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Piano strings with reduced inharmonicity

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$_{\scriptscriptstyle 1}$ Summary

2 Even on modern straight pianos, the inharmonicity of the lower strings is rather large especially for the first 3 octave. Consequently, the timber of these strings can 4 sometimes sound awful and chords on the first octave 5 be highly dissonant. The idea of the present study is 6 to show how this defect can be rectified using an in-7 homogeneous winding on the whole string in order to 8 minimize inharmonicity. The problem is solved using 9 an optimisation procedure considering a non uniform 10 linear density. Results show that the inharmonicity 11 of the first partials could be highly reduced by a non 12 uniform winding limited to a quarter of the string. 13

14 1 Introduction

So-called harmonic strings are largely used in mu-15 sic because a uniform string without stiffness, and 16 stretched between two fixed points, naturally have 17 harmonic eigenfrequencies. When considering the 18 string's stiffness the eigenfrequencies are no longer 19 harmonic. The consequence on instruments like harp-20 sichord and early pianos is limited. However, the de-21 velopment of piano making during the 19th century 22 saw a tendency of increasing string tension by a factor 23 4, and the mass in the same proportion (cf. [2]). A 24 consequence is that the inharmonicity cannot be con-25 sidered negligible anymore, which reflects on the tun-26 ing of the instrument and the timbre. For the lower 27 strings a solution has been found in order to increase 28 the mass of the string without increasing too much 29 its bending stiffness: the wound strings. Neverthe-30 less, the inharmonicity remains rather large especially 31 for the first octave of medium grand piano, and even 32 worse on an upright piano: according to Young the 33 inharmonicity of the bass strings on a medium piano 34 is twice that of a grand piano, and that of a straight 35 piano twice that of a medium ([12], [7]). Our study 36 focuses on straight piano because we consider that 37 designing strings with reduced inharmonicity would 38 improve quite a lot the musical quality of these in-39 struments. 40

41 The idea of the present study is to show how this

defect could be rectified by using an inhomogeneous 42 winding on the whole string, in order to minimise in-43 harmonicity. The string is thus considered to be inho-44 mogeneous that is with a non uniform linear density 45 ([8]). From a theoretical point of view, the problem 46 translates into finding an "optimal" non uniform lin-47 ear density for a stretched string with uniform stiff-48 ness. Here, the optimality condition amounts to being 49 as harmonic as possible. This problem is solved using 50 an optimisation procedure, initialised with the char-51 acteristics of a real string. The diameter of the opti-52 mised string is allowed to vary between the diameter 53 of the core (supporting the winding) and about twice 54 the diameter of the reference string. An area with 55 uniform winding will be kept to reduce the amount 56 of work during the manufacturing of the true string. 57 Moreover, it is proposed to limit the non uniform 58 winding to one side of the string. 59

2 Euler Bernoulli model

The model chosen for the string is linear and only involves tension, bending stiffness and mass per unit length. According to Chabassier, this model is sufficient for low frequencies (cf. [1], remark I.1.2, there is no need to add a shear term), small amplitudes. Moreover, even if this not completely true (see [3]), it is considered that the increase of stiffness due to the wrapping can be neglected. So, the stiffness is that of the core (cf. [1] I.1.5) and is therefore constant along the string. Finally, the model is taken without any losses as only the eigenfrequencies are of interest.

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2.1 Mathematical model

Let us consider the Euler-Bernoulli model without losses for a stiff string of length L.



Figure 1: Sketch of the string and notations

The displacement equation is then given in the

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Fourier domain by (cf. [4])

$$-\mu\omega^2 y = T\frac{\partial^2 y}{\partial x^2} - EI\frac{\partial^4 y}{\partial x^4} \tag{1}$$

where μ is the mass per unit length (function of x), T 75

is the tension, E is the string core's Young modulus, 76 $I = Ar^2/4$ with A the core section, r the core radius 77 and ω is the pulsation. 78

The string being simply supported at both ends. the boundary conditions are given by

$$y|_{x=0} = \left. y \right|_{x=L} = \left. \frac{\partial^2 y}{\partial x^2} \right|_{x=0} = \left. \frac{\partial^2 y}{\partial x^2} \right|_{x=L} = 0.$$

2.2Solution with constant μ 79

When $\mu(x) = \mu_0$ is constant, it is possible to find an explicit solution of (1). The eigenfrequencies of the oscillator are given by (cf. [11] §3.4)

$$f_n = n f_0 \sqrt{1 + Bn^2}$$
 with $f_0 = \frac{1}{2L} \sqrt{\frac{T}{\mu_0}}$ and $B = \frac{\pi^2 EI}{TL^2}$

