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Piano strings with reduced inharmonicity

J.P. Dalmont¹⁾, S. Maugeais²⁾

Le Mans Université, Avenue Olivier Messiaen,
72085 Le Mans Cedex 9, France

¹⁾ Laboratoire d'acoustique (LAUM, UMR CNRS 6613)

²⁾ Laboratoire Manceau de Mathématiques (LMM, EA 3263)

Summary

Even on modern straight pianos, the inharmonicity of the lower strings is rather large especially for the first octave. Consequently, the timber of these strings can sometimes sound awful and chords on the first octave be highly dissonant. The idea of the present study is to show how this defect can be rectified using an inhomogeneous winding on the whole string in order to minimize inharmonicity. The problem is solved using an optimisation procedure considering a non uniform linear density. Results show that the inharmonicity of the first partials could be highly reduced by a non uniform winding limited to a quarter of the string.

1 Introduction

So-called harmonic strings are largely used in music because a uniform string without stiffness, and stretched between two fixed points, naturally have harmonic eigenfrequencies. When considering the string's stiffness the eigenfrequencies are no longer harmonic. The consequence on instruments like harpsichord and early pianos is limited. However, the development of piano making during the 19th century saw a tendency of increasing string tension by a factor 4, and the mass in the same proportion (cf. [2]). A consequence is that the inharmonicity cannot be considered negligible anymore, which reflects on the tuning of the instrument and the timbre. For the lower strings a solution has been found in order to increase the mass of the string without increasing too much its bending stiffness: the wound strings. Nevertheless, the inharmonicity remains rather large especially for the first octave of medium grand piano, and even worse on an upright piano: according to Young the inharmonicity of the bass strings on a medium piano is twice that of a grand piano, and that of a straight piano twice that of a medium ([12], [7]). Our study focuses on straight piano because we consider that designing strings with reduced inharmonicity would improve quite a lot the musical quality of these instruments.

The idea of the present study is to show how this

defect could be rectified by using an inhomogeneous winding on the whole string, in order to minimise inharmonicity. The string is thus considered to be inhomogeneous that is with a non uniform linear density ([8]). From a theoretical point of view, the problem translates into finding an "optimal" non uniform linear density for a stretched string with uniform stiffness. Here, the optimality condition amounts to being as harmonic as possible. This problem is solved using an optimisation procedure, initialised with the characteristics of a real string. The diameter of the optimised string is allowed to vary between the diameter of the core (supporting the winding) and about twice the diameter of the reference string. An area with uniform winding will be kept to reduce the amount of work during the manufacturing of the true string. Moreover, it is proposed to limit the non uniform winding to one side of the string.

2 Euler Bernoulli model

The model chosen for the string is linear and only involves tension, bending stiffness and mass per unit length. According to Chabassier, this model is sufficient for low frequencies (cf. [1], remark I.1.2, there is no need to add a shear term), small amplitudes. Moreover, even if this not completely true (see [3]), it is considered that the increase of stiffness due to the wrapping can be neglected. So, the stiffness is that of the core (cf. [1] I.1.5) and is therefore constant along the string. Finally, the model is taken without any losses as only the eigenfrequencies are of interest.

2.1 Mathematical model

Let us consider the Euler-Bernoulli model without losses for a stiff string of length L .

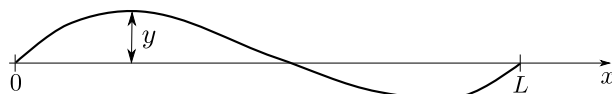


Figure 1: Sketch of the string and notations

The displacement equation is then given in the

Fourier domain by (cf. [4])

$$-\mu\omega^2 y = T \frac{\partial^2 y}{\partial x^2} - EI \frac{\partial^4 y}{\partial x^4} \quad (1)$$

75 where μ is the mass per unit length (function of x), T
76 is the tension, E is the string core's Young modulus,
77 $I = Ar^2/4$ with A the core section, r the core radius
78 and ω is the pulsation.

