - d - 1 + i}{i} \right) - \binom{s + d/2}{d/2}.$$

**Proof:** The counting argument for the facets of  $C'$ , based on Gale evenness, goes as in the proof of the number of facets of cyclic polytopes (cf. [19, Cor. 8.28]), with the difference that we discard the Gale  $d$ -tuples formed solely by the first  $d + s$  vertices, thus we discard exactly  $\binom{s+d/2}{d/2}$  of them.  $\square$

**Corollary 4.5** *The  $h$ -numbers of  $C'$  are given by:*

$$\begin{aligned} h_k(C') &= \binom{n - d - 1 + k}{k}, & \text{if } 0 \leq k \leq \lfloor (d - 1)/2 \rfloor; \\ h_{d-k}(C') &= \binom{n - d - 1 + k}{k} - \binom{s + k - 1}{k}, & \text{if } 1 \leq k \leq \lfloor d/2 \rfloor. \end{aligned}$$

**Proof:** The case of odd  $d$  was already established by 4.3 since  $C(d, n, s)$  is  $\lfloor (d - 1)/2 \rfloor$ -neighbourly. For the case of even  $d$  it remains to compute  $h_{d/2}(P')$ . Equating the corresponding expression in Proposition 4.4 with the expression of  $f_{d-1}$  in (1), after substituting the known values of  $h_k$  for  $k \neq d/2$ , gives

$$\begin{aligned} h_{d/2}(C') &= \binom{n - d/2 - 1}{d/2} + \sum_{i=0}^{d/2-1} \binom{s + i - 1}{i} - \binom{s + d/2}{d/2} \\ &= \binom{n - d/2 - 1}{d/2} - \binom{s + d/2 - 1}{d/2}, \end{aligned}$$

as desired.  $\square$

## 4.2 An upper bound theorem for almost simplicial polytopes

We are now in a position to state an upper bound theorem for almost simplicial polytopes  $P \in \mathcal{P}(d, n, s)$ .

**Theorem 4.6 (UBT for ASP)** *Any almost simplicial polytope  $P \in \mathcal{P}(d, n, s)$  satisfies*

$$h_k(P') \leq \binom{n-d-1+k}{k}, \quad \text{if } 0 \leq k \leq \lfloor (d-1)/2 \rfloor; \quad (3)$$

$$h_{d-k}(P') \leq \binom{n-d-1+k}{k} - \binom{s+k-1}{k}, \quad \text{if } 1 \leq k \leq \lfloor d/2 \rfloor. \quad (4)$$

Thus,

$$f_{i-1}(P) \leq f_{i-1}(C(d, n, s)) \quad \text{for } i = 1, 2, \dots, d,$$

for the almost cyclic  $d$ -polytope  $C(d, n, s)$ . Equality for some  $f_{i-1}$  with  $\lfloor (d-1)/2 \rfloor \leq i \leq d$  implies that  $P$  is  $\lfloor (d-1)/2 \rfloor$ -neighbourly.

**Proof of 4.6 via [1, Thm. 3.10]:** The inequalities on  $h_k(P')$  hold for  $0 \leq k \leq d-1$  by [1, Thm. 3.10], as  $P'$  is a special case of a homology ball whose boundary is an induced subcomplex. From Corollary 4.5 and (1) the inequality  $f_{i-1}(P) \leq f_{i-1}(C(d, n, s))$  follows. Equality for some  $f_{i-1}$  with  $d \geq i \geq \lfloor (d-1)/2 \rfloor$  implies, by (1), the equality  $h_k(P') = \binom{n-d-1+k}{k}$  for all  $0 \leq k \leq \lfloor (d-1)/2 \rfloor$ , and thus, again by (1), that  $P$  is  $\lfloor (d-1)/2 \rfloor$ -neighbourly.  $\square$

**Remark 4.7 (More maximizers.)** *As is the case with neighborly polytopes, we expect that there are many combinatorially distinct ASPs achieving the upper bounds in the UBT for ASP. We obtained another such construction, based on a certain perturbation of the Cayley polytopes constructed by Karavelas and Tzanaki [13, Sec.5]; the details are omitted from this extended abstract.*

**Remark 4.8** *We produced an alternative and elementary proof of the UBT for ASP, via shelling. Our proof follows ideas from the proof of the classical UBT by McMullen, cf. [19, Sec.8.4], and from a recent work of Karavelas and Tzanaki [13], and includes a special property of shellings of ASPs. For space limits we omit the details and refer to the full arXiv version.*

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