The inharmonicity factor comes from $\sqrt{1+Bn^2}$. 80 This equation shows the influence of string's stiffness 81 on inharmonicity. A shorter string with small tension 82 and high stiffness has a higher B, and therefore in-83 harmonic eigenfrequencies. For a typical piano string, 84 the inharmonicity is minimum for the second octave 85 and in the range of 10^{-4} . It increases with the fre-86 quency in upper octaves but also for the first octave 87 ([9]). For a grand piano for the first note A0, B is 88 less than 10^{-4} which leads to an inharmonicity of 20 89 cents for the 16th harmonic but for a straight piano 90 B can reach 10^{-3} which leads to an inharmonicity of 91 200 cents for the 16th harmonic (see [3]). With such 92 values of B, the sound of the lower string is awful and 93 chords on the first octave cause a lot of beatings in-94 ducing a high roughness. Therefore, we consider that 95 the reduction of the harmonicity factor would have a 96 beneficial influence on the sound of the medium and 97 straight piano, especially for the first two octaves. 98

3 Optimisation 99

The goal is to find a density function that corresponds to a harmonic string, i.e. a function μ that is a minimum for the "inharmonicity" function C (cf. [6]) defined by

$$C: \mu \mapsto \sum_{n=2}^{n_{max}} \left(\frac{\omega_n(\mu)}{n\omega_1(\mu)} - 1\right)^2$$

where the ω_n s are the eigenfrequencies of the equation 100 (1) sorted by increasing magnitude, n_{max} being the 101 number of harmonics considered in the optimisation. 102 This choice of inharmonicity function is quite natu-103 ral for an optimisation problem as it is a quadratic 104 function of the higher frequencies. 105

In general, it is not possible to work directly with 106 the function μ , and a discretisation of the space is needed so that an approximation in finite dimension can be used.

Numerical implementation 3.1

When mass is non uniform, solutions of equation (1)have to be approximated by a numerical method. The problem with non-constant μ is thus solved with a classical FEM in space (similar to the more complex setting of [1], §II.1) using Hermite's polynomials (cf. [5], 1.7) and a uniform discretisation of [0, L] for fixed N and 0 < i < N + 1 with $h = \frac{L}{N+1}$, $x_i = hi$. The projection of the operators "multiplication by μ ", $T\frac{\partial^2}{\partial x^2}$ and $EI\frac{\partial^4}{\partial x^4}$ then define three $2N \times 2N$ matrices $\mathbb{M}(\mu)$, \mathbb{T} and \mathbb{E} so that (1) can be approximated by

$$\omega^2 \mathbb{M}(\mu) U = (\mathbb{T} + \mathbb{E}) U \tag{2}$$

with $U = {}^{t}(y(x_1), y'(x_1), \cdots, y(x_N), y'(x_N)),$ 111 where the prime ' denotes the spatial derivative.

More precisely, the matrices \mathbb{T} and \mathbb{E} can be com-113 puted using Hermite's polynomials and leads to the 114 formulas 115

$$\mathbb{T} = \frac{T}{h^2} \begin{pmatrix} \ddots & & -\frac{6}{5} & \frac{h}{10} \\ & -\frac{h}{10} & -\frac{h^2}{30} \\ & -\frac{6}{5} & -\frac{h}{10} & \frac{12}{5} & 0 \\ & \frac{h}{10} & -\frac{h}{30} & 0 & \frac{4h^2}{15} \\ \end{pmatrix}, \ \mathbb{E} = \frac{EI}{h^4} \begin{pmatrix} \ddots & & -12 & 6h \\ & -6h & 2h^2 \\ & -12 & -6h & 24 & 0 \\ & 6h & 2h^2 & 0 & 8h^2 \\ & 6h & 2h^2 & 0 & 8h^2 \\ \end{pmatrix}$$
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and as well the matrix $\dot{\mathbb{M}}(\mu)$ is computed for functions μ constants on each interval $]x_i, x_{i+1}[$ by

