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# Piano strings with reduced inharmonicity

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## Summary

Even on modern straight pianos, the inharmonicity of the lower strings is rather large especially for the first octave. Consequently, the timber of these strings can sometimes sound awful and chords on the first octave be highly dissonant. The idea of the present study is to show how this defect can be rectified using an inhomogeneous winding on the whole string in order to minimize inharmonicity. The problem is solved using an optimisation procedure considering a non uniform linear density. Results show that the inharmonicity of the first partials could be highly reduced by a non uniform winding limited to a quarter of the string.

## 1 Introduction

So-called harmonic strings are largely used in music because a uniform string without stiffness, and stretched between two fixed points, naturally have harmonic eigenfrequencies. When considering the string's stiffness the eigenfrequencies are no longer harmonic. The consequence on instruments like harpsichord and early pianos is limited. However, the development of piano making during the 19th century saw a tendency of increasing string tension by a factor 4, and the mass in the same proportion (cf. [2]). A consequence is that the inharmonicity cannot be considered negligible anymore, which reflects on the tuning of the instrument and the timbre. For the lower strings a solution has been found in order to increase the mass of the string without increasing too much its bending stiffness: the wound strings. Nevertheless, the inharmonicity remains rather large especially for the first octave of medium grand piano, and even worse on an upright piano: according to Young the inharmonicity of the bass strings on a medium piano is twice that of a grand piano, and that of a straight piano twice that of a medium ([11], [7]). Our study focuses on straight piano because we consider that designing strings with reduced inharmonicity would improve quite a lot the musical quality of these instruments.

The idea of the present study is to show how this

defect could be rectified by using an inhomogeneous winding on the whole string, in order to minimise inharmonicity. The string is thus considered to be inhomogeneous that is with a non uniform linear density ([8]). From a theoretical point of view, the problem translates into finding an "optimal" non uniform linear density for a stretched string with uniform stiffness. Here, the optimality condition amounts to being as harmonic as possible. This problem is solved using an optimisation procedure, initialised with the characteristics of a real string. The diameter of the optimised string is allowed to vary between the diameter of the core (supporting the winding) and about twice the diameter of the reference string. An area with uniform winding will be kept to reduce the amount of work during the manufacturing of the true string. Moreover, it is proposed to limit the non uniform winding to one side of the string.

## 2 Euler Bernoulli model

The model chosen for the string is linear and only involves tension, bending stiffness and mass per unit length. According to Chabassier, this model is sufficient for low frequencies (cf. [1], remark I.1.2, there is no need to add a shear term), small amplitudes. Moreover, even if this not completely true (see [3]), it is considered that the increase of stiffness due to the wrapping can be neglected. So, the stiffness is that of the core (cf. [1] I.1.5) and is therefore constant along the string. Finally, the model is taken without any losses as only the eigenfrequencies are of interest.

### 2.1 Mathematical model

Let us consider the Euler-Bernoulli model without losses for a stiff string of length  $L$ .

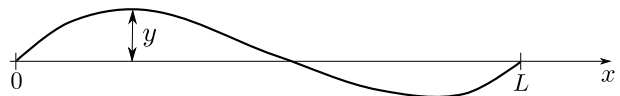


Figure 1: Sketch of the string and notations

The displacement equation is then given in the

Fourier domain by (cf. [4])

$$-\mu\omega^2 y = T \frac{\partial^2 y}{\partial x^2} - EI \frac{\partial^4 y}{\partial x^4} \quad (1)$$

where  $\mu$  is the mass per unit length (function of  $x$ ),  $T$  is the tension,  $E$  is the string core's Young modulus,  $I = Ar^2/4$  with  $A$  the core section,  $r$  the core radius and  $\omega$  is the pulsation.