The string being simply supported at both ends, the boundary conditions are given by

$$y|_{x=0} = y|_{x=L} = \frac{\partial^2 y}{\partial x^2} \Big|_{x=0} = \frac{\partial^2 y}{\partial x^2} \Big|_{x=L} = 0.$$

79 2.2 Solution with constant μ

When $\mu(x) = \mu_0$ is constant, it is possible to find an explicit solution of (1). The eigenfrequencies of the oscillator are given by (cf. [11] §3.4)

$$f_n = n f_0 \sqrt{1 + B n^2} \text{ with } f_0 = \frac{1}{2L} \sqrt{\frac{T}{\mu_0}} \text{ and } B = \frac{\pi^2 EI}{TL^2}.$$

80 The inharmonicity factor comes from $\sqrt{1 + B n^2}$.
81 This equation shows the influence of string's stiffness
82 on inharmonicity. A shorter string with small tension
83 and high stiffness has a higher B , and therefore in-
84 harmonic eigenfrequencies. For a typical piano string,
85 the inharmonicity is minimum for the second octave
86 and in the range of 10^{-4} . It increases with the fre-
87 quency in upper octaves but also for the first octave
88 ([9]). For a grand piano for the first note A0, B is
89 less than 10^{-4} which leads to an inharmonicity of 20
90 cents for the 16th harmonic but for a straight piano
91 B can reach 10^{-3} which leads to an inharmonicity of
92 200 cents for the 16th harmonic (see [3]). With such
93 values of B , the sound of the lower string is awful and
94 chords on the first octave cause a lot of beatings in-
95 ducing a high roughness. Therefore, we consider that
96 the reduction of the harmonicity factor would have a
97 beneficial influence on the sound of the medium and
98 straight piano, especially for the first two octaves.

99 3 Optimisation

The goal is to find a density function that corresponds to a harmonic string, i.e. a function μ that is a minimum for the ‘‘inharmonicity’’ function C (cf. [6]) defined by

$$C : \mu \mapsto \sum_{n=2}^{n_{max}} \left(\frac{\omega_n(\mu)}{n\omega_1(\mu)} - 1 \right)^2$$

100 where the ω_n s are the eigenfrequencies of the equation
101 (1) sorted by increasing magnitude, n_{max} being the
102 number of harmonics considered in the optimisation.
103 This choice of inharmonicity function is quite natu-
104 ral for an optimisation problem as it is a quadratic
105 function of the higher frequencies.

In general, it is not possible to work directly with
the function μ , and a discretisation of the space is
needed so that an approximation in finite dimension
can be used.

3.1 Numerical implementation

When mass is non uniform, solutions of equation (1) have to be approximated by a numerical method. The problem with non-constant μ is thus solved with a classical FEM in space (similar to the more complex setting of [1], §II.1) using Hermite's polynomials (cf. [5], 1.7) and a uniform discretisation of $[0, L]$ for fixed N and $0 < i < N + 1$ with $h = \frac{L}{N+1}$, $x_i = hi$. The projection of the operators ‘‘multiplication by μ ’’, $T \frac{\partial^2}{\partial x^2}$ and $EI \frac{\partial^4}{\partial x^4}$ then define three $2N \times 2N$ matrices $\mathbb{M}(\mu)$, \mathbb{T} and \mathbb{E} so that (1) can be approximated by

$$\omega^2 \mathbb{M}(\mu) U = (\mathbb{T} + \mathbb{E}) U \quad (2)$$

with $U = {}^t(y(x_1), y'(x_1), \dots, y(x_N), y'(x_N))$, where the prime ' denotes the spatial derivative.

More precisely, the matrices \mathbb{T} and \mathbb{E} can be computed using Hermite's polynomials and leads to the formulas

$$\mathbb{T} = \frac{T}{h^2} \begin{pmatrix} \ddots & & & & & & & & & & \\ & & & & & & & & & & \\ & & -\frac{6}{5} & \frac{h}{10} & & & & & & & \\ & & -\frac{h}{10} & -\frac{h^2}{30} & & & & & & & \\ & -\frac{6}{5} & -\frac{h}{10} & \frac{h^2}{5} & 0 & & & & & & \\ & \frac{h}{10} & -\frac{h}{30} & 0 & \frac{4h^2}{15} & & & & & & \\ & & & & & \ddots & & & & & \end{pmatrix}, \quad \mathbb{E} = \frac{EI}{h^4} \begin{pmatrix} \ddots & & & & & & & & & & \\ & & & & & & & & & & \\ & & & & & & & & & & \\ & & & & & & & & & & \\ & & & & & & -12 & 6h & & & \\ & & & & & & -6h & 2h^2 & & & \\ & -12 & -6h & 24 & 0 & & & & & & \\ & 6h & 2h^2 & 0 & 8h^2 & & & & & & \\ & & & & & & \ddots & & & & \end{pmatrix}$$