$$\mathbb{M} = \begin{pmatrix} \mu_{i-1/2} \frac{9}{20} & -\mu_{i-1/2} \frac{13h}{420} \\ \mu_{i-1/2} \frac{13h}{420} & -\mu_{i-1/2} \frac{13h}{420} \\ \mu_{i-1/2} \frac{13h}{420} & (\mu_{i-1/2} + \mu_{i+1/2}) \frac{13}{35} & (-\mu_{i-1/2} + \mu_{i+1/2}) \frac{11h}{210} \\ -\mu_{i-1/2} \frac{13h}{420} - \mu_{i-1/2} \frac{h^2}{140} & (-\mu_{i-1/2} + \mu_{i+1/2}) \frac{11h}{210} & (\mu_{i-1/2} + \mu_{i+1/2}) \frac{h^2}{105} \\ \end{pmatrix}$$

where $\mu_{i+1/2}$ denotes the value of μ on $]x_i, x_{i+1}[$.

Equation (2) is a generalised eigenvalue problem that can be solved using generalised Schur decomposition.

Optimisation algorithm 3.2

In the present study, the gradient algorithm is used 125 to find a minimum of the "inharmonicity" function C. 126 It works well for low stiffness and/or small numbers 127 n_{max} . When converging, it gives a solution having 128 arbitrarily low inharmonicity for the first n_{max} har-129 monics. 130

It uses the derivative of functions ω_n with respect to μ for $n \in \{1, \dots, n_{max}\}$. The value of this derivative is computed using perturbation theory and is approximated by the formula

$$\operatorname{grad}_{\mu}\omega_n = \frac{U_n \cdot U_n}{{}^t U_n \mathbb{M}(\mu) U_n}$$

where U_n is an eigenvector of equation (2) associated 131 to the eigenvalue $-\omega_n^2$, and $U_n U_n$ is the Hadamard 132 (entrywise) product of U_n with itself. 133

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The stopping condition is dictated by the inharmonicity of each partial, as it should be at least as good as that of the uniform string and at most $\leq \varepsilon$, for a fixed constant ε which, in practice, is taken to be 10^{-3} .

¹³⁹ 4 Examples and results with ¹⁴⁰ different strategies

Gradient algorithm looks for solutions in a vector 141 space, but most of the elements of this space are phys-142 ically irrelevant. It is therefore necessary to reduce 143 the search space and add conditions to find useful so-144 lutions. In particular, they must verify at least that 145 the density remains bigger than that of the core, as 146 it would otherwise weaken the string. On the other 147 hand it can be interesting to limit the inhomogeneous 148 part in order to make the manufacturing easier. These 149 considerations lead to different strategies which are 150 described below. 151

In the following the chosen nominal string is a C1 152 straight piano string corresponding to a frequency of 153 32.7 Hz: L = 1.035 m, $\mu_0 = 180 g/m$, T = 825 N and 154 $EI = 0.028 Nm^2$, which leads to an inharmonicity co-155 efficient $B = 3.13 \ 10^{-4}$. Here the string is considered 156 to be strictly uniform on all its length and it is con-157 sidered that the stiffness is that of the core only. In 158 practice it is likely that the inharmonicity coefficient 159 might be significantly higher ([3]). 160

¹⁶¹ 4.1 Minimum of constraints

This corresponds to the case were the only constraint 162 is that the density is at least equal to that of the 163 wire. Results depend on the number of harmonics 164 which are taken into account in the optimisation. On 165 figure 2 the density for different $n_{max} = 10, 15, 18,$ 166 is shown. The convergence is considered to be ob-167 tained when the maximum of harmonicity is $< 10^{-3}$. 168 The number of steps needed to obtain the conver-169 gence increases with the number of harmonics: for 170 $n_{max} = 10, 15, 18$ the number of steps are respectively 171 $n_{step} = 332, 848, 1188.$ 172

It appears that the density fluctuations show a 173 number of valleys equal to the number of harmon-174 ics involved in the optimisation. Physically it can 175 be interpreted as a way to slower the waves up to 176 the maximum frequency considered. Another impor-177 tant observation is that density fluctuations are all 178 the more important as the number of harmonics con-179 sidered is high: the amplitude varies roughly as the 180 square of the harmonic number. This is an important 181 182 limitation, because this limits the number of harmonics on which inharmonicity can be minimised. On the 183 given example, for $n_{max} = 18$ the density is localy 184 multiplied by about 2 which means that the diameter 185 of the string is locally increased by 40%. It can also 186



Figure 2: Result of convergence with no strategy for different values of n_{max} . Top: density profile; bottom: harmonicity as a function of harmonic number.

be noticed that the fluctuations are more important near the ends off the string which suggests that fluctuations in the middle of the string could be avoided.