The string being simply supported at both ends, the boundary conditions are given by

$$y|_{x=0} = y|_{x=L} = \frac{\partial^2 y}{\partial x^2}|_{x=0} = \frac{\partial^2 y}{\partial x^2}|_{x=L} = 0.$$

## 2.2 Solution with constant $\mu$

When  $\mu(x) = \mu_0$  is constant, it is possible to find an explicit solution of (1). The eigenfrequencies of the oscillator are given by (cf. [10] §3.4)

$$f_n = n f_0 \sqrt{1 + B n^2} \quad \text{with } f_0 = \frac{1}{2L} \sqrt{\frac{T}{\mu_0}} \text{ and } B = \frac{\pi^2 EI}{TL^2}.$$

The inharmonicity factor comes from  $\sqrt{1 + B n^2}$ . This equation shows the influence of string's stiffness on inharmonicity. A shorter string with small tension and high stiffness has a higher  $B$ , and therefore inharmonic eigenfrequencies. For a typical piano string, the inharmonicity is minimum for the second octave and in the range of  $10^{-4}$ . It increases with the frequency in upper octaves but also for the first octave ([9]). For a grand piano for the first note A0,  $B$  is less than  $10^{-4}$  which leads to an inharmonicity of 20 cents for the 16th harmonic but for a straight piano  $B$  can reach  $10^{-3}$  which leads to an inharmonicity of 200 cents for the 16th harmonic (see [3]). With such values of  $B$ , the sound of the lower string is awful and chords on the first octave cause a lot of beatings inducing a high roughness. Therefore, we consider that the reduction of the harmonicity factor would have a beneficial influence on the sound of the medium and straight piano, especially for the first two octaves.

## 3 Optimisation

The goal is to find a density function that corresponds to a harmonic string, i.e. a function  $\mu$  that is a minimum for the ‘‘inharmonicity’’ function  $C$  (cf. [6]) defined by

$$C : \mu \mapsto \sum_{n=2}^{n_{max}} \left( \frac{\omega_n(\mu)}{n\omega_1(\mu)} - 1 \right)^2$$

where the  $\omega_n$ s are the eigenfrequencies of the equation (1) sorted by increasing magnitude,  $n_{max}$  being the number of harmonics considered in the optimisation. This choice of inharmonicity function is quite natural for an optimisation problem as it is a quadratic function of the higher frequencies.

In general, it is not possible to work directly with the function  $\mu$ , and a discretisation of the space is needed so that an approximation in finite dimension can be used.

## 3.1 Numerical implementation

When mass is non uniform, solutions of equation (1) have to be approximated by a numerical method. The problem with non-constant  $\mu$  is thus solved with a classical FEM in space (similar to the more complex setting of [1], §II.1) using Hermite's polynomials (cf. [5], 1.7) and a uniform discretisation of  $[0, L]$  for fixed  $N$  and  $0 < i < N + 1$  with  $h = \frac{L}{N+1}$ ,  $x_i = hi$ . The projection of the operators ‘‘multiplication by  $\mu$ ’’,  $T \frac{\partial^2}{\partial x^2}$  and  $EI \frac{\partial^4}{\partial x^4}$  then define three  $2N \times 2N$  matrices  $\mathbb{M}(\mu)$ ,  $\mathbb{T}$  and  $\mathbb{E}$  so that (1) can be approximated by

$$\omega^2 \mathbb{M}(\mu) U = (\mathbb{T} + \mathbb{E}) U \quad (2)$$

with  $U = {}^t(y(x_1), y'(x_1), \dots, y(x_N), y'(x_N))$ , where the prime ' denotes the spatial derivative.

More precisely, the matrices  $\mathbb{T}$  and  $\mathbb{E}$  can be computed with the formulas

$$\mathbb{T} = \frac{T}{h^2} \begin{pmatrix} \ddots & & & -\frac{6}{5} & \frac{h}{10} \\ & \ddots & & -\frac{h}{5} & \frac{h^2}{30} \\ & & \ddots & \frac{12}{5} & 0 \\ & -\frac{6}{5} & -\frac{h}{10} & \frac{12}{5} & 0 \\ \frac{h}{10} & -\frac{h}{30} & 0 & \frac{4h^2}{15} & \ddots \end{pmatrix}, \quad \mathbb{E} = \frac{EI}{h^4} EI \begin{pmatrix} \ddots & & & & -12 & 6h \\ & \ddots & & & -6h & 2h^2 \\ & & \ddots & & -12 & -6h & 24 & 0 \\ & -12 & -6h & 24 & 0 & & & \\ 6h & 2h^2 & 0 & 8h^2 & \ddots & \ddots & \ddots & \end{pmatrix}$$