and as well the matrix $\mathbb{M}(\mu)$ is computed for functions μ constants on each interval $]x_i, x_{i+1}[$ by

$$\mathbb{M} = \begin{pmatrix} \ddots & & & & & & & & & & \\ & & & & & & & & & & \\ & & & & \mu_{i-1/2} \frac{9}{70} & & & & -\mu_{i-1/2} \frac{13h}{420} & & \\ & & & & \mu_{i-1/2} \frac{13h}{420} & & & & -\mu_{i-1/2} \frac{h^2}{140} & & \\ \mu_{i-1/2} \frac{9}{70} & \mu_{i-1/2} \frac{13h}{420} & (\mu_{i-1/2} + \mu_{i+1/2}) \frac{13}{35} & & & & & & & & \\ -\mu_{i-1/2} \frac{13h}{420} & -\mu_{i-1/2} \frac{h^2}{140} & (-\mu_{i-1/2} + \mu_{i+1/2}) \frac{11h}{210} & & & & & & & & \\ & & (\mu_{i-1/2} + \mu_{i+1/2}) \frac{h^2}{105} & & & & & & & & \ddots \end{pmatrix}$$

where $\mu_{i+1/2}$ denotes the value of μ on $]x_i, x_{i+1}[$.

Equation (2) is a generalised eigenvalue problem that can be solved using generalised Schur decomposition.

3.2 Optimisation algorithm

In the present study, the gradient algorithm is used to find a minimum of the ‘‘inharmonicity’’ function C . It works well for low stiffness and/or small numbers n_{max} . When converging, it gives a solution having arbitrarily low inharmonicity for the first n_{max} harmonics.

It uses the derivative of functions ω_n with respect to μ for $n \in \{1, \dots, n_{max}\}$. The value of this derivative is computed using perturbation theory and is approximated by the formula

$$\text{grad}_{\mu} \omega_n = \frac{U_n \cdot U_n}{{}^t U_n \mathbb{M}(\mu) U_n}$$

where U_n is an eigenvector of equation (2) associated to the eigenvalue $-\omega_n^2$, and $U_n \cdot U_n$ is the Hadamard (entrywise) product of U_n with itself.

134 The stopping condition is dictated by the inhar-
 135 monicity of each partial, as it should be at least as
 136 good as that of the uniform string and at most $\leq \varepsilon$,
 137 for a fixed constant ε which, in practice, is taken to
 138 be 10^{-3} .

139 4 Examples and results with 140 different strategies

141 Gradient algorithm looks for solutions in a vector
 142 space, but most of the elements of this space are phys-
 143 ically irrelevant. It is therefore necessary to reduce
 144 the search space and add conditions to find useful so-
 145 lutions. In particular, they must verify at least that
 146 the density remains bigger than that of the core, as
 147 it would otherwise weaken the string. On the other
 148 hand it can be interesting to limit the inhomogeneous
 149 part in order to make the manufacturing easier. These
 150 considerations lead to different strategies which are
 151 described below.

152 In the following the chosen nominal string is a C1
 153 straight piano string corresponding to a frequency of
 154 32.7Hz : $L = 1.035m$, $\mu_0 = 180g/m$, $T = 825N$ and
 155 $EI = 0.028Nm^2$, which leads to an inharmonicity co-
 156 efficient $B = 3.13 \cdot 10^{-4}$. Here the string is considered
 157 to be strictly uniform on all its length and it is con-
 158 sidered that the stiffness is that of the core only. In
 159 practice it is likely that the inharmonicity coefficient
 160 might be significantly higher ([3]).

161 4.1 Minimum of constraints

162 This corresponds to the case were the only constraint
 163 is that the density is at least equal to that of the
 164 wire. Results depend on the number of harmonics
 165 which are taken into account in the optimisation. On
 166 figure 2 the density for different $n_{max} = 10, 15, 18$,
 167 is shown. The convergence is considered to be ob-
 168 tained when the maximum of inharmonicity is $< 10^{-3}$.
 169 The number of steps needed to obtain the conver-
 170 gence increases with the number of harmonics: for
 171 $n_{max} = 10, 15, 18$ the number of steps are respectively
 172 $n_{step} = 332, 848, 1188$.