It can be seen on figure 2 that the diameter at the ends is much larger than that of the uniform string. ¹⁹⁰ In practice this will be probably difficult to manage. ¹⁹²

4.2 Non negative density fluctuations 193

The question is now whether it would be possible to 194 only add some masses to a uniform string in order 195 to apply local corrections. On the numerical point of 196 view this is obtained by replacing the gradient by its 197 positive part. The convergence is much slower than 198 for the previous case (4730 steps are needed for a max-199 imum inharmonicity of $\varepsilon = 10^{-3}$ against 332). How-200 ever, the comparison of figure 2 and figure 3 shows 201 that it should be possible to avoid valleys without in-202 creasing too much the amplitude of the hills. 203



Figure 3: Result of convergence with non negative fluctuation of density and $n_{max} = 10$. Top: density profile; bottom: harmonicity as a function of harmonic number.

No density fluctuation on a central 4.3204 part of given length 205

In order to limit the area of intervention, an idea is 206 to keep a uniform diameter on a portion of the string 207 that is as large as possible. On figure 4 results are 208 shown for a string that is kept uniform on one half of 209 its length (i.e. non constant on [0, a] and [L - a, L], 210 constant on [a, L-a] with a = L/4). It can be seen 211 that this constraint leads to a concentration of the 212 added mass near the end of the string. It is interesting 213 to notice that only 2 hills and 2 valleys on both sides 214 of the string are sufficient to obtain good results for 215 $n_{max} = 10.$ 216



Figure 4: Result of convergence with no fluctuation on one half of the string in the middle and $n_{max} =$ 10. Top: density profile; bottom: harmonicity as a function of harmonic number.

One sided density fluctuation 4.4217

Owing to the symmetrical nature of the problem, for 218 all the previous examples the algorithm converges to 219 a symmetrical solution. However there is a priori no 220 reason to keep a symmetrical string. Moreover, to 221 avoid the hammer hitting the string on a non uniform 222 region it would be good to limit the non uniform re-223 gion to only one end of the string. Figure 5 shows 224 that similar results are obtained when the fluctuation 225 are concentrated on one side. As one could expect, 226 this tends to increase the amplitude of the fluctua-227 tions. Results are given on figure 5 for $n_{max} = 10$ 228 and a = L/4. It is noticeable that the result tends to 229 a point mass near the end combined with two periods 230 of a "damped sinusoidal" variation of density. 231

4.5One sided non negative density 232 fluctuation 233

234 Now, the next step is to consider only non negative density fluctuations on a small part of the string. The 235 results given on figure 5 converge to what can be inter-236 preted as small masses. The second mass being rather 237 negligible it seems that two masses are sufficient to 271 238

correct the inharmonicity of the first ten harmonics. 239 The first mass is about 5 g and is centered at 3 cm from the end. The third one is 0.4 g and is centered at 24 cm from the end. In practice, the second mass is probably useless. So, it is surprising to realise that for $n_{max} = 10$ a single mass is probably sufficient to significantly improve the harmonicity of a string. 245



Figure 5: Result of convergence with and without positive density fluctuations on one quarter of the string and $n_{max} = 10$. Top: density profile; bottom: harmonicity as a function of harmonic number.

It is noticeable that the present result is not far 246 from what can be found in [10] in which a local over 247 winding is used to compensate for the inharmonicity 248 induced by the bare ends. 249

5 Conclusion

The present study shows that it might be possible 251 to build piano strings with an optimised non uniform 252 density leading to a reduced inharmonicity of the first 253 partials. Moreover the non uniform part of the string 254 could be limited to a short portion of the string near 255 the end (25 cm in the given example). Moreover, it 256 appears that a single mass a few centimeters from 257 the end might significantly improve the harmonicity. 258 Now, many questions arise. In practice, how to take 259 into account the effective inharmonicity of the string 260 including the influence of the sound board and that of 261 the winding? On the perceptive point of view, what 262 is the minimum number of harmonics to consider in 263 order to obtain a significant improvement? To answer 264 these questions, the next step is to find a process to 265 optimise the harmonicity on an actual string. Obvi-266 ously, the final design will be the result of exchanges 267 between piano tuners, pianists and engineers. Finally 268 we are deeply convinced that such harmonic strings 269 will make it possible to highly increase the musical 270 quality of the first two octaves of straight pianos but

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²⁷² also that of medium grand pianos.

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