and the matrix  $\mathbb{M}(\mu)$  is computed for functions  $\mu$  constants on each interval  $]x_i, x_{i+1}[$  by

$$\mathbb{M} = \begin{pmatrix} \ddots & & & & & & & \\ & \ddots & & & \mu_{i-1/2} \frac{9}{70} & & -\mu_{i-1/2} \frac{13h}{420} & \\ & & \ddots & & \mu_{i-1/2} \frac{13h}{420} & & -\mu_{i-1/2} \frac{h^2}{140} & \\ \mu_{i-1/2} \frac{9}{70} & \mu_{i-1/2} \frac{13h}{420} & (\mu_{i-1/2} + \mu_{i+1/2}) \frac{13}{35} & (-\mu_{i-1/2} + \mu_{i+1/2}) \frac{11h}{210} & \ddots & \ddots & \ddots & \\ -\mu_{i-1/2} \frac{13h}{420} & -\mu_{i-1/2} \frac{h^2}{140} & (-\mu_{i-1/2} + \mu_{i+1/2}) \frac{11h}{210} & (\mu_{i-1/2} + \mu_{i+1/2}) \frac{h}{105} & \ddots & \ddots & \ddots & \end{pmatrix}$$

where  $\mu_{i+1/2}$  denotes the value of  $\mu$  on  $]x_i, x_{i+1}[$ .

Equation (2) is a generalised eigenvalue problem that can be solved using generalised Schur decomposition.

## 3.2 Optimisation algorithm

In the present study, the gradient algorithm is used to find a minimum of the ‘‘inharmonicity’’ function  $C$ . It works well for low stiffness and/or small numbers  $n_{max}$ . When converging, it gives a solution having arbitrarily low inharmonicity for the first  $n_{max}$  harmonics.

It uses the derivative of functions  $\omega_n$  with respect to  $\mu$  for  $n \in \{1, \dots, n_{max}\}$ . The value of this derivative is computed using perturbation theory and is approximated by the formula

$$\text{grad}_\mu \omega_n = \frac{U_n \cdot U_n}{{}^t U_n \mathbb{M}(\mu) U_n}$$

where  $U_n$  is an eigenvector of equation (2) associated to the eigenvalue  $-\omega_n^2$ , and  $U_n \cdot U_n$  is the Hadamard (entrywise) product of  $U_n$  with itself.

The stopping condition is dictated by the inharmonicity of each partial, as it should be at least as good as that of the uniform string and at most  $\leq \varepsilon$ , for a fixed constant  $\varepsilon$  which, in practice, is taken to be  $10^{-3}$ .

## 4 Examples and results with different strategies

Gradient algorithm looks for solutions in a vector space, but most of the elements of this space are physically irrelevant. It is therefore necessary to reduce the search space and add conditions to find useful solutions. In particular, they must verify at least that the density remains bigger than that of the core, as it would otherwise weaken the string. On the other hand it can be interesting to limit the inhomogeneous part in order to make the manufacturing easier. These considerations lead to different strategies which are described below.

In the following the chosen nominal string is a C1 straight piano string corresponding to a frequency of 32.7Hz :  $L = 1.035m$ ,  $\mu = 180g/m$ ,  $T = 825N$  and  $EI = 0.028Nm^2$ , which leads to an inharmonicity coefficient  $B = 3.13 \cdot 10^{-4}$ . Here the string is considered to be strictly uniform on all its length and it is considered that the stiffness is that of the core only. In practice it is likely that the inharmonicity coefficient might be significantly higher ([3]).

### 4.1 Minimum of constraints

This corresponds to the case where the only constraint is that the density is at least equal to that of the wire. Results depend on the number of harmonics which are taken into account in the optimisation. On figure 2 the density for different  $n_{max} = 10, 15, 18$ , is shown. The convergence is considered to be obtained when the maximum of inharmonicity is  $< 10^{-3}$ . The number of steps needed to obtain the convergence increases with the number of harmonics: for  $n_{max} = 10, 15, 18$  the number of steps are respectively  $n_{step} = 332, 848, 1188$ .