173 It appears that the density fluctuations show a
 174 number of valleys equal to the number of harmonics
 175 involved in the optimisation. Physically it can
 176 be interpreted as a way to slower the waves up to
 177 the maximum frequency considered. Another impor-
 178 tant observation is that density fluctuations are all
 179 the more important as the number of harmonics con-
 180 sidered is high: the amplitude varies roughly as the
 181 square of the harmonic number. This is an important
 182 limitation, because this limits the number of harmo-
 183 nics on which inharmonicity can be minimised. On the
 184 given example, for $n_{max} = 18$ the density is locally
 185 multiplied by about 2 which means that the diameter
 186 of the string is locally increased by 40%. It can also

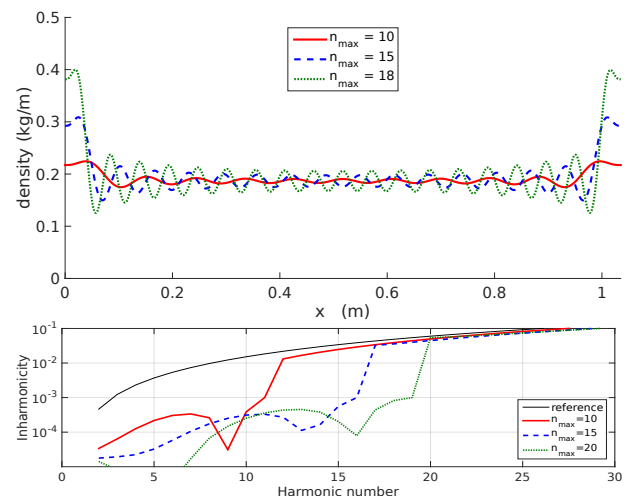


Figure 2: Result of convergence with no strategy for different values of n_{max} . Top: density profile; bottom: inharmonicity as a function of harmonic number.

187 be noticed that the fluctuations are more important
 188 near the ends of the string which suggests that fluc-
 189 tuations in the middle of the string could be avoided.

190 It can be seen on figure 2 that the diameter at the
 191 ends is much larger than that of the uniform string.
 192 In practice this will be probably difficult to manage.

193 4.2 Non negative density fluctuations

194 The question is now whether it would be possible to
 195 only add some masses to a uniform string in order
 196 to apply local corrections. On the numerical point of
 197 view this is obtained by replacing the gradient by its
 198 positive part. The convergence is much slower than
 199 for the previous case (4730 steps are needed for a max-
 200 imum inharmonicity of $\varepsilon = 10^{-3}$ against 332). How-
 201 ever, the comparison of figure 2 and figure 3 shows
 202 that it should be possible to avoid valleys without
 203 increasing too much the amplitude of the hills.

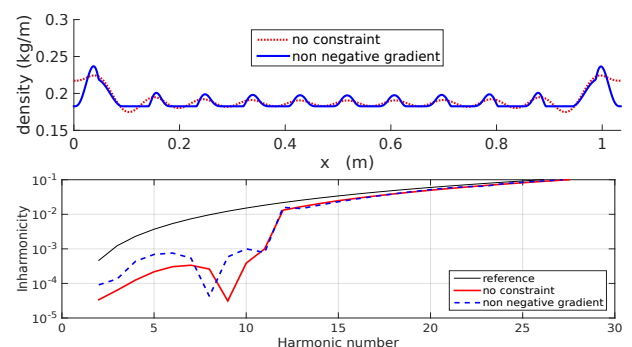


Figure 3: Result of convergence with non negative fluctuation of density and $n_{max} = 10$. Top: density profile; bottom: inharmonicity as a function of harmonic number.

4.3 No density fluctuation on a central part of given length

In order to limit the area of intervention, an idea is to keep a uniform diameter on a portion of the string that is as large as possible. On figure 4 results are shown for a string that is kept uniform on one half of its length (i.e. non constant on $[0, a]$ and $[L - a, L]$, constant on $[a, L - a]$ with $a = L/4$). It can be seen that this constraint leads to a concentration of the added mass near the end of the string. It is interesting to notice that only 2 hills and 2 valleys on both sides of the string are sufficient to obtain good results for $n_{max} = 10$.

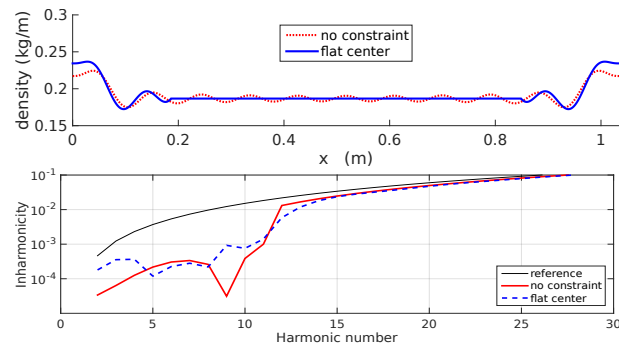


Figure 4: Result of convergence with no fluctuation on one half of the string in the middle and $n_{max} = 10$. Top: density profile; bottom: inharmonicity as a function of harmonic number.

4.4 One sided density fluctuation

Owing to the symmetrical nature of the problem, for all the previous examples the algorithm converges to a symmetrical solution. However there is a priori no reason to keep a symmetrical string. Moreover, to avoid the hammer hitting the string on a non uniform region it would be good to limit the non uniform region to only one end of the string. Figure 5 shows that similar results are obtained when the fluctuation are concentrated on one side. As one could expect, this tends to increase the amplitude of the fluctuations. Results are given on figure 5 for $n_{max} = 10$ and $a = L/4$. It is noticeable that the result tends to a point mass near the end combined with two periods of a "damped sinusoidal" variation of density.

4.5 One sided non negative density fluctuation

Now, the next step is to consider only non negative density fluctuations on a small part of the string. The results given on figure 5 converge to what can be interpreted as small masses. The second mass being rather negligible it seems that two masses are sufficient to

correct the inharmonicity of the first ten harmonics. The first mass is about 5 g and is centered at 3 cm from the end. The third one is 0.4 g and is centered at 24 cm from the end. In practice, the second mass is probably useless. So, it is surprising to realise that for $n_{max} = 10$ a single mass is probably sufficient to significantly improve the harmonicity of a string.

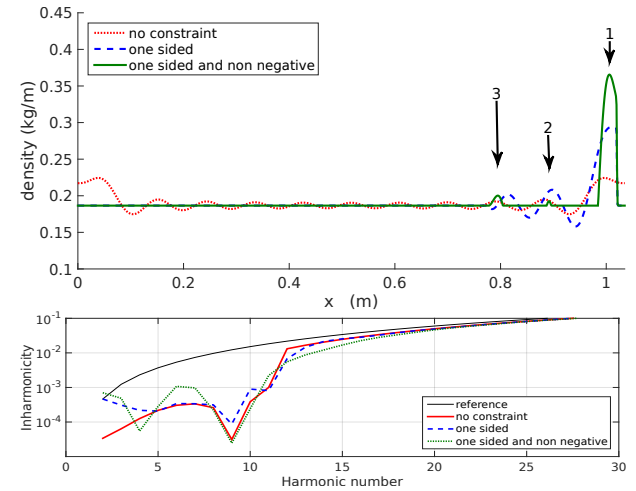


Figure 5: Result of convergence with and without positive density fluctuations on one quarter of the string and $n_{max} = 10$. Top: density profile; bottom: inharmonicity as a function of harmonic number.

It is noticeable that the present result is not far from what can be found in [10] in which a local over winding is used to compensate for the inharmonicity induced by the bare ends.

5 Conclusion

The present study shows that it might be possible to build piano strings with an optimised non uniform density leading to a reduced inharmonicity of the first partials. Moreover the non uniform part of the string could be limited to a short portion of the string near the end (25 cm in the given example). Moreover, it appears that a single mass a few centimeters from the end might significantly improve the harmonicity. Now, many questions arise. In practice, how to take into account the effective inharmonicity of the string including the influence of the sound board and that of the winding? On the perceptive point of view, what is the minimum number of harmonics to consider in order to obtain a significant improvement? To answer these questions, the next step is to find a process to optimise the harmonicity on an actual string. Obviously, the final design will be the result of exchanges between piano tuners, pianists and engineers. Finally we are deeply convinced that such harmonic strings will make it possible to highly increase the musical quality of the first two octaves of straight pianos but

272 also that of medium grand pianos.

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