It appears that the density fluctuations show a number of valleys equal to the number of harmonics involved in the optimisation. Physically it can be interpreted as a way to slow the waves up to the maximum frequency considered. Another important observation is that density fluctuations are all the more important as the number of harmonics considered is high: the amplitude varies roughly as the square of the harmonic number. This is an important limitation, because this limits the number of harmonics on which inharmonicity can be minimised. On the given example, for  $n_{max} = 18$  the density is locally multiplied by about 2 which means that the diameter of the string is locally increased by 40%. It can also

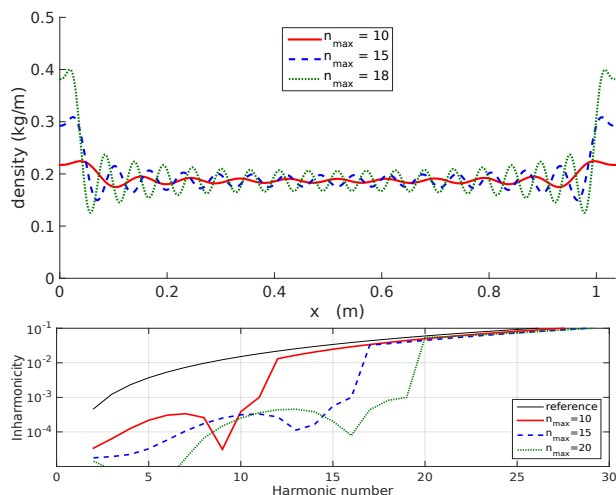


Figure 2: Result of convergence with no strategy for different values of  $n_{max}$ . Top: density profile; bottom: inharmonicity as a function of harmonic number.

be noticed that the fluctuations are more important near the ends of the string which suggests that fluctuations in the middle of the string could be avoided.

It can be seen on figure 2 that the diameter at the ends is much larger than that of the uniform string. In practice this will be probably difficult to manage.

### 4.2 Non negative density fluctuations

The question is now whether it would be possible to only add some masses to a uniform string in order to apply local corrections. On the numerical point of view this is obtained by replacing the gradient by its positive part. The convergence is much slower than for the previous case (4730 steps are needed for a maximum inharmonicity of  $\varepsilon = 10^{-3}$  against 332). However, the comparison of figure 2 and figure 3 shows that it should be possible to avoid valleys without increasing too much the amplitude of the hills.

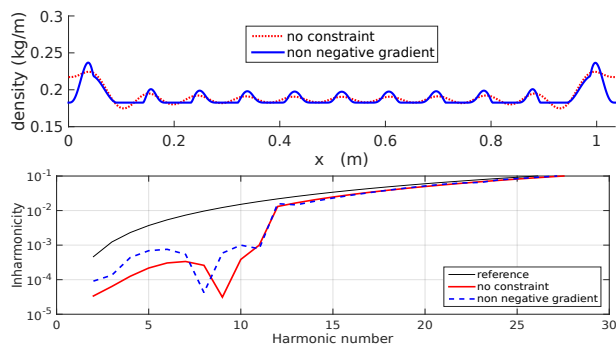


Figure 3: Result of convergence with non negative fluctuation of density and  $n_{max} = 10$ . Top: density profile; bottom: inharmonicity as a function of harmonic number.

### 4.3 No density fluctuation on a central part of given length

In order to limit the area of intervention, an idea is to keep a uniform diameter on a portion of the string that is as large as possible. On figure 4 results are shown for a string that is kept uniform on one half of its length (i.e. non constant on  $[0, a]$  and  $[L - a, L]$ , constant on  $[a, L - a]$  with  $a = L/4$ ). It can be seen that this constraint leads to a concentration of the added mass near the end of the string. It is interesting to notice that only 2 hills and 2 valleys on both sides of the string are sufficient to obtain good results for  $n_{max} = 10$ .

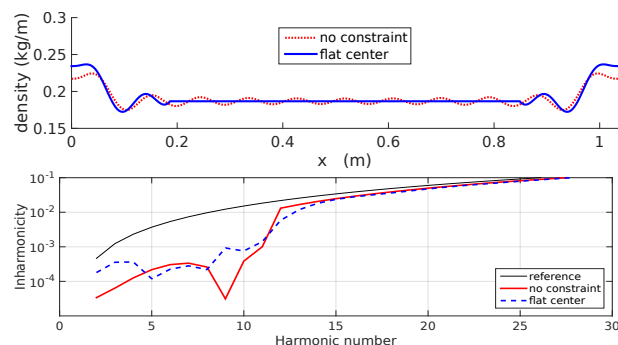


Figure 4: Result of convergence with no fluctuation on one half of the string in the middle and  $n_{max} = 10$ . Top: density profile; bottom: inharmonicity as a function of harmonic number.

### 4.4 One sided density fluctuation

Owing to the symmetrical nature of the problem, for all the previous examples the algorithm converges to a symmetrical solution. However there is a priori no reason to keep a symmetrical string. Moreover, to avoid the hammer hitting the string on a non uniform region it would be good to limit the non uniform region to only one end of the string. Figure 5 shows that similar results are obtained when the fluctuation are concentrated on one side. As one could expect, this tends to increase the amplitude of the fluctuations. Results are given on figure 5 for  $n_{max} = 10$  and  $a = L/4$ . It is noticeable that the result tends to a point mass near the end combined with two periods of a "damped sinusoidal" variation of density.

### 4.5 One sided non negative density fluctuation

Now, the next step is to consider only non negative density fluctuations on a small part of the string. The results given on figure 5 converge to what can be interpreted as small masses. The second mass being rather negligible it seems that two masses are sufficient to

correct the inharmonicity of the first ten harmonics. The first mass is about 5 g and is centered at 3 cm from the end. The third one is 0.4 g and is centered at 24 cm from the end. In practice, the second mass is probably useless. So, it is surprising to realise that for  $n_{max} = 10$  a single mass is probably sufficient to significantly improve the harmonicity of a string.

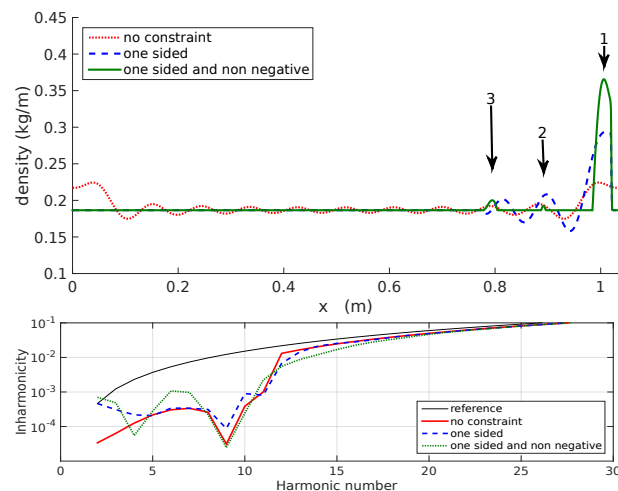


Figure 5: Result of convergence with and without positive density fluctuations on one quarter of the string and  $n_{max} = 10$ . Top: density profile; bottom: inharmonicity as a function of harmonic number.

## 5 Conclusion

The present study shows that it might be possible to build piano strings with an optimised non uniform density leading to a reduced inharmonicity of the first partials. Moreover the non uniform part of the string could be limited to a short portion of the string near the end (25 cm in the given example). Moreover, it appears that a single mass a few centimeters from the end might significantly improve the harmonicity. Now, many questions arise. In practice, how to take into account the effective inharmonicity of the string including the influence of the sound board and that of the winding? On the perceptive point of view, what is the minimum number of harmonics to consider in order to obtain a significant improvement? To answer these questions, the next step is to find a process to optimise the harmonicity on an actual string. Obviously, the final design will be the result of exchanges between piano tuners, pianists and engineers. Finally we are deeply convinced that such harmonic strings will make it possible to highly increase the musical quality of the first two octaves of straight pianos but also that of medium grand pianos.

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